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Harsanyi power solutions for graph-restricted games*

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Abstract

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game. A solution for TU-games assigns a set of payoff distributions (possibly empty or consisting of a unique element) to every TU-game. Harsanyi solutions are solutions that are based on distributing *dividends*. In this paper we consider games with limited communication structure in which the edges or links of an undirected graph on the set of players represent binary communication links between the players such that players can cooperate if and only if they are connected. For such games we discuss Harsanyi solutions whose dividend shares are based on power measures for nodes in corresponding communication graphs. Special attention is given to the Harsanyi degree solution which equals the *Shapley value* on the class of complete graph games (i.e. the class of TU-games) and equals the *position value* on the class of cycle-free graph games. Another example is the Harsanyi power solution that is based on the equal power measure, which turns out to be the Myerson value. Various applications of our results are provided.

Keywords: Cooperative TU-game, Harsanyi dividend, communication structure, power measure, position value, Myerson value, assignment games, auction games

AMS subject classification: 90B18, 91A12, 91A43
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1 Introduction

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair (N, v), where $N \subset \mathbb{N}$ is a finite set of players and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* on N such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is the worth of coalition S, i.e. the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate. Unless stated otherwise, we assume that $N = \{1, \ldots, n\}$, i.e. N is a set of n players, indexed by $i = 1, \ldots, n$.

A *payoff vector* $x \in \mathbb{R}^n$ of an n-player TU-game (N, v) is an n-dimensional vector giving a payoff $x_i \in \mathbb{R}$ to any player $i \in N$. A (single-valued) *solution* for TU-games is a mapping F that assigns to every game (N, v) a payoff vector $f(N, v) \in \mathbb{R}^n$. A payoff vector x for game (N, v) is *efficient* if it exactly distributes the worth $v(N)$ of the 'grand coalition' N, i.e. if $\sum_{i \in N} x_i = v(N)$. A solution f is efficient if the payoff vector $f(N, v)$ is efficient for any TU-game (N, v).

In its classical interpretation, a TU-game describes a situation in which the players in every coalition S of N can cooperate to form a feasible coalition and earn the worth $v(S)$. However, one can add certain restrictions on cooperation. One of the most well-known restrictions are the games with *limited communication structure* in which the members of some coalition S can realize the worth $v(S)$ if and only if they are connected nodes within a given *communication* graph on the set of players. These *graph-restricted* games were first studied in Myerson [24]. Solutions for graph-restricted games usually correspond to modified classical solutions for cooperative games.

In this paper we introduce *Harsanyi power solutions* for graph restricted games which are based on the, so called, *Harsanyi solutions* for TU-games. These Harsanyi solutions are proposed as solutions for TU-games in Vasil'ev [35], [37] (see also Derks, Haller and Peters [10], where a Harsanyi solution is called a *sharing value*). The idea behind a Harsanyi solution is that it distributes the Harsanyi *dividends* (see Harsanyi [16]) over the players in the corresponding coalitions according to a chosen *sharing system* which assigns to every coalition S a sharing vector which specifies for every player in S its share in the dividend of S. The payoff to each player i is thus equal to the sum of its shares in the dividends of all coalitions in which he is a member. A famous Harsanyi solution is the *Shapley value* (Shapley [28]) which distributes the dividend of each coalition equally among the players in that coalition.

In this paper we apply Harsanyi solutions to games with a limited communication graph. The novelty of our approach is that we associate sharing systems with some power measure for communication graphs. A *power measure* for (communication) graphs is a mapping which assigns a nonnegative real number to every node in any (communication) graph. These numbers represent the strength or power of those nodes in the graph. Given a power measure we define the corresponding sharing system such that the share vectors for every coalition are proportional to the power measure of the corresponding subgraphs. The resulting Harsanyi solution is called a *Harsanyi power solution*.

Out of a big variety of possible power measures (and corresponding power solutions), we give special attention to the *degree measure* that assigns to every player in a communication graph the number of players with whom it is directly connected. We show that on the class of cycle-free graph games, the corresponding Harsanyi power solution is equal to the position value, introduced in Borm, Owen and Tijs [4]. Applying the *equal power measure* that assigns equal power to all players, we obtain the Myerson value as introduced in Myerson [24] as the corresponding Harsanyi power solution. After weakening some of the axioms used in [4] to characterize the position- and Myerson value on the class of cycle-free graph games, we generalize these axioms to obtain axiomatic characterizations of all Harsanyi power solutions. Finally we discuss various applications, in particular assignment games, ATM-games and auction games.

The underlying paper is organized as follows. Section 2 is a preliminary section containing cooperative TU-games, communication graphs and (communication) graph games. In Section 3 we discuss the Harsanyi power solution induced by the degree measure. In Section 4 we consider the class of all Harsanyi power solutions and give axiomatic characterizations on the class of cycle-free graph games. In Section 5 we discuss applications. Finally, Section 6 concludes.

2 Preliminaries

2.1 Cooperative TU-games

A characteristic function v is *monotone* if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$. A characteristic function v is *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \subseteq N$. Throughout the paper we assume that any game is *zero-normalized*, i.e. $v(\{i\}) = 0$ for all $i \in N$ and we denote the collection of all zero-normalized characteristic functions on player set N by $\mathcal{G}^N$. A special class of monotone and convex games are unanimity games. For each non-empty $T \subseteq N$, the *unanimity* game (N, u^T) is given by $u^T(S) = 1$ if $T \subseteq S$, and $u^T(S) = 0$ otherwise. It is well-known that the unanimity games form a basis for $\mathcal{G}^N$ and that for each game[1] $v \in \mathcal{G}^N$ we have that

$$v = \sum_{S \in \Omega^N} \Delta^S(v) u^S,$$

where Ω^N is the collection of all non-empty subsets of N, and the Harsanyi *dividends* $\Delta^S(v)$ (see Harsanyi [16]) are given by

$$\Delta^S(v) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T), \quad S \in \Omega^N. \tag{2.1}$$

Equivalently, by applying the Möbius transformation, we have that

$$v(S) = \sum_{T \subseteq S} \Delta^T(v), \quad S \in \Omega^N. \tag{2.2}$$

Observe that every dividend $\Delta^{\{i\}}(v) = 0$, $i \in N$, because of the assumption that each game is zero-normalized.

In this paper we consider so-called *Harsanyi solutions* which have been proposed by Vasil'ev [35], see also Vasil'ev [37], and have been applied recently by van den Brink, van der Laan and Vasil'ev [9] to the class of line-graph games. First, a *sharing system* on N is a system $p = (p^S)_{S \in \Omega^N}$, where p^S is an $|S|$-dimensional vector assigning a nonnegative share p_i^S to every player $i \in S$ with $\sum_{j \in S} p_j^S = 1$, $S \in \Omega^N$. The collection of sharing systems on N is given by

$$P^N = \left\{ p = (p^S)_{S \in \Omega^N} \;\middle|\; p^S \in \mathbb{R}^{|S|} \text{ with } p^S \geq 0 \text{ and } \sum_{j \in S} p_j^S = 1, \text{ for each } S \in \Omega^N \right\}.$$

For a game (N, v) and sharing system $p \in P^N$, let the payoff vector $h^p(N, v) \in \mathbb{R}^n$ be given by

$$h_i^p(N, v) = \sum_{S \in \Omega^N, i \in S} p_i^S \Delta^S(v), \text{ for all } i \in N,$$

i.e. the payoff $h_i^p(N, v)$ to player $i \in N$ is the sum over all coalitions $S \in \Omega^N$, containing i, of the share $p_i^S \Delta^S(v)$ of player i in the Harsanyi dividend of coalition S. We therefore call the payoff vector $h^p(N, v)$ a *Harsanyi payoff vector*. A *Harsanyi solution* is a single-valued solution that assigns for a given sharing system $p \in P^N$ the Harsanyi payoff vector $h^p(N, v)$ to each game (N, v). Observe that, due to the equality $v(N) = \sum_{S \in \Omega^N} \Delta^S(v)$, for each sharing system $p \in P^N$ it holds that $\sum_{i \in N} h_i^p(N, v) = v(N)$, and thus each Harsanyi payoff vector is efficient.

[1] In case there is no confusion about the set of players N we sometimes identify a TU-game (N, v) by its characteristic function v.

An example of a Harsanyi solution is the Shapley value (see Shapley [28]) $\psi(N,v)$, defined by

$$\psi_i(N,v) = \sum_{S \in \Omega^N, i \in S} \frac{1}{|S|}\Delta^S(v), \text{ for all } i \in N,$$

i.e. the Shapley value is the Harsanyi solution that assigns to any game (N,v) the Harsanyi payoff vector which equally distributes the Harsanyi dividend of S over the players in S, i.e. it uses the sharing system p given by $p_i^S = \frac{1}{|S|}$, $S \in \Omega^N,\ i \in S$.

Before we proceed to discuss graph-games, we mention some results on Harsanyi solutions for TU-games. First, in Derks, Haller and Peters [10] a Harsanyi solution is called a *sharing value*. These authors discuss the relationship between the class of sharing values, *random order values*, see Weber [39], and *weighted Shapley values*, see Shapley [28], Kalai and Samet [21], Monderer, Samet and Shapley [22] and Hart and Mas-Colell [19]. In particular we have that for a vector of (positive) weights $\omega \in \mathbb{R}^n_{++}$, the Harsanyi solution given by the sharing system

$$p_i^S = \frac{\omega_i}{\sum_{j \in S} \omega_j}, \quad S \in \Omega^N,\ i \in S$$

is the weighted Shapley value with respect to the weight vector ω. Clearly, the class of weighted Shapley values is a subset of the class of Harsanyi solutions[2].

Second, Harsanyi solutions are related to the set-valued solution[3] that is known as the *Selectope*, see Derks, Haller and Peters [10], or *Harsanyi set*, see Vasil'ev and van der Laan [38], independently introduced by Hammer, Peled and Sorensen [15] and Vasil'ev [34], respectively. This solution assigns to any game the collection of all payoff vectors obtained by distributing the dividend of each coalition S over the players in S in any possible way, and thus is given by

$$H(N,v) = \{h^p(N,v) \mid p \in P^N\}.$$

Clearly, by definition we have that $H(N,v) \neq \emptyset$ for all (N,v), since every Harsanyi solution assigns a Harsanyi payoff vector to any game. In fact, a Harsanyi solution selects for any game the payoff vector in the Harsanyi set corresponding to a fixed sharing system. Note that a solution that always selects a payoff vector from the Harsanyi set need not be a Harsanyi solution, since for different games it might need different sharing systems to obtain a Harsanyi payoff vector.

