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# Networks of Collaboration in Oligopoly

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**Abstract**

In an oligopoly, prior to competing in the market, firms have an opportunity to form *pair-wise* collaborative links with other firms. These pair-wise links involve a commitment of resources and lead to lower costs of production of the collaborating firms. The collection of pair-wise links defines a collaboration network. We study the architecture of strategically stable networks.

Our analysis reveals that in a setting where firms are ex-ante identical, strategically stable networks are often asymmetric, with some firms having a large number of links while others have few links or no links at all. We characterize such asymmetric networks; the dominant group architecture, stars, and inter-linked stars are found to be stable. In asymmetric networks, the firms with many links have lower costs of production as compared to firms with few links. Thus collaboration links can have a major influence on the functioning of the market.

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# 1 Introduction

Empirical work suggests that R&D collaboration between firms is common.[1] This empirical work has also drawn attention to two striking features of collaboration relationships. The *first* feature is the difference in the number of links across the firms; some firms have a lot of links while others have relatively few links. These differences lead to *asymmetric* collaboration structures. The *second* feature is *intransitive* relations. Intransitive relationships arise when firms $i$ and $j$ have a link and firms $j$ and $k$ have a link, respectively, but $i$ and $k$ have no collaboration link. [2] This paper develops a simple model to understand the incentives of firms to form collaborative links and the nature of strategically stable networks.

Our model has the following structure: We consider an oligopoly setting in which firms form *pair-wise* collaborative links with other firms. These pair-wise links involve a commitment of resources on the part of the collaborating firms and yield lower costs of production for firms which form the link.[3] The collection of pair-wise links defines a collaboration network and induces a distribution of costs across the firms in the industry. Given these costs, firms then compete in the market. A distinctive feature of our model is that we allow a firm to form collaboration relations with other firms without seeking prior permission of current collaborators. This has important strategic effects and requires novel methods of analysis.[4]

We start by analyzing the case *where the costs of forming links are small.* In this setting, we are able to characterize the nature of strategically stable networks under fairly general conditions. We consider two types of market competition: moderate and aggressive. In a market with moderate competition, all firms make positive profits but lower cost firms make

[1]For instance, in the area of biotechnology the number of collaborations involving the world's largest firms rose from a total of around 100 in the pre-1987 period, to over 150 in the period 1988-1991. Moreover, the number of firms involved in these collaborations also increased sharply, doubling from one period to the next (Delapierre and Mytelka, 1998). A similar pattern is observed in the area of information technology (Hagedoorn and Schakenraad, 1990).

[2]Several authors have plotted the network of collaboration between firms; see e.g., Delapierre and Mytelka (1998). These plots indicate asymmetric collaborative structures as well as intransitive relations. An example of intransitive relations observed was the following. In the late 1980's and 1990's, Bristol-Mayers and Bayer had collaborative links, but Bayer also had collaborative links with Hoechst,, while there were no links between Bristol-Mayers and Hoechst.

[3]We interpret a link as a collaborative R&D project, which involves complementary facilities of the two firms. The project is costly and hence calls forth resources from the collaborating firms; it yields a process innovation which lowers the costs of production of the firms involved.

[4]The number of possible collaboration networks is very large. In a market with $n$ firms, there are $2^{n(n-1)/2}$ possible networks of collaboration. Thus, if $n = 10$, then there are over a billion possible networks of collaboration!

larger profits.[5] We first show that every pair of firms with the same costs must be linked. This implies that in the class of symmetric networks, i.e., networks where every firm has the same number of collaboration links, only the complete network can be stable.[6] We then develop sufficient conditions for this network to be the unique stable network (Theorem 3.1). We find that these conditions, though strong, are satisfied by a variety of standard models.[7]

Under aggressive competition, all but the lowest cost firms make zero profits. This allows for two cases of interest: first, in which a lowest cost firm makes profits only if it is the unique such firm, and second, if all lowest cost firms make positive profits. We find that in the first case, the empty network is the unique strategically stable network.[8] The latter case corresponds to a model of a patent race with the largest collaborating group winning the race. In this case, we provide a complete characterization of strategically stable networks (Theorem 3.2). In particular, we show that stable networks have an asymmetric architecture: the firms divide themselves into two groups, with one group containing at least three firms and having the feature that every pair of firms has a collaboration link, while the second group consists of isolated firms. We refer to this structure as the *dominant group architecture*.

We next consider the case where *costs of forming links are significant.* The analysis now focuses on the relationship between these costs and the nature of stable networks. For reasons of tractability we work with the linear demand Cournot model. We first derive a general property of the returns from link formation: firms have increasing returns from links. This implies, in particular, that the empty network and the complete network are the only symmetric networks that are stable. We then show that the only asymmetric network that is stable is the dominant group architecture, with the size of this group being sensitive to the cost of forming links. An interesting aspect of our analysis is a *non-monotonicity* in the sustainable size of the dominant group as the costs of forming links increase. Non-monotonicity manifests itself over an intermediate range of costs: over the initial part of this range, large as well as small dominant groups are strategically stable; however, over the latter part, only medium-sized dominant groups are stable and small and large groups are no longer sustainable (Proposition 4.1).

[5]Moderate competition accommodates quantity competition under homogeneous or differentiated demand, and price competition under differentiated demand.

[6]The complete network is one in which every pair of firms has a link.

[7]In particular, the homogeneous product model with quantity competition, and the differentiated product model with price and quantity competition fall in this category.

[8]The Bertrand model with homogeneous demand falls under the first case. The empty network is one in which there are no links.

