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On Harsanyi payoff vectors and the Weber set [1] [2]

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Abstract

The paper discusses the set of Harsanyi payoff vectors, also known as the Selectope. First, we reconsider some results on Harsanyi payoff vectors, published by Vasil'ev in the late 1970's, within a more general framework. In particular, these results state already that the set of Harsanyi payoff vectors is given by the core of an associated convex game, a result that recently has been proven by Derks *et. al.* (2000).

The marginal contribution vectors are examples of Harsanyi payoff vectors so that the Weber set, being the convex hull of the marginal contribution vectors, is a subset of the Harsanyi set, which denotes the set of Harsanyi payoff vectors. We provide two characterizations of those Harsanyi payoff vectors that are elements of the Weber set.

Key words: TU-games, Core, Harsanyi set, Weber set, Selectope.

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1 Introduction

A cooperative game with transferable utilities, or simply a game, describes a situation in which players can obtain certain payoffs by cooperation. A solution is a mapping which assigns to every game a set of payoff distributions over the players in the game. Well-known set-valued solutions are the Core and the Weber set, the set of all random order values. In this paper these sets are considered in the context of the set of payoff vectors obtained by all possible distributions of the Harsanyi dividends, see Harsanyi (1959) and (1963), of all coalitions amongst its members. For this reason, and in honor of Harsanyi, we call this set the *Harsanyi set*. In the seventies this set has been discussed independently by Vasil'ev (1978a, 1978b, and 1980), and by Hammer, Peled and Sorensen (1977) as the *Selectope*, being the convex hull of the so-called *selector values*. Vasil'ev's papers are in Russian but a discussion, in English, of most of his results in this context can be found in Vasil'ev (1981).

These papers show that the Harsanyi set encloses the core of the game, and Vasil'ev furthermore proved that this set has a core-type structure, a result that recently has been shown independently in Derks, Haller and Peters (2000). In Hammer *et al.* (1977) the inclusion of the core is shown with the help of a network flow model, whereas Vasil'ev (1978a) applies induction techniques. Here, we will copy the approach in Vasil'ev (1980) and Derks *et al.* (2000), which is based on a convenient adaptation of the game into a convex game. In this way, and with the help of the characterization of the extreme points of the core of convex games in Shapley (1971), not only the inclusion of the core in the Harsanyi set is proved but also its core-type structure is revealed. The question of which of the games have a Harsanyi set that coincide with the core, is solved independently both in Hammer *et al.* (1977) and Vasil'ev (1978b), as being the almost positive games. A characterization of the games where the Harsanyi set coincide with Weber set, is given in Derks *et al.* (2000).

From a historical viewpoint we recall several results already discussed in the papers of Vasil'ev. Some of these results are stated here in a more general framework. They concern the so-called Harsanyi imputations, being the individually rational payoff vectors in the Harsanyi set. We show that if these payoff vectors exist then they form an externally stable, core-typed structured, set.

The marginal contribution vectors are examples of Harsanyi payoff vectors so that the Weber set, being the convex hull of the marginal contribution vectors, is a subset of the Harsanyi set. We will provide two characterizations of the Harsanyi payoff vectors that give rise to elements of the Weber set. In these characterizations a combinatorial result, stated in Vasil'ev (2002), plays a central role. One of the characterizations solves a question concerning monotonicity of weight systems, raised in Derks *et al.* (2000).

2 Preliminaries

A cooperative game with transferable utilities, or simply a game, is a pair (N, v), where $N = \{1, \ldots, n\}$ is a finite set of players, and $v \colon 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$, is the characteristic function yielding for each subset S of N the payoff $v(S)$ that can be achieved if the players in S cooperate. Non-empty subsets of the player set are, therefore, called *coalitions*. For simplicity we denote $v(\{i\})$ by $v(i)$, $i \in N$, and we denote the collection of all non-empty subsets of N by $\Omega = \{S \subseteq N \mid S \neq \emptyset\}$. A *payoff vector* is a vector $x \in \mathbb{R}^n$ assigning payoff $x_i \in \mathbb{R}$ to player $i \in N$. For a payoff vector $x \in \mathbb{R}^n$ and $S \in \Omega$, we denote with $x(S) = \sum_{i \in S} x_i$ the total payoff to the players in coalition S.

In a game (N, v), or v for short, the main issue is the distribution of the worth $v(N)$ of the grand coalition among the players. A payoff vector x is therefore said to be *efficient* if the total payoff $x(N)$ equals $v(N)$; it is said to be *individually rational* if each player $i \in N$ gets at least its own worth $v(i)$. A payoff vector is called an *imputation* if it is both efficient and individual rational; the set of imputations of the game v is denoted by $I(v)$:

$$I(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N), \quad x_i \geq v(i), \ i \in N\}.$$

One may consider the elements of the imputation set as those distributions of the grand coalition worth that 'meet the needs' of the single players. An efficient payoff vector x that satisfies $x(S) \geq v(S)$ for each (multi-person) coalition $S \in \Omega$, is called *stable* for obvious reasons. The set of stable payoff vectors is called the *core* of the game v and is denoted by $C(v)$:

$$C(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N), \quad x(S) \geq v(S), \ S \in \Omega\}.$$

Unfortunately, the core, and also the imputation set, may be empty.

Let $\Pi(N)$ (or Π) denote the set of all permutations $\pi \colon N \to N$ on the player set N. For a permutation $\pi \in \Pi$, assigning rank number $\pi(i) \in N = \{1, 2, \ldots, n\}$ to player $i \in N$, define the set π^i to be $\{j \in N | \pi(j) \leq \pi(i)\}$; it denotes the set of all players with rank number at most equal to the rank number of i, including i itself. Then the *marginal contribution vector* $m^\pi(v) \in \mathbb{R}^n$ of game v and permutation π is given by

$$m_i^\pi(v) = v(\pi^i) - v(\pi^i \setminus \{i\}), \quad i \in N,$$

and thus assigns to player i its marginal contribution to the worth of the coalition consisting of all his predecessors in π.