[2] For consistency of weighted Shapley values, see Hart and Mas-Colell [19] and Derks, Haller and Peters [10].

[3] A set-valued solution for TU-games is a mapping F that assigns to every game (N,v) a set of payoff vectors $F(N,v) \subset \mathbb{R}^n$.

Another well-known set-valued solution is the *Core* (introduced in game theory by Gillies [12]), assigning to every game (N,v) the (possibly empty) set

$$C(N,v)=\left\{x\in\mathbb{R}^n \;\middle|\; \sum_{i\in N}x_i=v(N), \text{ and } \sum_{i\in S}x_i\geq v(S), \quad \text{for each } S\subset N\right\}.$$

It is well-known that $C(N,v)$ is non-empty if and only if v is balanced, see e.g. Bondareva [1] or Shapley [29]. It further holds that $C(N,v)\subseteq H(N,v)$ with equality if and only if v is almost positive[4], see e.g. Derks, Haller and Peters [10] or Vasil'ev and Van der Laan [38].

2.2 Notions in graph theory

An *undirected graph* is a pair (N,L) where N is the set of nodes[5] and L is a collection of edges, i.e. $L\subseteq\{\{i,j\}|i,j\in N,\ i\neq j\}$ is a collection of subsets of N such that each element of L contains precisely two elements of N. Because the elements of L represent the binary communication links between the players, in the sequel we will call them *links* instead of edges. If $\{i,j\}\in L$, then the nodes i and j are *adjacent* (neighboring) to each other and are *incident* with the link $\{i,j\}$. The set of nodes adjacent to i in graph (N,L) is called the *neighborhood* of i and will be denoted by $R_{(N,L)}(i)=\{j\in N\setminus\{i\}\mid\{i,j\}\in L\}$. The number of nodes adjacent to node $i\in N$ in graph (N,L) is known as i's *degree* in (N,L). This yields for every graph (N,L) the degree vector $d(N,L)\in\mathbb{R}^n$ given by $d_i(N,L)=|R_{(N,L)}(i)|$ for all $i\in N$.

A sequence of k different nodes $(i_1,\ldots,i_k)$ is a *path* in (N,L) if $\{i_h,i_{h+1}\}\in L$ for $h=1,\ldots,k-1$. Two nodes $i,j\in N$ are *connected* in graph (N,L) if there exists a path $(i_1,\ldots,i_k)$ with $i_i=i$ and $i_k=j$. A graph (N,L) is *connected* if any two nodes $i,j\in N$ are connected. For some $K\subseteq N$, the graph $(K,L(K))$ with $L(K)=\{l\in L|l\subseteq K\}$ is called a *subgraph* of (N,L). The notions of degree and neighborhood are straightforwardly extended to subgraphs. For given graph (N,L), a set of nodes K is said to be a connected subset of N when the subgraph $(K,L(K))$ is connected. A subset K of N is a *component* of N in (N,L) if the subgraph $(K,L(K))$ is maximally connected, i.e. $(K,L(K))$ is connected and for any $j\in N\setminus K$, the subgraph $(K\cup\{j\},L(K\cup\{j\}))$ is not connected. Clearly, for any graph (N,L), the collection of components of N forms a unique partition of N.

We introduce the following notation. For a graph (N,L) and set $K\subseteq N$, we denote by $C(K)$ the collection of all connected subsets of K in the subgraph $(K,L(K))$. Observe that for a subset K' of K, the subgraph $(K',L(K'))$ is a connected subgraph of $(K,L(K))$

[4]A TU-game is almost positive if $\Delta^S(v)\geq 0$ when $|S|\geq 2$.

[5]Since in this paper the nodes in a graph represent the players in a game we use the same notation for the set of nodes as the set of players.

if and only if it is a connected subgraph of (N, L). Hence

$$C(K) = \{K' \subseteq K \mid (K', L(K')) \text{ is a connected subgraph of } (N, L)\}.$$

Further, we denote by $C_m(K)$ the collection of all maximally connected subgraphs of $(K, L(K))$, i.e.

$$C_m(K) = \{K' \subseteq K \mid K' \text{ is a component of } K \text{ in } (K, L(K))\}.$$

Notice that a maximally connected subgraph of the subgraph $(K, L(K))$ does not need to be a maximally connected subgraph of (N, L). In fact, it follows straightforward from above that the collection $C_m(K)$ can also be written as

$$C_m(K) = \{K' \subseteq K \mid K' \in C(K) \text{ and } K' \cup \{j\} \notin C(K) \text{ for any } j \in K \setminus K'\}. \quad (2.3)$$

A sequence of nodes $(i_1, \ldots, i_{k+1})$ is a *cycle* in (N, L) if (i) $k \geq 3$, (ii) all nodes $i_1, \ldots, i_k$ are different elements of N, (iii) $i_{k+1} = i_1$ and (iv) $\{i_h, i_{h+1}\} \in L$ for $h = 1, \ldots, k$. A graph (N, L) is *cycle-free* when it does not contain any cycle. Finally, the *complete graph* on N is the graph (N, L^c) with $L^c = \{\{i, j\} \mid i, j \in N, \ i \neq j\}$ in which all nodes are adjacent to each other. For more notions on communication graphs and general graphs we refer to, respectively, van den Nouweland [25] and Harary [18].

2.3 Graph-restricted games and solutions

In this paper we assume the players in a cooperative TU-game (N, v) to be part of a communication structure that is represented by an undirected graph (N, L), with the player set N as the set of nodes and the collection L as the set of links representing the binary communication links between the players. We denote the class of all possible sets of links on N by $\mathcal{L}^N$ (i.e. if $L \in \mathcal{L}^N$ then (N, L) is a graph). Further, $\mathcal{L}^N_{CF}$ denotes the class of all sets $L \in \mathcal{L}^N$ such that (N, L) is a cycle-free graph on N. A game (N, v) with *communication graph* (N, L) is denoted shortly by (N, v, L) and is referred to as a *graph game*. In the graph game (N, v, L) players can cooperate if and only if they are able to communicate with each other, i.e. a coalition S can realize its worth $v(S)$ if and only if $(S, L(S))$ is a connected subgraph of (N, L). Whenever this is not the case, players in S can only realize the sum of the worths of the components of $(S, L(S))$. As introduced by Myerson [24], this yields the restricted game (N, v^L) given by

$$v^L(S) = \sum_{T \in C_m(S)} v(T), \quad S \subseteq N. \quad (2.4)$$

Borm, Owen and Tijs [4] refer to the restricted game (N, v^L) as the *point game* corresponding to (N, v, L). They also introduce the *link game* (L, r^L), whose set of players is the set of links L and whose characteristic function gives for every subset $E \subseteq L$ of links the worth that the 'grand coalition' N of the players in the game (N, v) earns when E is the set of all communication links[6]. So, the link game (L, r^L) induced by the graph game (N, v, L) is given by

$$r^L(E) = v^E(N) \text{ for all } E \subseteq L. \tag{2.5}$$

Two well-known single-valued solutions for graph games are the Myerson value and the position value. The *Myerson value* (Myerson [24]) of graph game (N, v, L), denoted by $\mu(N, v, L)$, is obtained by taking the Shapley value of the restricted game (N, v^L), i.e.

$$\mu(N, v, L) = \psi(N, v^L), \text{ for all } v \in \mathcal{G}^N \text{ and } L \in \mathcal{L}^N.$$

The *position value* (see [4]) of graph game (N, v, L), denoted by $\pi(N, v, L)$, is defined in two steps. First, the Shapley value of the link game (L, r^L) is determined. Second, the Shapley value of each link is distributed equally among the nodes incident with it. So,

$$\pi_i(N, v, L) = \sum_{l \in L_i} \frac{1}{2} \psi_l(L, r^L) \text{ for all } i \in N,$$

with $L_i = \{\{h, j\} \in L \mid i \in \{h, j\}\}$ and $\psi_l(L, r^L)$ being the Shapley value of the link l in the link game (L, r^L). In [4] a characterization is given for both the Myerson- and the position value on the class of cycle-free graph games. To give the axioms for these characterizations, we need the following three notions. First, link $l \in L$ is called *superfluous* in (N, v, L) if $v^E(N) = v^{E \cup \{l\}}(N)$ for all $E \subseteq L$. Second, graph game (N, v, L) is said to be *link anonymous* if there exists a function $g^L \colon \{0, 1, \ldots, |L|\} \to \mathbb{R}$ such that $r^L(E) = g^L(|E|)$ for all $E \subseteq L$, i.e. in the corresponding link game the value of a coalition of links depends only on the number of links in the coalition. Third, graph game (N, v, L) is called *point anonymous* if there exists a function $g^P \colon \{1, ..., |D(N, L)|\} \to \mathbb{R}$ such that $v^L(S) = g^P(|S \cap D(N, L)|)$ for all $S \subseteq N$, where $D(N, L) = \{i \in N \mid R_{(N,L)}(i) \neq \emptyset\}$ denotes the set of *non-isolated* nodes in (N, L), i.e. in the restricted (point) game the value of a coalition of players depends only on the number of non-isolated players in the coalition. We now state the following five axioms for a solution f on the class of graph games.

Component efficiency For every graph game (N, v, L) and every component S of N in (N, L) it holds that $\sum_{i \in S} f_i(N, v, L) = v(S)$.

Additivity For every pair of graph games $(N, v, L), (N, w, L)$ it holds that[7] $f(N, v + w, L) = f(N, v, L) + f(N, w, L)$.

[6]In [4] this game is called the arc game. Here we we follow e.g. Slikker [31] and call this game the link game.

[7]For two characteristic functions $v, w \in \mathcal{G}^N$ we define $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$.

Superfluous link property If $l \in L$ is a superfluous link in graph game (N, v, L), then $f(N, v, L) = f(N, v, L \setminus \{l\})$.

Degree measure property If graph game (N, v, L) is link anonymous, then there is an $\alpha \in \mathbb{R}$ such that $f(N, v, L) = \alpha d(N, L)$.[8]

Communication ability property If graph game (N, v, L) is point anonymous, then there is an $\alpha \in \mathbb{R}$ such that $f_i(N, v, L) = \alpha$ for all $i \in D(N, L)$, and $f_i(N, v, L) = 0$ for all $i \in N \setminus D(N, L)$.

For the proof of the following results, we refer to [4].