The property of increasing returns from link formation suggests that a firm with many links may have an incentive to induce a firm with few links to form a collaboration relationship by offering to subsidize its costs of link formation. This motivates an examination of stable networks when transfers are allowed between firms. Again we find that the only symmetric networks that can be stable are the empty and the complete network. In the class of asymmetric networks, the dominant group architecture continues to be stable. In addition, the only other asymmetric architectures that are stable are the *star* and *inter-linked stars* (Propositions 4.2-4.3). The results are interesting from an empirical point of view, since these intransitive network architectures have been oberved in practice.[9] The stability of the star and inter-linked star architectures also illustrates in a somewhat dramatic fashion how market dominance can arise in a setting with ex-ante identical firms. They also bring out clearly the role of transfers across links, since such structures would not be stable in the absence of transfers.

Our paper is a contribution to the study of group formation and cooperation in oligopolies. The model of collaborative networks we present is inspired by the recent work on strategic models of network formation; see e.g., Aumann and Myerson (1989), Bala and Goyal (2000), Dutta, van den Nouweland and Tijs (1995), Goyal (1993), Jackson and Wolinsky (1996), and Kranton and Minehart (2000).[10] We now place our work in relation to the existing literature on the endogenous formation of groups.

The work of Kranton and Minehart (2000) deals with networks between vertically related firms. In contrast, our paper studies collaborative ties between *horizontally* related firms, i.e., firms which compete in the market subsequently. This leads us to incorporate an explicit market competition element in our collaboration model. Our paper should thus be seen as complementary to their work. The analysis in our paper suggests that market competition has major implications for the nature of collaboration networks.

Issues relating to group formation and cooperation have been a central concern in economics, and game theory in particular. The traditional approach to these issues has been in terms of coalitions. In recent years, there has been considerable work on coalition formation in games; see e.g., Bloch (1995), Kalai, Postlewaite, and Roberts (1979), Ray and Vohra (1997), and Yi (1997,1998). For a survey of this work, refer to Bloch (1997). One application of this

[9]See for instance, Figures 4.3 and 4.4 in Delapierre and Mytelka (1998), which plot the architecture of the collaboration networks in the pharmaceutical and biotechnology industry.

[10]The present paper subsumes our earlier paper, Goyal and Joshi (1999).

theory is to the formation of groups in oligopolies. In this literature, group formation is modeled in terms of a *coalition structure* which is a partition of the set of firms. Each firm therefore can belong to one and only one element of the partition, referred to as a *coalition*.

In our paper, we consider two-player relationships. In this sense, our model is somewhat restrictive as compared to the work referred to above, which allows for groups of arbitrary size. However, the principal distinction concerns the nature of collaboration structures we examine. Our approach accommodates collaborative relations that are *non-exclusive*. From a conceptual point of view, this distinction is substantive. It means that we allow for relationships across coalitions. Thus, we consider a class of cooperative structures which is significantly different from those studied in the coalition formation literature. In particular, our approach leads to collaboration networks such as stars/inter-linked stars which are empirically observed but are ruled out in the coalition literature.

A direct comparison of the results in our paper with this literature is difficult since there are other substantive differences in the models such as the role of spillovers. We therefore discuss the results of Bloch and Yi in greater detail, after presenting our results, in Section 3.

The paper is organized as follows. In Section 2, we present the basic model. In Section 3, we analyze the formation of networks when the costs of forming links are small, while section 4 examines the case where costs of forming links are large. Section 5 concludes.

# 2 The Model

We consider a setting in which a set of firms first choose their collaboration links with other firms. These collaboration agreements are pair-wise and costly and help lower marginal costs of production. The firms then compete in the product market. We now develop the required terminology and provide some definitions.

## 2.1 Networks

Let $N = \{1, 2, ..., n\}$ denote a finite set of ex-ante identical firms. We shall assume that $n \geq 3$. For any $i, j \in N$, the pair-wise relationship between the two firms is captured by a binary variable, $g_{i,j} \in \{0, 1\}$; $g_{i,j} = 1$ means that a link is established between firms $i$

and $j$ while $g_{i,j} = 0$ means that no link is formed. A network $g = \{(g_{i,j})_{i,j \in N}\}$, is a formal description of the pair-wise collaboration relationships that exist between the firms. We let $\mathcal{G}$ denote the set of all networks. Let $g + g_{i,j}$ denote the network obtained by replacing $g_{i,j} = 0$ in network $g$ by $g_{i,j} = 1$. Similarly, let $g - g_{i,j}$ denote the network obtained by replacing $g_{i,j} = 1$ in network $g$ by $g_{i,j} = 0$.

A *path* in $g$ connecting firms $i$ and $j$ is a distinct set of firms $\{i_1, \ldots, i_n\}$ such that $g_{i,i_1} = g_{i_1,i_2} = g_{i_2,i_3} = \cdots = g_{i_n,j} = 1$. We say that a network is *connected* if there exists a path between any pair $i, j \in N$. A network, $g' \subset g$, is a *component* of $g$ if for all $i, j \in g'$, $i \neq j$, there exists a path in $g'$ connecting $i$ and $j$, and for all $i \in g$ and $j \in g$, $g_{i,j} = 1$ implies $g_{i,j} \in g'$. We will say that a component $g' \subset g$ is *complete* if $g_{i,j} = 1$ for all $i, j \in g'$. Finally, let $N_i(g) = \{j \in N; g_{i,j} = 1\}$ be the set of firms with whom firm $i$ has a link in the network $g$, and let $\eta_i(g) = |N_i(g)|$ be the cardinality of this set.

