TI 2003-101/1
Tinbergen Institute Discussion Paper

Maximin Play in Two-Person Bimatrix Games

Vitaly Pruzhansky

Department of Econometrics, Faculty of Economics and Business Administration, Vrije Universiteit Amsterdam, and Tinbergen Institute.

Tinbergen Institute
The Tinbergen Institute is the institute for economic research of the Erasmus Universiteit Rotterdam, Universiteit van Amsterdam, and Vrije Universiteit Amsterdam.

Tinbergen Institute Amsterdam
Roetersstraat 31
1018 WB Amsterdam
The Netherlands
Tel.: +31(0)20 551 3500
Fax: +31(0)20 551 3555

Tinbergen Institute Rotterdam
Burg. Oudlaan 50
3062 PA Rotterdam
The Netherlands
Tel.: +31(0)10 408 8900
Fax: +31(0)10 408 9031

MAXIMIN PLAY IN TWO-PERSON BIMATRIX GAMES

VITALY PRUZHANSKY

ABSTRACT. Since the seminal paper of Nash [7] game theoretic literature has focused mostly on equilibrium and not on maximin (minimax) strategies. We study the properties of these strategies in 2-player non-zero-sum strategic games, whose Nash equilibria are only mixed.

1. INTRODUCTION

The interest in this topic was sparkled by an example in Aumann and Maschler [3], which we reproduce here for convenience. Consider the following game.

	L	R
L	$1,0$	$0,1$
R	$0,3$	$1,0$

This game has a unique Nash equilibrium in mixed strategies: $p^* = \left(\frac{3}{4}, \frac{1}{4}\right)$ for player 1 and $q^* = \left(\frac{1}{2}, \frac{1}{2}\right)$ for player 2, where the first number in each bracket is the probability of each player selecting his strategy L. This equilibrium yields the following expected payoffs $v_1^* = \frac{1}{2}$ and $v_2^* = \frac{3}{4}$. The authors argue that "...if the equilibrium point concept is at all convincing, it should certainly be convincing here, where the equilibrium point is unique". However, further discussion shows that the same expected payoffs v_1^* and v_2^* could be *guaranteed* by having each player playing his minimax strategy, namely $\overline{p} = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\overline{q} = \left(\frac{1}{4}, \frac{3}{4}\right)$. On the other hand, randomizing according to the equilibrium probabilities p^* and q^* does *not* guarantee these values. If player 1 is not absolutely sure that his opponent randomizes according to q^*, he

Date: December 10, 2003.

I am thankful to A. Battinelli, J. Kamphorst, V. Protassov and especially to G. van der Laan for extensive comments.

will never commit himself to p^*. The same argument is valid for player 2, of course. The authors conclude that they do not know what to prescribe to play in such a situation, as minimax strategies are not in equilibrium.

This paper shows that the fact that maximin/minimax and equilibrium strategies yield the same expected payoff is not a coincidence. Section 3 characterizes exact conditions when this is the case, as well as related issues. Our main result - Value Equivalence Lemma - is very simple, and it is embarrassing that it was not shown before (at least we were not able to find a reference to anything similar). Section 4 discusses when maximin and equilibrium strategies are the same. For the sake of expositional clarity, these two sections consider only cases when completely mixed equilibria are unique. Section 5 presents extensions of the basic model and Section 6 demonstrates how in this case maximin strategies can be used as a refinement of mixed Nash equilibria.

2. Preliminaries

The primary object of this paper are two-person strategic (bimatrix) games possessing only mixed Nash equilibria, and in particular those that are *completely* mixed[1]. In a generic bimatrix game (A, B) players' payoffs will be given by two square, non-singular matrices of dimensions $n \times n$: A for player 1 and B for player 2. It will be convenient to consider these matrices as the sets of row or column vectors, each vector denoting a corresponding pure strategy of a player. We will index the rows of a matrix by subscripts and the columns by superscripts; thus, a_i (b_i) and a^j (b^j) denote the i-th column and the j-th row of matrix A (B). Sometimes players are also indexed by i and j; no confusion will result. Mixed strategies of both players will be identified with probability vectors in $\mathbb{R}^n$, where the k-th coordinate of such a vector stands for the probability of player's selecting his k-th pure strategy. The set of all mixed strategies of a player is denoted by Δ. Whenever matrix multiplication is involved, correct dimensions are assumed. We also suppress transpose signs, thus for any two vectors x and y, xy denotes their inner product. Throughout, without loss of generality, we will prove only the statements relating to player 1. Those for player

[1] As a consequence, every pure strategy of every player is rationalizable in the sense of Bernheim [4]. However, rationalizability alone does not imply the existence of a completely mixed equilibrium. See example on p. 1012 in [4].

2 are completely symmetric, with the only difference that the words 'rows' and 'columns' should be interchanged.