Proposition 2.1
(i) *The position value satisfies component efficiency, additivity, the superfluous link property and the degree measure property on the class of all graph games. Moreover, it is the unique solution on the class of cycle-free graph games satisfying these four properties.*
(ii) *The Myerson value satisfies component efficiency, additivity, the superfluous link property and the communication ability property on the class of all graph games. Moreover, it is the unique solution on the class of cycle-free graph games satisfying these four properties.*

The proposition states that both the position value and the Myerson value satisfy four of the five axioms on the class of all graph games, and that both values are the unique solution satisfying the four properties on the class of cycle-free graph games. Observe that the position value does not satisfy the communication ability property and that the Myerson value does not satisfy the degree measure property.

We conclude this section by weakening the latter two properties by replacing the link (respectively point) anonymity by link (point) unanimity. A graph game (N, v, L) is called *link unanimous* if it is link anonymous with $g^L(k) = v^L(N)$ if $k = |L|$, and $g^L(k) = 0$ for $k \in \{0, 1, \ldots, |L| - 1\}$, i.e. in the link game the worth of any coalition not containing all links is zero. A graph game (N, v, L) is *point unanimous* if it is point anonymous with $g^P(k) = v^L(N)$ if $k = |D(N, L)|$, and $g^P(k) = 0$ for $k \in \{1, ..., |D(N, L)| - 1\}$, i.e. in the point game the worth of any coalition not containing all non-isolated nodes is zero. We now have the next two axioms for a solution f.

Weak degree measure property If graph game (N, v, L) is link unanimous then there is an $\alpha \in \mathbb{R}$ such that $f(N, v, L) = \alpha d(N, L)$.

[8] Recall that $d(N, v)$ is the degree vector.

Weak communication ability property If graph game (N, v, L) is point unanimous then there is an $\alpha \in \mathbb{R}$ such that $f_i(N, v, L) = \alpha$ for all $i \in D(N, L)$, and $f_i(N, v, L) = 0$ for all $i \in N \setminus D(N, L)$.

It can be verified from the proofs given in [4] that in the two characterization statements (i) and (ii) of Proposition 2.1 the degree measure, respectively communication ability property can be replaced by the weaker properties stated above. Of course, the stronger properties still hold on the class of all graph games. In Section 4 we use these weaker variants to characterize a class of of Harsanyi solutions for cycle-free graph games containing both the Myerson and the position value.

3 The Harsanyi degree solution

Given a set of nodes N, a *power measure* on N is a function which assigns to any $S \subseteq N$ a nonnegative vector $\sigma(S, L(S)) \in \mathbb{R}_+^{|S|}$, yielding the nonnegative power $\sigma_i(S, L(S))$ of node $i \in S$ in the subgraph $(S, L(S))$. A *Harsanyi power solution* for graph games is a Harsanyi solution applied to the restricted game v^L such that the shares in the Harsanyi dividends are determined by some power measure for graphs. We first discuss the *Harsanyi degree solution*, being the power solution obtained by applying the degree measure d, which assigns to every subgraph $(S, L(S))$ of (N, L) the degree vector $d(S, L(S))$. The corresponding Harsanyi degree solution φ^d is

$$\varphi^d(N, v, L) = h^{p^d}(N, v^L)$$

with sharing system $p^d = (p^{d,S})_{S \in \Omega^N}$ given by

$$p_i^{d,S} = \frac{d_i(S, L(S))}{\sum_{j \in S} d_j(S, L(S))}, \text{ for } i \in S \text{ whenever } \sum_{j \in N} d_j(S, L(S)) \neq 0,$$

and $p_i^{d,S} = \frac{1}{|S|}$ for all $i \in S$ whenever $\sum_{j \in N} d_j(S, L(S)) = 0$. In fact, in the latter case the shares do not matter[9]. So, we distribute the (non-zero) dividends in the restricted game (N, v^L) proportional to the degree of the players in the corresponding subgraphs. Clearly, the Harsanyi degree solution yields the Shapley value on the class of complete graph games (which then also equals the Myerson value).

Proposition 3.1 *If $L = L^c$, then $\varphi^d(N, v, L) = \psi(N, v)$ for all $v \in \mathcal{G}^N$.*

[9]Explicit formulas for the dividends in the restricted game (N, v^L) in terms of the original game (N, v) can be found in Owen [27] for cycle-free graphs and Hamiache [17] for arbitrary graphs. In particular it holds that any unconnected coalition has zero dividend in the restricted game, implying that the dividend of a coalition S is zero if $\sum_{j \in N} d_j(S, L(S)) = 0$.

Proof

The proposition follows straightforward since (i) $v^L = v$ when $L = L^c$ and (ii) $d_i(S, L^c(S)) = |S| - 1$, and thus $p_i^{d,S} = \frac{1}{|S|}$ for all $i \in S \in \Omega^N$. □

It is also straightforward to verify that φ^d is component efficient and additive and also satisfies the (weak) degree measure property on the class of all graph games. However, it does not satisfy the superfluous link property for all graph games. Instead, it satisfies the weaker inessential link property. A link $l \in L$ is called *inessential* in graph game (N, v, L) if

$$\Delta^S(v^L) = 0, \text{ for each } S \in \Omega_l^N,$$

where Ω_l^N is the set of all non-empty subsets of N containing both nodes incident with link l, i.e. $S \in \Omega_l^N$ if and only if $l \subseteq S$.

Inessential link property If $l \in L$ is an inessential link in graph game (N, v, L), then $f(N, v, L) = f(N, v, L \setminus \{l\})$.

Lemma 3.2 *The Harsanyi degree solution satisfies the inessential link property.*

Proof

To prove the lemma, we first show that $v^{L\setminus l} = v^L$ when $l \in L$ is inessential, where $L \setminus l$ denotes $L \setminus \{l\}$. Using equation (2.4), we have that

$$v^L(S) = \sum_{T \in C_m^L(S)} v(T) \text{ and } v^{L\setminus\{l\}}(S) = \sum_{T \in C_m^{L\setminus\{l\}}(S)} v(T), \quad S \subseteq N. \tag{3.6}$$

When $S \notin \Omega_l^N$, then $C_m^{L\setminus l}(S) = C_m^L(S)$ and thus $v^{L\setminus l}(S) = v^L(S)$. Next consider the case that $S \in \Omega_l^N$ and let T^l be the component in $C_m^L(S)$ containing l. Since $C_m^L(S) \setminus T^l = C_m^L(S \setminus T^l)$ we have that the equations in (3.6) become

$$v^L(S) = v(T^l) + \sum_{T \in C_m^L(S\setminus T^l)} v(T), \quad S \subseteq N. \tag{3.7}$$

and

$$v^{L\setminus l}(S) = \sum_{T \in C_m^{L\setminus l}(S)} v(T) = \sum_{T \in C_m^{L\setminus l}(T^l)} v(T) + \sum_{T \in C_m^{L\setminus l}(S\setminus T^l)} v(T), \quad S \subseteq N. \tag{3.8}$$

Clearly the second right-hand term in equation (3.8) is equal to the second right-hand term of equation (3.7). So, it remains to show that

$$v(T^l) = \sum_{T \in C_m^{L\setminus l}(T^l)} v(T). \tag{3.9}$$

If T^l is connected in $(S, L(S) \setminus \{l\})$, then $C_m^{L\setminus l}(T^l) = \{T^l\}$ and thus equation (3.9) holds. If T^l is not connected, then $C_m^{L\setminus l}(T^l)$ contains precisely two subsets of T^l, say T^1 and T^2, because T^l is connected in $(S, L(S))$ and only link l is removed. Clearly, any of these three sets is connected in (N, L) and thus $v(R) = v^L(R)$ for $R \in \{T^l, T^1, T^2\}$. Now, suppose that

$$v(T^l) = v^L(T^l) \neq v^L(T^1) + v^L(T^2) = v(T^1) + v(T^2).$$

Since $T^1 \cup T^2 = T^l$, it then follows from the 'dividend' equation (2.2) that there exists at least one subset $R \subseteq T^l$ containing the nodes incident with link l such that $\Delta^R(v^L) \neq 0$, contradicting that link l is inessential. Hence $v(T^l) = v(T^1) + v(T^2)$, implying that $v^L = v^{L\setminus l}$.

To complete the proof, from $v^L = v^{L\setminus l}$ it follows that

$$\Delta^S(v^L) = \Delta^S(v^{L\setminus l}), \quad S \in \Omega^N.$$

Then for $S \notin \Omega_l^N$, we have that $L \setminus \{l\}(S) = L(S)$ and so $d_i(S, L(S)) = d_i(S, L \setminus \{l\}(S))$ for all $i \in S$, implying that the share of $i \in S$ in $\Delta^S(v^L)$ is equal to the share of i in $\Delta^S(v^{L\setminus l})$. When $S \in \Omega_l^N$, we have that $\Delta^S(v^L) = \Delta^S(v^{L\setminus l}) = 0$, so that the shares don't matter. Hence

$$\varphi^d(N, v, L) = \varphi^d(N, v, L \setminus \{l\}),$$

which proves the lemma. □

As was stated before, link $l \in L$ is superfluous if $v^E(N) = v^{E\cup\{l\}}(N)$ for all $E \subseteq L$. Since it is assumed that the games are zero-normalized, it follows straightforward from the definition of the restricted game that this condition holds if and only if $v^L = v^{L\setminus\{l\}}$. Hence, the next corollary follows immediately from the proof of Lemma 3.2.

Lemma 3.3 *If link $l \in L$ is inessential in graph game (N, v, L) then it is also superfluous in (N, v, L).*

Since the reverse is not true, Lemma 3.2 does not imply that the Harsanyi degree solution satisfies the superfluous link property, see also Example 5.4 on assignment games. However, if the graph is cycle-free, then link l is superfluous if and only if l is inessential.

Lemma 3.4 *If $L \in \mathcal{L}_{CF}^N$ then $l \in L$ is inessential in graph game (N, v, L) if and only if l is superfuous in (N, v, L).*

Proof

Because any inessential link is superfluous, we only need to proof that in a cycle-free graph game any superfluous link is inessential. Let l be a superfluous link in cycle-free graph game (N, v, L). Then $v^L = v^{L\setminus\{l\}}$ and thus $\Delta^S(v^L) = \Delta^S(v^{L\setminus\{l\}})$ for any S. Moreover, since (N, L) is cycle-free, we have that in graph $(N, L \setminus \{l\})$, every coalition $S \in \Omega_l^N$ is unconnected. Since any unconnected coalition has zero dividend in the restricted game (see Owen [27] or Hamiache [17]), we have that $\Delta^S(v^{L\setminus\{l\}}) = 0$ for any $S \in \Omega_l^N$, and thus it follows that $\Delta^S(v^L) = 0$ for any $S \in \Omega_l^N$. Hence l is inessential in (N, v, L). □

From Lemma 3.4 it follows immediately that the Harsanyi degree solution equals the position value on the class of cycle-free graph games.

