On the Extreme Points of Two Polytopes associated with a Digraph and Applications to Cooperative Games

René van den Brink [1]
Gerard van der Laan [1]
Valeri Vasil'ev [2]

[1] Vrije Universiteit Amsterdam, and Tinbergen Institute,
[2] Sobolev Institute of Mathematics, Novisibirsk, Russia.

On the Extreme Points of Two Polytopes Associated with a Digraph and Applications to Cooperative Games[1]

René van den Brink[2] Gerard van der Laan[3] Valeri Vasil'ev[4]

June 11, 2004

[2]J.R. van den Brink, Department of Econometrics and Tinbergen Institute, Free University, De Boelelaan 1105, 1081 HV   Amsterdam, The Netherlands. E-mail: jrbrink@feweb.vu.nl

[3]G. van der Laan, Department of Econometrics and Tinbergen Institute, Free University, De Boelelaan 1105, 1081 HV   Amsterdam, The Netherlands. E-mail: glaan@feweb.vu.nl

[4]V.A. Vasil'ev, Sobolev Institute of Mathematics, Prosp. Koptyuga 4, 630090 Novisibirsk, Russia, E-mail: vasilev@math.nsc.ru

Abstract

In this paper we describe the extreme points of two closely related polytopes that are assigned to a digraph. The first polytope is the set of all sharing vectors (elements from the unit simplex) such that each node gets at least as much as each of its successors. The second one is the set of all fuzzy vectors (elements of the unit cube) with participation rates of players subordinated to the relationships prescribed by the digraph. We also discuss some applications in cooperative game theory.

Keywords: polytope, directed graph, unit simplex, unit cube, cooperative game

1 Introduction

In this paper we describe the extreme points of two closely related polytopes that are assigned to a given *directed graph (digraph)* on the set $N \subseteq \mathbb{N}$ consisting of $n = |N|$ nodes. We assume $N = \{1, ..., n\}$ unless stated otherwise. In economics and game theory such digraphs often reflect organizational structures on the set of agents, respectively players, for instance the hierarchical structure within a firm. Within such an economic or game-theoretic setting, a *value vector* is an n-dimensional vector, in which the ith component gives some value to agent or player i, $i = 1, \ldots, n$, for instance the wage to the ith agent within the firm. In this paper a value vector is called a *sharing vector* if all components are nonnegative and the sum of the components is equal to one, i.e. the vector belongs to the unit simplex. In situations that the coalition of all n agents can distribute some worth, a sharing vector gives to each agent a nonnegative share of that worth. A value vector is called a *fuzzy vector* if all components are nonnegative and every component is between zero and one. The ith component of a fuzzy vector can be interpreted as the participation rate of an agent in some project, for instance the participation rate of an agent is the part of his wealth that he invests in some common investment project. Within this interpretation a fuzzy vector reflects a *fuzzy coalition* (a well-known concept within economics and game theory), i.e. the vector yields the participation rates of the agents in forming a coalition.

The first 'digraph polytope' that we consider in this paper - to be called the *sharing polytope* - assigns to a given digraph the set of all sharing vectors such that each node has a share that is at least as high as the share of each of its successors in the digraph. Within cooperative game theory, this type of polytopes appears to be very important to adapt the so-called Harsanyi set (see Vasil'ev and van der Laan (2002), Vasil'ev (2003)) to the case of cooperative games in which a digraph reflects a hierarchical structure between the players. In this paper we give a complete and rather simple description of the extreme points of the sharing polytope induced by the digraph. Rather surprisingly, it appears that the collection of all connected and comprehensive from above sets of nodes of the digraph play a key role in the description of the extreme points of the sharing polytope (given in Theorem 3.4 below). Putting differently, the result obtained may be considered as the linearization of the discrete optimization problems over these type of subgraphs of an arbitrary finite digraph (some results concerning the linearization of discrete optimization problems, related to graph theory, can be found in Schrijver (1986); see also the references therein).

The second digraph polytope - to be called the *fuzzy polytope* - is the set of all fuzzy vectors such that each node has a participation rate that is at least as high as the participation rate of each of its successors in the digraph. Within economics it is

well-known that domination (rejection) of certain outcomes by the fuzzy coalitions plays a crucial role in the cooperative characterization of the equilibria in pure competitive economies (see e.g. Aubin (1979)). Like in cooperative game theory, in certain economic situations it may be reasonable to take into account the hierarchical structure between the economic agents when considering the collection of all feasible fuzzy coalitions. Hence, it is of interest to know the structure of the fuzzy polytope, describing the collection of all feasible fuzzy coalitions for a given digraph on the set of agents, in order to get a more detailed description of the equilibria in, say, a graph-restricted market (a market with some elements of hierarchy). The characterization of the extreme points of the fuzzy polytope appears to be much more easy than for the sharing polytope. Nevertheless, the result (given in Theorem 4.2 below) is quite interesting. In particular, it turns out that the extreme points of the fuzzy polytope are generated by the collection of all comprehensive from above sets of nodes of the given digraph. Since, in contrast with the sharing polytope, for the fuzzy polytope connectedness of the subgraphs is not required, in general the number of its extreme points may be much higher than the number of extreme points of the sharing polytope. Again, like in case of Theorem 3.4, Theorem 4.2 together with the description of the fuzzy polyhedron itself may be considered as the solution of the linearization problem for the discrete optimization over the comprehensive from above subgraphs of a given digraph.

The paper is organized as follows. In Section 2 some graph-theoretic preliminaries are given. Section 3 is devoted to the main result, concerning the description of the extreme points of the sharing polytope (Theorem 3.4). Section 4 deals with the analog of Theorem 3.4 for the fuzzy polytope (Theorem 4.2). Section 5 contains some economic and game-theoretic applications of the results obtained.

2   Preliminaries

A directed graph or *digraph* is a pair (N, D) where $N \subseteq \mathbb{N}$ is a finite set of nodes and $D \subseteq N \times N$ is a binary relation on N. As mentioned in the introduction, we assume $N = \{1, ..., n\}$ unless stated otherwise. For $D \subseteq N \times N$ and $i \in N$ the nodes in $\{j \in N \mid (i, j) \in D\}$ are called the *successors* of i in D, and the nodes in $\{j \in N \mid (j, i) \in D\}$ are called the *predecessors* of i in D. For $i, j \in N$ a *path* between i and j in D is a sequence of nodes $(i_1, \ldots, i_m)$ such that $i_1 = i$, $i_m = j$, and $\{(i_k, i_{k+1}), (i_{k+1}, i_k)\} \cap D \neq \emptyset$ for $k = 1, \ldots, m - 1$. A set of nodes $T \subseteq N$ is *connected* in digraph (N, D) if there is a path between any two nodes in T that only uses arcs between nodes in T, i.e. if for every $i, j \in T$ there is a path $(i_1, \ldots, i_m)$ between i and j such that $\{i_1, \ldots, i_m\} \subseteq T$.

For some $S \subseteq N$, the digraph $(S, D(S))$ with $D(S) = \{(i, j) \in D \mid \{i, j\} \subseteq S\}$

is called the *subdigraph* of S in (N, D). A subset $S \subseteq N$ is a *component* of N if the subdigraph $(S, D(S))$ is maximally connected, i.e. $(S, D(S))$ is connected and for any $j \in N \setminus S$, the subdigraph $(S \cup \{j\}, D(S \cup \{j\}))$ is not connected. Clearly, for any digraph (N, D), the collection of components of N forms a partition of N. We call this partition the *decomposition* of N in (N, D).

Finally, the *transitive closure* of digraph (N, D) is the digraph $(N, tr(D))$ with $(i, j) \in tr(D)$ if and only if there exists a *directed path* from i to j in D, i.e. there exists a sequence of nodes $(i_1, \ldots, i_m)$ such that $i_1 = i$, $i_m = j$ and $(i_k, i_{k+1}) \in D$ for all $k \in \{1, ..., m-1\}$.