A pair of mixed strategies (p^*, q^*) is a Nash equilibrium of (A, B) if

$$p^* \in \arg\max_p \ pAq^*.$$
$$q^* \in \arg\max_q \ p^*Bq.$$

Nash equilibria are called completely mixed if p^* puts positive probability on every strategy of player 1 and q^* assigns positive probability to every strategy of player 2. We will say that the game (A, B) is completely mixed if it admits a completely mixed equilibrium. Let v_i^* be player i's payoff in a completely mixed equilibrium (p^*, q^*). The following fact is typical for (p^*, q^*). Given q^*, any two strategies of player 1 yield him the same equilibrium payoff v_1^*. Similarly, given p^*, any two strategies of player 2 yield him v_2^*. Formally

$$\begin{aligned} a_i q^* &= v_1^*, \\ p^* b^j &= v_2^*, \end{aligned} \tag{2.1}$$

for any $i, j = 1, ..., n$. It is well-known that a necessary condition for such an equilibrium to exist is that no pure or mixed strategy of player 1 (2) is dominated by a convex combination of his other strategies. Observe that due to the non-singularity of payoff matrices, generic game (A, B) cannot admit two different completely mixed equilibria.

A strategy $\overline{p} \in \Delta$ is a maximin[2] strategy of player 1, if and only if

$$\overline{p} \in \arg\max_p \left(\min_j \ pAe^j \right), \tag{2.2}$$

where e^j is the unit column vector in $\mathbb{R}^n$, i.e. its j-th coordinate is one and the rest are zero. Thus $\overline{p}$ maximizes the payoff that player 1 can guarantee regardless of the actions of player 2.

Similarly, $\overline{q} \in \Delta$ is a maximin strategy for player 2 if and only if

$$\overline{q} \in \arg\max_q \left(\min_i \ e_i Bq \right), \tag{2.3}$$

where e_i is the unit row vector in $\mathbb{R}^n$. Likewise, such strategy $\overline{q}$ maximizes the minimum of expected utility that player 2 can guarantee against any strategy of player 1. Since the functions $f(p) = \min\{pa^1, ..., pa^n\}$ and $g(q) = \min\{b_1 q, ..., b_n q\}$ are continuous in p and q respectively and the set Δ is compact, any bimatrix game has a pair

[2]We use the term *maximin* for both players. Of course in strictly competitive games, where one player's payoffs are the negative of the other one's, maximinimizing with negative payoffs is equivalent to minimaximizing. Thus, we are in line with the terminology of Aumann and Maschler [3].

of maximin strategies. It is not hard to show that maximin strategies of each player are not dominated and, thus, rationalizable, provided the game (A, B) is completely mixed. Hereafter, $\overline{v}_i$ denotes the expected payoff that player i can guarantee by playing a maximin strategy. Obviously, for any player i, $\overline{v}_i \leq v_i^*$ must hold.

There is a special type of strategies that will play a very important role in the subsequent analysis. These are the strategies that guarantee a player the same payoff, regardless of the pure strategy used by the opponent. For this reason, such strategies will be called column or row equalizers. Formally, a strategy $p \in \Delta$ is a *column* equalizer for player 1 if it enjoys

$$pa^j = u, \text{ for any } j = 1, ..., n, \tag{2.4}$$

and some $u \in \mathbb{R}$. Similarly, a vector $q \in \Delta$ is a *row* equalizer for player 2 if

$$b_i q = w, \text{ for any } i = 1, ..., n, \tag{2.5}$$

and some $w \in \mathbb{R}$.

Necessary conditions for the existence of equalizer strategies are similar to the ones for the existence of completely mixed equilibria. Namely, it is required that no convex combination of columns (rows) of A (B) dominate convex combinations of other columns (rows). However, in the case of equalizers one can state both sufficient and necessary conditions in one formula. Define matrices $\widetilde{A}$, $\widetilde{B}$, row vector $\widetilde{u}$ and column vector $\widetilde{w}$ as follows

$$\begin{aligned} \widetilde{A} &= \begin{pmatrix} a_{11} & \cdots & a_{1n} & 1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 1 \end{pmatrix}, \ \widetilde{B} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \\ 1 & \cdots & 1 \end{pmatrix} \\ \widetilde{u} &= (u, ..., u, 1), \ \widetilde{w} = (w, ..., w, 1). \end{aligned}$$

Then player 1 (2) possesses an equalizer strategy if and only if there exist a non-negative solution p' (q') to the following systems

$$p\widetilde{A} = \widetilde{u} \text{ and } \widetilde{B}q = \widetilde{w}.$$

Similarly to the uniqueness of completely mixed equilibrium strategies p^* and q^* in generic games, the uniqueness of equalizer strategies (once they exist) follows from non-degeneracy of the payoff matrices.

3. Main Results

We start out by characterizing conditions, under which the equalizer strategies are maximin.