We now define some networks that play a prominent role in our analysis. The *complete* network, $g^c$, is a network in which $g_{i,j} = 1$, $\forall i, j \in N$, and the *empty* network, $g^e$, is a network in which $g_{i,j} = 0$, $\forall i, j \in N$, $i \neq j$. Two other architectures will play a prominent role in our analysis: the *dominant group* architecture and the *star* architecture. The dominant group architecture is characterized by one complete non-singleton component and a set of singleton firms. Thus, there is a set of firms $N' \subset N$ with the property that $g_{i,j} = 1$ for every pair $i, j \in N'$ while for any $k \in N \backslash N'$, $g_{k,l} = 0$, $\forall\ l \in N \backslash \{k\}$. We will let $g^k$ denote the network in which there is one non-singleton component of size $k$ and the remaining $n - k$ firms are singletons. The star is a network in which there is a firm $i$ such that $g_{i,j} = 1$ for all $j \neq i$ and $g_{j,k} = 0$ for every pair of firms $j, k \neq i$.

## 2.2 Collaboration Links and Cost Reduction

A collaboration link in our framework can be interpreted as an agreement to jointly invest in cost-reducing R&D activity. We will suppose that a collaboration link requires a fixed investment, given by $f > 0$, from each firm. The firms are initially symmetric with zero fixed costs and identical constant returns-to-scale cost functions. Collaborations lower marginal costs of production. We will use the following specification:

$$c_i(g) = \gamma_0 - \gamma \eta_i(g; 1)\ , \quad i \in N. \tag{1}$$

where $\gamma_0$ is a positive parameter representing a firm's marginal cost when it has no links. In this case, firm $i$'s marginal costs are *linearly* declining in the number of links it has with other firms.[11]

In the general case, we will assume that firm $i$'s marginal cost in the network $g$ is a function of its collaboration links with other firms and is strictly decreasing in the number of these links:[12]

$$c_i(g) = c(\eta_i(g)); c(\eta_i(g) + 1) < c(\eta_i(g)), \quad i \in N. \tag{2}$$

To rule out uninteresting cases, we will always assume that $c_i(g) \geq 0$, $\forall i \in N$, $\forall g \in \mathcal{G}$. A network $g$, therefore, induces a marginal cost vector for the firms which is given by $c(g) = \{c_1(g), c_2(g), ..., c_n(g)\}$. Given this cost vector, and the specification of the demand functions in the product market, the firms compete in the second stage in the market. For every network $g$, we assume there is a well-defined Nash equilibrium in the second stage product market game. The profits of firm $i$, gross of the cost of forming links are given by $\pi_i(g)$.[13]

## 2.3 Stable Networks

A network $g$ is said to be stable if any firm that is linked to another in the network has a strict incentive to maintain the link and any two firms that are not linked have no strict incentive to form a link with each other. This definition is inspired by the notion of stability presented in Jackson and Wolinsky (1996). We need to adapt this general definition slightly to accommodate the different cases of fixed costs we consider. We, therefore, state the formal definitions in Sections 3 and 4.

The requirements above are very weak and should be seen as necessary conditions for a network to be stable. Our analysis illustrates that these weak requirements provide sufficient structure in an interesting class of network formation games.[14]

[11] This is a natural extension to the network framework of the specification used in Bloch (1995) where marginal cost of $i$ decreases linearly in the number of firms belonging to the same coalition as $i$.

[12] We are assuming that there are no spillovers across links in this model. We briefly address the issue of spillovers in Section 5.

[13] This implicitly assumes that there are no coordination problems in choosing across different equilibria at this stage.

[14] We have also examined an alternative *non-cooperative* formulation of the network formation game. In this formulation, every firm announces a set of links it intends to form with other firms. A link between two firms $i$ and $j$ is formed if both $i$ and $j$ announce an intention to form such a link. This announcement game

# 3 Small Costs of Link Formation

In this section, we will provide a fairly general analysis of network formation when the fixed costs of forming links are small. When two firms collaborate, they help lower each other's costs. There are two effects at work here: collaboration lowers a firm's cost but also lowers its competitor's cost. In addition, a collaboration between two firms generates competitive effects on non-participating firms. The precise nature of these effects depend on the nature of market competition. The main point of this section is to illustrate the influence of market competition on the architecture of strategically stable networks.

We will suppose that there are small but positive costs to forming links. This motivates the following simple definition of strategic stability. A network $g$ is *stable* if the following conditions are satisfied:

(i) For $g_{i,j} = 1$, $\pi_i(g) > \pi_i(g - g_{i,j})$ and $\pi_j(g) > \pi_j(g - g_{i,j})$

(ii) For $g_{i,j} = 0$, if $\pi_i(g + g_{i,j}) > \pi_i(g)$, then $\pi_j(g + g_{i,j}) \leq \pi_j(g)$

We have adapted this definition from Jackson and Wolinsky (1996). This definition of stability reflects two main ideas. First, while a link can be severed unilaterally, forming a link is a *bilateral* decision, i.e. a link is formed if and only if the two firms involved agree to form the link. Second, there are no transfers possible across links. Taken together with the idea of small but positive costs of link formation, this implies that both firms must make strictly greater profits by forming a link.

## 3.1 Example: Homogeneous Product Oligopoly

We begin by providing a complete characterization of collaboration networks in a homogeneous product oligopoly, i.e., a market where the outputs of the firms are perfect substitutes. In particular, we restrict attention to the following linear inverse market demand:[15]

of link formation was introduced in Dutta et al (1995). We assume that there are some positive but small costs to forming links. Given any network, the payoffs to a firm are then defined as in the model presented in Section 3. We examined the structure of networks that arise in Nash equilibrium in undominated strategies of this announcement game. Our analysis yields results analogous to those presented in Section 3 below.