3 Extreme points of the sharing polytope

In this section we give a characterization of the extreme points of the sharing polytope induced by a digraph (N, D). An element of the sharing polytope is a sharing vector in the unit simplex such that for every i the share of node i is at least as much than the shares of each of its successors. Let U^n be the n-dimensional unit simplex, i.e. $U^n = \{x \in \mathbb{R}^n \mid x_i \geq 0, \ i = 1, \ldots, n, \text{ and } \sum_{i=1}^n x_i = 1\}$.

Definition 3.1 *The sharing polytope associated with digraph (N, D), $N \neq \emptyset$, is the polytope P_D given by*

$$P_D = \{p \in U^n \mid p_i \geq p_j \text{ if } (i, j) \in D\}.$$

In applications in economics and game theory it is useful to have an explicit characterization of the extreme points of P_D. It appears that these extreme points are determined by the collection of all non-empty subsets S of N satisfying (i) S is connected in the digraph and (ii) for every node in S it holds that all its predecessors belong to S. To be more precise, we introduce the following formal definition and notations.

Definition 3.2 *A set of nodes $S \subseteq N$ is* **comprehensive from above** *in digraph (N, D) if $[j \in S$ and $(i, j) \in D]$ implies that $i \in S$.*

The collection of all comprehensive from above sets of nodes in digraph (N, D) is denoted by $A(D)$. For ease of simplicity, in the sequel of the paper we shortly call a subset comprehensive when it belongs to $A(D)$. Further we denote by $C(D)$ the collection of all connected sets of nodes in digraph (N, D). Observe that both $A(D)$ and $C(D)$ contain the empty set. We call a subset of nodes *complete* when it is non-empty, comprehensive and connected.

Definition 3.3 *A set of nodes S is* **complete** *in digraph (N, D) if S is not empty and belongs to $A(D) \cap C(D)$.*

The collection of all complete sets of nodes in digraph (N, D) is denoted by $AC(D)$. Finally, for any non-empty $S \subseteq N$ we define the vector $a^S \in \mathbb{R}^N$ by

$$
a_i^S = \begin{cases} \frac{1}{|S|} & \text{if } i \in S \\ 0 & \text{if } i \in N \setminus S. \end{cases}
$$

We are now ready to state the main result of this section, namely that the set $Ex(P_D)$ of extreme points of the sharing polytope P_D is equal to the set of vectors a^S obtained from complete sets in (N, D).

Theorem 3.4 *For every digraph (N, D) it holds that $Ex(P_D) = \{a^S | S \in AC(D)\}$.*

We prove this theorem in three steps reflected in the following three lemma's. First we show that for every complete set S, the vector a^S belongs to P_D.

Lemma 3.5 *For every digraph (N, D) and every $S \in A(D)$ it holds that $a^S \in P_D$.*

Proof. By definition of a^S we have $\sum_{i \in N} a_i^S = 1$ and $a_i^S \geq 0$ for all $i \in N$. Take $i, j \in N$ such that $(i, j) \in D$. If $\{i, j\} \subseteq S$ then $a_i^S = a_j^S = \frac{1}{|S|}$. If $i \in S$ and $j \in N \setminus S$ then $a_i^S = \frac{1}{|S|} > 0 = a_j^S$. If $i \in N \setminus S$ then by S being comprehensive from above in (N, D) it must hold that $j \in N \setminus S$, and thus $a_i^S = a_j^S = 0$. So, $a_i^S \geq a_j^S$ if $(i, j) \in D$. Thus we have shown that a^S meets all conditions to belong to P_D. $\qquad\square$

Next we show that for any complete S, the vector a^S is an extreme point of P_D.

Lemma 3.6 *For every digraph (N, D) and every $S \in C(D)$ with $a^S \in P_D$ it holds that $a^S \in Ex(P_D)$.*

Proof. Let $S \in C(D)$ be a subset of nodes with $a^S \in P_D$ and let $b, c \in P_D$ be such that $a^S = \frac{1}{2}(b + c)$. To establish the inclusion that $a^S \in Ex(P_D)$, it is sufficient to prove that $b = c = a^S$ for any such b and c. To this end, let us mention first, that due to the equalities $a_i^S = 0$ for all $i \in N \setminus S$ we have that $\frac{1}{2}(b_i + c_i) = 0$ for all $i \in N \setminus S$. Since $b, c \geq 0$ we then have

$$
b_i = c_i = 0 \text{ for all } i \in N \setminus S. \tag{3.1}
$$

Second, suppose that $i, j \in S$ are such that $(i, j) \in D$. Since $b, c \in P_D$ we have by definition of P_D that $b_i \geq b_j$ and $c_i \geq c_j$. Suppose that at least one of these two inequalities is strict.

4

Then $a_i^S = \frac{1}{2}(b_i + c_i) > \frac{1}{2}(b_j + c_j) = a_j^S$, which contradicts with $a_i^S = a_j^S = \frac{1}{|S|}$ for all $i, j \in S$. So,

$$b_i = b_j \text{ and } c_i = c_j \text{ for all } i, j \in S \text{ with } (i, j) \in D. \tag{3.2}$$

Third, suppose that $i, j \in S$ and $(i, j) \notin D$. Then, with connectedness of S it follows that there exists a sequence of nodes $(i_1, ..., i_m)$ such that $i_1 = i$, $i_m = j$, $\{i_1, ..., i_m\} \subseteq S$ and $\{(i_k, i_{k+1}), (i_{k+1}, i_k)\} \cap D \neq \emptyset$ for all $k \in \{1, ..., m-1\}$. Repeated application of (3.2) yields $b_i = b_{i_1} = ... = b_{i_m} = b_j$ and $c_i = c_{i_1} = ... = c_{i_m} = c_j$. Thus

$$b_i = b_j \text{ and } c_i = c_j \text{ for all } i, j \in S \text{ with } (i, j) \notin D. \tag{3.3}$$

With (3.1), (3.2), (3.3) and the fact that $b, c \in P_D$ (and thus $\sum_{i \in N} b_i = \sum_{i \in N} c_i = 1$) it then follows that $b_i = c_i = \frac{1}{|S|}$ for all $i \in S$. So, $b = c = a^S$. $\qquad \square$

Note that in Lemma 3.5 we do not require connectedness of S, while in Lemma 3.6 we do not require comprehensiveness of S. Obviously, both lemmas hold for complete S. To finish the proof of Theorem 3.4 we have to show that every extreme point of P_D can be obtained as a vector a^S for some complete S.

Lemma 3.7 *For every digraph (N, D) and every $a \in Ex(P_D)$ there exists an $S \in AC(D)$ such that $a = a^S$.*

Proof. Let $a \in Ex(P_D)$. Denote by $S^a = \{i \in N | a_i > 0\}$ the support of a. Note that $S^a \neq \emptyset$ since $\sum_{i \in N} a_i = 1$. We first prove that $S^a \in AC(D)$.

For $i, j \in N$ with $(i, j) \in D$, $a \in P_D$ implies that $a_i \geq a_j$. So, $[j \in S^a$ and $(i, j) \in D]$ implies that $i \in S^a$, i.e. S^a is comprehensive from above. To show that S^a is complete it then is sufficient to show that S^a is connected. On the contrary, suppose that S^a is not connected. Then there is a decomposition $\{S_1^a, ..., S_m^a\}$ of S^a in subdigraph $(S^a, D(S^a))$ with the number m of components at least equal to two. Since S_k^a is maximally connected in $(S^a, D(S^a))$, by comprehensiveness from above of S^a we have that every component S_k^a in this decomposition is comprehensive from above, as well. So, $S_k^a \in AC(D)$ for all $k \in \{1, ..., m\}$. Now, consider the vectors a^k, $k = 1, ..., m$, given by

$$a_i^k = \begin{cases} a_i & \text{if } i \in S_k^a \\ 0 & \text{if } i \in N \setminus S_k^a, \end{cases}$$

and define $\lambda_k = \sum_{i \in S_k^a} a_i^k$ for all $k \in \{1, ..., m\}$. By $\{S_1^a, ..., S_m^a\}$ being a decomposition of S^a and non-emptyness of S^a it follows by definition that the sets S_k^a, $k \in \{1, ..., m\}$, meet the conditions: (i) $S^a = \cup_{k=1}^m S_k^a$, (ii) $S_k^a \neq \emptyset$ for all $k \in \{1, ..., m\}$, and (iii) $S_k^a \cap S_l^a = \emptyset$ for all $k, l \in \{1, ..., m\}$ with $k \neq l$. But then $\lambda_k > 0$ for all $k \in \{1, ..., m\}$, and $\sum_{k=1}^m \lambda_k = 1$. Define

