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A Note on Ending Inventory Valuation in Multiperiod Production Scheduling

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In a recent paper, Fisher et al. (2001) present a method to mitigate end-effects in lot sizing by including a valuation term for end-of-horizon inventory in the objective function of the short-horizon model. Computational tests show that the proposed method outperforms the Wagner-Whitin algorithm and the Silver-Meal heuristic, under several demand patterns, within a rolling horizon framework. We replicate the computational tests also including a straightforward method that assumes the same knowledge about future demand as the ending inventory valuation method. Our results indicate that the superior performance reported by Fisher et al. is to a large extent due to the fact that their method assumes that quite accurate knowledge about future demand is available, whereas the traditional methods do not use any information about demand beyond the short model horizon. Moreover, when quite accurate knowledge about future demand is indeed available, our results suggest that for some demand patterns, ending inventory valuation is not the most effective way to use this knowledge. Furthermore, we point out a minor mistake in the results reported by Fisher et al.

(End Effects; Dynamic Lot Sizing; Ending Inventory Valuation)

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1. Introduction

Production planning decisions are usually made on a rolling horizon basis. Because of end-effects, the optimal solution of the short model horizon lot sizing problem may not be the optimal solution in the long run. In particular, in a lot sizing problem it is always optimal to leave zero inventory at the end of the model horizon. To deal with end-effects, Fisher et al. (2001) propose so-called ending inventory valuation, where ending inventory *I* at the end of the horizon *T* is assigned the value

$$V(I) = K - \frac{h}{2D}(EOQ - I)^2 , \qquad (1)$$

where *D* reflects the demand rate (the specific choice depends on the demand pattern; see Section 2), *K* is the setup cost, *h* the unit holding cost, and EOQ the economic order quantity ($EOQ = \sqrt{2KD/h}$). When production in a certain period covers more than the demand up to period *T*, cost are adjusted according to the valuation function. Now a similar recursion as the Wagner-Whitin algorithm (Wagner and Within 1958) can be constructed. Computational tests of Fisher et al. show that the proposed method outperforms the Wagner-Whitin algorithm and the well-known heuristic by Silver and Meal (1973), under several demand patterns, within a rolling horizon framework.

In this note we argue that the superior performance of the method of Fisher et al. is to a large extent due to the fact that it makes use of quite accurate knowledge about future demand, whereas the two traditional methods do not use any information about demand beyond the short model horizon. Hence, to apply the ending inventory valution approach, the required information needs to be available. In general, when reasonable (or even better) estimates of future data are available, it is quite natural to make use of these. In many practical situations this ican actually be done in a quite straightforward way. Indeed, for the lot sizing problem under consideration we can simply extend the short model horizon by using expected demand for the additional periods, as has been proposed before in the literature (see, for instance, Russel and Urban 1993). In this way, the future beyond the short model horizon is taken into account and end-effects may be decreased. This alternative method, to which we will refer as Extended Wagner-Whitin (EWW), can be applied on a rolling horizon basis in a straightforward way.

In the next section we will replicate the computational experiments of Fisher et al., now including EWW, and we will discuss the results. §3 contains some concluding remarks.

2. Comparison with the Extended Wagner-Whitin algorithm

As in Fisher et al. (2001) the performance of the Wagner-Within algorithm (WW), the Silver-Meal heuristic (SM), ending inventory valuation (EIV) and EWW is compared by simulation. The true horizon n is set to 300 periods and for every period in this horizon positive demand is generated. Now for each of the above methods a production plan is constructed using a rolling horizon scheme. The short model horizon T ranges from 2 to 20 periods. For the 300 period lot sizing problem the optimal solution is determined. For each setting of input parameters, we generate eight problem instances and for each lot sizing method the average percentage above optimal cost is computed.

The EIV production plan is constructed in such a way that when arriving at or past period $n - T$ during the rolling horizon scheme, the remaining lot sizing problem is solved in an optimal way so that the ending inventory always equals zero. In this way a fair comparison is made, because the WW algorithm also leaves ending inventory zero.