Lemma 1. *Let a completely mixed game* (A, B) *possess a column (row) equalizer* p (q). *Then such strategy* p (q) *is the unique maximin strategy of player 1 (2).*

Proof. We need to show that

$$\min_j pAe^j > \min_j xAe^j, \tag{3.1}$$

for any $x \in \Delta$, $x \neq p$. Take such an $x \in \Delta$, and let $y \in \mathbb{R}^n$ satisfy $x = p + y$. Note that $\sum_{i=1}^{n} y_i = 0$. Then

$$\min_j xAe^j = \min_j (pA + yA) e^j.$$

Since p is an equalizer strategy all coordinates of the vector pA are equal. Further, $y \neq 0$ by construction, and $\det(A) \neq 0$ by assumption. Thus inequality (3.1) holds true whenever the vector yA has at least one negative coordinate. This is indeed so, because (A, B) is completely mixed and thus there exists $q^* \in \Delta$ such that each coordinate of the vector Aq^* is equal to v_1^*. Thus

$$yAq^* = \left(\sum_{i=1}^{m} y_i \right) v_1^* = 0.$$

Hence, yA and q^* are orthogonal. Since q^* is completely mixed and lies in the positive orthant $\mathbb{R}^n_+$, at least one coordinate of yA must be negative. □

Following the result of this Lemma, we will use the words 'equalizer' and 'maximin' interchangeably whenever an equalizer strategy exists in a completely mixed game. Consequently, in this case equalizer strategies will be denoted by $\overline{p}$ and $\overline{q}$. Also note that the above proof hinges on the fact that a completely mixed equilibrium (p^*, q^*) exists. This condition cannot be relaxed. One can easily construct examples of games with a dominating strategy, which will also be a maximin strategy for a particular player. Even if an equalizer strategy exists in such a game it will typically be different from the maximin one.

The next Lemma shows that each player can guarantee himself the expected equilibrium payoff by playing the equalizer strategy in a completely mixed game.

Lemma 2. *Let* (p^*, q^*) *be a completely mixed equilibrium of* (A, B) *with payoffs* v_1^* *and* v_2^**. If there exists a column (row) equalizer* $\overline{p}$ $(\overline{q})$ *for player 1 (2), then such equalizer strategy guarantees the equilibrium payoff against any strategy of the opponent, i.e.*

$$\begin{aligned} \overline{p}Aq &= v_1^* \\ pB\overline{q} &= v_2^*, \end{aligned}$$

for all $p, q \in \Delta$.

Proof. Let the equalizer strategy $\overline{p}$ yield $\overline{v}_1$ to player 1. Then

$$\overline{v}_1 = \overline{p}Aq = \overline{p}Aq^*,$$

because, by definition, an equalizer strategy $\overline{p}$ makes player 1 indifferent between any strategy of the opponent. On the other hand, for all $p \in \Delta$

$$v_1^* = pAq^* = \overline{p}Aq^*,$$

as, given q^*, player 1 is indifferent between any of his strategies. Hence, the desired equality $\overline{v}_1 = v_1^*$ follows. □

The following Lemma states that if in a completely mixed game player's maximin strategy is not an equalizer, the guaranteed value $\overline{v}_i$ is strictly smaller than the expected equilibrium value v_i^*.

Lemma 3. *Let* (A, B) *be a completely mixed bimatrix game, such that* $\overline{p}$ *and* $\overline{q}$ *are not equalizers. Then the following is true:* $\overline{v}_1 < v_1^*$ $(\overline{v}_2 < v_2^*)$.

Proof. Let $\overline{p}$ be a maximin strategy of player 1. Because $\overline{p}$ is not an equalizer, for some j it will be the case that $\overline{p}a^j > \overline{v}_1$. Let us arrange the columns of A in such a way that for the first $k < n$ of them $\overline{p}a^j = \overline{v}$ holds, and for the other $n - k$ ones $\overline{p}a^j > \overline{v}_1$ holds. Therefore, $\overline{p}A = u$, where

$$u = (\underbrace{\overline{v}_1, ..., \overline{v}_1}_{k \text{ times}}, u_{k+1}, ..., u_n)$$

with

$$u_j > \overline{v}_1 \text{ for all } j = k + 1, ..., n.$$

In any completely mixed equilibrium (p^*, q^*) it holds that

$$v_1^* = pAq^* \text{ for all } p \in \Delta,$$

and thus

$$v_1^* = \overline{p}Aq^* = uq^* = \overline{v}_1 \left(\sum_{j=1}^{k} q_j^* \right) + \sum_{j=k+1}^{n} q_j^* u_j.$$

Using the fact that

$$\sum_{j=1}^{n} q_j^* = 1 \text{ and } q_j^* \geq 0 \text{ for all } j = 1, ..., n,$$

it follows that $v_1^* > \overline{v}_1$. □

With this background we can state the main result of this Section.

Lemma 4 (Value Equivalence)**.** *In a completely mixed bimatrix game (A, B) player 1 (2) can guarantee the expected payoff from a completely mixed equilibrium v_1^* (v_2^*) by playing a maximin strategy $\overline{p}$ $(\overline{q})$ if and only if such strategy is an equalizer.*

Proof. Necessity follows from Lemma 3 and sufficiency follows from Lemma 2. □

The equality between the player's equilibrium and guaranteed values also has an appealing geometric interpretation. If q^* is a strategy of player 2 in a completely mixed equilibrium, then by definition $a_i q^* = v_1^*$, for any $i = 1, ..., n$. Geometrically it means that all pure strategies of player 1, viewed as n points in $\mathbb{R}^n$, belong to the same hyperplane $h = \{x | xq^* = v_1^*\}$. On the other hand, for all $p \in \Delta$

$$pA \in conv\{a_1, ..., a_n\},$$

where $conv\{a_1, ..., a_n\}$ denotes the convex hull of rows of A, viewed as points in $\mathbb{R}^n$. Clearly, for any $x \in conv\{a_1, ..., a_n\}$ it is also true that $x \in h$. As the equalizer $\overline{p}$ makes player 1 indifferent among any of strategies of player 2, the vector $\overline{p}A$ has all its coordinates equal to $\overline{v}_1$. Moreover, it satisfies the equation of the hyperplane h

$$\overline{p}Aq^* = v_1^*.$$

Since $q^* \in \Delta$, it follows that

$$\overline{v}_1 = v_1^*.$$

Hence, player 1 can guarantee the equilibrium payoff from a completely mixed equilibrium (p^*, q^*) if and only if the diagonal ray in $\mathbb{R}^n$ (the locus of vectors with all coordinates equal) belongs to the convex cone[3] spanned by the rows of A.