$\overline{a}^k = \frac{1}{\lambda_k} a^k$ for every $k \in \{1, ..., m\}$, i.e. $\overline{a}_i^k = \frac{1}{\lambda_k} a_i$ if $i \in S_k^a$ and $\overline{a}_i^k = 0$ if $i \in N \setminus S_k^a$. Then $S_k^a \cap S_l^a = \emptyset$ for all $k, l \in \{1, ..., m\}$ with $k \neq l$, implies that $\overline{a}^k \neq \overline{a}^l$ for all $k, l \in \{1, ..., m\}$ with $k \neq l$, and, moreover, it implies that $|\{k \in \{1, ..., m\} | i \in S_k^a\}| = 1$ for all $i \in S^a$. But then $\sum_{k=1}^m \lambda_k \overline{a}_i^k = \sum_{k=1}^m a_i^k = a_i$ for all $i \in S^a$. Thus, $a = \sum_{k=1}^m \lambda_k \overline{a}^k$ with $\sum_{k=1}^m \lambda_k = 1$ and $\lambda_k \geq 0$, $k \in \{1, ..., m\}$.

To get a contradiction with $a \in Ex(P_D)$ it is sufficient to prove that $\overline{a}^k \in P_D$ for all $k \in \{1, ..., m\}$. Since, $\overline{a}^k \geq 0$ and $\sum_{i \in N} \overline{a}_i^k = 1$, it is sufficient to prove that $(i, j) \in D$ implies that $\overline{a}_i^k \geq \overline{a}_j^k$ for all $k \in \{1, ..., m\}$. To do so, fix some $(i, j) \in D$ and $k \in \{1, ..., m\}$. We distinguish the following two cases.

(i) Suppose that $j \in S_k^a$. Since $S_k^a \in AC(D)$ we have $i \in S_k^a$, and thus with $a \in P_D$ it follows that $\overline{a}_i^k = \frac{a_i}{\lambda_k} \geq \frac{a_j}{\lambda_k} = \overline{a}_j^k$.

(ii) Suppose that $j \notin S_k^a$. Since $\overline{a}^k \geq 0$ we have $\overline{a}_i^k \geq 0 = \overline{a}_j^k$.

So, we have shown that the vector a is equal to the convex combination $\sum_{k=1}^m \lambda_k \overline{a}^k$ with $m \geq 2$, $\lambda_k > 0$ for all $k \in \{1, ..., m\}$, $\overline{a}^k \in P_D$ for all $k \in \{1, ..., m\}$, and $\overline{a}^k \neq \overline{a}^l$ for all $k, l \in \{1, ..., m\}$ with $k \neq l$, which contradicts with $a \in Ex(P_D)$. Consequently, S^a is connected in (N, D). Since we already proved that S^a is comprehensive from above, we have shown that $S^a \in AC(D)$.

Next we prove that all components a_i of a with $i \in S^a$ are equal to each other, i.e. $a_i = a_j$ for all $i, j \in S^a$. On the contrary, suppose that this is not the case. Defining $\alpha = \min\{a_i | i \in S^a\}$ and $\beta = \max\{a_i | i \in S^a\}$ we then have $\alpha < \beta$. Consider the vectors a' and a'' given by

$$a_i' = \begin{cases} \alpha & \text{if } i \in S^a \\ 0 & \text{if } i \in N \setminus S^a, \end{cases} \quad \text{and} \quad a_i'' = \begin{cases} a_i - \alpha & \text{if } i \in S^a \\ 0 & \text{if } i \in N \setminus S^a. \end{cases}$$

Since $0 < \alpha < \beta$ we have $a', a'' \geq 0$ and $a = a' + a''$. Defining $\lambda' = \sum_{i \in S^a} a_i'$ and $\lambda'' = \sum_{i \in S^a} a_i''$ we obtain from $a \in P_D$ that $\lambda', \lambda'' \in (0, 1)$ and $\lambda' + \lambda'' = 1$. Moreover, we have $a = \lambda' \overline{a}' + \lambda'' \overline{a}''$ with $\overline{a}' = \frac{1}{\lambda'} a'$ and $\overline{a}'' = \frac{1}{\lambda''} a''$. For $i \in S^a$ with $a_i = \alpha$ we have $\overline{a}_i' = \frac{1}{\lambda'} \alpha > 0$ and $\overline{a}_i'' = 0$. So, $\overline{a}' \neq \overline{a}''$ and $\sum_{i \in N} \overline{a}_i' = \sum_{i \in N} \overline{a}_i'' = 1$. To get a contradiction with $a \in Ex(P_D)$, it is sufficient to show that $\overline{a}', \overline{a}'' \in P_D$. Take $(i, j) \in D$. If $j \notin S^a$, then due to $\overline{a}', \overline{a}'' \geq 0$ and $\overline{a}_j' = \overline{a}_j'' = 0$ we have that $\overline{a}_i' \geq \overline{a}_j'$ and $\overline{a}_i'' \geq \overline{a}_j''$. If $j \in S^a$ then, by the assumption that $a \in P_D$, we have $a_i \geq a_j$ and thus $\overline{a}_i' = \frac{1}{\lambda'} \alpha = \overline{a}_j'$ and $\overline{a}_i'' = \frac{1}{\lambda''}(a_i - \alpha) \geq \frac{1}{\lambda''}(a_j - \alpha) = \overline{a}_j''$. Hence, in both cases $\overline{a}', \overline{a}'' \in P_D$. This fact, together with $\overline{a}' \neq \overline{a}''$ and the above-mentioned equality $a = \lambda' \overline{a}' + \lambda'' \overline{a}''$ with $\lambda', \lambda'' \in (0, 1)$, is in contradiction with $a \in Ex(P_D)$.

Thus, $a_i = a_j$ for all $i, j \in S^a$. Since $\sum_{i \in N} a_i = 1$ we have $a_i = \alpha = \beta = \frac{1}{|S^a|}$ for all $i \in S^a$. Concluding we have $S^a \in AC(D)$ and $a = a^S$ with $S = S^a$. $\square$

6

The following example shows that both comprehensiveness and connectedness are necessary for the result of Theorem 3.4.

Example 3.8
Consider the digraph (N, D) with $N = \{1, 2, 3\}$ and $D = \{(1, 3), (2, 3)\}$. For this digraph we have $AC(D) = \{\{1\}, \{2\}, \{1, 2, 3\}\}$ and thus $P_D = \text{Conv}\{(1, 0, 0)^\top, (0, 0, 1)^\top, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^\top\}$ (see Figure 1).
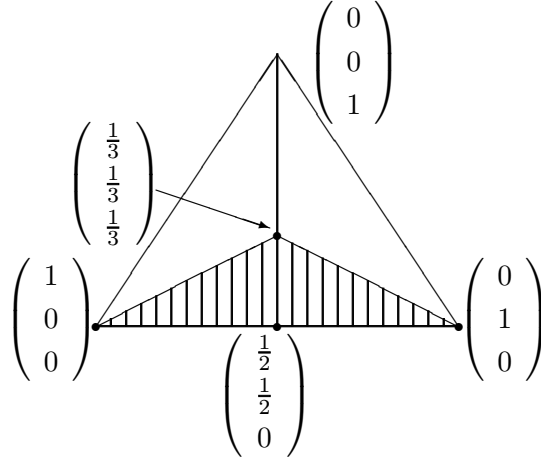

Figure 1

The set $\{2, 3\}$ is connected but is not comprehensive. Clearly, the vector $a^{\{2,3\}} = (0, \frac{1}{2}, \frac{1}{2})^\top$ does not belong to P_D since $a_1^{\{2,3\}} < a_3^{\{2,3\}}$ although $(1, 3) \in D$.

