TI 2002-009/4
Tinbergen Institute Discussion Paper

Equivalent Results in Minimax Theory

J.B.G. Frenk[1]
G. Kassay[2]
J. Kolumbán[2]

[1] *Econometric Institute, Faculty of Economics, Erasmus University Rotterdam, and Tinbergen Institute,*
[2] *Department of Mathematics, Babes Bolyai University, Cluj*

Tinbergen Institute

The Tinbergen Institute is the institute for economic research of the Erasmus Universiteit Rotterdam, Universiteit van Amsterdam and Vrije Universiteit Amsterdam.

Tinbergen Institute Amsterdam
Keizersgracht 482
1017 EG Amsterdam
The Netherlands
Tel.: +31.(0)20.5513500
Fax: +31.(0)20.5513555

Tinbergen Institute Rotterdam
Burg. Oudlaan 50
3062 PA Rotterdam
The Netherlands
Tel.: +31.(0)10.4088900
Fax: +31.(0)10.4089031

Equivalent Results in Minimax Theory.

J.B.G.Frenk
Econometric Institute, Erasmus University, Rotterdam

G.Kassay and J.Kolumbán
Department of Mathematics, Babes Bolyai University,Cluj

January 14, 2002

Abstract

In this paper we review known minimax results with applications in game theory and show that these results are easy consequences of the first minimax result for a two person zero sum game with finite strategy sets published by von Neumann in 1928. Among these results are the well known minimax theorems of Wald, Ville and Kneser and their generalizations due to Kakutani, Ky-Fan, König, Neumann and Gwinner-Oettli. Actually it is shown that these results form an equivalent chain and this chain includes the strong separation result in finite dimensional spaces between two disjoint closed convex sets of which one is compact. To show these implications the authors only use simple properties of compact sets and the well-known Weierstrass Lebesgue lemma.

1 Introduction.

Let A and B be nonempty sets and $f : A \times B \to \mathbb{R}$ a given function. A minimax result is a theorem which asserts that

$$\max_{a\in A} \min_{b\in B} f(a,b) = \min_{b\in B} \max_{a\in A} f(a,b). \tag{1}$$

In case min and/or max are not attained the min and/or max in the above expressions are replaced by inf and/or sup. The first minimax result was proved in a famous paper by von Neumann (cf.[23]) in 1928 for A and B unit simplices in finite dimensional vector spaces and f affine in both variables. In this paper it was also shown why such a result is of importance in game theory. After the break-through of linear programming the key role of relation (1) was also recognized in optimization theory. Therefore a lot of papers appeared in the literature after 1928 verifying the equality expressed by relation (1) under different conditions on the sets A and B and the function f. One might say that the conditions of von Neumann were generalized during the last 75 years and a nice overview of most of those generalizations is given in [29]. A careful review shows that the majority of these minimax results were either established by applying the Hahn Banach or a fixed point theorem. Although some of the minimax results considered in this paper are also proved initially by a fixed point argument we will only consider the most well known minimax results proven by a Hahn-Banach type argument. To start with this overview the minimax results needed in game theory assumed that the sets A and B represented sets of probability measures with finite support and the function f was taken to be affine in both variables. Later on, the condition on the function f was weakened and more general sets A and B were considered. As already observed, these results turned out to be useful in optimization theory and were derived by means of short or long proofs using a version of the Hahn Banach theorem in either finite or infinite dimensional vector spaces. With the famous minimax result in game theory proved by von Neumann in 1928 (cf.[23]) as a starting point we will show in this paper that several of these so-called generalizations published in the literature can be derived from each other using only elementary observations about compact sets and continuous functions on compact sets. Before introducing this chain of equivalent minimax results we need the following notation. Let $\mathcal{F}(A)$ denote the set of probability measures on A with finite support. If ϵ_a represents the one-point probability measure concentrated on a this means by definition that $\lambda \in \mathcal{F}(A)$ if and only if there exists some finite set $\{a_1, ..., a_n\} \subseteq A$ and a sequence $\lambda_i, 1 \leq i \leq n$ satisfying

$$\lambda = \sum\nolimits_{i=1}^{n} \lambda_i \epsilon_{a_i}, \sum\nolimits_{i=1}^{n} \lambda_i = 1 \text{ and } \lambda_i > 0, 1 \leq i \leq n. \tag{2}$$

If the set A is given by $\{a_1, ..., a_n\}$ then it is clear that

$$\mathcal{F}(A) = \{\lambda : \lambda = \sum\nolimits_{i=1}^{n} \lambda_i \epsilon_{a_i}, \sum\nolimits_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geq 0, 1 \leq i \leq n\}. \tag{3}$$

Moreover, the set $\mathcal{F}_2(A) \subseteq \mathcal{F}(A)$ denotes the set of two-point probability measures on A. This means that λ belongs to $\mathcal{F}_2(A)$ if and only if

$$\lambda = \lambda_1 \epsilon_{a_1} + (1 - \lambda_1)\epsilon_{a_2} \tag{4}$$

with $a_i, 1 \leq i \leq 2$ different elements of A and $0 < \lambda_1 < 1$ arbitrarily chosen. Finally, for each $0 < \alpha < 1$ the set $\mathcal{F}_{2,\alpha}(A)$ represents the set of two point probability measures with $\lambda_1 = \alpha$ in relation (4). Also on the set B similar spaces of probability measures with finite support are introduced. Within game theory any element of $\mathcal{F}(A)$, respectively $\mathcal{F}(B)$ represents a so-called mixed strategy of player 1, respectively player 2 (cf.[1], [26]) and to measure the payoff using those mixed strategies one needs to extend the so-called payoff function f to the Cartesian product of the sets $\mathcal{F}(A)$ and $\mathcal{F}(B)$. The extension $f_e : \mathcal{F}(A) \times \mathcal{F}(B) \to \mathbb{R}$ is defined by

$$f_e(\lambda, \mu) := \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j f(a_i, b_j) \tag{5}$$

with λ as in relation (2) and $\mu = \sum_{j=1}^{m} \mu_j \epsilon_{b_j}$. To start in a chronological order we first mention the famous bilinear minimax result in game theory for finite sets A and B due to von Neumann and published in 1928 (cf.[23]). Actually in this paper a more general minimax result was verified for f continuous, A and B unit simplices in finite dimensional vector spaces and f quasiconcave on A and quasiconvex on B. This more general result was later extended using fixed point arguments by von Neumann (cf.[24]), Nikaido (cf.[12]) and Sion (cf.[22]). However, most authors only mentioned the special bilinear case as the main result of von Neumann and this is probably due to the fact that in the book of von Neumann and Morgenstern (cf.[25]) the authors only mentioned this particular bilinear case.