[15] We analyze the general oligopoly model later in this section.

$$p \;=\; \alpha \;-\; \sum_{i\in N} q_i\,, \quad \alpha > 0 \tag{3}$$

The profits of the firms will depend on the nature of market competition. In this section we will assume that marginal cost of firm $i \in N$ is given by (1).

We start with the case of Cournot competition. Given any network $g$, the Cournot equilibrium output can be written as:

$$q_i(g) \;=\; \frac{(\alpha-\gamma_0) + n\gamma\eta_i(g,1) - \gamma\sum_{j\neq i}\eta_j(g,1)}{(n+1)}\,, \quad i \in N \tag{4}$$

In order to ensure that each firm produces a strictly positive quantity in equilibrium, we will assume that $(\alpha-\gamma_0) + (n-1)^2\gamma > 0$. Aggregate Cournot output for any given $g$ is:

$$Q(g) \;=\; \sum_{i\in N} q_i(g) \;=\; \frac{n(\alpha-\gamma_0) + \gamma\sum_{i\in N}\eta_i(g,1)}{(n+1)} \tag{5}$$

The second stage Cournot profits for firm $i \in N$ are given by $\pi_i(g) = q_i^2(g)$. In our study of stable networks, we will find it convenient to use a positive monotone transform of the firm's profits to write the payoffs as follows:

$$T_i(g) \;=\; (\alpha-\gamma_0) + n\gamma\eta_i(g,1) - \gamma\sum_{j\neq i}\eta_j(g,1)\,, \quad i \in N \tag{6}$$

We can now characterize the stable collaboration networks under quantity competition.

**Proposition 3.1** *Suppose there is quantity competition among the firms. If the marginal cost function satisfies* (1) *and demand satisfies* (3)*, then the complete network, $g^c$, is the unique stable network.*

The proofs of Propositions 3.1 and 3.2 are given in Appendix A. Figure 1a gives an example of a complete network. The intuition behind the above result is as follows. First note that if two firms form a link, then the costs of all other firms are unaffected, while the cost advantage to both firms forming a link is the same under (1). An inspection of the profit expression in (6) reveals that the positive effects on the profits of a firm $i$ from a link with another firm

$j$ are given by $n\gamma$, while the negative effects are given by $\gamma$. Thus link formation is clearly profit enhancing. This argument shows that any network other than the complete network cannot be stable. To see why the complete network is stable, note that no further links can be added, while the deletion of a link by a firm $i$, with (say) firm $j$ only increases the costs of firm $i$ and $j$ but leaves the costs of all other firms unaffected, lowering profits of firm $i$ by $(n-1)\gamma$. Thus it is not profitable to delete links either. This completes the argument.

It is useful to contrast our result with that of Bloch (1996) who, under a similar specification of demand and marginal cost, derives a stable coalition structure consisting of two asymmetrically-sized blocs in which the number of firms in the larger coalition is the integer closest to $3(n+1)/4$. This sharp difference in the results is in part due to the absence of spillovers in our setting. To see this, let us examine the incentives for link formation for a firm in a large component and an isolated firm. In our framework, the above arguments show that both the firms have an incentive to form a link. By contrast, in the setting of Bloch, due to the implicit assumption of perfect spillovers, the isolated firm gains access to a large amount of cost-reduction since it accesses all the firms in the component; similarly all the firms in the component also gain access to this (erstwhile) isolated firm. Thus the returns to the competing firms are much greater than in our setting, and it is possible that no firm in the large component wishes to form a link with an isolated firm, and an incomplete network can be stable. We discuss the issue of spillovers further in the concluding section.

The simultaneous open membership game in Yi (1998) obtains the grand coalition as the unique outcome of the game. This approach is similar to one in which the decision to join a coalition is one-sided. In such a game, in the presence of perfect spillovers, a member of a smaller group always has an incentive to join a larger group. In our paper, link formation is based on pair-wise incentive compatibility and there are no spillovers. Thus, our result provides an alternative explanation as to how a grand coalition may endogenously emerge in equilibrium.

We next take up the case of Bertrand competition. Given a network $g$, what are the payoffs of different firms under Bertrand competition? Standard considerations (exploiting the idea of a finite price grid) allow us to state that there exists an equilibrium, and in this equilibrium a firm will make profits only if it is the unique minimal cost firm in the market. In other words:

$$\pi_i(g) = 0, \text{ if } \; c_i(g) \geq c_j(g), \; \text{ for } i \neq j; \; \pi_i(g) > 0, \text{ if } \; c_i(g) < c_j(g), \forall\, j \neq i. \tag{7}$$

Since $g$ is arbitrary, the above expression allows us to specify the payoffs for all possible networks. What are the stable networks of collaboration in this setting of extreme competition? The following result provides a complete answer to this question:

**Proposition 3.2** *Suppose there is price competition among the firms. If the marginal cost function satisfies* (1) *and demand satisfies* (3)*, then the empty network, $g^e$, is the unique stable network.*

Figure 1b presents an example of an empty network. The intuition behind this result is simple. Suppose $g$ is a non-empty network and that firm $i$ has a link in this network. It is either the unique minimum cost firm, in which case its collaborators (of whom there must be at least one) have a incentive to delete their links. If, on the other hand, firm $i$ is not the unique minimum cost firm then it has a incentive to delete its links. Thus a network $g$ in which firm $i$ has a link cannot be stable. These arguments are very general; in particular, we do not make use of the linear structure of the demand or the cost function. This suggests that the absence of collaborative links is likely to obtain in general settings where competition is extreme. This result should be seen only as a benchmark case; incentives to form links are very different if we allow for a slight amount of differentiation in products (see Appendix A for an illustration of this).