What if the game (A, B) has other mixed strategy equilibria, that are not completely mixed? Specifically, let $(\widetilde{p}, \widetilde{q})$ be a mixed Nash equilibrium that assigns zero probability to some rows or columns of (A, B). The argument of the Value Equivalence Lemma still applies with full force to a completely mixed 'reduced' matrix game $\left(\widetilde{A}, \widetilde{B}\right)$ consisting

[3]Recall that $cone\{a_1, ..., a_n\} = \{a \in \mathbb{R}^n | \ a = \lambda_1 a_1 + ... + \lambda_n a_n, \ \lambda_1, ..., \lambda_n \geq 0\}$.

only of those rows and columns, to which $(\widetilde{p}, \widetilde{q})$ assigns positive probability[4]. Thus, once a player possesses an equalizer strategy, which is also maximin in $\left(\widetilde{A}, \widetilde{B}\right)$, it must guarantee him the same payoff as he receives in the equilibrium $(\widetilde{p}, \widetilde{q})$.

We will now formalize the conditions under which the equalizer and maximin strategies, in contrast to p^* and q^*, are not completely mixed.

Lemma 5. *Suppose that there exists a row i (column j), such that $a_{ij} = \alpha$ ($b_{ij} = \beta$) for all $i, j = 1, ..., n$, i.e. row i (column j) offers player 1 (2) the same payoff regardless the strategy of player 2 (1). Then $p = e_i$ ($q = e^j$) is an equalizer strategy of player 1 (2). If, moreover, the game (A, B) is completely mixed, then $\overline{p} = e_i$ ($\overline{q} = e^j$) is also a maximin strategy of player 1 (2).*

Proof. The statement that $p = e_i$ is an equalizer is self-evident. The other statement follows from Lemma 1. □

If the game (A, B) is *not* completely mixed, players may have many equalizers, not all of them being maximin. Also observe that if there are multiple rows of A such that $a_{ij} = \alpha_i$ and $a_{kj} = \alpha_k$ for all $j = 1, ..., n$, then any convex combination of the pure strategies e_i and e_k is a maximin strategy on itself, provided (A, B) possesses a completely mixed equilibrium, of course. (The same logic holds for player 2).

Maximin strategy of player 1 (2) will be pure if one of the hyperplanes in the set $\{pa^1, ..., pa^n\}$ ($\{b_1 q, ..., b_n q\}$) lies below[5] all others. Formally, there exists $k, l = 1, ..., n$ such that

$$\begin{aligned} pa^l &\leq pa^j, \\ b_k q &\leq b_i q, \end{aligned} \tag{3.2}$$

for all $i, j = 1, ..., n$ and all $p, q \in \Delta$. These conditions imply that the l-th column (k-th row) of matrix A (B) is dominated by all other columns (rows). Thus, if player 1 (2) were to minimize the payoff of player 2 (1), then he would choose his k-th (l-th) strategy. Clearly, then a pure best response to the l-th (k-th) strategy of the opponent will be a maximin strategy of player 1 (2). Note that if at least one of the inequalities in (3.2) is strict for all possible $p, q \in \Delta$, such maximin strategies will not be equalizers.

Pure maximin strategies possess the following interesting property in 2×2 (completely mixed) games. Define the function $\delta : \mathbb{R}^n \to \mathbb{R}$ as

$$\delta(x) = \max(x) - \min(x),$$

[4]Note that the reduced game may not be generic in the sense of Section 2.

[5]Of course, the region we have to look at is restricted to Δ.

where $\max(x)$ and $\min(x)$ are, respectively, the values of maximal and minimal coordinates of the vector $x \in \mathbb{R}^n$. Then the following is easily checked.

Lemma 6. *Let (A, B) be a 2×2 completely mixed bimatrix game with maximin strategies $\overline{p} = e_k$ and $\overline{q} = e^l$. Then $\delta(a_k) \leq \delta(a_i)$ and $\delta\left(b^l\right) \leq \delta\left(b^j\right)$ for $i, j = 1, 2$.*

Proof. Omitted. □

These $\overline{p} = e_k$ and $\overline{q} = e^l$ do not have to be equalizers for the result to hold. If they *are* equalizers, however, then $\delta(a_k) = \delta\left(b^l\right) = 0$. The Lemma also requires that (A, B) be completely mixed (cf. Prisoners Dilemma).

Unfortunately Lemma 6 cannot be generalized for games of higher dimensions (even if they are completely mixed), as Example 1 shows.