The well known *Shapley value*, introduced by Shapley (1953), has been characterized as being the average of the marginal contribution vectors over all permutations. It is an element of the convex hull of the marginal contribution vectors of v, denoted by $W(v)$, and is called the *Weber set*. Contrary to the core, the Weber set is always non-empty. It contains the core as a subset, as shown by Weber (1988); it may, however, have no points in common with the imputation set (see Martinez and Rafels, 1998); of course this only occurs in case the core is empty.

The core and the Weber set coincide if and only if the game v fulfills the inequalities $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for every pair of coalitions $S,\ T \in \Omega$ (Shapley, 1971 and Ichiishi, 1981). A game is called *convex* when these inequalities are fulfilled. Examples of convex games are the so called *unanimity games* u^S, $S \in \Omega$, defined by $u^S(T) = 1$ if $T \supseteq S$, and $u^S(T) = 0$ otherwise.

The *dividends* $\Delta^S(v)$, $S \in \Omega$, of the game v, as defined by Harsanyi (1959, 1963), follow recursively from the system of equations

$$v(S) = \sum_{\{T \in \Omega | T \subset S\}} \Delta^T(v), \quad S \in \Omega.$$

It follows that $v = \sum_{S \in \Omega} \Delta^S(v) u^S$, i.e., each game v can be written as a linear combination of the unanimity games. The unanimity games are linearly independent, so that they form a basis for the vector space of characteristic functions; see e.g. Shapley (1953), Rosenmüller (1981), Owen (1982). The in linear spaces more standard basis of unit vectors is formed by the so-called *unity games* $\mathbf{1}^S$, $S \in \Omega$, defined by $\mathbf{1}^S(T) = 1$ if $T = S$ and $\mathbf{1}^S(T) = 0$ for every coalition $T \neq S$.

3 The Harsanyi set

In this section we consider the set of all payoff vectors obtained by distributing the dividend of each coalition S over the players in S. To facilitate this distribution we make use of the weight systems $p = (p_i^S)_{S \in \Omega, i \in S}$, assigning for each $S \in \Omega$ a weight p_i^S to every player $i \in S$. A weight system p is called a *sharing system* if all weights are non-negative, and the weights p_i^S, $i \in S$, sum up to 1 for each coalition $S \in \Omega$, i.e., the collection of sharing systems is given by

$$P = \{p = (p_i^S)_{S \in \Omega, i \in S} \mid p \geq 0,\ \sum_{j \in S} p_j^S = 1,\ \text{for each } S \in \Omega\}.$$

For a game v and sharing system $p \in P$, let the payoff vector $\phi^p(v) \in \mathbb{R}^n$ be given by

$$\phi_i^p(v) = \sum_{\{S \in \Omega | i \in S\}} p_i^S \Delta^S(v), \quad i \in N,$$

i.e., the payoff $\phi_i^p(v)$ to player $i \in N$ is the sum over all coalitions $S \in \Omega$, containing i, of the share p_i^S of player i in the Harsanyi dividend $\Delta^S(v)$ of coalition S. We therefore call the payoff vector $\phi^p(v)$ a *Harsanyi payoff vector*. Observe that, due to the equality $v(N) = \sum_{S \in \Omega} \Delta^S(v)$, for each sharing system p it holds that $\sum_{i \in N} \phi_i^p(v) = v(N)$, and thus each Harsanyi payoff vector is efficient. Examples of Harsanyi payoff vectors are the marginal contribution vectors. To show this, for a permutation $\pi \in \Pi$, let the weight system p^π be given by

$$(p^\pi)_i^S = \begin{cases} 1 & \text{if } i \in S \text{ and } S \subseteq \pi^i, \\ 0 & \text{otherwise,} \end{cases}$$

i.e., for $i \in S$, $(p^\pi)_i^S$ has value 1 for the unique player i of S having the highest rank number in π and has value 0 for all other players in S. So, for each $S \in \Omega$ it holds that $\sum_{j \in S} (p^\pi)_j^S = 1$. Therefore p^π is a sharing system and thus $\phi^{p^\pi}(v)$ a Harsanyi payoff vector. Further,

$$\begin{aligned} \phi_i^{p^\pi}(v) &= \sum_{\{S \subseteq \pi^i | i \in S\}} \Delta^S(v) = \sum_{S \subseteq \pi^i} \Delta^S(v) - \sum_{S \subseteq \pi^i \setminus \{i\}} \Delta^S(v) \\ &= v(\pi^i) - v(\pi^i \setminus \{i\}) = m_i^\pi(v), \quad i \in N, \end{aligned}$$

showing that $m^\pi(v) = \phi^{p^\pi}(v)$. Let $H(v)$ denote the set of all Harsanyi payoff vectors of the game v, i.e.,

$$H(v) = \{\phi^p(v) \mid p \in P\}.$$

This set has been introduced as the so-called *selectope* in Hammer *et al.* (1977). Independently, Harsanyi payoff vectors and the set $H(v)$ have been proposed by Vasil'ev (1978a, 1978b, and 1980). Here, we prefer to call the set $H(v)$ the *Harsanyi set* instead of selectope, because we want to stress the property of distributing the Harsanyi dividends instead of the role of the selectors as recently discussed in Derks *et al.* (2000).

In the remaining of this section we will recall some results on the Harsanyi set and reposition them in the historical context. These results concern the relationship between the core and the Harsanyi set, and the geometrical structure of the Harsanyi set. In particular, we show that the Harsanyi set encloses the core, and has a core-type structure.