Propositions 3.1 and 3.2 suggest that the nature of market competition has a significant influence on incentives for collaboration and the architecture of stable networks. We now study this influence under more general conditions.

## 3.2 Moderate Competition

We consider the case where all firms make positive profits but lower cost firms make higher profits. Such a situation is described as moderate competition. Formally, this situation is reflected in the following assumption:

**Assumption MC** *Fix some $g$. $\pi_i(g) > 0$ for all $i \in N$; $\pi_i(g) = \pi_j(g)$ if $c_i(g) = c_j(g)$, while $\pi_i(g) > \pi_j(g)$ if $c_i(g) < c_j(g)$.*

The next assumption concerns the payoffs of similar cost firms.

**Assumption SY1** *Fix some $g$. Suppose that for a pair of firms $i$ and $j$, $c_i(g) = c_j(g)$. (i) If $g_{i,j} = 0$ then $\pi_i(g + g_{i,j}) > \pi_i(g) > 0$ and $\pi_j(g + g_{i,j}) > \pi_j(g) > 0$. (ii) If $g_{i,j} = 1$ then $\pi_i(g - g_{i,j}) < \pi_i(g)$ and $\pi_j(g - g_{i,j}) < \pi_j(g)$.*

Yi (1998, Lemma 3) demonstrates that (SY1) holds under a set of reasonable restrictions on general homogeneous demand (downward-sloping, concave) and costs (total cost is convex in own output, total and marginal cost are strictly decreasing with the number of links) along with a joint restriction on demand and costs. These conditions ensure that a favorable cost shock to a pair of symmetric firms will increase their net profits. Yi (1998, Section 5) also shows that (SY1) is valid for symmetrically differentiated demand (where firm $i$'s payoff depends only on the aggregate output of the rival firms). Symmetry in the presence of moderate competition implies the following property of stable networks.

**Proposition 3.3** *Suppose that* (SY1) *and* (2) *hold. Consider a stable network, $g$. If $\eta_i(g) = \eta_j(g)$, then $g_{i,j} = 1$.*

**Proof** Let $g$ be stable. If $\eta_i(g) = \eta_j(g) = n$, then by definition $g_{i,j} = 1$. Therefore, consider the case where $\eta_i(g) = \eta_j(g) < n$ and $g_{i,j} = 0$. Under (2) the costs of $i$ and $j$ are identical if $\eta_i(g) = \eta_j(g)$. Under assumption (SY1)(i), it follows that $\pi_i(g + g_{i,j}) > \pi_i(g)$ and $\pi_j(g + g_{i,j}) > \pi_j(g)$. This violates requirement (ii) of stability and contradicts the hypothesis that $g$ is stable.

$\square$

Proposition 3.3 has several interesting implications for the nature of stable networks. The *first* implication is that a stable network cannot have two or more singleton components. This implies in particular that the empty network cannot be stable. The *second* implication is that the star/hub-spokes network is not stable. This is because in all these networks, there are at least two firms $i$ and $j$ who have the same number of links but $g_{i,j} = 0$. By Proposition 3.3, such firms have an incentive to form a link.[16] A *third* implication of this result is that if a stable network contains two or more complete components then they must be of unequal size. The result above thus implies that if all firms have the same cost, then every pair of firms must be linked; thus, the only symmetric network that can be stable is the complete network. Our next result derives conditions under which the complete network is the unique stable network.

[16]Our analysis in section 4 will illustrate that this property also obtains for large costs of forming links but is no longer valid when we allow for transfers across firms.

**Theorem 3.1** *Suppose that hypotheses (MC) and (SY1) hold. Then the complete network, $g^c$, is stable. If in addition, for every network $g$ and any link $g_{i,j} = 0$ it is true that $\pi_i(g + g_{i,j}) > \pi_i(g)$ and $\pi_j(g + g_{i,j}) > \pi_j(g)$ then the complete network, $g^c$, is the unique stable network.*

**Proof** We provide a proof of the first statement. The second statement is immediate and a proof is omitted. In $g^c$, $\eta_i(g^c) = n - 1$, $\forall i \in N$. Therefore, all firms have the same cost and this is the minimum cost. There are no links to add so requirement (ii) of stability is automatically satisfied. We check requirement (i) next. Suppose we set $g_{i,j} = 0$ for some pair $i$ and $j$. In the ensuing network, $g^c - g_{i,j}$, assumption (SY1)(ii) implies that both firms $i$ and $j$ loose strictly. This implies that requirement (i) is also satisfied. Thus $g^c$ is stable.

$\square$

The additional monotonicity condition in Theorem 3.1 may seem strong. However, it is satisfied by a variety of standard oligopoly models. *First*, we note that it satisfied by the standard model of a differentiated oligopoly, with linear demand and linearly reducing costs (as in expression (1)). The calculations for both price and quantity competition are given in Appendix A. *Second*, we note that the monotonicity condition is also satisfied if each of the firms is a monopoly in its own market. This is true since the only 'costs' of forming links in our model arise out of the greater competitiveness of a firm whose costs are lowered. However, if the other firms are in unrelated markets then there is no 'cost' to forming additional links while there are benefits in terms of of lowering marginal costs of production. It is then immediate that in such a case every pair of firms has an incentive to form links and thus the unique stable network is the complete network. *Finally*, it can be shown that it is satisfied by Cournot oligopoly under fairly general demand conditions. Suppose that the inverse demand, $p(Q)$, satisfies the following general specification: *$p(Q)$ is a twice continuously differentiable function with $p'(Q) < 0$ and $p''(Q) \leq 0$.* We have shown that if inverse demand satisfies this condition and the cost reduction is linear, then the additional monotonicity condition on profits of the firms is also satisfied.[17]

[17]Due to space constraints, we have omitted them from the paper. The details of these derivations are available from the authors upon request.