Example 1. *Let (A, B) be a bimatrix game, in which the payoffs of player 1 are given by*

$$A = \begin{pmatrix} 5.5 & 2 & 1 \\ 4 & 3 & 0 \\ 9 & 0 & 0 \end{pmatrix}$$

The fact that (A, B) is completely mixed (for a suitably defined matrix B) can be verified by computing $q^* = \left(\frac{6}{17}, \frac{10}{17}, \frac{1}{17}\right)$ and $v_1^* = \frac{54}{17}$. Moreover, $\overline{p} = e_1$ and $\overline{v}_1 = 1$. However, $\delta(a_1) = 4.5$ and $\delta(a_2) = 4$.

The last Lemma in this Section provides a useful insight into the nature of equilibrium payoffs in games possessing several mixed Nash equilibria with different supports.

Lemma 7. *Let the game (A, B) possess a completely mixed equilibrium (p^*, q^*) and another equilibrium $(\widetilde{p}, \widetilde{q})$, which is not completely mixed. Suppose player i's payoffs in these two equilibria are v_i^* and $\widetilde{v}_i$, respectively. If player i possesses an equalizer strategy in (A, B), then $v_i^* \leq \widetilde{v}_i$.*

Proof. Without loss of generality let us suppose that the equilibrium strategy $\widetilde{q}$ assigns positive probability to the first $k < n$ pure strategies of player 2. The following two relations follow from the self-evident properties of maximin strategies

$$\begin{aligned} \max_p \left(\min_{j=1,\ldots,n} pAe^j \right) &\leq v_1^*, \\ \max_p \left(\min_{j=1,\ldots,k} pAe^j \right) &\leq \widetilde{v}_1. \end{aligned}$$

Moreover,

$$\max_p \left(\min_{j=1,\ldots,n} pAe^j \right) \leq \max_p \left(\min_{j=1,\ldots,k} pAe^j \right).$$

If player 1 possesses an equalizer strategy in $G(A, B)$, then by Lemma 2

$$v_1^* = \max_p \left(\min_{j=1,\ldots,n} pAe^j \right).$$

Hence,

$$v_1^* \leq \widetilde{v}_1.$$

□

Observe that this result does not depend on the existence of an equalizer for player i in the 'reduced' game $\left(\widetilde{A}, \widetilde{B}\right)$, that is produced by deleting from (A, B) those rows and columns, to which $(\widetilde{p}, \widetilde{q})$ assigns probability zero. In particular, note that $(\widetilde{p}, \widetilde{q})$ may be a pure strategy Nash equilibrium.

Obviously, Lemma 7 can be easily generalized for a sequence of mixed Nash equilibria $\{(p_k^*, q_k^*)\}_{k=1}^K$, such that $\text{supp}\,(p_k^*, q_k^*) \subset \text{supp}\,\left(p_{k+1}^*, q_{k+1}^*\right)$ for all $k = 1, ..., K$, where $\text{supp}\,(p_k^*, q_k^*)$ denotes the set of pure strategies $\{a_1, ..., a_k\} \times \{b^1, ..., b^k\}$ to which the equilibrium (p_k^*, q_k^*) assigns positive probabilities. One only needs the requirement that all reduced games, whose equilibria are indexed by $k = 2, ..., K$, possess equalizers.

Combining Value Equivalence Lemma with Lemma 7 yields the following: if the maximin strategy of player i guarantees him expected equilibrium payoff v_i^*, then v_i^* must be the *lowest* equilibrium payoff to player i the game.

4. Equivalence Between Equilibrium and Maximin Strategies

We now turn to a related issue: when is an equilibrium strategy of a player equivalent to his maximin strategy? In the simplest setting, this case includes dominance solvable games, eg. Prisoners' Dilemma. On a more general level, such games can have several pure strategy equilibria. These games are interesting because they give rise to provoking questions about the inability of pre-play communication to select among pure Nash equilibria[6] (see Aumann [2]) or uncertainty aversion (see Marinacci [6]). Another peculiarity is related to the fact that the

[6]It can be easily shown that in 2×2 games the statement of player i "player j prefers me to play the strategy x no matter what", that is crucial to Aumann's conclusion, implies that player j has a pure maximin strategy.

payoffs in correlated equilibria for the games in this class can lie outside the convex hull of their Nash equilibria payoffs (see Aumann [1] for an example).

In this Section, however, we will be interested in the equivalence between completely mixed Nash equilibrium and maximin strategies that *are* equalizers. It will be shown that this question can be answered algebraically, without even computing the strategies themselves. As before, we restrict attention to player 1. Observe that $\overline{p}$ defines a hyperplane $\overline{h} = \{x|x\overline{p} = v_1^*\}$, to which it is orthogonal[7], and p^* is a normal to another hyperplane $h^* = \{y|yp^* = v_2^*\}$. The question of equality between equalizer and mixed strategies simply reduces to characterizing the conditions when $\overline{h}$ and h^* are parallel, or, as a special case of parallelism, coincide everywhere.