Proposition 3.5 *If $L \in \mathcal{L}_{CF}^N$ then $\varphi^d(N, v, L) = \pi(N, v, L)$ for any $v \in \mathcal{G}^N$.*

Proof

From Proposition 2.1 we have that on the class of cycle-free graph games the position value is characterized by component efficiency, additivity, the degree measure property and the superfluous link property. It is straightforward to verify that φ^d satisfies the first three properties on the class of of all graph games, while Lemmas 3.2 and 3.4 show that it satisfies the superfluous link property on the class of cycle-free graph games. Hence the Harsanyi degree solution is equal to the position value on this class. □

This proposition shows that to define and compute the position value for cycle-free graph games we do not need to introduce the link game as done in [4], since it is a Harsanyi solution applied to the Myerson restricted (point) game. For arbitrary graph games the position value is not a Harsanyi solution and the characterization in [4] does not work either. In fact, the position value even may give a payoff vector outside the Harsanyi set of the corresponding restricted game, implying that the position value may differ from the Harsanyi degree solution if the graph contains a cycle.

Proposition 3.6 *Let $L \in \mathcal{L}^N$. Then $\pi(N, v, L) \in H(N, v^L)$ for all $v \in \mathcal{G}^N$ if and only if $L \in \mathcal{L}_{CF}^N$.*

Proof

The 'if' part follows from Proposition 3.5 and the fact that by definition $\varphi^d(N, v, L)$ is a Harsanyi payoff vector of (N, v^L) for any $L \in \mathcal{L}^N$.

To prove the 'only if' part, suppose that $L \in \mathcal{L}^N \setminus \mathcal{L}_{CF}^N$. Then (N, L) contains a minimal cycle, i.e. for some $k \geq 3$ there is a cycle $(i_1, \ldots, i_{k+1})$ such that $\{i_j, i_m\} \in L$ if and only if $m = j + 1$, $j = 1, \ldots, k$. Take $v = u^{\{i_1, i_2\}}$ being the unanimity game of two neighboring nodes in the cycle. Since $\{i_1, i_2\} \in L$, we have that $v^L = v = u^{\{i_1,i_2\}}$, and thus all $i \in N \setminus \{i_1, i_2\}$ are null players in (N, v^L), i.e. $v^L(S) = v^L(S \setminus \{i\})$ for all $i \in N \setminus \{i_1, i_2\}$. Since null players earn a zero payoff in any Harsanyi payoff vector we have

$$x_i = 0 \text{ for all } x \in H(N, v^L) \text{ and } i \in N \setminus \{i_1, i_2\}. \tag{3.10}$$

However, since the cycle $(i_1, \ldots, i_{k+1})$ is minimal we have that $r^L(E) - r^L(E \setminus \{i_2, i_3\}) = 1$ for $E = \{\{i_j, i_{j+1}\} \mid j = 2, \ldots, k\}$. Hence the link $\{i_2, i_3\}$ is not a null player in the link game (L, r^L). Moreover, since $v = u^{\{i_1,i_2\}}$ is monotone, it follows that also r^L is monotone, and thus the Shapley value of the link game satisfies $\psi_{\{i_2,i_3\}}(L, r^L) > 0$ and $\psi_l(L, r^L) \geq 0$ for all $l \in L$. But then $\pi_{i_3}(N, v, L) \geq \frac{1}{2}\psi_{\{i_2,i_3\}}(L, r^L) > 0$. With (3.10) it then follows that $\pi(N, v, L) \notin H(N, v^L)$. □

We end this section by giving an example which motivates that the Harsanyi degree solution may be preferred above the position value and the Myerson value for certain graph games.

Example 3.7 Consider the graph game (N, v, L) with $N = \{1, 2, 3, 4\}$, $v = u^{\{1,2,3\}}$ and $L = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}\}$.[10] The payoffs assigned to this graph game by the position value, the Myerson value and the Harsanyi degree solution, respectively, are

$$\pi(N, v, L) = \frac{1}{24}(11, 4, 7, 2), \ \mu(N, v, L) = \frac{1}{3}(1, 1, 1, 0) \text{ and } \varphi^d(N, v, L) = \frac{1}{4}(2, 1, 1, 0).$$

In this graph game player 1 clearly has a more central position than players 2 and 3. Since $v = u^{\{1,2,3\}}$, it is therefore reasonable that player 1 should have a higher payoff than players 2 and 3. The Myerson value does not satisfy this requirement. Further, player 4 is a null player in the restricted game v^L and should therefore receive a zero payoff. However, according the position value player 4 gets a positive payoff. From these three solutions only the Harsanyi degree solution meets both requirements that player 1 receives more than the players 2 and 3 and that player 4 gets zero payoff. □

4 Axiomatizations of Harsanyi power solutions for cycle-free graph games

4.1 Harsanyi power solutions

In this section we generalize the Harsanyi degree solution by allowing to take any arbitrarily given power measure yielding in any $(S, L(S))$ positive power to any node in S having at least one neighbor in $(S, L(S))$. In the following we call such a measure a *positive* power

[10] This graph game is taken from [4], Example 6.1.

measure. Given a set of players N and a positive power measure σ on N, we now define the corresponding Harsanyi power solution for graph games as the solution

$$\varphi^\sigma(N, v, L) = h^{p^\sigma}(N, v^L)$$

with sharing system $p^\sigma = (p^{\sigma,S})_{S \in \Omega^N}$ given by[11]

$$p_i^{\sigma,S} = \frac{\sigma_i(S, L(S))}{\sum_{j \in S} \sigma_j(S, L(S))} \text{ for all } i \in S \text{ whenever } \sum_{j \in N} \sigma_j(S, L(S)) \neq 0.$$

So, we distribute any (non-zero) dividend of a coalition in the restricted game (N, v^L) proportional to the power measure of the corresponding subgraph. In the previous section we already discussed the Harsanyi degree solution which is based on the degree measure. Examples of some other positive power measures for undirected graphs are the following[12].

1. The *β-measure* is given by

$$\beta_i(S, L(S)) = \sum_{j \in R_{(S,L(S))}(i)} \frac{1}{|R_{(S,L(S))}(j)|}, \text{ for all } i \in S \text{ and } S \subseteq N.$$

This measure is introduced in van den Brink and Gilles [8] for directed graphs, and applied to undirected graphs in Borm, van den Brink, Hendrikx and Owen [3]. According to the β-measure, any node i is assigned an amount $\frac{1}{|R_{(S,L(S))}(j)|}$ of power from each of its neighbours, or equivalently, each node distributes one unit of power equally amongst its neighbours (if any).

2. The *positional power measure*[13] is introduced in Herings, van der Laan and Talman [20] for directed graphs. Applied to undirected graphs it yields the power measure σ given by

$$\sigma_i(S, L(S)) = |R_{(S,L(S))}(i)| + \frac{1}{|S|} \sum_{j \in R_{(S,L(S))}(i)} \sigma_j(S, L(S)), \text{ for all } i \in S \text{ and } S \subseteq N,$$

i.e. the positional power of node i is equal to the number of neighbours of i (as in the degree measure) plus a fraction $\frac{1}{|S|}$ of the total power of its neighbours[14].

---

[11]Recall that unconnected coalitions have zero dividend, so the shares $p^{\sigma,S}$ do not matter when $\sum_{j \in N} \sigma_j(S, L(S)) = 0$.

[12]Other examples are centrality measures as considered in, e.g. Monsuur and Storcken [23].

[13]Despite its name, the positional power measure is not related to the position value.

[14]Observe that for $S \subseteq N$, this power measure requires to solve an $|S|$-dimensional system of equations. It is shown in Herings, van der Laan and Talman [20] that this system has a unique non-negative solution (with positive numbers for the nodes having at least one neighbour) and therefore the measure is well-defined.

3. The *equal power measure* is the straightforward power measure given by $\gamma_i(S, L(S)) = \frac{1}{|S|}$ for all $i \in S$ and $S \subseteq N$. This measure gives equal power to every node irrespective of the links in the graph.

It follows straightforward that for the equal power measure γ the corresponding Harsanyi power solution is the Myerson value.

Proposition 4.1 *For every graph game (N, v, L) it holds that $\varphi^\gamma(N, v, L) = \mu(N, v, L)$.*

Proof

For graph game (N, v, L) we have that

$$\begin{aligned}\varphi_i^\gamma(N, v, L) &= h_i^{p^\gamma}(N, v^L) = \sum_{S \subseteq N, i \in S} \frac{\gamma_i(S, L(S))}{\sum_{j \in S} \gamma_j(S, L(S))} \Delta^S(v^L) \\ &= \sum_{S \subseteq N, i \in S} \frac{1}{|S|} \Delta^S(v^L) = \psi_i(N, v^L) = \mu(N, v, L).\end{aligned}$$

□

4.2 An axiomatization using the weak σ-measure property

Next, we consider cycle-free graph games. On this class of graph games we generalize the characterization of the position value as given by Borm, Owen and Tijs [4] (see (i) of Proposition 2.1) but with the degree measure property replaced by the weak degree measure property. In order to do this, we adapt the weak degree measure property for any positive power measure σ on N.

Weak σ-measure property If graph game (N, v, L) is link unanimous then there is an $\alpha \in \mathbb{R}$ such that $f(N, v, L) = \alpha\sigma(N, L)$.

To show that φ^σ satisfies the weak σ-measure property we first prove the following lemma (recall that $D(N, L)$ denotes the set of non-isolated nodes in (N, L)).

Lemma 4.2 *If (N, v, L) is link unanimous, then $v^L = cu^{D(N,L)}$ for some $c \in \mathbb{R}$.*

Proof

If $D(N, L) = \emptyset$, then $L = \emptyset$ and thus v^L is the null-game, i.e. $v^L(S) = 0$ for all $S \subseteq N$. Clearly, then $v^L = cu^{D(N,L)}$ with $c = 0$.[15]

[15] Although we did not define the unanimity game of the empty set, multiplying it by zero yields the null-game.

Next, consider the case that $D(N,L) \neq \emptyset$ and let (N,v,L) be a link unanimous graph game, so $v^{L\setminus\{l\}}(N) = 0$ for all $l \in L$. Then v^L satisfies the following two sufficient properties.