Theorem 1 *If A and B are finite sets then it follows that*

$$\max_{\lambda \in \mathcal{F}(A)} \min_{\mu \in \mathcal{F}(B)} f_e(\lambda, \mu) = \min_{\mu \in \mathcal{F}(B)} \max_{\lambda \in \mathcal{F}(A)} f_e(\lambda, \mu).$$

The next minimax result due to Ville (cf.[19]) and published in 1938 is a generalization of Theorem 1 and serves as an important tool in infinite antagonistic game theory (cf.[26]).

Theorem 2 *If A and B are compact sets in metric spaces and the function $f : A \times B \to \mathbb{R}$ is continuous then it follows that*

$$\sup_{\lambda \in \mathcal{F}(A)} \inf_{\mu \in \mathcal{F}(B)} f_e(\lambda, \mu) = \inf_{\mu \in \mathcal{F}(B)} \sup_{\lambda \in \mathcal{F}(A)} f_e(\lambda, \mu).$$

In 1941 Kakutani (cf.[28]) proved the following minimax theorem arising from his famous generalization of the Brouwer fixed point theorem.

Theorem 3 *If A and B are compact convex sets in normed linear spaces and the function $f : A \times B \to \mathbb{R}$ is continuous and $a \to f(a, b)$ is concave in A for every $b \in B$ and $b \to f(a, b)$ is convex in B for every $a \in A$ then it follows that*

$$\max_{a \in A} \min_{b \in B} f(a, b) = \min_{b \in B} \max_{a \in A} f(a, b).$$

Another generalization of Theorem 1 due to Wald (cf.[3]) and published in 1945 is given by the next result. This result plays a fundamental role in the theory of statistical decision functions (cf.[4]).

Theorem 4 *If A is an arbitrary nonempty set and B is a finite set then it follows that*

$$\sup_{\lambda\in\mathcal{F}(A)}\min_{\mu\in\mathcal{F}(B)} f_e(\lambda,\mu)=\min_{\mu\in\mathcal{F}(B)}\sup_{\lambda\in\mathcal{F}(A)} f_e(\lambda,\mu).$$

In 1952 Kneser (cf.[10]) proved in a two page note a very general minimax result useful in game theory. Its proof is ingenious and very elementary and uses only some simple computations and the well-known result that any upper semicontinuous function attains its maximum on a compact set (Weierstrass-Lebesgue lemma).

Theorem 5 *If A is a nonempty convex, compact subset of a topological vector space and B is a nonempty convex subset of a vector space and the function $f: A\times B\to\mathbb{R}$ is affine in both variables and upper semicontinuous on A for every $b\in B$ then it follows that*

$$\max_{a\in A}\inf_{b\in B} f(a,b)=\inf_{b\in B}\max_{a\in A} f(a,b). \tag{6}$$

One year later, in 1953, generalizing the proof and result of Kneser, Ky Fan (cf.[20]) published his celebrated minimax result. To show his result Ky Fan introduced the following class of functions. As in [31] and [32] we call this class of functions the class of Ky Fan convex (Ky Fan concave) functions. In the literature (see for example [6]) these functions are also called convexlike (concavelike).

Definition 6 *The function $f: A\times B\to\mathbb{R}$ is called Ky Fan concave on A if for every $\lambda\in\mathcal{F}_2(A)$ there exists some $a_0\in A$ satisfying*

$$f_e(\lambda,\epsilon_b)\leq f(a_0,b)$$

for every $b\in B$. The function $f: A\times B\to\mathbb{R}$ is called Ky Fan convex on B if for every $\mu\in\mathcal{F}_2(B)$ there exists some $b_0\in B$ satisfying

$$f_e(\epsilon_a,\mu)\geq f(a,b_0)$$

for every $a\in A$. Finally, the function $f: A\times B\to\mathbb{R}$ is called Ky Fan concave-convex on $A\times B$ if f is Ky Fan concave on A and Ky Fan convex on B.

By induction it is easy to show that one can replace in the above definition $\mathcal{F}_2(A)$ and $\mathcal{F}_2(B)$ by $\mathcal{F}(A)$ and $\mathcal{F}(B)$. Although rather technical the above concept has a clear interpretation in game theory. It means that the payoff function f has the property that any arbitrary mixed strategy is dominated by a pure strategy. Eliminating the linear structure in Kneser's proof Ky Fan (cf.[20]) showed the following result.

Theorem 7 *If A is a compact subset of a topological space and the function $f : A \times B \to \mathbb{R}$ is Ky Fan concave-convex on $A \times B$ and upper semicontinuous on A for every $b \in B$ then it follows that*

$$\max_{a \in A} \inf_{b \in B} f(a, b) = \inf_{b \in B} \max_{a \in A} f(a, b).$$

Unaware of Ky-Fan's more general minimax result Peck and Dulmage (cf.[27]) proved in 1957 the following minimax result. It is curious to note that also Peck and Dulmage generalized the proof of Kneser.