In the EWW algorithm the short model horizon T is extended up to the true horizon n. This means, for instance, that in period 1 a lot sizing problem consisting of 300 periods is solved and in period 150 a lot sizing problem consisting of 151 periods is solved. Note that this method does not assume more knowledge about future demand than Fisher et al., who also assume that expected demand is known for every future period.

In the following subsections the results of the replicated computational experiments are discussed for different demand patterns.

2.1 Stationary demand

In this case demand is generated from a normal distribution with mean μ = 100 and standard deviation σ = 0, 10, 22 and 43. The setup cost K is set to 800 and the holding cost h is normalized to 1. So the expected production cycle equals $\sqrt{2K/Dh}$ = 4 periods, where D is the long run demand. For the EIV method it is assumed that the long run demand rate is known and so D is set to 100 and EOQ is set to 400 in (1). Also, for EWW, demand for every period beyond the short model horizon is set equal to 100.

Table 1 shows the results. Although the overall performance of EIV is better than EWW, it is important to observe that, especially for model horizons up to 10 periods, the difference between the performance of WW and EWW is typically much larger than the difference between the performance of EWW and EIV. This indicates that the superior performance of EIV in comparison to WW is to a large extent due to the fact that EIV uses information about demand beyond the short model horizon, whereas WW does not. A similar remark can be made with respect to the difference in performance between EIV and SM.

Table 1: Percentage deviation from optimality for normally distributed demand

Model	σ = 0				σ = 10				σ = 22				σ = 43			
Horizon	WW	SM	EIV	EWW	WW	SM	EIV	EWW	WW	SM	EIV	EWW	WW	SM	EIV	EWW
2	28.57	28.57	0.00	0.00	29.33	29.33	15.53	17.17	31.59	31.59	18.25	20.82	37.10	37.10	21.40	24.99
3	4.76	4.76	0.00	0.00	5.30	5.30	15.58	12.24	7.20	7.20	18.13	15.04	11.42	11.64	21.48	19.53
4	0.00	0.00	0.00	0.00	0.54	0.64	0.85	0.54	2.48	2.30	2.05	2.19	6.48	4.07	3.85	4.33
5	2.86	0.00	0.00	0.00	3.37	0.56	0.47	0.59	4.93	1.12	0.82	1.24	7.48	2.30	1.64	2.26
6	4.76	0.00	0.00	0.00	5.21	0.59	0.43	0.48	5.14	0.98	0.68	1.23	5.44	1.67	1.28	1.27
7	4.76	0.00	0.00	0.00	1.61	0.57	0.33	0.48	2.27	0.98	0.70	1.15	2.28	1.65	0.66	0.98
8	0.00	0.00	0.00	0.00	0.56	0.57	0.33	0.53	0.92	0.98	0.35	0.71	1.31	1.65	0.30	0.43
9	0.00	0.00	0.00	0.00	1.09	0.57	0.30	0.37	1.14	0.98	0.25	0.48	1.13	1.65	0.26	0.47
10	4.67	0.00	0.00	0.00	1.69	0.57	0.22	0.47	1.19	0.98	0.28	0.36	1.01	1.65	0.30	0.34
11	4.67	0.00	0.00	0.00	0.60	0.57	0.18	0.40	0.85	0.98	0.20	0.36	0.67	1.65	0.12	0.17
12	0.00	0.00	0.00	0.00	0.36	0.57	0.19	0.36	0.42	0.98	0.17	0.31	0.29	1.65	0.14	0.19
13	0.00	0.00	0.00	0.00	0.44	0.57	0.19	0.29	0.32	0.98	0.11	0.26	0.29	1.65	0.07	0.12
14	4.57	0.00	0.00	0.00	0.75	0.57	0.12	0.25	0.42	0.98	0.12	0.15	0.29	1.65	0.04	0.08
15	4.57	0.00	0.00	0.00	0.44	0.57	0.13	0.20	0.26	0.98	0.08	0.11	0.15	1.65	0.06	0.06
16	0.00	0.00	0.00	0.00	0.25	0.57	0.12	0.24	0.24	0.98	0.07	0.09	0.09	1.65	0.03	0.07
17	0.00	0.00	0.00	0.00	0.38	0.57	0.09	0.18	0.20	0.98	0.05	0.13	0.08	1.65	0.05	0.06
18	4.57	0.00	0.00	0.00	0.43	0.57	0.08	0.15	0.15	0.98	0.05	0.05	0.08	1.65	0.01	0.02
19	4.57	0.00	0.00	0.00	0.24	0.57	0.05	0.18	0.10	0.98	0.05	0.06	0.02	1.65	0.01	0.01
20	0.00	0.00	0.00	0.00	0.12	0.57	0.08	0.14	0.05	0.98	0.03	0.04	0.06	1.65	0.01	0.03