1. Let $S \subset N$ be such that $D(N,L) \not\subseteq S$. Take $i \in D(N,L) \setminus S$. Denoting by $C'_m(N)$ the set of components in $(N, L(S))$ we have that

$$v^{L(S)}(N) = \sum_{T \in C'_m(N)} v(T) \; = \; \sum_{T \in C'_m(S)} v(T) + \sum_{i \in N \setminus S} v(\{i\}) \; = \; v^{L(S)}(S) \; = \; v^L(S),$$

where the second equality follows from the fact that all nodes outside S are singletons in the set of components in $(N, L(S))$ and the third equality follows from v being zero-normalized. (The other equalities follow from the definition of the restricted game.) From (N,v,L) being link unanimous it further follows that $v^{L(S)}(N) = 0$ since $L(S)$ is a proper subset of L because S does not contain $D(N,L)$. So, we conclude that

$$v^L(S) = v^{L(S)}(N) = 0 \;\text{ if }\; D(N,L) \not\subseteq S. \tag{4.11}$$

2. Let $S, T \subseteq N$ be such that $D(N,L) \subseteq S$ and $D(N,L) \subseteq T$. Then $D(N,L) \subseteq S \cap T$. By definition of v^L we have that

$$v^L(S) = \sum_{Z \in C_m(S)} v(Z) = \sum_{Z \in C_m(S), |Z|=1} v(Z) + \sum_{Z \in C_m(S), |Z| \geq 2} v(Z).$$

Since v is zero-normalized (and thus the first sum is zero) and $Z \subseteq D(N,L)$ if $|Z| \geq 2$ and is a component of S in $(S, L(S))$, this equation becomes

$$\begin{aligned} v^L(S) \;&=\; \sum_{Z \in C_m(S), Z \subseteq D(N,L)} v(Z) = \sum_{Z \in C_m(S), Z \subseteq S \cap T} v(Z) - \sum_{Z \in C_m(S), Z \subseteq (S \cap T), Z \not\subseteq D(N,L)} v(Z) \\ &=\; \sum_{Z \in C_m(T), Z \subseteq D(N,L)} v(Z) \;=\; v^L(T). \end{aligned}$$

So,

$$v^L(S) = v^L(T) \;\text{ if }\; D(N,L) \subseteq S \;\text{ and }\; D(N,L) \subseteq T. \tag{4.12}$$

The lemma follows from the properties (4.11) and (4.12). □

The next proposition characterizes φ^σ on the class of cycle-free graph games.

Proposition 4.3 *For a positive power measure σ, the Harsanyi power solution φ^σ satisfies component efficiency, additivity, the inessential link property and the weak σ-measure property on the class of all graph games. Moreover, it is the unique solution on the class of cycle-free graph games satisfying these four properties.*

Proof

First we show the first part of the proposition.

1. Since $\Delta^S(v^L) = 0$ if S is not connected in (N, L), for every component $T \in C_m(N)$ in (N, L) we have

$$\begin{aligned} \sum_{i \in T} \varphi_i^\sigma(N, v, L) &= \sum_{i \in T} h_i^{p^\sigma}(N, v^L) = \sum_{i \in T} \sum_{S \subseteq N, i \in S} p_i^{\sigma,S} \Delta^S(v^L) \\ &= \sum_{i \in T} \sum_{S \subseteq T, i \in S} p_i^{\sigma,S} \Delta^S(v^L) = \sum_{S \subseteq T} \sum_{i \in S} p_i^{\sigma,S} \Delta^S(v^L) \\ &= \sum_{S \subseteq T} \Delta^S(v^L) = v^L(T), \end{aligned}$$

showing that φ^σ satisfies component efficiency.

2. For $v, w \in \mathcal{G}^N$ and $L \in \mathcal{L}^N$ it holds that

$$(v+w)^L(S) = \sum_{T \in C_m(S)} (v+w)(T) = \sum_{T \in C_m(S)} (v(T) + w(T)) = v^L(S) + w^L(S)$$

and thus $\Delta^S((v+w)^L) = \Delta^S(v^L) + \Delta^S(w^L)$ for all $S \subseteq N$. Then

$$\begin{aligned} \varphi_i^\sigma(N, v+w, L) &= \sum_{S \subseteq N, i \in S} p_i^{\sigma,S} \Delta^S((v+w)^L) \\ &= \sum_{S \subseteq N, i \in S} p_i^{\sigma,S} (\Delta^S(v^L) + \Delta^S(w^L)) = \varphi_i^\sigma(N, v, L) + \varphi_i^\sigma(N, w, L), \end{aligned}$$

showing that φ^σ satisfies additivity.

3. Analogously as to the proof of Lemma 3.2, it follows that φ^σ satisfies the inessential link property.

4. If (N, v, L) is link unanimous then by Lemma 4.2 we have that $v^L = cu^{D(N,L)}$ for some $c \in \mathbb{R}$. Then the weak σ-measure property is satisfied by definition of φ^σ.

For the second part of the proposition, recall from Lemma 3.4 that in a cycle-free graph game a link is inessential if and only if it is superfluous and thus φ^σ satisfies the superfluous link property on the class of cycle-free graph games. Then the proof that φ^σ is the unique solution satisfying the four properties is similar to the uniqueness proof for the position value in [4] and is therefore omitted. □

Note that φ^σ satisfies component efficiency, additivity, the weak σ-measure property and the inessential link property for any graph game, but that these four properties characterize φ^σ only on the class of cycle-free graph games. Also note that the Propositions 4.1 and 4.3 imply the following corollary, yielding a characterization of the Myerson value on the class cycle-free graph games. Observe that the weak equal power measure property states that in a link unanimous graph game the payoffs of the non-isolated players are equal.

Corollary 4.4 *The Myerson value satisfies component efficiency, additivity, the inessential link property and the weak equal power measure property on the class of all graph games. Moreover, it is the unique solution on the class of cycle-free graph games satisfying these four properties.*

In case the power measure is symmetric[16] the corresponding Harsanyi power solution φ^σ extends the Shapley value to the class of graph games in the sense that it yields the Shapley value of game (N, v) whenever the graph (N, L^c) is the complete graph.

Proposition 4.5 *If σ is symmetric then $\varphi^\sigma(N, v, L^c) = \psi(N, v)$ for all $(N, v) \in \mathcal{G}^N$.*

PROOF
If σ is symmetric then $\frac{\sigma_i(S,L^c(S))}{\sum_{j\in S}\sigma_j(S,L^c(S))} = \frac{1}{|S|}$ for all $i \in S \subseteq N$. Moreover, $\Delta^S(v) = \Delta^S(v^{L^c})$ for all $S \in \Omega^N$. Thus $\varphi_i^\sigma(N, v, L^c) = \sum_{S\subseteq N, i\in S} \frac{\sigma_i(S,L^c(S))}{\sum_{j\in S}\sigma_j(S,L^c(S))}\Delta^S(v^{L^c}) = \sum_{S\subseteq N, i\in S} \frac{1}{|S|}\Delta^S(v) = \psi_i(N, v)$ for all $i \in N$. □

4.3 An axiomatization using the weak σ-communication ability property

In the previous subsection we characterized the Harsanyi power solutions on the class of cycle-free graph games using the weak σ-measure property. As noted in Section 2.3, Borm, Owen and Tijs [4] used the degree measure property to characterize the position value, and the communication ability property to characterize the Myerson value on the class of cycle-free graph games. Since we showed that the Myerson value is also a Harsanyi power solution, we also characterized that value using some degree measure property, namely the weak equal power measure property. In a similar way we can generalize the communication ability property to characterize the Harsanyi power solutions (including the position value) on the class of cycle-free graph games.

Lemma 4.6 *If (N, v, L) is point unanimous, then $v^L = cu^{D(N,L)}$ for some $c \in \mathbb{R}$.*

PROOF
Let (N, v, L) be a point unanimous graph game. If $D(N, L) = \emptyset$, then v^L is the null-game, and thus $v^L = cu^{D(N,L)}$ with $c = 0$. Suppose that $D(N, L) \neq \emptyset$. Then $v^L(S) = 0$ if $S \not\subseteq D(N, L)$. If $S, T \subseteq N$ are such that $D(N, L) \subseteq S$ and $D(N, L) \subseteq T$, then by definition of point unanimity we have that $v^L(S) = v^L(T) = g^P(|D(N, L)|)$. Thus, $v^L = cu^{D(N,L)}$ for some $c \in \mathbb{R}$. □

[16] A power measure σ on N is *symmetric* if for $i, j \in S \subseteq N$ with $R_{(S,L(S))}(i) \setminus \{j\} = R_{(S,L(S))}(j) \setminus \{i\}$ it holds that $\sigma_i(S, L(S)) = \sigma_j(S, L(S))$.

Note that this lemma implies that the weak communication ability property can be reformulated by saying that in a point unanimous graph game the dividends of the restricted game are distributed among the players in the corresponding coalition proportional to the equal power measure γ. (In a similar way the communication ability property, using point anonymous graph games, can be reformulated.)

On the class of cycle-free graph games the characterization of the Myerson value in [4], (see (ii) of Proposition 2.1) can be generalized for any positive power measure σ on N by adapting the (weak) communication ability property as follows.

Weak σ-communication ability property If graph game (N, v, L) is point unanimous then there is an $\alpha \in \mathbb{R}$ such that $f(N, v, L) = \alpha\sigma(N, L)$.

Proposition 4.7 *For a positive power measure σ, the Harsanyi power solution φ^σ satisfies component efficiency, additivity, the inessential link property and the weak σ-communication ability property on the class of all graph games. Moreover, it is the unique solution on the class of cycle-free graph games satisfying these four properties.*

PROOF

If (N, v, L) is point unanimous then with Lemma 4.6 and the definition of φ^σ, it follows that φ^σ satisfies the weak σ-communication ability property.

The second part of the proposition is proved similar to the uniqueness proof for the Myerson value in [4], taking into consideration that Lemma 3.4 says that in a cycle-free graph game a link is inessential if and only if it is superfluous and thus φ^σ satisfies the superfluous link property on the class of cycle-free graph games. The proof is therefore omitted. □

Clearly, the Myerson value is obtained by taking the equal power measure, and thus the weak equal power-communication ability property. Taking the degree measure, and thus the weak degree-communication ability property, yields another characterization of the position value. Whereas Borm, Owen and Tijs [4] characterize the position value using the weak degree measure property and the Myerson value using the weak communication ability property[17], we generalized both characterizations so that they both include characterizations of the position value and the Myerson value, taking the appropriate power measure. So, the view that the difference between the position value and Myerson value (on cycle-free graph games) is about using the weak degree measure property (which is defined using link unanimous graph games) or the weak communication ability property (which is defined using point anonymous graph games) has to be reconsidered. Both values

[17]Although [4] states the stronger versions of these axioms, in their proofs they only apply the weak versions.

satisfy the weak σ-measure property and the weak σ-communication ability property, but the difference is with respect to which power measure σ to use, the degree measure or the equal power measure.