To show this, for a game v we consider the corresponding game v_H, defined by

$$\begin{aligned} v_H(S) &= v(S) + \sum_{\{T \in \Omega | T \cap S \neq \emptyset,\ T \setminus S \neq \emptyset,\ \Delta^T(v) < 0\}} \Delta^T(v) \\ &= \sum_{\{T \in \Omega | T \cap S \neq \emptyset,\ \Delta^T(v) < 0\}} \Delta^T(v) + \sum_{\{T \in \Omega | T \subseteq S,\ \Delta^T(v) > 0\}} \Delta^T(v), \quad S \subseteq N. \end{aligned}$$

The first expression says that $v_H(S)$ is equal to the worth $v(S)$ and the sum of all negative dividends of coalitions T containing at least one player in S and at least one player outside S. From this it follows immediately that the game v majorizes the game v_H, i.e., $v(S) \geq v_H(S)$ for all $S \in \Omega$, and $v(N) = v_H(N)$. The second expression shows that $v_H(S)$ is the sum of all positive dividends of coalitions T containing only players in S and all negative dividends of coalitions T having at least one player in common with S. Since a Harsanyi payoff vector distributes the dividends of a coalition T among the players in T, it follows immediately from the second expression that in a Harsanyi payoff vector the payoff to the players in S is minimized when distributing all the negative payoffs of the coalitions T having at least one player in common with S among the players in $S \cap T$, i.e., for each $S \in \Omega$ it holds that $x(S) \geq v_H(S)$ for all $x \in H(v)$. We therefore call v_H the *Harsanyi mingame*. The Harsanyi payoff vectors are core allocations of the Harsanyi mingame. More precisely, we have the next theorem.

Theorem 3.1 (Vasil'ev 1980, Derks *et al.* 2000).
For each game v, the Harsanyi mingame v_H is convex, and $H(v) = C(v_H)$.

Proof.
First, we observe that a game v with the property that for each two coalitions $S, T \in \Omega$, $S \subseteq T$, there exists a core allocation x of v with $x(S) = v(S)$ and $x(T) = v(T)$, has to be convex: for any $S, T \in \Omega$ and core allocation x with $x(S \cap T) = v(S \cap T)$ and $x(S \cup T) = v(S \cup T)$ we have

$$v(S \cap T) + v(S \cup T) = x(S \cap T) + x(S \cup T) = x(S) + x(T) \geq v(S) + v(T).$$

We apply this to the Harsanyi mingame: with little effort one shows that for each two coalitions $S, T \in \Omega$, $S \subseteq T$, there is a Harsanyi payoff vector x in the game v such that $x(S) = v_H(S)$ and $x(T) = v_H(T)$. We mentioned already that the Harsanyi payoff vectors are core allocations of v_H, so that with our first observation we conclude that v_H has to be convex.

Left to prove is the inclusion $C(v_H) \subseteq H(v)$. Consider the Harsanyi payoff vectors x^π, with π any permutation of N, defined by

$$x_i^\pi = \sum_{\{S | i \in S, S \subseteq \pi^i, \Delta^S(v) > 0\}} \Delta^S(v) + \sum_{\{S | i \in S, S \subseteq \pi^{-i}, \Delta^S(v) < 0\}} \Delta^S(v), \quad i \in N,$$

with π^i the already defined set of predecessors of i in π, and $\pi^{-i} = \{j \mid \pi(j) \geq \pi(i)\}$ the set of successors of i. It is straightforward that x^π is a Harsanyi payoff vector, for which $x^\pi(\pi^i) = v_H(\pi^i)$ holds for all $i \in N$. Observe that these equalities also hold for the marginal contribution vector $m^\pi(v_H)$: $\sum_{i \in \pi^i} m_i^\pi(v) = v_H(\pi^i)$, $i \in N$. From this it follows that $x^\pi = m^\pi(v)$, and we conclude that the marginal contribution vectors of v_H are elements of $H(v)$. Since the core of a game is contained in the Weber set, the convex hull of the marginal contribution vectors of the game, we finally conclude that $C(v_H) \subseteq H(v)$. □

The theorem shows that the Harsanyi set $H(v)$ of a game v has a core-typed structure, and can be found as the core $C(v_H)$ of the corresponding Harsanyi mingame v_H. Moreover, since v_H is convex, it follows that the Harsanyi set is equal to the Weber set $W(v_H)$ of v_H and thus equal to the convex hull of the marginal contribution vectors of v_H. In the proof we saw that such a vector, with respect to a permutation π, assign to each player i all positive dividends $\Delta^S(v)$ of coalitions S where i is the last member, and all negative dividends of coalitions where player i is the first member with respect to the ranking π. For a discussion on this kind of *greedy allocations* we refer to Derks *et al.* 2000).

Since the game v majorizes its Harsanyi mingame, and therefore the core of v must be a subset of the core of v_H, we have the following corollary.

Corollary 3.2 (Hammer *et al.* 1977, Vasil'ev 1978a).
For each game v we have $C(v) \subseteq H(v)$.

Finally, let v be called *almost positive* if the dividends of the multi-person coalitions are non-negative. Then we have the following theorem.

Theorem 3.3 (Hammer *et al.* 1977, Vasil'ev 1978b).
The Harsanyi set and the core of a game v coincide if and only if v is almost positive.

Proof.
Let $C(v_H) = C(v)$, and choose, for arbitrary coalition S, a payoff vector $x \in C(v_H)$ such that $x(S) = v_H(S)$. Since $x \in C(v)$ we must have $v_H(S) = x(S) \geq v(S) \geq v_H(S)$, so that we conclude that $v_H = v$. With the definition of v_H in mind it follows that v has to be almost positive.

On the other hand, if v is almost positive, then $v_H = v$, and thus also $C(v_H) = C(v)$.
□

4 The Intersection of the Harsanyi set and the Imputation set

It is very desirable that a payoff vector is efficient and, if possible, also individual rational and thus belongs to the Imputation set $I(v)$. In Martinez and Rafels (1998) it is shown that the intersection of the Weber set and the Imputation set can be empty. This may also be the case for the intersection of the Harsanyi set and the Imputation set. On the other hand, in Derks *et al.* (2000) it was shown that the inclusion of the Harsanyi set in the Imputation set is equivalent to the coincidence of the Harsanyi set and the core, and thus occurs if and only if the game is almost positive.

In this section we show that whenever the intersection is non-empty, it is equal to the core of a well-defined convex game. Further, we show that the intersection, if not empty, is externally stable. These results have first been proved in Vasilev (1980, 1988), and are reconsidered here within a more general framework.