Note: **μ = 100, K = 800, h = 1**

WW: Wagner Whitin, SM: Silver Meal, EIV: Ending inventory valuation, EWW: Extended Wagner Whitin

Both EWW and EIV produce optimal quantities in the case of a constant demand rate. The same holds for the SM heuristic if the model horizon is sufficiently large. The WW method, however, shows a striking pattern. Moreover, this pattern differs from what is reported by Fisher et al. (no error for T larger than 3). It can easily be verified that the latter results are incorrect. In the optimal production plan production takes place every 4 periods incurring cost 1400/4 = 350 per period, while for model horizon T = 5 it is optimal to produce for 5 periods incurring total cost of 1800/5 = 360 per period, which is 2.87% above minimal cost. For T = 6 it is optimal to produce twice for 3 periods in periods 1 and 4 so that the first production decision equals 3 periods incurring cost 1100/3 = 367 per period, which is 4.76% above minimal cost. The same computations can be made for the other model horizons. Note that in some cases multiple optimal solutions exist (e.g., for T = 7: produce for 3 and then for 4 periods or produce for 4 and then for 3 periods) and that it depends on the way of implementing the Wagner-Whitin algorithm, which solution is selected. In our case the algorithm is implemented such that the first production decision is as short as possible. Hence, contrary to what Table 1 in Fisher et al. may suggest, when demand is constant and the short term models are solved exactly, it is not true that using a larger value of T will never lead to long term solutions that are worse. Moreover, it is not true that a long term optimal solution will always result if T can be chosen large enough.

A counterintuitive pattern in Table 1 is that when the standard deviation of demand increases WW, EIV and EWW perform better for longer model horizons. Although Fisher et al. also observe this phenomenon, they do not give an explanation. We note, however, that Federgruen and Tzur (1994) already observed that minimal forecast horizon tend to be smaller in case of more variability and that they also gave a plausible explanation for this latter phenomenon.

2.2 Linearly increasing and linearly decreasing demand

Now demand is generated by adding a normally distributed random variable to a linearly increasing or linearly decreasing trend, i.e. demand in period t equals

$$d_t = \mu + \sigma x_t + tf(t-1), \; t = 1, \ldots, 300 \qquad (2)$$

where x_t is a normally distributed random variable and tf is the trend factor. We generate problem instances with μ = 100, σ = 10, K = 800, h = 1 and tf = 1, 10, 20 and 40. To determine the ending inventory in the EIV method in period t, D is replaced by the expected demand in period $t + T$, which equals $\mu + tf(t + T - 1)$ and EOQ is replaced by the corresponding economic order quantity. For EWW, we estimate demand beyond the short model horizon by substituting $x_t = 0$ in (2).

We only present the results for linearly deceasing demand and trend factors 1 and 10. For trend factors 20 and 40 the tables mainly consist of zeros because when t in (2) is sufficiently large, demand is large relative to the setup cost, so it is always optimal to produce each period and each method produces a (near) optimal solution. The results for linearly decreasing demand and linearly increasing demand are similar.