Theorem 8 *If A is a nonempty compact convex subset of a topological vector space and B is a nonempty convex subset of a vector space and the function $f : A \times B \to \mathbb{R}$ is concave-convex on $A \times B$ and upper semicontinuous on A for every $b \in B$ then it follows that*

$$\max_{a \in A} \inf_{b \in B} f(a, b) = \inf_{b \in B} \max_{a \in A} f(a, b).$$

Several years after the publication of Ky Fan's minimax theorem König (cf.[11]) showed in 1968 under the same topological conditions the result of Ky Fan for a larger class of functions. To prove this result König introduced the following class of functions. As in [31] and [32] we call this class of functions the class of König convex (König concave) functions. In the literature (see for example [6]) these functions are also called α-concavelike (β-convexlike).

Definition 9 *The function $f : A \times B \to \mathbb{R}$ is called König concave on A if there exists some $0 < \alpha < 1$ such that for every $\lambda \in \mathcal{F}_{2,\alpha}(A)$ there exists some $a_0 \in A$ satisfying*

$$f_e(\lambda, \epsilon_b) \leq f(a_0, b)$$

for every $b \in B$. The function $f : A \times B \to \mathbb{R}$ is called König convex on B if there exists some $0 < \beta < 1$ such that for every $\mu \in \mathcal{F}_{2,\beta}(B)$ there exists some $b_0 \in B$ satisfying

$$f_e(\epsilon_a, \mu) \geq f(a, b_0)$$

for every $a \in A$. Finally, the function $f : A \times B \to \mathbb{R}$ is called König concave-convex on $A \times B$ if f is König concave on A and König convex on B.

The above definition means that the payoff function f has the property that one can find some $0 < \alpha < 1$ such that any two-point mixed strategy with probability α of selecting one action is dominated by a pure strategy. Using the same topological conditions as in Theorem 7 König (cf.[11]) proved relation (6) for the larger class of König concave-convex functions by means of a version of the Hahn Banach theorem due to Mazur-Orlicz (see Theorem 1.1 of [30]). Actually König proved relation (6) for the class of König concave-convex functions on $A \times B$ with $\alpha = \beta = \frac{1}{2}$ but observed in a remark at the end of his paper that the same proof can also be given for an arbitrary $0 < \alpha < 1$ and $0 < \beta < 1$. The result of König was again generalized in 1977 to a more general class of functions by Neumann (cf.[21]), in 1980 by Fuchssteiner and König (cf.[5]) and in 1986 by Jeyakumar (cf.[33]). To list their result we need to introduce the class of closely concave-closely convex functions. For an explanation of the name for these functions the reader should consult [14].

Definition 10 *The function $f : A \times B \to \mathbb{R}$ is called closely concave on A if for every $\epsilon > 0$ and $\lambda \in \mathcal{F}_2(A)$ there exists some $a_0 \in A$ satisfying*

$$f_e(\lambda, \epsilon_b) \leq f(a_0, b) + \epsilon$$

for every $b \in B$. The function $f : A \times B \to \mathbb{R}$ is called closely convex on B if for every $\epsilon > 0$ and $\mu \in \mathcal{F}_2(B)$ there exists some $b_0 \in B$ satisfying

$$f_e(\epsilon_a, \mu) \geq f(a, b_0) - \epsilon$$

for every $a \in A$. Finally, the function $f : A \times B \to \mathbb{R}$ is called closely concave-closely convex on $A \times B$ if f is closely concave on A and closely convex on B.

Actually Fuchssteiner and König introduced the class of functions $f : A \times B \to \mathbb{R}$ having the next property: there exists some $0 < \alpha < 1$ such that for every $\lambda \in \mathcal{F}_{2,\alpha}(A)$ and every $\epsilon > 0$ one can find some $a_0 \in A$ satisfying

$$f_e(\lambda, \epsilon_b) \leq f(a_0, b) + \epsilon \tag{7}$$

Although this class of functions looks more general than the class of closely convex functions on B it can be shown (cf.[14], [35]) that any function f satisfying relation (7) for some $0 < \alpha < 1$ also satisfies this property for any α belonging to a dense subset of $(0, 1)$. This implies that such a function is actually closely convex on B and so we are dealing with the same class of functions. Again by induction it is easy to show in Definition 10 that one can replace

$\mathcal{F}_2(A)$ and $\mathcal{F}_2(B)$ by $\mathcal{F}(A)$ and $\mathcal{F}(B)$. Also this class of payoff functions has a clear game theoretic interpretation. Using the same topological conditions as in Theorem 7 Neumann (cf.[21]), Fuchssteiner and König (cf.[5]) and Jeyakumar (cf.[33]) proved relation (6) for the larger class of closely concave-closely convex functions. Another, seemingly unrelated, result was shown by Gwinner and Oettli (cf.[16]) in 1994. Technically speaking this result is not a minimax result and to list their result we introduce for the arbitrary sets A and B the sets $\mathbb{R}^A$, respectively $\mathbb{R}^B$ of real valued functions on A, respectively B. Consider now the set $D \subseteq \mathbb{R}^A$ and $C \subseteq \mathbb{R}^B$ given by

$$D := \{u \in \mathbb{R}^A : \exists_{b \in B}\ f(a,b) \leq u(a) \text{ for every } a \in A\} \tag{8}$$

and

$$C := \{v \in \mathbb{R}^B : \exists_{a \in A}\ f(a,b) \geq v(b) \text{ for every } b \in B\}. \tag{9}$$

and endow $\mathbb{R}^B$ and $\mathbb{R}^A$ with the product topology π (cf.[7]). If $co(C)$ and $co(D)$ denote the convex hull of the sets C and D and $cl(co(C))$ the closure of the set $co(C)$ with respect to the product topology π then the main result of Gwinner and Oettli (cf.[16]) is given by the following theorem.

Theorem 11 *For any sets A and B it follows that*

$$\inf_{u \in co(D)} \sup_{a \in A} u(a) = \sup_{v \in cl(co(C))} \inf_{b \in B} v(b).$$

Finally in 1996 Kassay and Kolumbán (cf.[9]) introduced the following class of functions. To list their definition we denote by $< B >$ the set of all finite subsets of B.