4.1 Existence of Harsanyi imputations

Let $H'(v)$ denote the intersection of the Harsanyi set and the Imputation set, i.e.,

$$H'(v) = \{x \in H(v)|\ x_i \geq v(i),\ i \in N\} = \{x \in C(v_H)|\ x_i \geq v(i),\ i \in N\}.$$

We call its elements Harsanyi imputations.

Let w be a game on player set N, and $z \in \mathbb{R}^N$ an arbitrarily chosen vector. We denote the portion of the core of w that lies above z by $C^z(w)$, i.e.,

$$C^z(w) = \{x \in C(w)|\ x \geq z\}.$$

Clearly, we have $H'(v) = C(v_H) \cap I(v) = C^z(w)$, with $w = v_H$ and $z_i = v(i)$, $i \in N$. We show that $C^z(w)$ is either empty, or is equal to the core of the game w^z defined by

$$w^z(S) = \max_{T \subseteq S}\ \{w(T) + z(S \setminus T)\}, \quad S \subseteq N.$$

Intuitively, the game w^z is observed in case the players are offered to choose between cooperation in the game w or to be paid according to the (not necessarily efficient) payoff vector z.

Theorem 4.1
Let w be a game on player set N and $z \in \mathbb{R}^N$. If there is a coalition T with $w(N) < w(T) + z(N \setminus T)$, then $C^z(w) = \emptyset$; otherwise, $C^z(w) = C(w^z)$.

Proof.
First, consider the case that there exists a coalition $T \subseteq N$ such that $w(N) < w(T) + z(N \setminus T)$. Then for each vector $y \in \mathbb{R}^n$ satisfying $y \geq z$ and $y(T) \geq w(T)$ it holds that $y(N) = y(T) + y(N \setminus T) \geq w(T) + z(N \setminus T) > w(N)$, so that $C^z(w)$ is empty.

Second, let $w(N) \geq w(T) + z(N \setminus T)$ for all $T \subseteq N$. Then it follows that

$$w^z(N) = \max_{T \subseteq N}\ \{w(T) + z(N \setminus T)\} = w(N).$$

Now, suppose that $y \in C^z(w)$. Then we have, for $S \subseteq N$,

$$y(S) = y(T) + y(S \setminus T) \geq w(T) + z(S \setminus T), \quad \text{for all } T \subseteq S,$$

and thus $y(S) \geq w^z(S)$. This shows that $y \in C(w^z)$. On the other hand, for each $y \in C(w^z)$ we have

$$y(S) \geq w^z(S) \geq \max\ \{w(S),\ z(S)\}, \quad S \subseteq N.$$

Together with $y(N) = w^z(N) = w(N)$ this proves that $y \in C(w)$ and $y \geq z$, i.e., $y \in C^z(w)$. Hence $C^z(w) = C(w^z)$. □

The proof shows that the equality $w(N) = w^z(N)$ implies $C^z(w) = C(w^z)$. It should be noticed that $w(N) = w^z(N)$ does not imply that $C^z(w)$ is non-empty. For example, let w be a monotonic game with an empty core (for instance, $w(S) = 1$ for all non-empty coalitions S), and $z = 0$. Then $w^z = w$, implying $w(N) = w^z(N)$. However, $C^z(w) = C(w^z) = C(w) = \emptyset$.

It is interesting to examine which of the properties of w are invariant under the transformation into w^z. In particular, the convexity property is of interest in our context, since we want to apply the previous theorem on the Harsanyi mingame v_H, which is convex. So, suppose w is a convex game, $z \in \mathbb{R}^n$ an arbitrarily chosen vector and let S, T be two arbitrary non-empty coalitions. Then there exist coalitions $U \subseteq S$ and $V \subseteq T$ such that $w^z(S) = w(U) + z(S \setminus U)$ and $w^z(T) = w(V) + z(T \setminus V)$. Substracting the equality $z(U) + z(V) = z(U \cup V) + z(U \cap V)$ from the equality $z(S) + z(T) = z(S \cup T) + z(S \cap T)$ gives

$$z(S) - z(U) + z(T) - z(V) = z(S \cup T) - z(U \cup V) + z(S \cap T) - z(U \cap V).$$

Taking into account that $U \subseteq S$ and $V \subseteq T$ this equality reduces to

$$z(S \setminus U) + z(T \setminus V) = z((S \cup T) \setminus (U \cup V)) + z((S \cap T) \setminus (U \cap V)).$$

Therefore,

$$\begin{aligned} w^z(S) + w^z(T) &= w(U) + z(S \setminus U) + w(V) + z(T \setminus V) \\ &= w(U \cup V) + w(U \cap V) + z((S \cup T) \setminus (U \cup V)) + z((S \cap T) \setminus (U \cap V)) \\ &\leq w^z(S \cup T) + w^z(S \cap T), \end{aligned}$$

and so, also w^z is convex.

Now, consider the convex Harsanyi mingame v_H and take $z \in \mathbb{R}^n$ with $z_i = v(i)$, $i \in N$. Then $(v_H)^z$ is convex and thus $C((v_H)^z)$ is not empty. Applying Theorem 4.1 with $w = v_H$ and $z_i = v(i)$, $i \in N$, we obtain the following result.

Corollary 4.2 (Vasil'ev 1980).
The set $H'(v)$ of Harsanyi imputations of a game v is empty if and only if there is a coalition T with $v(N) < v_H(T) + \sum_{i \notin T} v(i)$. If non-empty, then $H'(v)$ is equal to the core of the convex game $(v_H)^z$, with $z_i = v(i)$, $i \in N$.