The set $\{1, 2\}$ is comprehensive from above but is not connected. Clearly, the vector $a^{\{1,2\}} = (\frac{1}{2}, \frac{1}{2}, 0)^\top$ belongs to P_D, but is not an extreme point of P_D. $\qquad\square$

We end this section by discussing some special cases and examples. First, we consider a directed *tree* (N, D), i.e. (i) N is connected in (N, D), (ii) $(i, i) \notin D$ for all $i \in N$, (iii) $|\{j \in N | (j, i) \in D\}| \leq 1$ for all $i \in N$, and (iv) there is a unique node (the root) $i \in N$ with $\{j \in N | (j, i) \in D\} = \emptyset$. Denoting $T_S = S \cup \{j \in N | (j, i) \in tr(D) \text{ for some } i \in S\}$ for all $S \subseteq N$, it then follows that $AC(D) = \{T_S | S \subseteq N\}$. Theorem 3.4 then yields the following corollary.

Corollary 3.9 *If (N, D) is a directed tree then $Ex(P_D) = \bigcup_{S \subseteq N} \{t^S\}$, with $t^S \in \mathbb{R}^n$ given by $t_j^S = \frac{1}{|T_S|}$ if $j \in T_S$, and $t_j^S = 0$ otherwise.*

Example 3.10

Consider the digraph (N, D) with $N = \{1, 2, 3, 4\}$ and $D = \{(1, 2), (1, 3), (3, 4)\}$. Then

$$
T_S = \begin{cases}
\{1\} & \text{if } S = \{1\} \\
\{1, 2\} & \text{if } S \in \{\{2\}, \{1, 2\}\} \\
\{1, 3\} & \text{if } S \in \{\{3\}, \{1, 3\}\} \\
\{1, 2, 3\} & \text{if } S \in \{\{2, 3\}, \{1, 2, 3\}\} \\
\{1, 3, 4\} & \text{if } S \in \{\{4\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}\} \\
\{1, 2, 3, 4\} & \text{if } S \in \{\{2, 4\}, \{1, 2, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}.
\end{cases}
$$

Hence $AC(D) = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$. The corresponding extreme points of the sharing polytope are: $(1, 0, 0, 0)^\top$, $(\frac{1}{2}, \frac{1}{2}, 0, 0)^\top$, $(\frac{1}{2}, 0, \frac{1}{2}, 0)^\top$, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)^\top$, $(\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3})^\top$, $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^\top$. $\qquad\square$

Next we consider two special cases of directed trees. First, we consider line-graphs. Digraph (N, D) with $N = \{1, ..., n\}$ is a line-graph if $D = \{(i, i+1) | i \in \{1, ..., n-1\}\}$. (Of course, the labeling of the nodes can be taken different.) Then $AC(D) = \{T_{\{i\}} | i \in N\}$ with $T_{\{i\}} = \{1, ..., i\}$. Theorem 3.4 (and Corollary 3.9) yield the following corollary.

Corollary 3.11 *If (N, D) is a line-graph then $Ex(P_D) = \bigcup_{i \in N} \{t^{\{i\}}\}$, with $t^{\{i\}} \in \mathbb{R}^n$ given by $t_j^{\{i\}} = \frac{1}{i}$ if $j \leq i$, and $t_j^{\{i\}} = 0$ if $j > i$.*

Example 3.12

Consider the line-graph (N, D) with $N = \{1, 2, 3\}$ and $D = \{(1, 2), (2, 3)\}$. Then $AC(D) = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$ and thus $P_D = \text{Conv}\{(1, 0, 0)^\top, (\frac{1}{2}, \frac{1}{2}, 0)^\top, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^\top\}$ (see Figure 2). $\qquad\square$
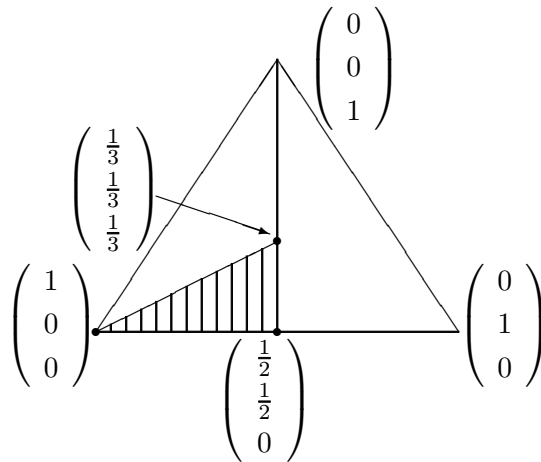

Figure 2

Second, we consider star-graphs, i.e. digraphs (N, D) with $N = \{1, ..., n\}$ such that there is a node i_0 with $D = \{(i_0, i) | i \in N \setminus \{i_0\}\}$. Since $T_S = S \cup \{i_0\}$, we have $AC(D) = \{T \subseteq N | i_0 \in T\}$. Theorem 3.4 (and Corollary 3.9) yield the following corollary.

Corollary 3.13 *If (N, D) is a star-graph with the center i_0, then $Ex(P_D) = \bigcup_{\substack{S \subseteq N \\ i_0 \in S}} \{t^S\}$, with $t^S \in \mathbb{R}^n$ given, as earlier, by $t_j^S = \frac{1}{|S|}$ if $j \in S$, and $t_j^S = 0$ otherwise.*

Example 3.14
Consider the star graph (N, D) with $N = \{1, 2, 3\}$ and $D = \{(1, 2), (1, 3)\}$. Then $AC(D) = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$, and thus $P_D = \text{Conv}\{(1, 0, 0)^\top, (\frac{1}{2}, \frac{1}{2}, 0)^\top, (\frac{1}{2}, 0, \frac{1}{2})^\top, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^\top\}$ (see Figure 3). $\square$
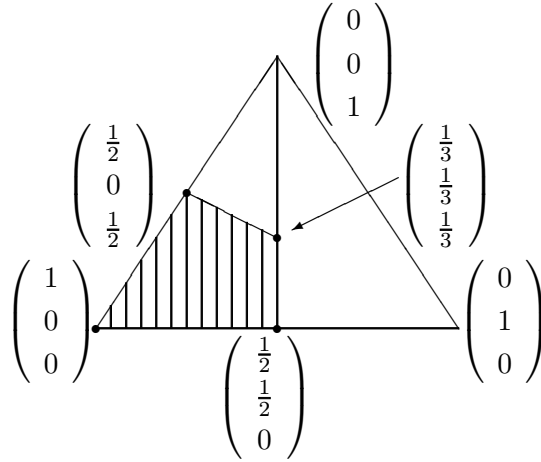

Figure 3

4 Extreme points of the fuzzy polytope

In the previous section we associated to every digraph (N, D) the sharing polytope P_D, being a subset of the unit simplex. Instead of restricting ourselves to the unit simplex, in this section we restrict ourselves to the *unit cube* $I^n = [0, 1]^n = \{p \in \mathbb{R}^n | 0 \le p_i \le 1$ for all $i \in N\}$. Doing so, we obtain the fuzzy polytope.

Definition 4.1 *The fuzzy polytope associated with digraph (N, D), $N \ne \emptyset$, is the polytope K_D given by*

$$K_D = \{p \in I^n \mid p_i \ge p_j \text{ if } (i, j) \in D\}.$$

To give a similar characterization of the extreme points of K_D as given in the previous section for P_D, for any $S \subseteq N$, including the empty set, we define the vector $e^S \in \mathbb{R}^n$ by

$$e_i^S = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \in N \setminus S. \end{cases}$$

9

It appears that the set $Ex(K_D)$ of extreme points of K_D are exactly the vectors e^S that correspond to comprehensive subsets S of N in the digraph (N, D) (including the empty set), i.e. the sets $S \in A(D)$. Observe that in contrast with the sharing polytope it is not required that the comprehensive set is connected.