5 Applications

5.1 Assignment games

The assignment game, introduced by Shapley and Shubik [30], is a game in which the player set N is partitioned in two sets, say the set V of *sellers* and the set W of *buyers*. Any pair $\{i,j\}$, $i \in V$, $j \in W$, can realise a nonnegative surplus $a_{i,j}$ from trade. However, any seller $i \in V$ can trade with only one buyer $j \in W$. A *matching* on a subset $S \subseteq N$ of players is a collection M of subsets $\{i,j\} \subset N$, $i \in V \cap S$, $j \in W \cap S$, such that for any $i \in V \cap S$ it holds that $|\{\{h,j\} \in M | h = i\}| \leq 1$ and for any $j \in W \cap S$ it holds that $|\{\{i,h\} \in M | h = j\}| \leq 1$, i.e. any seller $i \in V \cap S$ and any buyer $j \in W \cap S$ is in at most one element of the collection M of subsets $\{i,j\}$ of N. For $S \subseteq N$, let $\mathcal{M}(S)$ be the set of all matchings on S. Then the maximum surplus that can be obtained by a coalition $S \subseteq N$ is given by

$$v(S) = \max_{M \in \mathcal{M}(S)} \sum_{\{i,j\} \in M} a_{i,j}$$

with $v(S) = 0$ when $\mathcal{M}(S) = \emptyset$, i.e. when $S \subseteq V$ or $S \subseteq W$.

We now consider the communication graph on N in which the links reflect all matching possibilities, so the graph on N is the bipartite graph (N, L) with $\{i,j\} \in L$ if and only if $i \in V$ and $j \in W$. Clearly, by definition of (N, v, L), the characteristic function v^L of the point game (N, v^L) is equal to v. Since in the bipartite graph, any coalition only containing either sellers or buyers is unconnected, any such a coalition has zero dividend in the point game. Thus $\Delta^S(v^L) = 0$ if $S \subseteq V$ or $S \subseteq W$. A connected coalition contains at least one seller and at least one buyer and the dividend of such a coalition S in the point game (N, v^L) is given by

$$\Delta^S(v^L) = \sum_{\{T \subseteq S | \min(|T \cap V|,\ |T \cap W|) \geq 1\}} (-1)^{|S|-|T|} v(T).$$

Example 5.1
Consider the assignment game with one seller $V = \{1\}$ and two buyers $W = \{2, 3\}$. So, the bipartite graph is given by $L = \{\{1,2\}, \{1,3\}\}$. Further, let $a_{1,2} = 1$ and $a_{1,3} = 2$.

Then the assignment game (N, v) is given by

$$v(S) = \begin{cases} 1 & \text{if } S = \{1,2\}, \\ 2 & \text{if } S \in \{\{1,3\},\{1,2,3\}\}, \\ 0 & \text{otherwise.} \end{cases}$$

and the dividends of the point game (N, v^L) by

$$\Delta^S(v^L) = \begin{cases} 1 & \text{if } S = \{1,2\}, \\ 2 & \text{if } S = \{1,3\}, \\ -1 & \text{if } S = \{1,2,3\}, \\ 0 & \text{otherwise.} \end{cases}$$

For the coalitions with nonzero dividends, the degree measures of the subgraphs are given by $d_1(\{1,2\}, L(\{1,2\})) = d_2(\{1,2\}, L(\{1,2\})) = d_1(\{1,3\}, L(\{1,3\})) = d_3(\{1,3\}, L(\{1,3\})) = d_2(N,L) = d_3(N,L) = 1$ and $d_1(N,L) = 2$, which yields the Harsanyi degree solution $\varphi^d(N,v,L) = (1, \frac{1}{4}, \frac{3}{4})$. Since, the graph is cycle-free, this solution is equal to the position value. As an alternative solution, the β-measure yields $\beta_1(\{1,2\}, L(\{1,2\})) = \beta_2(\{1,2\}, L(\{1,2\})) = \beta_1(\{1,3\}, L(\{1,3\})) = \beta_3(\{1,3\}, L(\{1,3\})) = 1$, $\beta_1(N,L) = 2$ and $\beta_2(N,L) = \beta_3(N,L) = \frac{1}{2}$, and the resulting Harsanyi power solution is given by $\varphi^\beta(N,v,L) = (\frac{5}{6}, \frac{1}{3}, \frac{5}{6})$.

Example 5.2 In an assignment game with two sellers $V = \{1,2\}$ and two buyers $W = \{3,4\}$, the bipartite graph is given by $L = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$. Further, let $a_{1,3} = 1$, $a_{1,4} = 3$, $a_{2,3} = 4$ and $a_{2,4} = 5$. Then the nonzero dividends of the point game (N, v^L) are given by

$$\Delta^S(v^L) = \begin{cases} a_{i,j} & \text{if } S = \{i,j\},\ i \in V,\ j \in W, \\ -\min[a_{1,j},\ a_{2,j}] & \text{if } S = \{1,2,j\},\ j \in W, \\ -\min[a_{i,3},\ a_{i,4}] & \text{if } S = \{i,3,4\},\ i \in V, \\ 3 & \text{if } S = V \cup W. \end{cases}$$

For these coalitions with nonzero dividends, the degree measures of the subgraphs are given by $d_i(S, L(S)) = d_j(S, L(S)) = 1$ if $S = \{i,j\}$, $i \in V$, $j \in W$, $d_1(S, L(S)) = d_2(S, L(S)) = \frac{1}{2} d_j(S, L(S)) = 1$ if $S = \{1,2,j\}$ with $j \in W$, $d_3(S, L(S)) = d_4(S, L(S)) = \frac{1}{2} d_i(S, L(S)) = 1$ if $S = \{i,3,4\}$ with $i \in V$, and $d_i(N,L) = d_j(N,L) = 2$ for $i \in V$, $j \in W$ if $N = V \cup W$. From this it follows that by distributing the dividends according to the degrees, the Harsanyi degree solution is given by $\varphi^d(N,v,L) = (\frac{5}{4}, \frac{9}{4}, \frac{3}{2}, 2)$.

However, in this case the graph is not cycle-free and so the Harsanyi degree solution is not equal to the position value. It follows that the Shapley value of the link game (L, r^L) is given by $\psi_{\{1,3\}}(L, r^L) = \frac{1}{3}$, $\psi_{\{1,4\}}(L, r^L) = 2$, $\psi_{\{2,3\}}(L, r^L) = \frac{5}{2}$ and $\psi_{\{2,4\}}(L, r^L) = \frac{13}{6}$. This yields the position value $\pi(N,v,L) = (\frac{7}{6}, \frac{7}{3}, \frac{17}{12}, \frac{25}{12})$. □

Although Example 5.2 shows that in the assignment game the position value is not equal to the Harsanyi degree solution, in both solutions the total payoff to the sellers is equal to the total payoff to the buyers. Clearly, in the communication graph as defined above each link is a link between a seller and a buyer. So, the position value is obtained by distributing the Shapley payoff of each link in the link game (L, r^L) equally between the seller and the buyer. For the Harsanyi degree solution we have that in each connected coalition the sum of the degrees of the sellers is equal to the sum of the degrees of the buyers, so any dividend is equally shared between sellers and buyers. Since both solutions are component efficient, we have the following corollary.

Corollary 5.3 *Let (N, v) with $N = V \cup W$ be an assignment game and let (N, L) be the corresponding bipartite graph with $L = \{\{i, j\} | i \in V,\ j \in W\}$. Then*

$$\sum_{i \in V} \varphi_i^d(N, v, L) = \sum_{j \in W} \varphi_j^d(N, v, L) = \sum_{i \in V} \pi_i(N, v, L) = \sum_{j \in W} \pi_j(N, v, L).$$

Moreover, $\varphi^d(N, v, L) = \pi(N, v, L)$ if $|V| = 1$ or $|W| = 1$.

The next example shows that the Harsanyi degree solution does not satisfy the superfluous link property.

Example 5.4 Consider the assignment game given in Example 5.1 and suppose now that also the two buyers 2 and 3 can communicate, i.e. the communication graph is given by $L^c = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\} = L \cup \{\{2, 3\}\}$ with L the graph in Example 5.1. Clearly $v^E(N) = v^{E \cup \{2,3\}}(N)$ for all $E \subset L^c$, so $\{2, 3\}$ is superfluous in L^c. Hence, according to the superfluous link property we have that $\pi(N, v, L^c) = \pi(N, v, L) = (1, \frac{1}{4}, \frac{3}{4})$. However, when distributing the dividends according to the degree measure, each player has degree 2 in the grand coalition N, so the Harsanyi degree solution becomes $\varphi^d(N, v, L^c) = (\frac{7}{6}, \frac{1}{6}, \frac{4}{6})$ (being the Shapley value of v), which is not equal to $\varphi^d(N, v, L)$. The communication possibility between the two buyers decreases their payoffs. It shows that communication might be harmful, because it may give bigger shares in negative dividends. □

We end this subsection by considering the case that buyers and sellers cannot trade directly with each other, but need intermediaries to connect them. We do this by assuming that the set N is partitioned in three sets: a set V of sellers, a set W of buyers and a set I of intermediaries. Now the communication graph on N is the graph (N, L) in which every intermediary is connected to every buyer and seller, i.e. $L = \{\{i, j\} \mid i \in I,\ j \in V \cup W\}$.