In Rafels and Tijs (1997) it is shown that the Weber set of a game v has a nonempty intersection with the set of imputations when v fulfills the very mild condition $v(N) \geq$

$v(T) + \sum_{i \notin T} v(i)$ for all coalitions $T \subseteq N$. For instance, this sufficient condition obviously holds for the large class of 0-monotonic games (v is called 0-monotonic if $v(S) \geq v(T) + \sum_{i \in S \setminus T} v(i)$ for all $T \subseteq S$, $S \subseteq N$). Since $v_H(T) \leq v(T)$ for all $T \subseteq \Omega$, the corollary shows that the set of Harsanyi imputations is not empty *if and only if* v fulfills the weaker condition that $v(N) \geq v_H(T) + \sum_{i \notin T} v(i)$ for all coalitions $T \subseteq N$.

4.2 External Stability

In Rafels and Tijs (1997) it is also shown that the intersection of the Weber set and the imputation set is externally stable (in the sense of Von Neumann-Morgenstern (1944)) under the condition $v(N) \geq v(T) + \sum_{i \notin T} v(i)$ for all coalitions $T \subseteq N$. In this subsection we investigate the external stability of the set $H'(v)$ of Harsanyi imputations, when not empty.

For an imputation $x \in I(v)$, we say that $y \in I(v)$ *dominates* x when there exists a coalition S such that $y_i > x_i$ for all $i \in S$ and $\sum_{i \in S} y_i \leq v(S)$. A set $Y \subseteq I(v)$ is *externally stable* if for each $x \in I(v) \setminus Y$ there exists $y \in Y$ such that y dominates x.

We first prove a more general external stability result in which the notion of a large core plays a central role. We say that a game v has a *large core* if for each vector $x \in \mathbb{R}^n$ majorizing v in the sense that $v(S) \leq x(S)$ for all coalitions $S \in \Omega$, there is an imputation $y \in C(v)$ such that $y \leq x$. Convex games are known to have a large core.

Theorem 4.3
For each game v and convex game v' with $v' \leq v$, the intersection $C(v') \cap I(v)$ is either empty or an externally stable (w.r.t. v) subset of $I(v)$.

Proof.
If $C(v') \cap I(v)$ is empty, the theorem is true. Obviously, the theorem is also true when $I(v) \setminus C(v') = \emptyset$, So, it remains to consider the case $C(v') \cap I(v) \neq \emptyset$ and $I(v) \setminus C(v') \neq \emptyset$. According to Theorem 4.1 we have

$$I(v) \cap C(v') = C(v'')$$

where $v'' = (v')^z$, with $z_i = v(i)$, $i \in N$, i.e.,

$$v''(S) = \max_{T \subseteq S} \{v'(T) + \sum_{i \in S \setminus T} v(i)\}, \quad S \subseteq N.$$

Take an element y of $I(v) \setminus C(v')$. To prove the theorem, we construct an imputation y' of v in $I(v) \cap C(v')$ that dominates y.

Since $y \notin C(v')$, there is at least one coalition S with $y(S) < v'(S)$ and thus also $y(S) < v''(S)$, since $v' \leq v''$. Choose a coalition $\widehat{S}$ such that $y(\widehat{S}) < v''(\widehat{S})$ and $y(T) \geq v''(T)$

for each subcoalition T of $\widehat{S}$. Now, increase all coefficients y_i, $i \in \widehat{S}$, by the same amount $|\widehat{S}|^{-1}(v''(\widehat{S}) - y(\widehat{S}))$, and all other coefficients by a sufficiently large amount such that the thus obtained allocation, say x, majorizes v'', i.e., $v''(T) \leq x(T)$ for all $T \in \Omega$. Since v'' is convex because v' is convex, v'' has a large core. This implies the existence of a vector $y' \in C(v'')$ such that $y' \leq x$. Since

$$v''(\widehat{S}) \leq y'(\widehat{S}) \leq x(\widehat{S}) = \sum_{i \in \widehat{S}} (y_i + |\widehat{S}|^{-1}(v''(\widehat{S}) - y(\widehat{S}))) = v''(\widehat{S}),$$

it follows that $v''(\widehat{S}) = y'(\widehat{S}) = x(\widehat{S})$. With $y' \leq x$ we conclude that $y'_i = x_i$ for all $i \in \widehat{S}$, and thus $y'_i = x_i > y_i$ for all $i \in \widehat{S}$.

Finally, we have to show that $y'(\widehat{S}) \leq v(\widehat{S})$. From $y' \in C(v'')$ we gain $y' \in C(v') \cap I(v)$. Further, $v''(\widehat{S}) = v'(\widehat{T}) + \sum_{i \in \widehat{S} \setminus \widehat{T}} v(i)$ for some $\widehat{T} \subset \widehat{S}$. When $\widehat{T} = \widehat{S}$, then $v''(\widehat{S}) = v'(\widehat{S})$. Suppose $\widehat{T} \neq \widehat{S}$. It follows from $y(T) \geq v''(T)$ for each proper subcoalition T of $\widehat{S}$ and $y_i \geq v(i)$ for all i, that $v''(\widehat{S}) \leq y(\widehat{T}) + \sum_{i \in \widehat{S} \setminus \widehat{T}} y_i = y(\widehat{S}) < v'(\widehat{S}) \leq v''(\widehat{S})$; we arrive at a contradiction. Hence $\widehat{T} = \widehat{S}$ and thus $v''(\widehat{S}) = v'(\widehat{S}) \leq v(\widehat{S})$, so that $y'(\widehat{S}) = v''(\widehat{S}) \leq v(\widehat{S})$. Since $y'_i > y_i$ for all $i \in \widehat{S}$ it follows that y' dominates y, proving that $C(v') \cap I(v)$ is an externally stable subset of $I(v)$. □

To apply this theorem to the set of Harsanyi imputations, recall that $H'(v) = C(v_H) \cap I(v)$, the Harsanyi mingame v_H is convex, and $v_H \leq v$. This gives us the following result:

Corollary 4.4 (Vasil'ev 1988).
The set of Harsanyi imputations of a game is either empty or externally stable.

Together with Corollary 4.2 it follows that the set of Harsanyi imputations is externally stable if and only if $v(N) \geq v_H(T) + \sum_{i \notin T} v(i)$ for all coalitions $T \subseteq N$.