Theorem 4.2 *For every digraph (N, D) it holds that $Ex(K_D) = \{e^S | S \in A(D)\}$.*

Proof. We first prove that for any $S \in A(D)$ it holds that $e^S \in Ex(K_D)$. Since the vectors e^S are extreme points of the cube I^n, and $K_D \subseteq I^n$, the only thing we have to check is the implication $S \in A(D) \Rightarrow e^S \in K_D$. To do so, fix an arbitrary $S \in A(D), (i, j) \in D$, and note that by comprehensiveness from above of S it holds that: $j \in S$ implies $i \in S$, and hence, $e_i^S \geq e_j^S$. As to the case $j \notin S$, the inequality required is trivially valid due to the nonnegativity of e^S.

As to the inverse inclusion $Ex(K_D) \subseteq \{e^S \mid S \in A(D)\}$, let us mention first that each $a \in Ex(K_D)$ is a zero-one vector, i.e. $a_i \in \{0, 1\}$ for any $a \in Ex(K_D)$ and $i \in N$. Suppose, to the contrary, that $N(a) = \{i \in N \mid a_i \in (0, 1)\} \neq \emptyset$ for some $a \in Ex(K_D)$. Take $\mu = \max\{a_i \mid a_i \in N(a)\}$, $N_\mu(a) = \{i \in N(a) \mid a_i = \mu\}$, and

$$\nu = \begin{cases} \max\{a_i \mid i \in N(a) \setminus N_\mu(a)\}, & \text{if } N(a) \neq N_\mu(a), \\ 0, & \text{if } N(a) = N_\mu(a). \end{cases}$$

Fix some $\delta \in (0, 1)$ such that both numbers $\mu + \delta$ and $\mu - \delta$ belong to the interval $(\nu, 1)$, and define the vectors a' and a'' as follows

$$a_i' = \begin{cases} \mu + \delta, & \text{if } i \in N_\mu(a), \\ a_i, & \text{if } i \in N \setminus N_\mu(a), \end{cases}$$

$$a_i'' = \begin{cases} \mu - \delta, & \text{if } i \in N_\mu(a), \\ a_i, & \text{if } i \in N \setminus N_\mu(a). \end{cases}$$

It is clear that by construction of a' and a'' it holds: $a', a'' \in I^n$, and $a_i' \geq a_j' \Leftrightarrow a_i'' \geq a_j'' \Leftrightarrow a_i \geq a_j$ for any $i, j \in N$. Since $a \in K_D$, both vectors a' and a'' belong to K_D. Since, obviously, $a' \neq a''$ and $a = \frac{1}{2}(a' + a'')$, we get a contradiction with our assumption $a \in Ex(K_D)$. Thus, for any $a \in Ex(K_D)$ it holds $N(a) = \emptyset$ and, consequently, extreme points of the polytope K_D are zero-one vectors.

Consider an arbitrary $a \in Ex(K_D)$ and put $S(a) = \{i \in N \mid a_i = 1\}$. Since $a = e^{S(a)}$, the only thing we have to prove is the inclusion $S(a) \in A(D)$. But the latter follows immediately from the inclusion $a \in K_D$. $\square$

Since the extreme points of the fuzzy polytope correspond to the comprehensive subsets of N in (N, D) and the extreme points of the sharing polytope correspond to the complete

subsets (i.e. non-empty, comprehensive and connected subsets) of N in (N, D), there may be a considerable difference between the numbers of the extreme points of the digraph polytopes P_D and K_D. To illustrate this, let us consider the following example.

Example 4.3
Let (N, D) be a reverse star with $N = \{1, \ldots, n\}$ such that there is a node i_0 with $D = \{(i, i_0) \mid i \in N \setminus \{i_0\}\}$. Due to Theorem 3.4 and Theorem 4.1 we have $Ex(P_D) = \{a^{\{i\}} \mid i \in N \setminus \{i_0\}\} \cup \{a^N\}$, and $Ex(K_D) = \{e^S \mid S \subseteq N \setminus \{i_0\}\} \cup \{e^N\}$. Hence, the cardinality of $Ex(K_D)$ equals $2^{n-1} + 1$, while $Ex(P_D)$ contains only n elements. $\qquad\square$

5 Applications to cooperative games

In this section we consider some applications in cooperative game theory of the two polytopes induced by a digraph. Therefore we first introduce some concepts from cooperative game theory.

A *cooperative game with transferable utility*, or simply a TU-game, is a pair (N, v) with $N = \{1, \ldots, n\}$ a finite set of players and $v: 2^N \to \mathbb{R}$ a *characteristic function* on N satisfying $v(\emptyset) = 0$. For any *coalition* $S \subseteq N$, $v(S)$ is the worth of coalition S, i.e. the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate.[1] Many economic and decision situations can be described as cooperative TU-games, for instance auction situations, assignment problems, linear production situations, sequencing situations, water distribution problems, landing fee problems, and joint inventory situations.[2] We denote the collection of all characteristic functions on player set N by $\mathcal{G}^N$. A special class of games are the unanimity games. For each nonempty $T \subseteq N$, the *unanimity* game (N, u^T) is given by $u^T(S) = 1$ if $T \subseteq S$, and $u^T(S) = 0$ otherwise. It is well-known that the unanimity games form a basis for $\mathcal{G}^N$ and that for each $v \in \mathcal{G}^N$ we have that

$$v = \sum_{S \in \Omega^N} \Delta^S(v) u^S,$$

where Ω^N is the collection of all nonempty subsets of N, and the coefficients $\Delta^S(v)$ are the so-called Harsanyi *dividends* (see Harsanyi, 1959), which can be found recursively from the system

$$v(S) = \sum_{T \subseteq S} \Delta^T(v), \quad S \in \Omega^N. \tag{5.4}$$

---

[1] In the sequel we assume without loss of generality that (N, v) is zero-normalised, i.e. $v(\{i\}) = 0$ for all $i \in N$.

[2] More complex economic situations, like exchange economies, can be modelled as cooperative games without transferable utility.

A *payoff vector* of an n-person TU-game is an n-dimensional vector $x \in \mathbb{R}^n$, giving payoff x_i to player i, $i = 1, \ldots, n$. A payoff vector is called an imputation if it is efficient (meaning that the total payoff is equal to $v(N)$ [3]) and individually rational (meaning that each player i gets at least its own worth $v(\{i\}) = 0$). So, the set $\mathrm{Im}(v)$ of imputations is given by

$$\mathrm{Im}(v) = \left\{ x \in \mathbb{R}^n_+ \ \middle| \ \sum_{i \in N} x_i = v(N) \right\}.$$

A *solution* F on $\mathcal{G}^N$ assigns a set $F(v) \subset \mathbb{R}^n$ of payoff vectors to every characteristic function $v \in \mathcal{G}^N$. [4] A well-known set-valued solution is the *Core*, introduced in game theory by Gillies (1953). The core assigns to every $v \in \mathcal{G}^N$ the (possibly empty) subset of imputations given by

$$C(v) = \left\{ x \in \mathrm{Im}(v) \ \middle| \ \sum_{i \in S} x_i \geq v(S) \text{ for each } S \subseteq N \right\},$$

i.e. $C(v)$ is the set of undominated (meaning that each coalition gets at least its own worth) imputations. So, the core is the set of imputations that are stable against any possible deviation by coalitions. It is well-known that $C(v)$ is non-empty if and only if the game (N, v) satisfies the so-called balancedness condition (see e.g. Bondareva (1963) and Shapley (1967)).