Let us first see when $\overline{h}$ and h^* do not intersect. Observe that any point $x \in \overline{h}$ can be expressed as a linear combination of n vectors a^j, $j = 1, ..., n$, that themselves belong to $\overline{h}$

$$x = \sum_{j=1}^{n} \lambda_j a^j, \text{ where } \sum_{j=1}^{n} \lambda_j = 1.$$

Similarly, any point $y \in h^*$ can be expressed as

$$y = \sum_{j=1}^{n} \mu_j b^j, \text{ where } \sum_{j=1}^{n} \mu_j = 1.$$

Clearly, then $\overline{h}$ and h^* have no point in common if and only if the following linear system has no solution

$$\text{(4.1)} \qquad \begin{cases} A\lambda = B\mu \\ \sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{n} \mu_j = 1. \end{cases}$$

If, however, (4.1) holds for any λ and μ satisfying the above constraint, then $\overline{h}$ and h^* completely coincide. We summarize this result in the following Proposition.

Proposition 1. *Let (A, B) be a completely mixed bimatrix game. Then there exists a probability vector $p \in \Delta$ for player 1 that is both his equalizer (maximin) and mixed equilibrium strategy if the system (4.1)*

[7]Strictly speaking, it must be $\overline{h} = \{x|x\overline{p} = \overline{v}_1\}$, but here Lemma 2 applies and, thus, $v_1^* = \overline{v}_1$.

either has no solutions, or holds for any $\lambda, \mu \in \mathbb{R}^n$ *that satisfy* $\sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{n} \mu_j = 1.$

It is an easy exercise to show that for player 2 the corresponding system is

$$\begin{cases} \lambda A = \mu B \\ \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mu_i = 1. \end{cases}$$

5. Extensions

So far we presented all results for the case when payoff matrices were square and non-singular. The main reason for doing so was to ensure the existence and uniqueness of completely mixed equilibria[8]. In this Section we extend the previous results to the case of arbitrary $m \times n$ payoff matrices. Once we allow for this extension and preserve existence, we have to face the problem of non-uniqueness of Nash equilibrium strategies or equalizers. One rationale for considering these, somewhat degenerate games, is that an interesting refinement of mixed Nash equilibria arises precisely in this case. The next Section is devoted to this issue.

Some of the previous results (Lemma 1) slightly change. For instance, it may generically occur that the player with the highest number of strategies has multiple equalizers, all of which are also maximin (see Example 2 below). Lemmas 2 through 5 and 7 are not affected by the extension. An immediate Corollary of Lemma 2, which we formulate only for player 1 is the following.

Corollary 1. *Let* (p^*, q^*) *and* $(\widetilde{p}, \widetilde{q})$ *be two distinct completely mixed Nash equilibria of* (A, B), *yielding payoffs* v_1^* *and* $\widetilde{v}_1$ *to player* 1. *If player 1 possesses an equalizer strategy* $\overline{p}$, *then* $v_1^* = \widetilde{v}_1$.

Observe that for the Corollary to hold it is not essential whether $m < n$ or vice versa .

Lemma 6 can be generalized for those non-generic games, in which at least one player has only two pure strategies. Specifically, suppose that payoff matrices have dimensions $n \times 2$ $(2 \times n)$, and, in addition to that, the game (A, B) does not have a unique Nash equilibrium in pure

[8]Generically in this case payoff matrices must be square and non-singular. I am thankful to G. van der Laan for drawing my attention to this point.

strategies[9]. It is possible to show that under these assumptions the maximin strategy of player 1 (2), say $\overline{p} = e_k$ $\left(\overline{q} = e^l\right)$, minimizes the payoff dispersion of those pure strategies i (j), which are not (weakly) dominated by $\overline{p}$ $(\overline{q})$. Thus, $\delta\left(a_k\right) \leq \delta\left(a_i\right)$ or $\delta\left(b^l\right) \leq \delta\left(b^j\right)$. Again, the result does not depend on whether the maximin strategies are equalizers or not.

The question of equality between equilibrium and maximin (equalizer) strategies naturally becomes more complicated when such strategies are not unique. The reason is that we need to consider equality not between a pair of strategies but between two sets of strategies. Denote the set of equalizers of player 1 by $\overline{P}$ and the set of his equilibrium mixed strategies by P^*. Using the logic of the previous Section, we see that every $\overline{p} \in \overline{P}$ defines a hyperplane $\overline{h}$, and every $p^* \in P^*$ is a normal to another hyperplane h^*. Let us denote the sets of two types of these hyperplanes in $\mathbb{R}^m$ by $\overline{H}$ and H^*, respectively. Therefore, every $h \in \overline{H}$ contains n vectors that are the columns of A, and every $h \in H^*$ contains n vectors, that are the columns of B.

The sets $\overline{P}$ and P^* are identically equal to each other whenever the following two conditions hold:

i) for every $\overline{p} \in \overline{P}$ there exists $h^* \in H^*$, such that $\overline{p}$ is a normal to h^*.

ii) for every $p^* \in P^*$ there exists $\overline{h} \in \overline{H}$, such that p^* is a normal to $\overline{h}$.