Observe further that, by Corollary 4.4, the Harsanyi imputation set should contain all the non-dominated imputations. These imputations constitute the well-known Domination core. The core is always contained in the Domination core, but not vice versa in general; it may happen that the core is empty whereas the Domination core is non-empty. Therefore, Corollary 4.4 is a refinement of Corollary 3.2.

5 Characterizing the Weber set by Harsanyi payoff vectors

In this section we consider the relation between the set $H(v)$ of Harsanyi payoff vectors and the Weber set $W(v)$. In Derks *et al.* (2000) the games are characterized for which these two sets coincide. For this, a combinatorial approach was needed that is elaborated in Derks and Peters (2002). In this section we will focus on the characterization of those sharing systems for which the corresponding Harsanyi payoff vector is an element of the Weber set.

An element of $W(v)$ is a weighted combination of the marginal contribution vectors, and as such it assigns to each player a weighted combination of his marginal contributions to the coalitions. In order to capture this kind of payoff vector consider a weight system $q = (q_i^S)_{S \in \Omega, i \in S}$ with arbitrarily (possibly negative) values q_i^S for $S \subset \Omega$ and $i \in S$, and define the following payoff vector in an arbitrary game v:

$$\psi_i^q(v) = \sum_{\{S \in \Omega | i \in S\}} q_i^S(v(S) - v(S \setminus \{i\})), \quad i \in N.$$

Efficiency of a payoff vector $\psi_i^q(v)$ is equivalent to

$$q^N(N) = 1 \text{ and } q^S(S) = \textstyle\sum_{j \notin S} q_j^{S \cup j} \text{ for each } S \in \Omega,\ S \neq N, \tag{1}$$

where $q^S(T)$ denotes, as usual, the summation $\sum_{i \in T} q_i^S$, with $S, T \in \Omega$, $T \subseteq S$. For a proof we refer to Weber (1988). The extreme elements of the polytope Q^* consisting of the non-negative weight systems that fulfill (1),

$$Q^* = \{q = (q_i^S)_{S \in \Omega, i \in S} \mid q \geq 0,\ q^N(N) = 1,\ q^S(S) = \sum_{j \notin S} q_j^{S \cup \{j\}},\ S \in \Omega \setminus \{N\}\},$$

are described first in Vasil'ev (2002, in Russian) (see also Vasil'ev and van der Laan (2002) for a similar approach, in English). These are of the following form, with π running through all permutations of N,

$$(q^\pi)_i^S = \begin{cases} 1 & \text{if } \ S = \pi^i, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

This combinatorial result appears to be of crucial importance in the remaining of the paper.

Theorem 5.1 (Vasil'ev 2002).
The extreme points of the polytope Q^* are the elements q^π, $\pi \in \Pi$.

Proof. We sketch the outline of the proof as provided in Derks (2002), to which we refer for further details. The elements q^π, $\pi \in \Pi$ are contained in Q^* and have the property that

for each $q \in Q^*$ there is $\pi \in \Pi$ and scalar $\epsilon > 0$ such that $q \geq \epsilon q^\pi$.

Further, the set Q^* is of the form $F(A, b) = \{x \in \mathbb{R}_+^k \mid Ax = b\}$, with $b \in \mathbb{R}^m$ and A an $(m \times k)$-matrix. With little effort it can be shown that for any (finite) subset F of $F(A, b)$ satisfying

for each $x \in F(A, b)$ there is $y \in F$ and scalar $\epsilon > 0$ such that $x \geq \epsilon y$

it holds that F contains all extreme elements of $F(A,b)$. From this we conclude that the extreme points of Q^* are among the elements $q^\pi,\ \pi \in \Pi$. By symmetry reasoning all these elements are extreme. □

The payoff vectors $\psi^{q^\pi}(v)$ corresponding to the permutations $\pi \in \Pi$ are the marginal contribution vectors of the game v:

$$\psi_i^{q^\pi}(v) = \sum_{\{S\in\Omega \mid i\in S\}} (q^\pi)_i^S(v(S) - v(S\setminus\{i\})) = (v(\pi^i) - v(\pi^i\setminus\{i\})) = m_i^\pi(v).$$

This gives the next theorem.

Theorem 5.2
For each game v the Weber set $W(v)$ is given by

$$W(v) = \{\psi^q(v) \mid q \in Q^*\}. \tag{3}$$

Proof.
One easily shows that for each two $q, \widehat{q} \in Q^*$ and scalar $0 \le \lambda \le 1$ we have

$$\psi^{\lambda q+(1-\lambda)\widehat{q}}(v) = \lambda\psi^q(v) + (1-\lambda)\psi^{\widehat{q}}(v),$$

so that with the help of Theorem 5.1 we may conclude that $\{\psi^q(v) \mid q \in Q^*\}$ is a convex set, spanned by the elements $\psi^q(v)$, with q extreme in Q^*, i.e., $\{\psi^q(v) \mid q \in Q^*\}$ is the convex hull of the elements $\psi^{q^\pi}(v)$, $\pi \in \Pi$, and these payoff vectors are equal to the marginal contribution vectors, so that (3) holds. □

To examine the relation between Harsanyi payoff vectors $\phi_i^p(v) = \sum_{\{S\in\Omega|i\in S\}} p_i^S\Delta^S(v)$, with $p \in P$, and the payoff vectors $\psi^q(v)$ in the Weber set, we first show that for any weight system $p = (p_i^S)_{S\in\Omega, i\in S}$ (p is not restricted to be a sharing system from P) there exists a weight system $q = (q_i^S)_{S\in\Omega, i\in S}$, and reversely, such that

$$\phi^p(v) = \psi^q(v), \quad \text{for all games } v.$$

To show this we use the so called Moebius transformation, which describes a unity game as a weighted sum of the unanimity games: $\mathbf{1}^S = \sum_{\{T|S\subseteq T\}} (-1)^{|T|-|S|}u^T$, $S \in \Omega$.