### 3.3 Aggressive Competition

The notion of aggressive competition should be seen as a generalization of Bertrand competition with homogeneous products. We will say that competition among firms is aggressive if all but the lowest cost firms make zero profits. This section provides a complete characterization of strategically stable networks under aggressive competition.

There are two sub-cases: one, the lowest cost firm makes positive profits only if it is the unique such firm, and two, all the lowest cost firms make positive profits. The former case is written as follows:

**Assumption B** *Fix some $g$. If $c_i(g) \geq c_j(g)$, then $\pi_i(g) = 0$, while if $c_i(g) < c_j(g)$ for all $j \in N \backslash \{i\}$ then $\pi_i(g) > 0$.*

This specification generalizes the Bertrand competition to allow for general demand functions and also general cost reduction functions. The arguments in the proof of Proposition 3.2 generalize in a straightforward way to show that the empty network is the unique strategically stable network under Assumption (B).

We now take up the case where every lowest cost firm make positive profits. By way of motivation, consider a set of firms that are competing to apply for a patent for a cost reducing process technology. Suppose that each of the firms has some useful complementary knowledge. If they collaborate, then this knowledge can be jointly used to lower costs. Moreover, only the lowest cost technology is patented. Once the patent is available, it is randomly allotted to one of the firms who have the lowest cost technology. Price competition then ensures that only this firm makes profits. The positive profits should be seen as the (ex-ante) expected profits from collaboration.

**Assumption AC** *Fix some $g$. If $c_i(g) > c_j(g)$, then $\pi_i(g) = 0$, while if $c_i(g) \leq c_j(g)$ for all $j \in N \backslash \{i\}$ then $\pi_i(g) > 0$.*

In our analysis we shall use the following symmetry assumption with respect to the lowest cost firms.

**Assumption SY2** *Fix some $g$. Suppose that for a pair of firms $i$ and $j$, $c_i(g) = c_j(g) = \min_{k \in N} c_k(g)$. (i) If $g_{i,j} = 0$ then $\pi_i(g + g_{i,j}) > \pi_i(g) > 0$ and $\pi_j(g + g_{i,j}) > \pi_j(g) > 0$. (ii) If $g_{i,j} = 1$ then $\pi_i(g - g_{i,j}) < \pi_i(g)$ and $\pi_j(g - g_{i,j}) < \pi_j(g)$.*

Assumption (SY2) is weaker than Assumption (SY1) since it applies only to the minimum cost firms. Once again, sufficient conditions on demand and costs under which (SY2) holds are provided in Yi (1998, Lemma 3 and Section 5). Symmetry in the presence of aggressive competition has strong implications for collaboration. This is demonstrated in the following result.

**Theorem 3.2** *Let* $n \geq 4$. *Suppose* (AC) *and* (SY2) *hold and marginal cost is specified by* (2). *Then a network is stable if and only if it is a dominant group network* $g^k$, *with* $k \in \{3, 4, \cdots, n\}$.

The proof of this result is given in Appendix A. Figure 2 presents the dominant group networks in a market with 5 firms. We provide a sketch of the arguments. *First*, we show that any non-singleton component in a stable network must be complete. In proving this property, we also establish that all firms in a non-singleton component must have the same costs and that these costs must be the minimum in the given network. *Second*, we show that there can be at most one non-singleton component in a stable network. These two properties reduce the set of candidates for stable networks dramatically to a subset of dominant group networks.[18]

We note that the number of stable networks is very small as compared to the number of total networks. For example, when $n$ is 3, 4, 5 or 6, the total number of networks is given by 8, 64, 1024 and 32768, respectively. By contrast, the number of stable networks is given by 3, 5, 16, and 42, respectively. Thus the two simple requirements of stability lead to a strong restriction on the class of networks.

# 4 Large Costs of Link Formation

In general, R & D collaboration agreements will involve commitment of funds. This leads us to study the model where the costs of forming links are substantial. We suppose that each link imposes a cost of $f > 0$ on each of the two firms forming the link. No costs are incurred if the link is not formed. The main results in this section pertain to the relationship between the costs of forming links and the architecture of strategically stable networks.

[18] The above result is stated for $n \geq 4$. It is easily seen that in case of $n = 3$ an analogous result obtains: a stable network is either complete or has two components, one component with two firms and the other component with a singleton firm. We have stated the result for $n \geq 4$ as it allows for a simpler statement.

In the analysis so far, we have worked with the assumption of negligible costs. This has allowed us to study incentives of link formation simply in terms of the 'sign' of the terms. In the presence of large costs of forming links, an assessment of the incentives to form links requires an explicit measurement of the benefits of links. This complicates the analysis considerably, and to get our main points across easily, we study the Cournot model with linear demand. We believe that some of our main insights hold more generally and we will clarify the scope of the analysis below.