If there is a continuum of either $\overline{p}$ or p^*, or both, then the question of equivalence between $\overline{P}$ and P^* is far from trivial: their intersection may be another set, or a particular vector, or empty. It may be much simpler to compute the sets of these strategies directly and see if they are equal, than to answer this question analytically. There are, however, two special cases, to which we would like to attract attention. The first one arises if $B = cA$, where c is a constant[10]. The second - when $B = A + Z$, where all columns of Z are equal, i.e. $z^j = z^k$ for all $j, k = 1, ..., n$. The reader can easily verify that in these two instances every p that solves (2.4) also satisfies (2.1), and vice versa, provided, of course, that the game (A, B) is completely mixed. Because of linearity, for any combination of the above two cases it also holds that the sets $\overline{P}$ and P^* are identically equal.

The equivalence between equilibrium and equalizer strategies in non-generic games turns out to play an interesting role for exchangeability of Nash equilibria. Recall that two equilibria (p', q') and (p'', q'') are

[9]This requirement substitutes the condition that (A, B) be completely mixed.

[10]Note that strictly competitive games fall in this category with $c = -1$.

said to be exchangeable if the pairs (p', q'') and (p'', q') are also equilibria. It is known from Chin *et al.* [5] that exchangeable equilibria exist if and only if the set of all Nash equilibria of a game is convex. When equilibrium strategies are also equalizers we have the following strengthening of this result:

Proposition 2. *Let (A, B) possess two distinct (not necessarily completely mixed) equilibria (p', q') and (p'', q''), and let players' expected payoffs in these equilibria be v_i' and v_i'' respectively for $i = 1, 2$. If the strategies p', p'', q', q'' are also equalizers, then the following statements are true:*

i) $v_i' = v_i''$ *for $i = 1, 2$; and all equilibria are exchangeable.*
ii) *for any $\alpha, \beta \in [0, 1]$ the pair $(\alpha p' + (1-\alpha)p'', \beta q' + (1-\beta)q'')$ is an equilibrium too.*

Proof. To prove (i) we have

$$\begin{aligned} p'Aq' &= p'Aq'' \leq p''Aq'' \\ p''Aq'' &= p''Aq' \leq p'Aq', \end{aligned}$$

where equalities follow because p' and p'' are equalizers, and the inequalities reflect the fact that p' and p'' are best replies to q' and q'' respectively. Hence, $p'Aq' = p''Aq''$, i.e. $v_1' = v_1''$. Moreover, for player 1 it holds that

$$\begin{aligned} p''Aq' &= p'Aq' \geq pAq' \\ p'Aq'' &= p''Aq'' \geq pAq'' \end{aligned}$$

for all $p \in \Delta$. That is, p'' is a best reply to q' and p' is a best reply to q''. Similar statements can be obtained for player 2. Thus, the pairs (p', q'') and (p'', q') are Nash equilibria too.

To prove (ii), note that for all $\alpha, \beta \in [0, 1]$

$$\beta\left[\alpha p'Aq' + (1-\alpha)\, p''Aq'\right] \geq \beta[pAq']$$

and

$$(1-\beta)\left[\alpha p'Aq'' + (1-\alpha)\, p''Aq''\right] \geq (1-\beta)\left[pAq''\right].$$

It is then a simple algebra to show that

$$(\alpha p' + (1-\alpha)p'')\, A\, (\beta q' + (1-\beta)q'') \geq pA\, (\beta q' + (1-\beta)q'').$$

for any $p \in \Delta$. □

One special class of games where Proposition 2 applies is the case of zero-sum games, where it is well known that mixed equilibrium strategies are also equalizers.

Proposition 1 of the previous Section can also be applied in the non-generic case, however, the interpretation will be somewhat different.

The fact that among the sets $\overline{H}$ and H^* there exist two hyperplanes $\overline{h}$ and h^* that do not intersect (or coincide) will imply that the intersection of $\overline{P}$ and P^* is non-empty. That is, there exists at least one strategy, which is both an equilibrium strategy and an equalizer. Moreover, the above proposition will describe the necessary conditions only. The fact that they are not sufficient will be seen from the extension of Example 2 in the next Section. The intuition for this reasoning is that if all possible $\overline{h}$ and h^* intersect, then clearly their normals cannot be the same. However, if some hyperplanes do not intersect, then one more requirement arises that is not stated by (4.1), namely that their normals belong to the unit simplex. Sometimes this requirement is not satisfied.

6. Maximin Strategies as a Refinement of Mixed Nash Equilibria

Just on its own, the existence of an equalizer strategy for player i strengthens his position in the following two aspects. First, as follows from Corollary 1, i's expected equilibrium payoff does not depend on the particular mixed equilibrium strategy chosen by the opponent, even if the latter possesses infinitely many such strategies. Second, as follows from the Value Equivalence Lemma, player i can guarantee himself the expected equilibrium payoff, should he use the equalizer strategy. These properties become especially attractive when equalizer and equilibrium strategies coincide. In generic bimatrix games this coincidence is rare because both equalizer and equilibrium strategies are unique. However, in games with a continuum of both types of strategies matters become more interesting. In these cases equalizer strategies can be considered as a refinement tool that strengthens those mixed equilibria that *guarantee* the expected equilibrium value to the player and rules out those that do *not* do so. The following simple example demonstrates this idea.