Example 5.5 We consider Example 5.1 with a single intermediary player, labeled 4. So $I = \{4\}$ and $L = \{\{i, 4\} \mid i = 1, 2, 3\}$. Now the point game (N, v^L) follows from the assignment game (N, v) in Example 5.1 and is given by

$$v^L(S) = \begin{cases} 1 & \text{if } S = \{1,2,4\}, \\ 2 & \text{if } S \in \{\{1,3,4\},\{1,2,3,4\}\}, \\ 0 & \text{otherwise.} \end{cases}$$

and the dividends of the point game (N, v^L) by

$$\Delta^S(v^L) = \begin{cases} 1 & \text{if } S = \{1,2,4\}, \\ 2 & \text{if } S = \{1,3,4\}, \\ -1 & \text{if } S = \{1,2,3,4\}, \\ 0 & \text{otherwise.} \end{cases}$$

For any coalition S with nonzero dividend we have that the degree of i is 1 if $i \neq 4$, while the degree of player 4 is equal to $|S| - 1$. From this it follows that the Harsanyi degree solution is given by $\varphi^d(N, v, L) = (\frac{7}{12}, \frac{1}{12}, \frac{4}{12}, 1)$. Since, the graph is cycle-free, this solution is equal to the position value. □

Observe that the graph (N, L) is not cycle-free when $|I| \geq 2$ and thus the position value will differ from the Harsanyi degree solution when there are multiple intermediaries. However, also if $|I| \geq 2$, any link in the graph connects one of the intermediaries with either a buyer or a seller. So, in both the position value as the Harsanyi degree solution the total payoff to the intermediaries will be equal to the total payoff to the sellers and the buyers. Since both solutions are component efficient, we have the following corollary.

Corollary 5.6 *Let (N, v) with $N = V \cup W \cup I$ be an assignment game with intermediaries and let (N, L) be the corresponding graph with $L = \{\{i, j\} \mid i \in I,\ j \in V \cup W\}$. Then*

$$\sum_{i \in I} \varphi_i^d(N, v, L) = \sum_{j \in V \cup W} \varphi_j^d(N, v, L) = \sum_{i \in I} \pi_i(N, v, L) = \sum_{j \in V \cup W} \pi_j(N, v, L).$$

Moreover, $\varphi^d(N, v, L) = \pi(N, v, L)$ if $|I| = 1$.

5.2 ATM games

In this subsection we consider ATM games as introduced recently in Bjorndal, Hamers and Koster [2]. An ATM-game models a situation of n banks on a single location, where some banks have an Automated Teller Machine (money dispenser) and others do not. The banks may agree to cooperate, meaning that customers of banks not having an ATM are allowed to make use of the ATMs of the other banks, resulting in cost savings because using ATMs is a relatively cheap way of cash withdrawals.

We first consider a situation that there is only one single bank having an ATM. Specifically, let the banks not having an ATM be indexed by $i = 2, ..., n$ and let player 1 be the only bank that has an ATM. The number of visits of customers of bank $i \neq 1$ to the ATM of bank 1 is given by ω_i. We assume that each visit yields a cost saving of one. So, coalition $S = \{1, i\}$ can realize the non-negative worth ω_i, $i = 2, \ldots, n$ and thus the characteristic function of the game is given by

$$v(S) = \begin{cases} \sum\limits_{i \in S \setminus \{1\}} \omega_i, & \text{if } 1 \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, it follows straightforward that

$$\Delta^S(v) = \begin{cases} \omega_i, & \text{if } S = \{1, i\},\ i = 2, \ldots, m, \\ 0, & \text{otherwise,} \end{cases}$$

i.e. only the two-player coalitions $\{1, i\}$, $i \neq 1$, have positive dividends. All other coalitions have zero dividend. From [26] (Theorem 4.3) we know that for such games with non-zero dividends only for two-player coalitions we have that the Shapley value ψ coincides with the τ-value (see Tijs [32]) and the nucleolus η of (N, v). The game (N, v) is also a so-called *peer group game*, see Brânzei, Fragnelli and Tijs [5], such that the underlying tree is given by the communication graph (N, L) on N with $L = \{\{1, i\} \mid i = 2, \ldots, n\}$, i.e. (N, L) is the graph on N such that there is a link between the single ATM bank and any other bank[18]. Clearly, since only the two player coalions $\{1, i\}$, $i \neq 1$, have non-zero dividends, it follows that $v^L = v$. Moreover, in any two player coalition $\{1, i\}$, $i \neq 1$, both players have degree one, so that according to both the Harsanyi degree solution and the Hatsanyi β-measure solution the dividend of such a coalition is shared equally between the two players in the coalition. Thus these two solutions are equal to each other and are also equal to the Shapley value of (N, v) and the Myerson value of (N, v, L). Since the graph is cycle-free, we also have that the position value equals the Harsanyi degree solution. Hence we have that

$$\pi(N, v, L) = \varphi^d(N, v, L) = \varphi^\beta(N, v, L) = \mu(N, v, L)$$

and also

$$\mu(N, v, L) = \psi(N, v) = \tau(N, v) = \eta(N, v) \in C(N, v),$$

where the latter inclusion follows from the fact that all dividends are nonnegative and therefore the game is convex. Observe that all these solutions satisfy the *equal split property*, see Bjorndal, Hamers and Koster [2], i.e. the cost savings ω_i obtained from the

[18]In fact, in [5] a peer group game is a game on a *directed* graph, which in this example is given by $D = \{(1, i) \mid i = 2, \ldots, n\}$.

cooperation between bank i, $i \neq 1$, and ATM bank 1 is equally distributed between i and 1. In each solution the payoff to the ATM bank 1 is equal to $\sum_{i=2}^{n} \frac{1}{2}\omega_i$ and the payoff to bank i is $\frac{1}{2}\omega_i$, $i = 2, \ldots, n$.

Let us now consider the case that there are multiple banks having an ATM. We suppose that there is only one single bank without an ATM. Let $\{1, ..., n-1\}$ be the set of banks who possess ATMs and let bank n be the bank without ATM. The value of any coalition containing bank n and at least one other bank equals the total number of customers ω_n of bank n. So, the characteristic function is given by

$$v(S) = \begin{cases} \omega_n, & \text{if } n \in S \text{ and } |S| \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that the dividends are given by

$$\Delta^S(v) = \begin{cases} \omega_n, & \text{if } n \in S \text{ and } |S| \geq 2 \text{ is odd}, \\ -\omega_n, & \text{if } n \in S \text{ and } |S| \geq 2 \text{ is even}, \\ 0, & \text{if } S \subseteq N \setminus \{n\}. \end{cases}$$

Note that the game is equivalent to the assignment game with $n-1$ sellers (the banks with ATMs) and one buyer (bank n without ATM), which can realise the surplus ω_n with anyone of the sellers. Let (N, L) be the corresonding bipartite graph with $L = \{\{i, n\}) \mid i = 1, \ldots, n-1\}$, i.e. L is the set of links between the single bank without ATM and any other bank. Observe that $v = v^L$. Since the graph is cycle free, it follows that the Harsanyi degree solution is equal to the position value. Furthermore, from Corollary 5.3 it follows that the payoff to player n according to these solutions is given by

$$\pi_n(N, v, L) = \varphi_n^d(N, v, L) = \frac{\omega_n}{2}.$$

Hence, by symmetry and component efficiency of φ^d and π it follows that

$$\pi_i(N, v, L) = \varphi_i^d(N, v, L)) = \frac{\omega_n}{2(n-1)}, \quad i = 1, \ldots, n-1.$$

It is instructive to find out what distribution of the total value ω_n other solution concepts prescribe. First, we consider the Shapley value. The marginal contribution of player n is always ω_n, unless the permutation of all players is such that he enters first, which occurs in $(n-1)!$ cases. Hence,

$$\psi_n(N, v) = \frac{n! - (n-1)!}{n!}\omega_n = \frac{n-1}{n}\omega_n.$$

Again by the symmetry property and efficiency of the Shapley value, we obtain

$$\psi_i(N, v) = \frac{1}{n(n-1)}\omega_n, \quad i = 1, \ldots, n-1.$$

Since $v = v^L$, the Myerson value equals the Shapley value: $\mu(N, v, L) = \psi(N, v)$. When $n \geq 3$, the core of the game consists of a single point $x^* \in \mathbb{R}^n$, namely

$$x_n^* = \omega_n \text{ and } x_i^* = 0, \quad i = 1, \ldots, n-1.$$

Furthermore, the game is easily checked to be quasi-balanced (see Tijs [32]), so that both the τ-value and nucleolus η satisfy

$$\tau(N, v) = \eta(N, v) = x^*.$$

Summarizing these observations, we obtain the following relations (in case $n \geq 3$)

$$\begin{aligned} \omega_n = \tau_n(N,v) = \eta_n(N,v) &> \psi_n(N,v) = \mu_n(N,v,L) > \varphi_n^d(N,v,L), \\ 0 = \tau_i(N,v) = \eta_i(N,v) &< \psi_i(N,v) = \mu_i(N,v,L) < \varphi_i^d(N,v,L) = \pi_i(N,v,L),\ i \neq n. \end{aligned}$$

The Harsanyi degree solution shares the cost savings equally between the banks with ATMs and the single bank without ATM, whereas the τ-value and the nucleolus assign all the value to the buyer, the bank without ATM, saying that this bank can use the money dispensers of the other banks for free. The Shapley value is between the nucleolus and the Harsanyi degree solution, converging to the former when n goes to infinity.

In Bjorndal, Hamers amd Koster [2] a single solution concept is proposed for both situations with one ATM and situations with multiple ATMs (in that case we may suppose that there is a single bank without ATM). They call this solution the *equal split solution*. Despite this name, this solution, is given by the nucleolus, i.e. it shares the surplus equally between the ATM-bank and the others in case of a single ATM, but it gives all the surplus to the bank without ATM in case of multiple ATMs. In view of the analysis above, the only solution giving an equal split in both situations is the Harsanyi degree solution. In reality, banks cooperate using each other's ATM's by agreeing on a fee between banks for each visit to an ATM, i.e. for each visit that a customer makes to an ATM of another bank, the bank of this customer has to pay a fee to the owner of the ATM. This fee does not depend on the location, i.e. it does not depend on whether or not there is a single ATM. The equal split solution ES (the nucleolus) results in different fees for the two situations. The Harsanyi degree solution yields a uniform fee, namely equal to half of the cost savings, so fee $f = \frac{1}{2}$ is to be paid for each visit. In case of a single ATM this fee results in the Harsanyi degree payoff. In case of multiple ATMs, usually the customers are free to choose which ATM they want to use. Since the total number of visits is ω_n, a fee of $f = \frac{1}{2}$ for any visit gives a total payoff of $\frac{1}{2}\omega_n$ to the owners of the ATMs. When the customers of bank n choose randomly between the available ATMs, the Harsanyi degree solution gives the expected payoff to the ATM-banks.