Definition 12 *The function $f : A \times B \to \mathbb{R}$ is called weakly concavelike on A if for every I belonging to $< B >$ it follows that*

$$\sup_{\lambda \in \mathcal{F}(A)} \min_{b \in I} f_e(\lambda, \epsilon_b) \leq \sup_{a \in A} \min_{b \in I} f(a,b).$$

Since ϵ_a belongs to $\mathcal{F}(A)$ it is easy to see that f is weakly concavelike on A if and only if for every $I \in < B >$ it follows that

$$\sup_{\lambda \in \mathcal{F}(A)} \min_{b \in I} f_e(\lambda, \epsilon_b) = \sup_{a \in A} \min_{b \in I} f(a,b)$$

and this equality also has an obvious interpretation within game theory. The main result of Kassay and Kolumban (cf.[9]) is given by the following theorem.

Theorem 13 *If A is a compact subset of a topological space and the function $f : A \times B \to \mathbb{R}$ is weakly concavelike on A and upper semicontinuous on A for every $b \in B$ then it follows that*

$$\inf_{\mu \in \mathcal{F}(B)} \max_{a \in A} f_e(\epsilon_a, \mu) = \max_{a \in A} \inf_{b \in B} f_e(a, b).$$

At first sight this result might not be recognized as a minimax result. However, it is easy to verify for every $a \in A$ that

$$\inf_{b \in B} f(a, b) = \inf_{\mu \in \mathcal{F}(B)} f_e(\epsilon_a, \mu). \tag{10}$$

By relation (10) an equivalent formulation of Theorem 13 is now given by

$$\inf_{\mu \in \mathcal{F}(B)} \max_{a \in A} f_e(\epsilon_a, \mu) = \max_{a \in A} \inf_{\mu \in \mathcal{F}(B)} f_e(\epsilon_a, \mu)$$

and so the result of Kassay and Kolumban is actually a minimax result. Finally we list the following well-known strong separation result in convex analysis.

Theorem 14 *If $A \subseteq \mathbb{R}^n$ is a closed convex set and $B \subseteq \mathbb{R}^n$ a compact convex set and the intersection of A and B is empty then there exists some $s_0 \in \mathbb{R}^n$ satisfying*

$$\sup\{s_0^\top a : a \in A\} < \inf\{s_0^\top b : b \in B\}.$$

In the next section we will show that all these results are easy consequences of each other and so they form an equivalent chain of results.

2 Analysis.

In this section we will verify by means of the next chain of implications that the minimax results mentioned in the introduction can be easily derived from each other.

von Neumann $\Rightarrow^{th16}$ Wald $\Rightarrow^{th19}$ Gwinner-Oettli $\Rightarrow^{th20}$
Kassay-Kolumbán $\Rightarrow^{th21}$ Neumann-Jeyakumar $\Rightarrow^{th22}$ König $\Rightarrow^{th22}$
Ky-Fan $\Rightarrow^{th23}$ Peck-Dulmage $\Rightarrow^{th23}$ Kneser $\Rightarrow^{th24}$
strong separation $\Rightarrow^{th25}$ Ville $\Rightarrow^{th26}$ Kakutani $\Rightarrow^{th27}$ von Neumann.

Some of these implications are obvious. To prove the other implications we only use an easy consequence of the finite intersection property of compact sets given by Lemma 15, the Weierstrass-Lebesgue lemma and the well-known result that any continuous function on a compact set is uniformly continuous. Observe that the strong separation result itself is an easy consequence of the Weierstrass-Lebesgue lemma (cf.[15]) and this shows that all these minimax results can be proved using only some elementary properties of compact sets. This verifies that these minimax results are elementary results in mathematics which do not need for its proof the Hahn-Banach theorem in infinite dimensional vector spaces and hence Zorn's lemma. (see the original proof of König of his minimax result). The connection with the separation result for disjoint convex sets in finite dimensions was already discussed for the Peck-Dulmage minimax result by Joó (cf.[13]). (actually the so-called level set method developed by Joó deserves much more attention. The proof in [13] can be adapted without using the separation result to give an elementary proof of Sion's minimax theorem), for Ky-Fan's minimax result by Borwein and Zhuang (cf.[17]), for König's minimax result by Kassay (cf.[8]) and for Fuchssteiner-König's minimax result by Wen Song (cf.[35]). Also Jeyakumar (cf.[33]) used a finite dimensional separation result for convex sets to verify the Neumann-Jeyakumar minimax result and the same was done by Kassay and Kolumban (cf.[9]) to prove their minimax result. However, in none of these papers the easy implications between the above minimax results was established. To keep the paper self contained a short proof of Lemma 15 is included. Observe for every set Y the set $< Y >$ denotes the set of all finite subsets of Y.

Lemma 15 *If the set X is compact and the function $h : X \times Y \to \mathbb{R}$ is upper semicontinuous on X for every $y \in Y$ then $\max_{x\in X} \inf_{y\in Y} h(x, y)$ is well defined and*

$$\max_{x\in X} \inf_{y\in Y} h(x, y) = \inf_{Y_0 \in <Y>} \max_{x\in X} \min_{y\in Y_0} h(x, y).$$

Proof. Since the function h is upper semicontinuous on X for every $y \in Y$ we obtain that $p(x) := \inf_{y\in Y} h(x, y)$ is upper semicontinuous on X and so by the Weierstrass-Lebesgue lemma (see Corollary 1.2 of [18]) and X compact the function p attains its maximum on X. This shows that $\max_{x\in X} \inf_{y\in Y} h(x, y)$ is well defined and to check the equality it is sufficient to verify that

$$\alpha := \max_{x\in X} p(x) \geq \inf_{Y_0\in<Y>} \max_{x\in X} \min_{y\in Y_0} h(x, y) := \beta.$$