We incorporate the fixed costs of forming links in the payoffs as follows. Fix a network $g$. The *net* profit of each firm $i \in N$ is given by: $\Pi_i(g) = \pi_i(g) - \eta_i(g)f$, while the *gross* profit is given by $\pi_i(g) = q_i^2(g)$. Given a network $g$, let $g_{-i}$ denote the network in which all of firm $i$'s links are deleted. We can now define a stable network as follows.

**Definition 4.1** *Let $f$ be the fixed cost of link formation. A network $g$ is stable, if the following conditions hold.*

1. *For $g_{i,j} = 1$, $\pi_i(g) - \pi_i(g - g_{i,j}) \geq f$, $\pi_j(g) - \pi_j(g - g_{i,j}) \geq f$*
2. *For $g_{i,j} = 0$, $\pi_i(g + g_{i,j}) - \pi_i(g) > f \implies \pi_j(g + g_{i,j}) - \pi_j(g) < f$*
3. *For every $i \in N$, $\pi_i(g) - \eta_i(g)f \geq \pi_i(g_{-i})$.*

In words, the first two conditions require respectively that in a stable network, any firm that is linked to another has no incentive to sever the link, and any two firms that are not linked should have no incentive to establish a collaboration link. These two conditions constitute a "marginal" check for stability. The third condition is an "aggregate", or "global", check for stability which requires that a firm should find it profitable to maintain its collaboration links in the network rather than not having any links. This condition can be seen as an individual rationality condition for participation in the network.

We focus on the homogeneous demand model with linear demand. We first note that with small costs of forming links the empty network was the unique stable network under price competition. Clearly, the same result will obtain once we assume that there are large costs of forming links. Therefore, in the rest of the analysis in this section, we will focus our attention on quantity competition.

Our first result establishes that gross profits of a firm exhibit *increasing returns* to the number of links which the firm has with other firms.

**Lemma 4.1** *Consider any network $g$ and distinct firms $i, j, k \in N$ such that $g_{i,j} = g_{i,k} = 0$. Then:*

$$\pi_i(g + g_{i,j} + g_{i,k}) - \pi_i(g + g_{i,j}) \; > \; \pi_i(g + g_{i,j}) - \pi_i(g) \tag{8}$$

**Proof** First of all note that the Cournot output of firm $i$ is strictly increasing with each additional link: $q_i(g + g_{i,j}) - q_i(g) \; = \; \gamma(n-1)/(n+1) \; > \; 0$

Recall that for any network $g$, the gross profit of $i$ is $\pi_i(g) = q_i^2(g)$. It follows that:

$$\begin{aligned} \pi_i(g + g_{i,j} + g_{i,k}) - \pi_i(g + g_{i,j}) \;&=\; \frac{\gamma(n-1)}{(n+1)}\left[q_i(g + g_{i,j} + g_{i,k}) + q_i(g + g_{i,j})\right] \\ &>\; \frac{\gamma(n-1)}{(n+1)}\left[q_i(g + g_{i,j}) + q_i(g)\right] \\ &=\; \pi_i(g + g_{i,j}) - \pi_i(g) \end{aligned} \tag{9}$$

This proves the result.

□

We note that by virtue of increasing returns in gross profits, condition (3) implies condition (1) in the definition of stability. Therefore, it suffices to verify conditions (2) and (3) when checking the stability of any network.

We now develop a complete characterization of the architecture of stable networks. We start by noting a 'transitivity' implication of the increasing returns property.

**Lemma 4.2** *Let $g$ be a network which is stable under fixed cost $f$ of link formation. Let $i$ and $j$ be two distinct firms. Then $g$ satisfies the following property: suppose there exists a firm $k \neq i, j$ such that $g_{i,k} = 1$ and a firm $l \neq i, j$ such that $g_{j,l} = 1$; then, $g_{i,j} = 1$.*

**Proof** The proof is by contradiction. Suppose that $g$ is stable but $g_{i,j} = 0$. Since $g$ is stable, it follows that $\pi_i(g) - \pi_i(g - g_{i,k}) \geq f$. From the property of increasing returns (Lemma 4.1)it follows that $\pi_i(g + g_{i,j}) - \pi_i(g) > \pi_i(g) - \pi_i(g - g_{i,k}) \geq f$. Thus firm $i$ has an incentive to form a link with firm $j$. The only property we have used is that firm $i$ has a link with some other firm. In this respect the situation of firm $j$ is symmetric. Therefore, using an identical argument, we can show that firm $j$ has an incentive to form a link with firm $i$. This establishes that $g$ is not stable, a contradiction.

$\square$

This result has a number of interesting implications. *Firstly*, it implies that any stable network $g$ can have at most one non-singleton component, $g'$. Furthermore, $g'$ must be complete, i.e. all firms in this component must have links with each other. Thus, a stable network will have the dominant group architecture. Recall that $g^k$ denotes the network in which there is one non-singleton complete component of size $k$ and the remaining $n-k$ firms are singletons. *Secondly*, this result implies that there are only two possible *symmetric* stable networks: the empty and the complete.

We now provide a complete characterization of stable networks. Recall that $g^k$ refers to a dominant group network in which the dominant group has $k$ firms.