Another well-known solution is the *Selectope*, see Derks, Haller and Peters (2000), also called the *Harsanyi Set*, see Vasil'ev and van der Laan (2002), independently introduced by Hammer, Peled and Sorensen (1977) and Vasil'ev (1978a,b), respectively. This set is the collection of all payoff vectors obtained by distributing the Harsanyi dividends of every coalition S over the players in S in any possible way. To state this precisely, a sharing system on N is a system $q = (q^S)_{S \in \Omega^N}$, where q^S is an $|S|$-dimensional vector assigning a nonnegative share q_i^S to every player $i \in S$ with $\sum_{i \in S} q_i^S = 1$, $S \in \Omega^N$. The collection of all sharing systems on player set N is given by

$$Q^N = \left\{ q = (q^S)_{S \in \Omega^N} \ \middle| \ q \geq 0, \text{ and } \sum_{j \in S} q_j^S = 1 \text{ for each } S \in \Omega^N \right\}.$$

Next, for $v \in \mathcal{G}^N$ and $q \in Q^N$, let the payoff vector $h^q(v) \in \mathbb{R}^n$ be given by

$$h_i^q(N, v) = \sum_{\{S \in \Omega^N | i \in S\}} q_i^S \Delta^S(v), \quad i \in N,$$

i.e. the payoff $h_i^q(v)$ to player $i \in N$ is the sum over all coalitions $S \in \Omega^N$ containing i, of the share $q_i^S \Delta^S(v)$ of player i in the Harsanyi dividend of coalition S. We therefore

---

[3]We assume that the total payoff is maximized if all players cooperate together.

[4]A solution is called single-valued if to every game it assigns exactly one payoff distribution, otherwise it is a set-valued solution.

call such a vector $h^q(v)$ a *Harsanyi payoff vector*. Observe that, due to the equality $v(N) = \sum_{S \in \Omega^N} \Delta^S(v)$, it holds that $\sum_{i \in N} h_i^q(N, v) = v(N)$ for each $q \in Q^N$, and thus each Harsanyi payoff vector is efficient.[5] Now the Selectope or Harsanyi Set is the set $H(v)$ assigning to every $v \in \mathcal{G}^N$ the set of all Harsanyi payoff vectors, i.e.

$$H(v) = \{h^q(N, v) \mid q \in Q^N\}.$$

Clearly, by definition we have that $H(v) \neq \emptyset$ for each v. Note that, for example, the Shapley value, being the single-valued solution which distributes the Harsanyi dividend of every coalition S equally over the players in S, always belongs to the Harsanyi set[6]. It further holds that $C(v) \subseteq H(v)$ with equality if and only if v is almost positive (i.e. $\Delta^S(v) \geq 0$ when $|S| \geq 2$), see e.g. Derks *et.al.* (2000) or Vasil'ev and Van der Laan (2002).

5.1 Harsanyi set for games with ordered players

We are now ready to apply the results of the previous section to *TU-games with ordered players* as studied in van den Brink, van der Laan and Vasil'ev (2004). Here we consider the modified Harsanyi set for a game with ordered players. The Harsanyi set of the game with ordered players (N, v, D) is the collection of all Harsanyi payoff vectors such that the distribution of the dividends takes into account the hierarchical order of the players by requiring that in any coalition S containing two players i and j, the share of player j in the dividend of coalition S is not more than the share of player i in $\Delta^S(v)$ if $(i, j) \in D$. To formalize this, for every $S \in \Omega^N$, let P_D^S be the sharing polytope as defined in Section 3, that is associated to the subdigraph $(S, D(S))$, i.e. P_D^S is the subset of the unit simplex in the $|S|$-dimensional space given by

$$P_D^S = \left\{ p \in \mathbb{R}_+^S \;\middle|\; \sum_{i \in S} p_i = 1, \text{ and } p_i \geq p_j \text{ if } (i, j) \in D(S) \right\},$$

where $\mathbb{R}^S$ is the restriction of $\mathbb{R}^n$ to the components with respect to the players in S. The Harsanyi set $H(v, D)$ of the game with ordered players (N, v, D) is now defined as follows.

Definition 5.1 *The Harsanyi set of the game with ordered players (N, v, D) is the set $H(v, D)$ given by*

$$H(v, D) = \{h^q(N, v) \mid q \in Q^N \text{ such that } q^S \in P_D^S \text{ for each } S \in \Omega^N\}.$$

---

[5]However, $h^q(v)$ does not need to be an imputation.

[6]The Shapley value ψ assigns to any $v \in \mathcal{G}^n$ the payoff vector $\psi(v)$ given by $\psi(v) = \sum_{\{S \in \Omega^N | i \in S\}} \frac{1}{|S|} \Delta^S(v)$ for all $i \in N$.

Clearly, since the collection of possible sharing vectors is restricted by the requirement that for any coalition S the sharing vector $q^S \in P_D^S$, it follows immediately that $H(v, D) \subseteq H(v)$ for any digraph (N, D) (with equality when $D = \emptyset$). Further, note that the set $H(v, D)$ is a polytope. We now apply Theorem 3.4 to find the extreme points of this polytope. To do so, we use the fact that the Harsanyi set is a disjoint additive solution, see van den Brink *et. al.* (2004).[7] From the disjoint additivity it follows that for any game (N, v, D) it holds that

$$H(v, D) = \sum_{S \in \Omega^N} \Delta^S(v) H(u^S, D),$$

i.e. the Harsanyi set of the game (N, v, D) is the weighted sum of the Harsanyi sets of the unanimity games with ordered players with weights equal to the Harsanyi dividends. From formula (5.4) it follows that the dividends of a unanimity game u^S are given by $\Delta^T(u^S) = 1$ if $T = S$ and $\Delta^T(u^S) = 0$ otherwise. Now, for $p \in \mathbb{R}^n$, let p^S be the restriction to $\mathbb{R}^S$, i.e. p^S is the $|S|$-dimensional vector containing the components p_i, $i \in S$, of p. Then it follows that

$$H(u^S, D) = \{p \in U^n \mid p_i = 0 \text{ if } i \notin S, \text{ and } p^S \in P_D^S\}, \quad S \in \Omega^N,$$

i.e. $H(u^S, D)$ is the set of all vectors p in U^n with zero components for the players not in S and with the restriction p^S in the sharing polytope P_D^S associated to $(S, D(S))$ as defined above. Now, for $S \in \Omega^N$, let $AC^S(D)$ be the collection of all complete subsets of players in the subgraph $(S, D(S))$. Then Theorem 3.4 yields the following corollary.

Corollary 5.2 *For $S \in \Omega^N$ and $D \in \mathcal{D}^N$, the extreme points of $H(u^S, D)$ are given by $a^T \in \mathbb{R}^n$, $T \in AC^S(D)$.*

So, the corollary gives the extreme points of the Harsanyi sets of the unanimity games. By using the disjoint additivity, this may be applied to calculate the extreme points of $H(v, D)$ in more general situations, as illustrated in the next example.

Example 5.3
Consider the game with ordered players (N, v, D) with $N = \{1, 2, 3\}$, $v = u^{\{1,2\}} + u^{\{1,3\}}$ and $D = \{(1, 3), (2, 3)\}$ as given in Example 3.8. For this game with ordered players we find $AC^{\{1,2\}}(D) = \{\{1\}, \{2\}\}$ and $AC^{\{1,3\}}(D) = \{\{1\}, \{1, 3\}\}$. According to Corollary 5.2 the set of extreme points of $H(u^{\{1,2\}}, D)$ is $\{(1, 0, 0)^\top, (0, 1, 0)^\top\}$ and the set of extreme

---

[7]For games with ordered players, a solution $\mathcal{F}$ satisfies disjoint additivity if $\mathcal{F}(N, v + w, D) = \mathcal{F}(N, v, D) + \mathcal{F}(N, w, D)$, whenever $\Delta_v(T)\Delta_w(T) = 0$ for all $T \in \Omega^N$. (When $D = \emptyset$, we obtain disjoint additivity for standard TU-games, which is a weak version of additivity.)

points of $H(u^{\{1,3\}}, D)$ is $\{(1,0,0)^\top, (\frac{1}{2}, 0, \frac{1}{2})^\top\}$. Summing up $H(u^{\{1,2\}}, D)$ and $H(u^{\{1,3\}}, D)$ yields

$$H(v, D) = \mathrm{Conv}\left\{(2,0,0)^\top, (1,1,0)^\top, (\frac{1}{2}, 1, \frac{1}{2})^\top, (1\frac{1}{2}, 0, \frac{1}{2})^\top\right\},$$
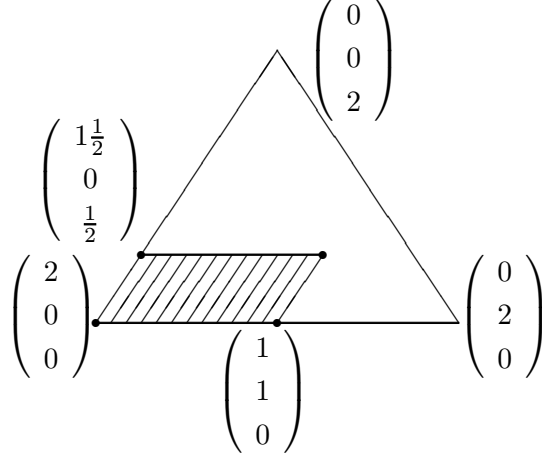
see Figure 4. $\qquad\qquad\square$