If we assume by contradiction that $\alpha < \beta$ there exists some finite γ satisfying $\alpha < \gamma < \beta$ and this implies by the definition of α that

$$\cap_{y\in Y}\{x \in X : h(x, y) \geq \gamma\} = \emptyset. \tag{11}$$

Since X is compact and h upper semicontinuous on X for every $y \in Y$ we obtain that the set $\{x \in X : h(x,y) \geq \gamma\}$ is compact for every $y \in Y$ and by relation (11) and the finite intersection property of compact sets (cf.[34]) we obtain for some $Y_0 \in < Y >$ that

$$\cap_{y \in Y_0} \{x \in X : h(x,y) \geq \gamma\} = \emptyset.$$

This implies $\min_{y \in Y_0} h(x,y) < \gamma$ for every $x \in X$ and by the first part $\max_{x \in X} \min_{y \in Y_0} h(x,y) < \gamma < \beta$. This contradicts the definition of β and so $\alpha \geq \beta$. □

Since for every $\mu \in \mathcal{F}(B)$ and $J \subseteq A$ it is easy to see that

$$\sup_{\lambda \in \mathcal{F}(J)} f_e(\lambda, \mu) = \sup_{a \in J} f_e(\epsilon_a, \mu) \tag{12}$$

we are now ready to derive Wald's minimax result from von Neumann's minimax result. Observe Wald (cf.[3]) uses in his paper von Neumann's minimax result and the Lebesgue dominated convergence theorem to derive his result.

Theorem 16 *von Neumann's minimax result $\Rightarrow$ Wald's minimax result.*

Proof. If $\alpha := \sup_{\lambda \in \mathcal{F}(A)} \min_{\mu \in \mathcal{F}(B)} f_e(\lambda, \mu)$ then clearly

$$\alpha = \sup_{J \in <A>} \max_{\lambda \in \mathcal{F}(J)} \min_{\mu \in \mathcal{F}(B)} f_e(\lambda, \mu). \tag{13}$$

Since the set B is finite we may apply von Neumann's minimax result in relation (13) and this implies in combination with relation (12) that

$$\begin{aligned} \alpha &= \sup_{J \in <A>} \min_{\mu \in \mathcal{F}(B)} \max_{\lambda \in \mathcal{F}(J)} f_e(\lambda, \mu) \\ &= \sup_{J \in <A>} \min_{\mu \in \mathcal{F}(B)} \max_{a \in J} f_e(\epsilon_a, \mu) \\ &= -\inf_{J \in <A>} \max_{\mu \in \mathcal{F}(B)} \min_{a \in J} -f_e(\epsilon_a, \mu). \end{aligned} \tag{14}$$

The finiteness of the set B also implies that the set $\mathcal{F}(B)$ is compact and the function $\mu \rightarrow f_e(\epsilon_a, \mu)$ is continuous on $\mathcal{F}(B)$ for every $a \in A$. This shows in relation (14) that we may apply Lemma 15 with the set X replaced by $\mathcal{F}(B)$, Y by A and $h(x,y)$ by $-f_e(\epsilon_a, \mu)$ and so it follows that

$$\alpha = \min_{\mu \in \mathcal{F}(B)} \sup_{a \in A} f_e(\epsilon_a, \mu). \tag{15}$$

Finally by relation (12) with J replaced by A the desired result follows from relation (15). □

In order to show that Wald's minimax result implies the Gwinner-Oettli result we first need to rewrite the Gwinner-Oettli result by means of the following elementary lemmas. Remember the sets C and D are given by relations (8) and (9).

Lemma 17 *It follows that*

$$\inf_{u\in co(D)} \sup_{a\in A} u(a) = \inf_{\mu\in\mathcal{F}(B)} \sup_{a\in A} f_e(\epsilon_a,\mu)$$
$$= \inf_{I\in<B>} \min_{\mu\in\mathcal{F}(I)} \sup_{a\in A} f_e(\epsilon_a,\mu).$$

Proof. To show the first equality it is clear by relation (8) for every $\mu \in \mathcal{F}(B)$ that the function $u \in \mathbb{R}^A$ given by $u(a) := f_e(\epsilon_a,\mu)$ belongs to $co(D)$ and so we obtain

$$\inf_{u\in co(D)} \sup_{a\in A} u(a) \leq \inf_{\mu\in\mathcal{F}(B)} \sup_{a\in A} f_e(\epsilon_a,\mu). \tag{16}$$

Moreover, for any $u \in co(D)$ there exist a sequence of functions $u_j \in D$ and a positive sequence $\mu_j, 1 \leq j \leq m$ such that

$$u = \sum_{j=1}^m \mu_j u_j, \sum_{j=1}^m \mu_j = 1. \tag{17}$$

Since $u_j \in D$ one can find some $b_j \in B$ satisfying $f(a,b_j) \leq u_j(a)$ for every $a \in A$ and introducing now $\mu \in \mathcal{F}(B)$ given by $\mu = \sum_{j=1}^m \mu_j\epsilon_{b_j}$ we obtain by relation (17) that $u(a) \geq f_e(\epsilon_a,\mu)$ for every $a \in A$. This implies

$$\inf_{u\in co(D)} \sup_{a\in A} u(a) \geq \inf_{\mu\in\mathcal{F}(B)} \sup_{a\in A} f_e(\epsilon_a,\mu) \tag{18}$$

and by relations (16) and (18) the first equality follows. The second equality is a direct consequence of the continuity of the function $\mu \to f_e(\epsilon_a,\mu)$ on $\mathcal{F}(I)$ for every $a \in A$ and the Weierstrass-Lebesgue lemma. $\square$

Another elementary observation is given by the following consequence of the product topology on $\mathbb{R}^B$.