Table 2: Percentage deviation from optimality for a demand distribution with linearly decreasing trend

Model	tf = 1				tf = 10			
Horizon	WW	SM	EIV	EWW	WW	SM	EIV	EWW
2	5.14	5.14	18.57	9.59	0.46	0.46	1.34	0.78
3	3.39	0.59	3.19	1.27	1.27	0.15	0.29	0.22
4	1.68	0.22	0.17	0.13	0.23	0.10	0.12	0.01
5	3.03	0.29	0.22	0.14	1.07	0.09	0.06	0.01
6	0.33	0.24	0.10	0.13	0.03	0.09	0.04	0.01
7	1.54	0.24	0.14	0.12	0.72	0.09	0.02	0.01
8	0.60	0.24	0.11	0.13	0.13	0.09	0.04	0.00
9	0.76	0.24	0.06	0.07	0.72	0.09	0.03	0.00
10	0.47	0.24	0.07	0.06	0.03	0.09	0.03	0.01
11	0.53	0.24	0.07	0.08	0.58	0.09	0.01	0.00
12	0.10	0.24	0.07	0.06	0.05	0.09	0.02	0.02
13	0.51	0.24	0.08	0.06	0.39	0.09	0.02	0.01
14	0.16	0.24	0.04	0.04	0.03	0.09	0.01	0.00
15	0.25	0.24	0.06	0.06	0.32	0.09	0.02	0.01
16	0.13	0.24	0.03	0.02	0.01	0.09	0.01	0.01
17	0.26	0.24	0.02	0.02	0.47	0.09	0.01	0.00
18	0.09	0.24	0.05	0.02	0.02	0.09	0.01	0.01
19	0.22	0.24	0.04	0.02	0.31	0.09	0.01	0.00
20	0.12	0.24	0.06	0.05	0.00	0.09	0.01	0.01

Note: μ **= 100,** σ **=10, K = 800, h=1.**

The results for decreasing demand can be found in Table 2. We see in this table that for trend factor 1 EWW outperforms WW and SM for almost all model horizons and that EWW performs (slightly) better than EIV for most model horizons. For trend factor 10 the results are even more favorable for EWW. Hence, these results suggest that when demand faces a linearly decreasing trend EWW is a better choice than EIV. An explanation for the fact that EWW performs better than EIV is that when arriving in period t in the rolling horizon scheme, EIV actually assumes a constant demand after period $t + T - 1$, while EWW takes the expected trend into account.

2.3 Seasonal demand

For this case demand is generated according to the following formula:

$$d_t = \mu + \sigma x_t + a \sin[2\pi/c(t + c/4)],\ t = 1,\ldots, 300, \qquad (3)$$

where a is the amplitude of the seasonal component and c is the length of the seasonal cycle. To account for the seasonal demand pattern in period t for determining ending inventory in the EIV algorithm, define the periodic order quantity POQ as EOQ/μ. Now D in (1) is replaced by the average demand in the first POQ periods immediately following time $t + T - 1$ and EOQ is replaced by the economic order quantity corresponding with this average demand. To compare the different solution methods the same settings are used as in Fisher et al. (2001), i.e. $\mu = 100$, $\sigma = 10$, $K = 800$, $h = 1$, $a = 20, 40, 60, 80$ and $c = 12$. Again for each lot sizing method the average deviation from optimality is computed using eight iterations for each parameter setting.

The results of the simulations are shown in Table 3, where we see that EWW outperforms EIV, especially when the amplitude of the seasonal component increases. Again we explain the better performance of EWW by the fact that it incorporates the seasonal component in a better way. EIV averages the first POQ periods of expected demand to account for its cyclical behavior, but EWW incorporates the actual expected cyclical behavior.