Figure 4

5.2 Core for fuzzy games with ordered players

In the approach above we adapted the Harsanyi set for standard TU-games to games with ordered players by taking the hierarchical structure on the set of players to restrict the set of allowable payoff distributions. When applying the core solution to games with ordered players it seems to be more reasonable to restrict the set of allowable deviating coalitions. In particular, in situations that the digraph (N, D) reflects a hierarchical structure, it may happen that a player needs permission of his predecessors to cooperate with other players, i.e. a (multiple player) coalition $S \subseteq N$ may form if and only if for every player in the coalition it holds that all its predecessors in the digraph (N, D) belong to the coalition. So, the set of feasible deviating coalitions is given by the collection of all singletons and all comprehensive multiple player coalitions $A(D)$. Since it is still reasonable to assume that any single player can obtain at least its own worth, the core for a game (N, v, D) with ordered players now becomes

$$C(v, D) = \left\{x \in \mathrm{Im}(v) \,\middle|\, \sum_{i \in S} x_i \geq v(S) \text{ for each } S \in A(D)\right\}, \qquad (5.5)$$

i.e. it is the subset of the imputation set that can not be dominated by a comprehensive coalition. In contrast to $H(v, D) \subseteq H(v)$, we now have $C(v) \subseteq C(v, D)$, since there are more inequalities to be satisfied in $C(v)$ than in $C(v, D)$.

15

In the sequel we are going to present an application of Theorem 4.2 by considering the analog of the set $C(v, D)$ in case of *fuzzy games* with ordered players. The class of cooperative games with *fuzzy coalitions* (shortly fuzzy games) and the corresponding concept of the core for such games is introduced in Aubin (1979). A fuzzy coalition on a player set $N = \{1, \ldots, n\}$ is a vector $s \in I^n = [0, 1]^n$, where component s_i is the *participation rate* of player i. A fuzzy game is a pair (N, f) where the characteristic function f assigns a value $f(s)$ to any fuzzy coalition $s \in I^n$ with $f(s) = 0$ when $s_i = 0$ for all $i \in N$. Observe that $s = e^S$ denotes the case that the players in S have participation rate equal to one and the others zero. So, the restriction of f to the vectors $e^S \in I^n$ induces a standard TU-game (N, v^f), also called *crisp* game, with characteristic function $v^f \in \mathcal{G}^N$ given by $v^f(S) = f(e^S)$, $S \subseteq N$. Again we assume in the sequel that the game is zero-normalized, thus $f(e^{\{i\}}) = v^f(\{i\}) = 0$ for all $i \in N$. The core (Aubin, 1979) of a fuzzy game (N, f), to be denoted by $F(f)$, is the set of all imputations $x \in \mathbb{R}^n$ which are stable against any possible deviation by fuzzy coalitions, i.e.

$$F(f) = \left\{ x \in \text{Im}(f) \ \middle| \ \sum_{i \in N} s_i x_i \geq f(s) \text{ for each } s \in I^n \right\},$$

where the set of imputations is given by $\text{Im}(f) = \left\{ x \in \mathbb{R}_+^n \ \middle| \ \sum_{i \in N} x_i = f(e^N) = v^f(N) \right\}$.

We now consider a fuzzy game with ordered players (N, f, D), where D is the digraph on N reflecting the ordering. According to the approach above, in such a situation it may be reasonable to restrict the set of feasible deviating coalitions (involving multiple players) by requiring that coalition $s \in I^n$ (with at least two positive components) may only deviate when $s_i \geq s_j$ if $(i, j) \in D$, i.e. in a feasible coalition with multiple positive participation rates the participation rate of i is at least as high as the participation rate of j when j is dominated by i. This yields the core for fuzzy games with ordered players given by

$$F(f, D) = \left\{ x \in \text{Im}(f) \ \middle| \ \sum_{i \in N} s_i x_i \geq f(s) \text{ for each } s \in K_D \right\},$$

where K_D is the fuzzy polytope as defined in Section 4.

To demonstrate an application of Theorem 4.2 to the description of the core $F(f, D)$, we introduce first a special class of fuzzy games, which may be of interest in itself. We say that a fuzzy game (N, f) is *proper convex* if $f(\alpha s + (1 - \alpha)t) \leq \alpha f(s) + (1 - \alpha)f(t)$ for all $\alpha \in [0, 1]$ and $s, t \in I^n$. Putting differently, a fuzzy game (N, f) is proper convex if $f : I^n \to \mathbb{R}$ is a convex function.

Below, an equivalence result, related to the cores of the fuzzy and ordinary games with ordered players is given. In particular, this result provides a simplified description of the core $F(f, D)$ in case (N, f) is a proper convex fuzzy game.

16

Corollary 5.4 *Let (N, f, D) be a proper convex fuzzy game with ordered players. Then*

$$F(f, D) = C(v^f, D) = \left\{ x \in \text{Im}(v^f) \;\middle|\; \sum_{i \in S} x_i \geq v^f(S) \text{ for each } S \in A(D) \right\}.$$

Proof. By definition of $F(f, D)$ and $C(v^f, D)$, we have $F(f, D) \subseteq C(v^f, D)$. To prove the opposite inclusion pick some $x \in C(v^f, D)$ and consider an arbitrary fuzzy coalition $s \in K_D$. Due to Theorem 4.2 we have a representation

$$s = \sum_{S \in A(D)} \lambda_S e^S \tag{5.6}$$

with $\lambda_S \geq 0$, $S \in A(D)$, satisfying equality $\sum_{S \in A(D)} \lambda_S = 1$. Hence, it holds

$$\sum_{i \in N} s_i x_i = \sum_{i \in N} \sum_{S \in A(D)} \lambda_S e_i^S x_i = \sum_{S \in A(D)} \sum_{i \in S} \lambda_S x_i = \sum_{S \in A(D)} \lambda_S \Big(\sum_{i \in S} x_i \Big). \tag{5.7}$$

From $x \in C(v^f, D)$ it follows that $\sum_{i \in S} x_i \geq v^f(S)$ for any $S \in A(D)$. Consequently, by equality (5.7) we have

$$\sum_{i \in N} s_i x_i \geq \sum_{S \in A(D)} \lambda_S v^f(S).$$

The latter inequality, together with the convexity of f and representation (5.6) yields

$$\sum_{i \in N} s_i x_i \geq \sum_{S \in A(D)} \lambda_S f(e^S) \geq f\Big(\sum_{S \in A(D)} \lambda_S e^S \Big) = f(s).$$

Hence, for any $s \in K_D$ it holds $\sum_{i \in N} s_i x_i \geq f(s)$, which proves the inclusion $x \in F(f, D)$. Due to the arbitrariness of $x \in C(v^f, D)$ it follows that $C(v^f, D) \subseteq F(f, D)$. $\qquad \square$

The corollary says that the core $F(f, D)$ of a proper convex fuzzy game (N, f, D) with ordered players coincides, in fact, with the (ordinary) core $C(v^f, D)$ of the game (N, v^f, D) with ordered players. Consequently, an imputation is in the core of the proper convex fuzzy game with ordered players if and only if it is stable against any deviation by a comprehensive from above crisp coalition.