Lemma 18 *If cl denotes the closure with respect to the product topology* π *then it follows that*

$$\sup_{v\in cl(co(C))} \inf_{b\in B} v(b) = \inf_{I\in<B>} \sup_{v\in co(C)} \min_{b\in I} v(b)$$
$$= \inf_{I\in<B>} \sup_{\lambda\in\mathcal{F}(A)} \min_{b\in I} f_e(\lambda,\epsilon_b).$$

Proof. To show the first equality introduce for every I belonging to $< B >$ the mapping $h_I : \mathbb{R}^B \to \mathbb{R}$ given by

$$h_I(v) := \min_{b\in I} v(b).$$

Since the neighbourhood base of an arbitrary $w_0 \in \mathbb{R}^B$ in the product topology π is given by the sets (cf.[7])

$$W(I,\epsilon,w_0) := \{w \in \mathbb{R}^B : |w(b) - w_0(b)| < \epsilon \text{ for every } b \in I\}$$

with I belonging to $< B >$ and $\epsilon > 0$ it is easy to verify that the function h_I is continuous on $(\mathbb{R}^B, \pi)$. This shows by contradiction that

$$\sup_{v \in cl(co(C))} h_I(v) = \sup_{v \in co(C)} h_I(v)$$

and so with $\alpha := \sup_{v \in cl(co(C))} \inf_{b \in B} v(b)$ we obtain

$$\alpha \leq \inf_{I \in <B>} \sup_{v \in cl(co(C))} h_I(v) = \inf_{I \in <B>} \sup_{v \in co(C)} h_I(v). \quad (19)$$

To show the reverse inequality we assume by contradiction that

$$\alpha < \inf_{I \in <B>} \sup_{v \in co(C)} \min_{b \in I} v(b).$$

If this holds there exists some $\epsilon > 0$ such that for every I belonging to $< B >$ one can find some $v_I \in co(C)$ satisfying

$$\min_{b \in I} v_I(b) > \alpha + \epsilon. \quad (20)$$

Introduce now the function $w_I := \min\{\underline{\alpha + \epsilon}, v_I\}$ and $\underline{\alpha + \epsilon}$ denoting the constant function on $\mathbb{R}^B$ with value everywhere equal to $\alpha + \epsilon$. It is now obvious that the function $\gamma_I : B \to \mathbb{R}$ given by

$$\gamma_I := v_I - w_I \quad (21)$$

is nonnegative for every $b \in B$. Since we also know that $v_I \in co(C)$ it follows that there exists some functions $v_{I,j} \in C, 1 \leq j \leq m_I < \infty$ satisfying

$$v_I = \sum_{j=1}^{m_I} \mu_{I,j} v_{I,j},\ \mu_{I,j} > 0 \text{ and } \sum_{j=1}^{m_I} \mu_{I,j} = 1 \quad (22)$$

and this implies by relations (21) and (22) that

$$w_I = \sum_{j=1}^{m_I} \mu_{I,j}(v_{I,j} - \gamma_I) \quad (23)$$

By the nonnegativity of the function γ_I and the definition of C it follows using $v_{I,j} \in C, 1 \leq j \leq m_I$ that also $v_{I,j} - \gamma_I$ belongs to C for every $1 \leq j \leq m_I$ and so by relation (23) we obtain $w_I \in co(C)$. Clearly the set $\{I : I$ belongs to $< B >\}$ is a directed set with partial ordering $\subseteq$ and so we consider the net $\{w_I : I \in < B >\} \subseteq co(C)$. By the definition of the product topology π and relation (20) we obtain that w_I converges in the product topology to $\underline{\alpha + \epsilon}$ and this shows that $\underline{\alpha + \epsilon}$ belongs to $cl(co(C))$. Hence it follows that

$$\alpha = \sup_{v \in cl(co(C))} \inf_{b \in B} v(b) \geq \alpha + \epsilon$$

and we obtain a contradiction. This verifies the first equality and the second equality can be proved similarly as Lemma 17. □

In the next theorem we show that Wald's minimax result implies the result of Gwinner and Oettli.

Theorem 19 *Wald's minimax result $\Rightarrow$ result of Gwinner and Oettli.*

Proof. Introducing $\alpha := \inf_{u \in co(D)} \sup_{a \in A} u(a)$ it follows by Lemma 17 and relation (12) that

$$\begin{aligned} \alpha &= \inf_{I \in <B>} \min_{\mu \in \mathcal{F}(I)} \sup_{a \in A} f_e(\epsilon_a, \mu) \\ &= \inf_{I \in <B>} \min_{\mu \in \mathcal{F}(I)} \sup_{\lambda \in \mathcal{F}(A)} f_e(\lambda, \mu). \end{aligned}$$

Since every element of $< B >$ is a finite set we may apply Wald's minimax result and this shows

$$\begin{aligned} \alpha &= \inf_{I \in <B>} \sup_{\lambda \in \mathcal{F}(A)} \min_{\mu \in \mathcal{F}(I)} f_e(\lambda, \mu) \\ &= \inf_{I \in <B>} \sup_{\lambda \in \mathcal{F}(A)} \min_{b \in I} f_e(\lambda, \epsilon_b). \end{aligned} \tag{24}$$

Applying now Lemma 18 yields the desired result. $\square$

We will now verify that the Gwinner-Oettli result implies the Kassay-Kolumbán minimax result.

Theorem 20 *Gwinner-Oettli result $\Rightarrow$ Kassay-Kolumbán minimax result.*

Proof. If $\alpha := \inf_{\mu \in \mathcal{F}(B)} \max_{a \in A} f_e(\epsilon_a, \mu)$ we obtain by Lemma 17 and the result of Gwinner and Oettli that

$$\alpha = \sup_{v \in cl(co(C))} \inf_{b \in B} v(b). \tag{25}$$

Applying now Lemma 18 and f is weakly concavelike on A it follows by relation (25) that

$$\alpha = \inf_{I \in <B>} \sup_{a \in A} \min_{b \in I} f(a, b). \tag{26}$$

Also, since f is upper semicontinuous on A for every $b \in B$ and A is compact we know by relation (26) and the Weierstrass-Lebesgue lemma that $\alpha = \inf_{I \in <B>} \max_{a \in A} \min_{b \in I} f(a, b)$ and using Lemma 15 we obtain $\alpha = \max_{a \in A} \inf_{b \in B} f(a, b)$ showing the desired result. $\square$

We will now show that the Kassay-Kolumbán minimax result implies the Neumann-Jeyakumar minimax result.