Example 2. *Let us be given the following game*

	L	M	R
T	$4, 1$	$0, 3$	$2, 1$
B	$1, 2$	$3, \frac{1}{2}$	$\frac{3}{2}, 2$

It is possible to show that all (mixed) Nash equilibria (p^*, q^*) are given by the following pair of strategies

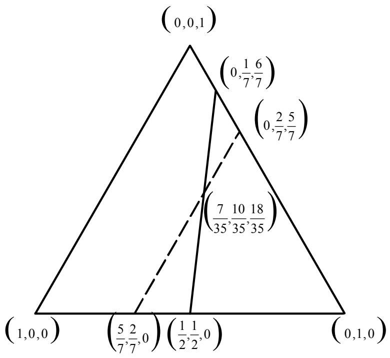
$$p^* = \left(\frac{3}{7}, \frac{4}{7}\right),$$
$$q^* = \left(x, \frac{5}{7}x + \frac{1}{7}, \frac{6}{7} - \frac{12}{7}x\right) \text{ for any } x \in \left[0, \frac{1}{2}\right],$$

and yield the following expected payoffs

$$v_1^* = \frac{12}{7} + \frac{4}{7}x,$$
$$v_2^* = \frac{11}{7}.$$

Column a^3 of matrix A is dominated by a convex combination $\lambda a^1 + (1-\lambda) a^2$ for any $\lambda \in \left(\frac{1}{2}, \frac{3}{4}\right)$, thus player 1 does not have an equalizer strategy. His unique maximin strategy $\overline{p}$ turns out to be equal to p^*. However, it guarantees him just $\overline{v}_1 = \frac{12}{7}$, which is strictly less than in any completely mixed equilibrium.

Player 2 has an equalizer strategy $\overline{q} = \left(y, \frac{2}{7}, \frac{5}{7} - y\right)$for any $y \in \left[0, \frac{5}{7}\right]$. There is a unique point $\widetilde{q} = \left(\frac{7}{35}, \frac{10}{35}, \frac{18}{35}\right)$, such that $q^* = \overline{q} = \widetilde{q}$. This $\widetilde{q}$ is found as the intersection between the sets of $\overline{q}$ and q^* on the unit simplex in $\mathbb{R}^3$. On the picture below the set of Nash equilibrium strategies of player 2 is shown by a solid line, and the set of his equalizers by a broken one.



At the intersection point $\widetilde{q}$, $x = \frac{1}{5}$. If the strategy $\widetilde{q}$ seems to be the most desirable strategy for player 2, then the most likely expected payoff for player 1 is $v_1^* = \frac{12}{7} + \frac{4}{7} \cdot \frac{1}{5} = \frac{64}{35}$.

We now amend the above example to demonstrate the claim of Section 5 that Proposition 1 represents only necessary but not sufficient condition for a player to have a strategy that is both maximin and equilibrium. Let us suppose that player 1's payoff matrix now is

$$A = \begin{pmatrix} 4 & 0 & 5 \\ 1 & 3 & 0 \end{pmatrix},$$

and leave matrix B without changes. It can be verified that the system

$$\begin{cases} \lambda A = \mu B \\ \sum_i^2 \lambda_i = \sum_i^2 \mu_i = 1 \end{cases}$$

does not have a solution, and it can be mistakenly concluded that there is a $q \in \Delta$ such that $q = q^* = \overline{q}$. Indeed, there exists $\widetilde{q} = q^* = \overline{q}$, but it is not a part of the unit simplex in $\mathbb{R}^3$, as the following straightforward computations show

$$\begin{aligned} q^* &= \left(x, \frac{5}{8} - \frac{1}{4}x, \frac{3}{8} - \frac{3}{4}x\right) \text{ for } x \in \left[0, \frac{1}{2}\right], \\ \overline{q} &= \left(y, \frac{1}{3}, \frac{2}{3} - y\right) \text{ for } y \in \left[0, \frac{2}{3}\right], \\ \widetilde{q} &= \left(\frac{7}{6}, \frac{1}{3}, \frac{-1}{2}\right). \end{aligned}$$

References

[1] Aumann, R. (1974) "Subjectivity and Correlation in Randomized Strategies", *Journal of Mathematical Economics*, 1, 67-96.

[2] Aumann, R. "Nash Equilibria are not Self-Enforcing" in *Economic Decision-Making: Games, Econometrics and Optimization*, J.J. Gabszewicz, J.F. Richard and L.A. Wolsey, Eds., Amsterdam, North-Holland 1990, pp.201-206.

[3] Aumann, R., M. Maschler (1972) "Some Thoughts on the Minimax Principle", *Management Science,* 18, 54-63.

[4] Bernheim, B. (1984) "Rationalizable Strategic behavior", *Econometrica,* 52, 1007-27.

[5] Chin, H., T.Parthasarathy, T.E.S.Raghavan (1974) "Structure of Completely Mixed Equilibria in n-person Non-cooperative Games", *International Journal of Game Theory,* 3, 1-19.

[6] Marinacci, M. (2000) "Ambiguous Games", *Games and Economic Behavior*, 31, 191-219.

[7] Nash, J. (1950) "Equilibrium Points in n-person Games", *Proceedings of the National Academy of Sciences* (USA), 36, 48-49.