5.3 Auction games

Consider a second-price sealed bid auction with n bidders. Suppose that their private valuations are arranged in a non-increasing order $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_{n-1} \geq \theta_n > 0$, and let us assume that the seller attaches utility $\theta_{n+1} \in [0, \theta_n)$ to the object which serves as a reservation price. If all n bidders collude and reveal their private valuations[19], they can earn as mush as $\theta_1 - \theta_{n+1}$. How should they share this surplus?

The communication structure of this game can be represented as a line-graph $L = \{\{i, i+1\} \mid i \in \{1, \ldots, n-1\}\}$. Any coalition not including player 1 (player with the highest private valuation) generates zero worth. For any coalition S that includes player 1, the worth of S, is equal to $\theta_1 - \theta_{k+1}$, where $k+1 = \min\{j \in N | \; j \notin S\}$. So, the worth of S including 1 is determined by its largest connected part $[1, k]$, where $[i, j] = \{i, i+1, \ldots, j-1, j\}$ denotes the coalition of consecutive players from i to j. By applying Theorem 3.1 from van den Brink, van der Laan and Vasil'ev [9] we have

$$\Delta^S(v) = \begin{cases} \theta_k - \theta_{k+1}, & \text{if } S = [1, k], \\ 0, & \text{otherwise.} \end{cases}$$

Observe that all dividends are non-negative, thus this auction game is totally positive[20]. Consequently, the set of Harsanyi payoff vectors coincides with the core of the game. Moreover, since the graph is cycle-free the Harsanyi degree solution coincides with the position value.

For any connected coalition $S = [i, j]$ in a line-graph, the degree measure is given by

$$d_k(S, L(S)) = \begin{cases} 1, & \text{if } k \in \{i, j\} \\ 2, & \text{if } k \in S \setminus \{i, j\}. \end{cases} \tag{5.13}$$

Applying the degree measure, given by (5.13) it can be verified that

$$\varphi_1^d(N, v) = \Delta^{\{1\}}(v) + \frac{1}{2}\sum_{k=2}^{n} \frac{\Delta^{[1,k]}(v)}{k-1} = (\theta_1 - \theta_2) + \frac{1}{2}\left(\sum_{k=2}^{n} \frac{\theta_k - \theta_{k+1}}{k-1}\right),$$

$$\varphi_i^d(N, v) = \frac{1}{2(i-1)}\Delta^{[1,i]}(v) + \sum_{k=i+1}^{n} \frac{\Delta^{[1,k]}(v)}{k-1} = \frac{\theta_i - \theta_{i+1}}{2(i-1)} + \sum_{k=i+1}^{n} \frac{\theta_k - \theta_{k+1}}{k-1}$$

for any $i \in \{2, ..., n-1\}$, and

$$\varphi_n^d(N, v) = \frac{1}{2(n-1)}\Delta^{[1,n]}(v) = \frac{\theta_n - \theta_{n+1}}{2(n-1)}.$$

[19] In second-price auctions collusion is not at all unrealistic. In fact, there is a simple incentive-compatible mechanism that induces bidders to disclosure their private information about valuations and fosters collusive behavior, see Graham and Marshall [13].

[20] Totally positiveness of the game implies its convexity.

It is interesting to note that the Shapley value ψ was proposed as a solution for this type of games by Graham, Marshall and Richard [14]. For players 1 and n the latter is equal to

$$\psi_1(N, v) = \Delta^{\{1\}}(v) + \sum_{k=2}^{n} \frac{\Delta^{[1,k]}(v)}{k}$$

and

$$\psi_n(N, v) = \frac{\Delta^{[1,n]}(v)}{n}.$$

It is easily shown that if $n \geq 3$ it holds that $\psi_1(N, v) > \varphi_1^d(N, v)$, and $\psi_n(N, v) > \varphi_n^d(N, v)$ for the bidder with the lowest valuation. Thus, the Harsanyi degree solution gives more to 'central' players at the expense of the 'end' ones.

6 Concluding remarks

In this paper we studied Harsanyi power solutions for graph games, i.e. cooperative TU-games in which the cooperation possibilities are restricted because the players belong to a limited communication (graph) structure. In such solutions the sharing system that is used in distributing the Harsanyi dividends in the restricted game is determined by a power measure for (communication) graphs. Although any positive power measure can be applied, we gave special attention to the degree measure and the equal power measure. On the class of cycle-free graph games, the Harsanyi power solution that is based on the degree measure is equal to the position value. This is not the case for arbitrary graph games. The Harsanyi power solution that is based on the equal power measure is always equal to the Myerson value. We argued that for some graph games the Harsanyi degree solution seems better than both the position value and the Myerson value since it assigns zero payoff to null players that are not connecting any non-null player (a property that is not satisfied by the position value) and it assigns higher payoffs to more central players (which is not done by the Myerson value).

We gave two axiomatic characterizations of the Harsanyi power solutions on the class of cycle-free graph games. One axiomatization uses the weak σ-degree measure property, and the other uses the weak σ-communication ability property. Both give characterizations for the position value and the Myerson value as special cases. So, the difference between the position value and the Myerson value (on cycle-free graph games) is not about using the weak degree measure property or the weak communication ability property but about which power measure to use.

We also applied the Harsanyi power solution to some specific classes of games, in particular to assignment games, ATM games and auction games. It was shown that in assignment games, both the position and the Harsanyi degree solution give half of the payoffs to the buyers and half to the sellers. These two solutions differ in the distribution of the payoffs among the buyers and among the sellers (except when there is only one buyer or seller, in which case the corresponding graph is cycle-free). If we allow for intermediaries then in both solutions half of the payoffs always go to the intermediaries. In ATM games with one bank owning an ATM the corresponding game is a peer group game, and we saw that the position value, Harsanyi degree solution and Myerson value coincide with many known solutions (such as the τ-value and nucleolus) and assign to every bank without ATM half of the cost reduction its customers generate by using the ATM while the ATM bank obtains the other half. This is also a Core element. In case there are more banks with an ATM then the corresponding game is not a peer group game, but again the bank without an ATM obtains half of its cost reduction, while the other half is equally split among the ATM banks. In auction games we saw that applying the Harsanyi degree solution to the corresponding line-graph game distributes the extra valuation of a player above the next highest valuation among this player and its 'predecessors' (i.e. the players with higher valuation) in such a way that this player and player 1 (with highest valuation) get a share that is half of the shares of the intermediary players.

Finally, we would like to mention that the results of this chapter can be restated for asymmetric directed graphs. A *directed graph* or *digraph* is a pair (N, D) where N is the set of nodes and $D \subseteq N \times N = \{(i,j) \mid i,j \in N, i \neq j\}$ is a binary relation on N consisting of ordered pairs called directed links or *arcs*. The digraph (N, D) is called *asymmetric* if $(i,j) \in D$ implies that $(j,i) \notin D$. Suppose that the players in a TU-game (N, v) are organized according to a directed graph (N, D) on the player set N and denote such a *digraph game* shortly by (N, v, D). The asymmetry of (N, D) reflects the idea that one player incident with a link has more control over that link than the other player. Again we assume that the directed graph is a communication graph in the sense that the restricted game of a digraph game (N, v, D) is the Myerson restricted game v^{L^D} of the corresponding undirected graph (N, L^D), with $L^D = \{\{i,j\} \mid (i,j) \in D\}$. So, cooperation in a coalition S is possible if and only if there is a path in the associated undirected graph L^D between any pair of players of S.[21] However, in distributing the dividends of the restricted game we take account of the direction of the arcs by using a power measure for directed graphs. Similar as with undirected graphs, a power measure for directed graphs on a set of nodes N is a function that assigns to any $S \subseteq N$ and digraph D on N a nonnegative vector $\sigma(S, D(S)) \in \mathbb{R}_+^{|S|}$, yielding the power $\sigma_i(S, D(S))$ of node $i \in S$ in the directed subgraph $(S, D(S))$, where $D(S) = \{(i,j) \in D | \{i,j\} \subseteq S\}$. We say that

[21] This is different from the approach in games with a permission structure where the direction of the arc is relevant in determining feasability of coalitions, see Gilles, Owen and van den Brink [11], van den Brink and Gilles [7] as well as van den Brink [6].

a measure is *positive* if in any subgraph $(S, D(S))$ it assigns positive power to any node in S who has at least one follower in $(S, D(S))$, where the set of followers of $i \in N$ in digraph (N, D) is given by $F_{(N,D)}(i) = \{j \in N \mid (i,j) \in D\}$. Given a set of players N and a positive power measure σ on $\mathcal{D}^N$ the corresponding Harsanyi power solution for digraph games is defined as $\varphi^\sigma(N, v, D) = h^{p^\sigma}(N, v^{L^D})$ with (for the connected coalitions) the sharing system $p^\sigma = (p^{\sigma,S})_{S \in 2^N}$ given by

$$p_i^{\sigma,S} = \frac{\sigma_i(S, D(S))}{\sum_{j \in S} \sigma_j(S, D(S))} \text{ for all } i \in S \text{ whenever } \sum_{j \in N} \sigma_j(S, D(S)) \neq 0.$$

Again, S is unconnected[22] (and thus has zero dividend in the restricted game) when no player in S has a follower. So, the shares do not matter when $\sum_{j \in S} \sigma_j(S, D(S)) = 0$ and we could take any sharing vector, for example $p_i^{\sigma,S} = \frac{1}{|S|}$ for all $i \in S$. For cycle-free digraph games[23] we obtain similar characterizations as for undirected graph games in Section 4. Similarly to the proof of Proposition 4.3, it can be shown that for a positive power measure for digraphs, the Harsanyi power solution φ^σ is the unique solution for cycle-free digraph games that satisfies appropriate modifications of the properties of component efficiency, additivity, the inessential arc property and the weak σ-measure property to the class of digraph games. In a similar way the characterization using a weak σ-communication ability property can be adapted for digraph games.

An example of a positive power measure for digraphs that can be used is the *outdegree measure* d^{out}, which assigns to every node its *outdegree*, i.e. its number of followers $d_i^{out}(N, D) = |F_{(N,D)}(i)|$. The corresponding Harsanyi power solution is the *Harsanyi outdegree solution* for digraph games according to which the dividend of a coalition S in the restricted game (N, v^{L^D}) is distributed proportional to the outdegree of the players in the corresponding digraph restricted to S. Other examples of positive power measures for digraphs are the *β-measure* which distributes the power of every node $i \in N$ that has ingoing arcs equally among its predecessors, or the *positional power measure* which assigns to every node i the number of followers of i (i.e. the nodes to which i has an outgoing arc) plus a fraction $\frac{1}{|S|}$ of the total power of its followers in distributing the dividend of coalition $S \subseteq N$.

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