Theorem 21 *Kassay-Kolumbán minimax result $\Rightarrow$ Neumann-Jeyakumar minimax result.*

Proof. We first show that any function $f : A \times B \to \mathbb{R}$ which is closely concave on A is also weakly concavelike on A. To verify this we first observe by induction that f is closely concave on A if and only if for every $\epsilon > 0$ and $\lambda \in \mathcal{F}(A)$ there exists some $a_0 \in A$ satisfying

$$f_e(\lambda, \epsilon_b) \leq f(a_0, b) + \epsilon \tag{27}$$

for every $b \in B$. This implies for every $n \in \mathbb{N}$ and $\lambda \in \mathcal{F}(A)$ that there exists some $a_n \in A$ satisfying $f_e(\lambda, \epsilon_b) \leq f(a_n, b) + n^{-1}$ for every $b \in B$ and so for every I belonging to $< B >$ and $n \in \mathbb{N}$ we obtain

$$\min_{b \in I} f_e(\lambda, \epsilon_b) \leq \sup_{a \in A} \min_{b \in I} f(a, b) + n^{-1}.$$

Therefore $\min_{b \in I} f_e(\lambda, \epsilon_b) \leq \sup_{a \in A} \min_{b \in I} f(a, b)$ and since $\lambda \in \mathcal{F}(A)$ is arbitrary it follows that f is weakly concavelike on A. By the Kassay-Kolumbán minimax result we obtain therefore

$$\max_{a \in A} \inf_{b \in B} f(a, b) = \inf_{\mu \in \mathcal{F}(B)} \max_{a \in A} f_e(\epsilon_a, \mu) \tag{28}$$

Also, since f is closely convex on B we obtain as in the first part of this proof that for every $n \in \mathbb{N}$ and $\mu \in \mathcal{F}(B)$ there exists some $b_n \in B$ satisfying

$$f_e(\epsilon_a, \mu) \geq f(a, b_n) - n^{-1}$$

for every $a \in A$ and so for every $n \in \mathbb{N}$ and $\mu \in \mathcal{F}(B)$ it follows that

$$\max_{a \in A} f_e(\epsilon_a, \mu) \geq \inf_{b \in B} \max_{a \in A} f(a, b) - n^{-1}.$$

This shows

$$\inf_{\mu \epsilon \mathcal{F}(B)} \max_{a \in A} f_e(\epsilon_a, \mu) \geq \inf_{b \in B} \max_{a \in A} f(a, b)$$

and by relation (28) the desired result follows. □

Since any closely concave-closely convex function on $A \times B$ is König concave-convex on $A \times B$ (cf.[14]) and any Ky-Fan concave-convex function on $A \times B$ is König concave-convex on $A \times B$ we obtain immediately the next implication.

Theorem 22 *Neumann-Jeyakumar minimax result⇒König's minimax result⇒Ky-Fan minimax result.*

Also it is clear that Ky-Fan's minimax result is a generalization of the Peck-Dulmage minimax result and the Peck-Dulmage minimax result is a generalization of Kneser's minimax result and so the following result is obvious.

Theorem 23 *Ky-Fan minimax result ⇒Peck-Dulmage minimax result⇒Kneser's minimax result.*

In the next result we verify that the strong separation result given by Theorem 14 is an easy consequence of Kneser's minimax result.

Theorem 24 *Kneser's minimax result⇒strong separation result.*

Proof. Since $A \subseteq \mathbb{R}^n$ is a closed convex set and $B \subseteq \mathbb{R}^n$ is a compact convex set we obtain that $H := A - B$ is a closed convex set. It is now easy to see that the strong separation result as given in Theorem 14 holds if and only if there exists some $s_0 \in \mathbb{R}^n$ satisfying $\sigma_H(s_0) := \sup\{s_0^\top x : x \in H\} < 0$. To verify this we assume by contradiction that $\sigma_H(s) \geq 0$ for every $s \in \mathbb{R}^n$. This clearly implies $\sigma_H(s) \geq 0$ for every s belonging to the compact Euclidean unit ball E and applying Kneser's minimax result we obtain

$$\sup_{h \in H} \inf_{s \in E} s^\top h = \inf_{s \in E} \sup_{h \in H} s^\top h \geq 0. \tag{29}$$

Since by assumption the intersection of A and B is nonempty we obtain that 0 does not belong to $H := A - B$ and this implies using H is closed that $\inf_{h \in H} \|h\| > 0$. By this observation we obtain for every $h \in H$ that $-h\|h\|^{-1}$ belongs to E and so for every $h \in H$ it follows that $\inf_{s \in E} s^\top h \leq -\|h\|$. This implies that

$$\sup_{h \in H} \inf_{s \in E} s^\top h \leq \sup_{h \in H} -\|h\| = -\inf_{h \in H} \|h\| < 0$$

and we obtain a contradiction with relation (29). Hence there must exist some $s_0 \in \mathbb{R}^n$ satisfying $\sigma_H(s_0) < 0$ and we are done. □

In the next result we verify that Ville's minimax result is a consequence of the strong separation result.

Theorem 25 *strong separation result⇒ Ville's minimax result.*

Proof. It follows immediately that

$$\inf_{\mu \in \mathcal{F}(B)} \sup_{\lambda \in \mathcal{F}(A)} f_e(\lambda, \mu) \geq \sup_{\lambda \in \mathcal{F}(A)} \inf_{\mu \in \mathcal{F}(B)} f_e(\lambda, \mu)$$

and so we only have to verify that the reverse inequality holds. By relation (12) it is now sufficient to show that

$$\inf_{\mu \in \mathcal{F}(B)} \sup_{a \in A} f_e(\epsilon_a, \mu) \leq \sup_{\lambda \in \mathcal{F}(A)} \inf_{b \in B} f_e(\lambda, \epsilon_b)$$