Theorem 5.3
Let p and q be two weighting systems. Then the following assertions are equivalent:

1). $\phi^p(v) = \psi^q(v)$ *for all games v;* (4)

2). $$q_i^S = \sum_{\{T|S\subseteq T\}} (-1)^{|T|-|S|}p_i^T, \quad \textit{for each } S \in \Omega \textit{ and } i \in S; \tag{5}$$

3). $$p_i^S = \sum_{\{T|S\subseteq T\}} q_i^T, \quad \textit{for each } S \in \Omega \textit{ and } i \in S. \tag{6}$$

Proof.
1) implies **2)**: By applying ϕ^p and ψ^q to the unanimity games, respectively unity games, one simply derives that for each $S \in \Omega$ and $i \in S$ we have $\phi_i^p(u^S) = p_i^S$ and $\psi_i^q(\mathbf{1}^S) = q_i^S$. Since $\phi^p(v) = \psi^q(v)$ for all v it follows that

$$\begin{aligned} q_i^S &= \psi_i^q(\mathbf{1}^S) \\ &= \psi_i^q\left(\sum_{\{T|S\subseteq T\}} (-1)^{|T|-|S|} u^T\right) \\ &= \sum_{\{T|S\subseteq T\}} (-1)^{|T|-|S|} \psi_i^q(u^T) \\ &= \sum_{\{T|S\subseteq T\}} (-1)^{|T|-|S|} \phi_i^p(u^T) \\ &= \sum_{\{T|S\subseteq T\}} (-1)^{|T|-|S|} p_i^T. \end{aligned}$$

2) implies **3)**: Notice that for given set $U \subseteq N$ it holds that $\sum_{\{T|T\subseteq U\}} (-1)^{|U|-|T|}$ equals 1 if U is the empty set and 0 for all other U. Then, for $S \in \Omega$ and $i \in S$, it follows from assertion **2)** that

$$\begin{aligned} \sum_{\{T|S\subseteq T\}} q_i^T &= \sum_{\{T|S\subseteq T\}} \sum_{\{U|T\subseteq U\}} (-1)^{|U|-|T|} p_i^U \\ &= \sum_{\{U|S\subseteq U\}} \sum_{\{T|S\subseteq T\subseteq U\}} (-1)^{|U|-|T|} p_i^U \\ &= \sum_{\{U|S\subseteq U\}} p_i^U \left(\sum_{\{T|T\subseteq U\setminus S\}} (-1)^{|U\setminus S|-|T|}\right) \\ &= p_i^S. \end{aligned}$$

3) implies **1)**: The unanimity games form a basis of the game space. So, it is sufficient to show that the linear solutions ϕ^p and ψ^q coincide on the set of unanimity games. Therefore, consider an arbitrary unanimity game u^S. Since $\Delta^T(u^S) = 1$ if $T = S$ and $\Delta^T(u^S) = 0$ otherwise, it follows that $\phi_i^p(u^S) = 0$ if i is not in S. Also, a player i outside S is a null player in u^S and thus $\psi_i^q(u^S) = 0$. Hence $\phi_i^p(u_S) = \psi_i^q(u^S) = 0$ if $i \notin S$. Further, for $i \in S$, we have $\phi_i^p(u^S) = p_i^S$. and $\psi_i^q(u_S) = \sum_{\{T\in\Omega|i\in T\}} q_i^T(u^S(T) - u^S(T \setminus \{i\})) = \sum_{\{T|S\subseteq T\}} q_i^T$. Now it follows from **3)** that $\phi^p(u^S) = \psi^q(u^S)$. □

Strongly related results can also be found in Vasil'ev (1988) and Dragan (1994). For example, in the latter paper it is showed that for each linear solution ϕ the weight systems p and q, defined by $q^S = \phi(\mathbf{1}^S)$ and $p^S = \phi(u^S)$ for all $S \in \Omega$, fulfill the equations (5) and (6).

Theorem 5.3 simply states that any solution given by a weighted sum of dividends can be transformed to a solution given by a weighted sum of marginal contributions and vice versa. The equations (5) and (6) are the transformation rules between the weight systems p and q. These results will be used in the next theorem for characterizing the Weber set of a game v as a subset of the Harsanyi set. To do so, we define the subset P^* of the set P of sharing systems by

$$P^* = \{p \in P| \sum_{\{T|S\subseteq T\}} (-1)^{|T|-|S|} p_i^T \geq 0,\ S \in \Omega,\ i \in S\}.$$

We now have the following lemma.

Lemma 5.4
Let p and q be two weight systems such that for all $S \in \Omega$ and $i \in S$ it holds that $p_i^S = \sum_{\{T|S\subseteq T\}} q_i^T$. Then $q \in Q^*$ if and only if $p \in P^*$.

Proof.
Suppose $q \in Q^*$. We prove that $p \in P^*$. First, from $q \geq 0$ it follows immediately that $p \geq 0$. Second, from expression (3) we know that $\psi^q(v) \in W(v)$ and thus $\sum_{i\in N} \psi_i^q(v) = v(N)$. Applying this to the unanimity game $v = u^S$ we obtain that

$$\begin{aligned} 1 &= \sum_{i\in N} \psi_i^q(u^S) = \sum_{i\in S} \psi_i^q(u^S) = \sum_{i\in S} \sum_{\{T|S\subseteq T\}} q_i^T (u^S(T) - u^S(T \setminus \{i\}) \\ &= \sum_{i\in S} \sum_{\{T|S\subseteq T\}} q_i^T = \sum_{\{T|S\subseteq T\}} q^T(S). \end{aligned}$$

Hence, for all $S \in \Omega$ it follows that

$$\sum_{i\in S} p_i^S = \sum_{i\in S} \sum_{\{T|S\subseteq T\}} q_i^T = 1,$$

and thus $p \in P$. Finally, from Theorem 5.3 we have that $\sum_{\{T|S\subseteq T\}} (-1)^{|T|-|S|} p_i^T = q_i^S \geq 0$, for all $S \in \Omega$ and $i \in S$ and thus $p \in P^*$.