5.3 Related literature on games with hierarchical structure

To conclude this paper we want to stress the difference between this approach in which the hierarchical structure restricts the sets of feasible coalitions and the approach as developed in Gilles, Owen and van den Brink (1992), Gilles and Owen (1994), van den Brink and Gilles (1996) and van den Brink (1997) for TU-games with a *permission structure*. In fact, in the approach above we adapted the solution to the situation of ordered players, while

in this literature the game is adapted and standard solution concepts are applied to this adapted game. Both approaches have in common that also in this literature cooperation is restricted in the sense that players need permission from their predecessors before they are allowed to cooperate. However, this permission structure is used to obtain a restricted game in the sense of graph-restricted games as introduced in Myerson (1977) in the context of (undirected) communication graphs. According to the *conjunctive permission approach* it is assumed that each player needs permission from all its predecessors before it is allowed to cooperate with other players[8]. So, a coalition S can only realise the worth of the largest subset of S that contains all its predecessors, i.e. a coalition S can only realise the worth of its largest comprehensive subcoalition, denoted by $\sigma_D(S)$, and given by

$$\sigma_D(S) = \cup\{T \in A(D) \mid T \subseteq S\}.$$

Observe that (i) $\sigma_D(S)$ is a (possibly) empty subset of S in $A(D)$, (ii) $\sigma_D(S) = S$ if and only if $S \in A(D)$ and thus (iii) $\sigma_D(N) = N$, since $N \in A(D)$. The latter property implies that $v^D(N) = v(N)$. For a crisp game with ordered players (N, v, D), the *conjunctive restriction* of v on D is the characteristic function $v^D \colon 2^N \to \mathbb{R}$ given by

$$v^D(S) = v(\sigma_D(S)) \text{ for all } S \subseteq N.$$

So, (N, v^D) is a (crisp) game obtained from (N, v, D) by taking into account the conjunctive permission structure restricting the cooperation possibilities.[9] A subclass of such games is the class of *peer-group games*, see Brânzei, Fragnelli and Tijs (2002).

Now, all solution concepts for TU-games can be applied to the restricted game (N, v^D) to obtain a solution for the game with ordered players (N, v, D). Applying the Core as a solution yields the solution given by

$$C'(v, D) = C(v^D) = \left\{ x \in \operatorname{Im}(v^D) \;\middle|\; \sum_{i \in S} x_i \geq v^D(S) \text{ for each } S \subseteq N \right\}.$$

Since $v^D(N) = v(N)$ for every digraph D, we have that $\operatorname{Im}(v^D) = \operatorname{Im}(v)$ and thus $C'(v, D)$ is a subset of the set of imputations of the game (N, v). When (N, v) is *monotone*[10], we have that $v^D(S) = v(\sigma_D(S)) \leq v(S)$ because $\sigma_D(S) \subseteq S$, and thus $C(v) \subseteq C'(v, D)$.

Concerning the difference between $C(v, D)$ as defined in equation (5.5) and $C'(v, D) = C(v^D)$, observe that the former is obtained by restricting the set of feasible deviating coalitions (thus by adapting the solution concept), while the latter applies the standard core solution to the adapted characteristic function v^D. Remarkably, the following proposition says that both sets are equal.

---

[8]Alternatively, in the *disjunctive permission approach* it is assumed that each player needs permission from at least one of its predecessors before it is allowed to cooperate.

[9]Recall that $v(\emptyset) = 0$, so that (N, v^D) is zero-normalised if (N, v) is zero-normalised.

[10]A TU-game (N, v) is monotone if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$.

Proposition 5.5 *Let (N, v, D) be a zero-normalised game with ordered players. Then $C(v^D) = C(v, D)$.*

Proof. Since $\sigma_D(S) = S$ if (and only if) $S \in A(D)$, any restriction in $C(v, D)$ also appears in $C(v^D)$. Hence $C(v^D) \subseteq C(v, D)$. To show the reverse, let $x \in C(v, D)$, i.e. $\sum_{i \in S} x_i \geq v(S)$ for each $S \in A(D)$. Therefore $\sum_{i \in \sigma_D(S)} x_i \geq v(\sigma_D(S))$ for all $S \subseteq N$. Since $\sigma_D(S) \subseteq S$, it follows with $x_i \geq 0$ for all $i \in N$ that $\sum_{i \in S} x_i \geq \sum_{i \in \sigma_D(S)} x_i \geq v(\sigma_D(S)) = v^D(S)$ for all S, implying that $x \in C(v^D)$. $\square$

The proposition shows that an imputation is in the core $C(v^D)$ of the restricted game (N, v^D) if and only if x is stable in (N, v) against any deviation by a comprehensive from above coalition in the graph (N, D). So, this may lead to a substantial reduction of the number feasibility constraints to be checked whether or not an imputation is in the core of (N, v^D).

References

[1] Aubin, J.P. (1979) Mathematical Methods of Games and Economic Theory, Amsterdam-New York-Oxford: North-Holland.

[2] Bondareva, O. (1963), Some applications of linear programming methods to the theory of cooperative games, Problemy Kibernetiki, 10, 119-139 (in Russian).

[3] Brânzei, R., D. Dimitrov, and S. Tijs (2002), Convex fuzzy games and participation monotonic allocation schemes, Fuzzy Sets and Systems, 139, 267-281.

[4] Brânzei, R., V. Fragnelli, and S. Tijs (2002), Tree connected peer group situations and peer group games, Mathematical Methods of Operations Research, 55, 93-106.

[5] Brink, R. van den (1997), An Axiomatization of the Disjunctive Permission Value for Games with a Permission Structure, Int. J. Game Theory 26, 27-43.

[6] Brink, R. van den, and Gilles, R. P. (1996), Axiomatizations of the Conjunctive Permission Value for Games with Permission Structures, Games and Economic Behav. 12, 113-126.

[7] Brink, R. van den, G.van der Laan and V.A.Vasil'ev (2003), Harsanyi solutions in line-graph games, Tinbergen Discussion Paper 03-076/1, Tinbergen Institute and Free University, Amsterdam.

[8] Brink, R. van den, G.van der Laan and V.A.Vasil'ev (2004), On the distribution of dividends in games with ordered players, Mimeo, Tinbergen Institute and Free University, Amsterdam.

[9] Derks, J., H. Haller, and H. Peters (2000), The Selectope for Cooperative TU-Games, International Journal of Game Theory, 29, 23-38.

[10] Gilles, R. P., and Owen, G. (1994), Cooperative Games and Disjunctive Permission Structures, Department of Economics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia.

[11] Gilles, R. P., Owen, G., and Brink, R. van den (1992), Games with Permission Structures: The Conjunctive Approach, Int. J. Game Theory 20, 277-293.

[12] Gillies D.B. (1953), Some Theorems on n-Person Games, Princeton University Press, Princeton, NJ.

[13] Hammer, P.L., U.N. Peled, and S. Sorensen (1977), Pseudo-Boolean Functions and Game Theory. I. Core Elements and Shapley Value, Cahiers du CERO, 19, 159-176.

[14] J.C. Harsanyi (1959), A bargaining model for cooperative n-person games, in: A.W. Tucker and R.D. Luce (eds.), *Contributions to the Theory of Games IV* (Princeton University Press, Princeton NJ, pp. 325-355.

[15] R.B. Myerson (1977), Graphs and cooperation in games, Mathematics of Operations Research 2, 225-229.

[16] Shapley, L. (1967), On balanced sets and cores, Naval Research Logistics Quarterly, 14, 453-460.

[17] A. Schrijver (1986), Theory of Linear and Integer Programming, Chichester-New York-Brisbane-Toronto-Singapore: John Wiley & Sons.

[18] Vasil'ev, V.A. (1978a), Polynomial cores for cooperative games, Optimizacija, 21, 5-29 (in Russian).

[19] Vasil'ev, V.A. (1978b), Support function of the core of a convex game, Optimizacija, 21, 30-35 (in Russian).

[20] V.A. Vasil'ev (2003), Extreme points of the Weber polyhedron, Discretnyi Analiz i Issledovaniye Operatsyi, Ser.1, v.10, no.2, 17-55 (in Russian).

[21] V.A. Vasil'ev and G. van der Laan (2002), The Harsanyi set for cooperative TU-games, Siberian Advances in Mathematics v.12, no.2, 97-125.