Introducing $\beta := \sup_{\lambda \in \mathcal{F}(A)} \inf_{b \in B} f_e(\lambda, \epsilon_b)$ suppose now by contradiction that there exists some $\gamma > 0$ satisfying

$$\sup_{a \in A} f_e(\epsilon_a, \mu) \geq \beta + \gamma \tag{30}$$

for every $\mu \in \mathcal{F}(B)$. Since the sets A and B are compact ant the function f is continuous it is well-known (cf.[2]) that the function f is uniformly continuous on $A \times B$. This implies that there exists some $\delta > 0$ such that

$$\sup_{b \in B} |f_e(\epsilon_x, b) - f_e(\epsilon_y, b)| < \frac{\gamma}{2}$$

for every $x, y \in A$ satisfying $\varrho(x, y) < \delta$ with ρ the metric on A and so it follows that

$$\sup_{\mu \in \mathcal{F}(B)} |f_e(\epsilon_x, \mu) - f_e(\epsilon_y, \mu)| < \frac{\gamma}{2} \tag{31}$$

for every $x, y \in A$ satisfying $\varrho(x, y) < \delta$. By the compactness of A we also know that there exists some finite set $\{a_1, ..., a_p\} \subseteq A$ satisfying $A \subseteq \cup_{i=1}^p (a_i + \delta E)$ with E denoting the unit open ball and this shows by relation (31) and (30) that

$$\max_{1 \leq i \leq p} f_e(\epsilon_{a_i}, \mu) \geq \sup_{a \in A} f_e(\epsilon_a, \mu) - \frac{\gamma}{2} \geq \beta + \frac{\gamma}{2}. \tag{32}$$

for every $\mu \in \mathcal{F}(B)$. Introducing now the set $S \subseteq \mathbb{R}^p$ given by

$$S := \{(f(a_1, b), ..., f(a_p, b)) : b \in B\}$$

we obtain by the continuity of f and B compact that S is compact and hence the convex hull $co(S)$ of S is compact. Also by relation (32) we obtain that $\max_{1 \leq i \leq p} z_i \geq \beta + \frac{\gamma}{2}$ for every $(z_1, ..., z_p) \in co(S)$ and so the intersection of the compact conves set $co(S)$ and the closed convex set $V := \{(z_1, ..., z_p) : \max_{1 \leq i \leq p} z_i \leq \beta + \frac{\gamma}{4}\}$ is empty. By the strong separation theorem one can now find some vector $(\lambda_1, ..., \lambda_p) \geq 0$ with $\sum_{i=1}^p \lambda_i = 1$ such that for $\lambda := \sum_{i=1}^p \lambda_i \epsilon_{a_i}$ it follows that

$$\beta + \frac{\gamma}{4} < \inf_{b \in B} f_e(\lambda, \epsilon_b).$$

This contradicts the definition of β and so $\beta \leq \alpha$ verifying Ville's minimax result. □

In the next result we verify that Kakutani's minimax result result is an easy consequence of Ville's minimax result.

Theorem 26 *Ville's minimax result ⇒Kakutani's minimax result.*

Proof. Since for every $a \in A$ the function $b \to f(a, b)$ is convex on the compact convex set B it follows for every $\mu \in \mathcal{F}(B)$ given by $\mu = \sum_{j=1}^{m} \mu_j \epsilon_{b_j}$ that

$$\sup_{a\in A} f_e(\epsilon_a, \mu) \geq \sup_{a\in A} f(a, \sum_{j=1}^{m} \mu_j b_j) \geq \inf_{b\in B} \sup_{a\in A} f(a, b).$$

This implies in combination with relation (12) that

$$\begin{aligned} \inf_{\mu\in\mathcal{F}(B)} \sup_{\lambda\in\mathcal{F}(A)} f_e(\epsilon_a, \mu) &= \inf_{\mu\in\mathcal{F}(B)} \sup_{a\in A} f_e(\epsilon_a, \mu) \\ &\geq \inf_{b\in B} \sup_{a\in A} f(a, b). \end{aligned} \tag{33}$$

Similarly we obtain by the concavity of the function $a \to f(a, b)$ on the compact convex set A for every $b \in B$ that

$$\begin{aligned} \sup_{\lambda\in\mathcal{F}(A)} \inf_{\mu\in\mathcal{F}(B)} f_e(\lambda, \mu) &= \sup_{\lambda\in\mathcal{F}(A)} \inf_{b\in B} f_e(\lambda, \epsilon_b) \\ &\leq \sup_{a\in A} \inf_{b\in B} f(a, b). \end{aligned} \tag{34}$$

Applying now Ville's minimax result and relations (33) and (34) yields

$$\inf_{b\in B} \sup_{a\in A} f(a, b) \leq \sup_{a\in A} \inf_{b\in B} f(a, b).$$

Since trivially the reverse inequality holds and f is continuous on the compact set $A \times B$ Kakutani's minimax result holds. □

Observe now for any finite sets $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_m\}$ that von Neumann's minimax result can also be written as

$$\max_{\lambda\in\Delta_n} \min_{\mu\in\Delta_m} h(\lambda, \mu) = \min_{\mu\in\Delta_m} \max_{\lambda\in\Delta_n} h(\lambda, \mu)$$

with $h(\lambda, \mu) := \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j f(a_i, b_j)$. Since the function h is affine in both variables and Δ_n, respectively Δ_m denote the compact unit simplices of $\mathbb{R}^n$, respectively $\mathbb{R}^m$ it is clear by the above representation that von Neumann's minimax result is a special case of Kakutani's minimax result.

Theorem 27 *Kakutani's minimax result ⇒von Neumann's minimax result.*

This completes the proofs of the different implications. To conclude the paper we finally list some conclusions. In this paper we have shown that a number of minimax results are easy consequences of each other. This shows that one can construct a chain of minimax results and so to prove one of those results one needs to prove that minimax result of which its proof is most elementary. As such the authors believe that in all the minimax papers in the references the most elementary proof is given by Kneser. Similarly one can argue that all those papers proving generalizations of the original first minimax result of von Neumann by means of different proofs are elementary implications of this result. This also shows that von Neumann already captured in 1928 the basic minimax result which can be proved by means of a finite dimensional separating hyperplane argument. In that respect it is curious to note that von Neumann was the handling editor of the celebrated paper of Ky Fan in which the arguments of Kneser were generalized.

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