Table 3: Percentage deviation from optimality for a seasonal demand distribution

Model	A = 20				a = 40				a = 60				a = 80			
Horizon	**WW**	**SM**	**EIV**	**EWW**	**WW**	**SM**	**EIV**	**EWW**	**WW**	**SM**	**EIV**	**EWW**	**WW**	**SM**	**EIV**	**EWW**
2	29.82	29.82	17.09	16.13	31.09	31.09	16.68	16.21	34.98	34.98	15.89	15.29	40.77	40.77	14.96	15.92
3	5.85	5.85	17.59	14.29	6.85	6.85	18.69	10.75	9.93	9.93	17.17	6.51	14.89	14.89	17.99	4.34
4	0.95	0.97	3.49	0.87	2.04	2.02	5.31	4.12	4.76	4.40	4.63	4.04	9.62	8.37	5.22	4.06
5	3.77	0.85	0.64	0.41	4.19	1.72	1.27	0.64	6.69	3.96	2.34	3.74	9.64	7.84	4.92	3.76
6	4.25	0.78	0.46	0.44	2.52	1.41	0.99	0.31	1.97	2.32	1.61	0.25	2.92	2.89	2.43	0.88
7	2.07	0.78	0.67	0.52	2.52	1.33	0.90	0.28	2.32	3.22	1.55	0.11	1.98	3.71	2.01	0.24
8	0.62	0.78	0.38	0.33	1.15	1.96	0.64	0.26	2.13	3.12	0.55	0.11	3.00	3.60	0.70	0.14
9	1.10	0.78	0.27	0.26	1.07	1.69	0.53	0.21	1.70	3.49	0.24	0.05	2.08	4.96	0.54	0.20
10	1.74	0.78	0.22	0.25	1.52	169	0.40	0.18	1.04	3.25	0.44	0.07	0.83	5.15	0.36	0.10
11	0.80	0.78	0.27	0.18	0.99	1.69	0.28	0.11	0.81	3.25	0.20	0.05	0.64	4.93	0.22	0.10
12	0.40	0.78	0.22	0.17	0.40	1.69	0.21	0.14	0.23	3.25	0.27	0.08	0.30	4.93	0.48	0.09
13	0.55	0.78	0.17	0.16	0.49	1.69	0.18	0.07	0.44	3.25	0.34	0.05	0.38	4.93	0.47	0.05
14	0.77	0.78	0.16	0.12	0.64	1.69	0.16	0.08	0.47	3.25	0.15	0.05	0.71	4.93	0.20	0.06
15	0.43	0.78	0.11	0.06	0.40	1.69	0.12	0.08	0.39	3.25	0.10	0.06	0.63	4.93	0.15	0.05
16	0.29	0.78	0.15	0.11	0.21	1.69	0.13	0.06	0.24	3.25	0.18	0.02	0.46	4.93	0.23	0.03
17	0.25	0.78	0.09	0.07	0.26	1.69	0.09	0.03	0.18	3.25	0.11	0.03	0.23	4.93	0.17	0.02
18	0.39	0.78	0.10	0.08	0.20	1.69	0.11	0.03	0.13	3.25	0.03	0.02	0.08	4.93	0.10	0.01
19	0.27	0.78	0.07	0.04	0.18	1.69	0.10	0.05	0.13	3.25	0.06	0.01	0.14	4.93	0.08	0.02
20	0.15	0.78	0.04	0.03	0.11	1.69	0.05	0.03	0.07	3.25	0.03	0.00	0.11	4.93	0.04	0.02

Note: μ = 100, σ =10, b = 12, K = 800, h = 1

3. Concluding remarks

In our opinion, the results that Fisher et al. find for their ending inventory valuation method are not surprising once one realizes that this method uses additional information. It seems to make more sense to compare it with other methods that make use of the same information, such as EWW. For non-stationary demand, the performance of EWW actually appears to be superior to EIV. Note that EWW can easily be applied in practice, since it will often suffice to extend the short model horizon with only a limited number of periods instead of the "complete future". This follows from the computational study of Federgruen and Tzur (1994) which indicates that minimal forecast horizons tend to be small. Furthermore, computation times are not an issue since the extended lot sizing problems can be solved in linear time (Federgruen and Tzur 1991, Wagelmans et al. 1992).

Of course, one could argue that the simulation experiments do not describe a very realistic situation, if only because the parameters of the demand patterns are not known but have to be estimated, for instance by using smoothing methods (Makridakis and Wheelwright. 1986). We have run additional experiments to investigate if this would change the performance of EIV and EWW. Our conclusion from the results of these experiments is that the performance of the two methods does not change significantly. For details on the results of these and other additional experiments, we refer to Van den Heuvel (2002).

Finally, it is worth metnioning that Stadtler (2001) proposes an alternative method for lot sizing in a rolling horizon environment, that produces very good results without the assumption of the availability of good estimates of future demand.

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