Next, suppose that $p \in P^*$. We prove that $q \in Q^*$. First, from $p \in P^*$ it follows with the second assertion of Theorem 5.3 that $q_i^S = \sum_{\{T|S\subseteq T\}} (-1)^{|T|-|S|} p_i^T \geq 0$ for all $S \in \Omega$ and $i \in S$. Second, since p is a sharing system it follows that

$$q^N(N) = \sum_{i\in N} \sum_{\{T|N\subseteq T\}} (-1)^{|T|-|S|} p_i^T = \sum_{i\in N} (-1)^{|N|-|N|} p_i^N = 1.$$

Moreover, for a coalition $S \neq N$ it holds that

$$\begin{aligned} q^S(S) &= \sum_{i\in S} \sum_{\{T|S\subseteq T\}} (-1)^{|T|-|S|)} p_i^T \\ &= \sum_{\{T|S\subseteq T\}} (-1)^{|T|-|S|} p^T(S) \\ &= \sum_{\{T|S\subseteq T\}} (-1)^{|T|-|S|} (1 - p^T(T \setminus S)) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\{T|S\subseteq T\}} (-1)^{|T|-|S|} - \sum_{\{T|S\subseteq T\}} \sum_{j\in T\setminus S} (-1)^{|T|-|S|} p_j^T \\
&= 0 + \sum_{j\notin S} \sum_{\{T|S\cup j\subseteq T\}} (-1)^{|T|-|S|-1} p_j^T \\
&= \sum_{j\notin S} q_j^{S\cup j}.
\end{aligned}
$$

This shows that $q \in Q^*$. □

We now arrive at the following conclusion, stating that the Weber set of a game is the subset of the set of Harsanyi payoff vectors when restricting the set of feasible sharing systems to P^*.

Theorem 5.5
For each game v, the Weber set $W(v)$ is the subset of $H(v)$ given by

$$W(v) = \{\phi^p(v)\ |p \in P^*\}.$$

Proof.
The theorem follows from expression (3), Theorem 5.3 and Lemma 5.4. □

We conclude this paper by proving that each sharing system $p \in P^*$ is monotonic, implying that each payoff vector in the Weber set is a Harsanyi payoff vector resulting from a monotonic sharing system. A sharing system $p \in P$ is called *monotonic* if $p_i^S \geq p_i^T$ for all players $i \in N$ and coalitions $S,\ T \in \Omega$ with $i \in S$ and $S \subseteq T$. Further, for $p \in P$, $i \in N$ and coalitions $S,\ T \in \Omega$ with $i \in S$ and $S \subseteq T$, denote

$$M_i^p(S,T) = \sum_{\{U|S\subseteq U\subseteq T\}} (-1)^{|U|-|S|} p_i^U.$$

Then $p \in P$ is called *strong monotonic* if $M_i^p(S,T) \geq 0$, for all $i \in S$ and $S,\ T \in \Omega$ with $S \subseteq T$. Now, strong monotonicity implies monotonicity: clearly, when p is strong monotonic, then for all $S \in \Omega$ and $T = S \cup \{j\}$ for some $j \notin S$, it follows that

$$M_i^p(S, S\cup\{j\}) = (-1)^0 p_i^S + (-1)^1 p_i^{S\cup\{j\}} \geq 0$$

and thus $p_i^S \geq p_i^{S\cup\{j\}}$, implying that p is monotonic. Surprisingly, we have the following result.

Theorem 5.6
A weight system $p \in P$ is strong monotonic if and only if $p \in P^*$.

Proof.
First, when $p \in P$ is strong monotonic, we have for all $S \in \Omega$ and $i \in S$ that

$$M_i^p(S,N) = \sum_{\{U|S\subseteq U\}} (-1)^{|U|-|S|} p_i^U \geq 0.$$

So, when $p \in P$ is strong monotonic, it follows that $p \in P^*$.

To prove the reverse, observe that for $S,\ T \in \Omega$ and player $i \in S \subseteq T$ we have, for any $j \notin T$,

$$\begin{aligned} M_i^p(S,T) &= \sum_{\{U|S\subseteq U\subseteq T\}} (-1)^{|U|-|S|} p_i^U \\ &= \sum_{\{U|S\subseteq U\subseteq T\cup\{j\}, j\notin U\}} (-1)^{|U|-|S|} p_i^U \\ &= \sum_{\{U|S\subseteq U\subseteq T\cup\{j\}\}} (-1)^{|U|-|S|} p_i^U - \sum_{\{U|(S\cup\{j\}\subseteq U\subseteq T\cup\{j\}\}} (-1)^{|U|-|S|} p_i^U \\ &= M_i^p(S, T\cup\{j\}) + M_i^p(S\cup\{j\}, T\cup\{j\}). \end{aligned}$$

Applying this rule two times, for each $k \notin T \cup \{j\}$, we obtain

$$\begin{aligned} M_i^p(S,T) &= M_i^p(S, T\cup\{j,k\}) + M_i^p(S\cup\{k\}, T\cup\{j,k\}) \\ &\quad + M_i^p(S\cup\{j\}, T\cup\{j,k\}) + M_i^p(S\cup\{j,k\}, T\cup\{j,k\}). \end{aligned}$$

Let $h_1, h_2, \ldots, h_{n-|T|}$ be a sequence of elements not in T. Then, continuing as above by subsequently adding h_ℓ to $T \cup \{h_1, \ldots, h_{\ell-1}\}$ for $\ell = 1, \ldots, n-|T|$, we obtain

$$M_i^p(S,T) = \sum_{\{U|U\subseteq N\backslash T\}} M_i^p(S\cup U, N).$$

Since the right hand terms are all nonnegative, when $p \in P^*$, it follows that $M_i^p(S,T) \geq 0$, $i \in S \subseteq T$, and thus p is strong monotonic. □

This result, together with Theorem 5.5, provides us a new characterization of the random order values within the set of Harsanyi payoff vectors.

Corollary 5.7
For each game v, the Harsanyi payoff vector $x \in H(v)$ belongs to the Weber set $W(v)$ if and only if there exists a strong monotonic sharing system p such that $x = \phi^p(v)$.

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