**Proposition 4.1** *Suppose that marginal cost satisfies (1), demand satisfies (3) and that firms compete in quantities. Then there exist numbers $F_0$, $F_1$, $F_2$, and $F_3$, where $F_0 < F_1 < F_2 < F_3$, with the following property: (1). For $f < F_0$, $g^c$ is the unique stable network. (2) For $F_0 \leq f < F_1$, a network $g^k$ is stable if and only if $k \in \{k(f), ..., n\}$, with $k(f) > 1$. (3) For $F_1 \leq f < F_3$, a network $g^k$ is stable if and only if $k \in \{\underline{k}(f), ..., \bar{k}(f)\}$, with $1 \leq \underline{k}(f) < \bar{k}(f) < n$. (4) For $f > F_2$, $g^e$ is a stable network. Moreover, if $f > F_3$ then $g^e$ is the unique stable network.*

**Proof** Consider a dominant group network, $g^k$. A firm in the non-singleton component of size $k$ has no incentive to delete all its links if:

$$Y(k) \;\equiv\; \frac{(n-1)\gamma}{(n+1)^2}\left[2(\alpha-\gamma_0)+(k-1)(n+3-2k)\gamma\right] \;\geq\; f \tag{10}$$

If the above condition is satisfied, then by virtue of the property of increasing returns (Lemma 4.1), a firm in the non-singleton component would always want to form a link with an isolated firm. Therefore, if $g$ is stable, then the isolated firm should have no incentive to form a link with a firm in the non-singleton component. This requires:

$$X(k) \;\equiv\; \frac{(n-1)\gamma}{(n+1)^2}\left[2(\alpha-\gamma_0)+(n-1)\gamma-2k(k-1)\gamma\right] \;<\; f \tag{11}$$

A network $g^k$ is stable if and only if it satisfies (10) and (11). By inspection, we see that $X(k)$ is declining in $k$. Further, $X(n-1) = F_0$. Regarding $Y(k)$, it is initially increasing and then decreasing in $k$. Note that $F_1 = Y(n)$, $F_2 = Y(2) = X(1)$, and $F_3 = Y(k^*)$. Further, $F_0 < F_1 < F_2 < F_3$. The proof now follows from Figures 3 and 4.

□

Figure 4 illustrates the nature of stable architectures, as the cost of forming links $f$ varies. We *first* note that the cost of forming collaboration links has a significant impact on the structure of the collaboration network. In particular, for low costs, the complete network is uniquely stable, for moderate costs only networks with relatively large dominant groups are stable, for high costs, only medium size dominant groups are stable (small and large groups are not sustainable), while for very high costs, the empty network is uniquely stable. Hence, the effect of R&D costs on the size of the dominant group is *non-monotonic*. The intuition for this pattern is as follows: when costs are low, the incentive constraint of the isolated firm to form a link is binding. The marginal payoff to an isolated firm from an additional link is declining in the size of the dominant group. Hence, as the costs of forming R&D collaboration links increase, smaller groups are sufficient to discourage the isolated firm from forming a link. However, beyond a certain cost level, the incentive constraint for a firm in the dominant group to retain its links is binding. The returns from links to a firm in a dominant group are non-monotonic in the size of the dominant group: they are increasing for group sizes until some critical value $k^*$, and then declining. This implies that for high cost levels, small and large dominant groups are not stable.

*Secondly*, we note that the architecture with dominant groups reflects sharp asymmetries in the market outcome. This suggests that the possibilities of collaboration lead to very asymmetric outcomes in spite of ex-ante identical firms. And, most importantly, for a large class of parameters, such asymmetric networks are the *only* stable networks. This means that in these circumstances, firms only want a certain number of collaborators; this creates an incentive to preempt and form links early.

*Finally*, we note that the key property which drives the characterization results of this section is increasing returns in gross profits. This property ensures that only the empty and complete networks can be stable in the class of symmetric networks, and only the dominant group architecture can be stable in the class of asymmetric networks. Our analysis, therefore, implies that only three architectures - empty, complete and dominant group - will be candidates for stability in any general model of network formation where the reduced form payoff to each player displays increasing returns.

## 4.1 Transfers

The property of increasing returns suggests that firms with many links may have an incentive to make transfers to firms who are poorly linked to induce them to form links. These considerations motivate an analysis of the nature of stable networks when transfers are allowed across firms. This section provides a characterization of stable networks when transfers are allowed. We find that stars (and variants of this architecture) are strategically stable in this setting.

Let $t_i = \{t_i^1, ...t_i^n\}$ be the transfers offered by firm $i$ to other firms. We shall suppose that $t_i^j \geq 0$, for all $i, j \in N$, and that $t_i^i = 0$, for all $i \in N$. We modify the concept of stability to accommodate the possibility of transfers. The concept of strategic stability we use is defined as follows.

**Definition 4.2** *A network $g$ is stable against transfers if:*

1. *For all $g_{i,j} = 1$, $[\pi_i(g) - \pi_i(g - g_{ij})] + [\pi_j(g) - \pi_j(g - g_{i,j})] > 2f$*
2. *For all $g_{i,j} = 0$, $[\pi_i(g + g_{i,j}) - \pi_i(g)] + [\pi_j(g + g_{i,j}) - \pi_j(g)] < 2f$*
3. *There exist transfers $t_i \in R^n$, $i = 1, 2, ...n$ such that*

$$\pi_i(g) - \eta_i(g)f + \sum_{j \in N_i(g)} (t_j^i - t_i^j) \geq \pi_i(g_{-i}), \tag{12}$$

We first note that increasing returns from collaboration obtain in this setting as before. This in turn implies that there can be at most one non-singleton component in a stable network. A related implication of increasing returns is the following result on the local structure of a stable network. The following lemma shows that if firm $i$ has a link with firm $j$ in a network $g$ which is stable against transfers, then it must also have a link with every firm $k$ which has as many links as $j$ in the network $g - g_{i,j}$.

**Lemma 4.3** *If $g$ is stable against transfers, then it satisfies the following property: suppose $g_{i,j} = 1$ for distinct $i, j \in N$; then, $g_{i,k} = 1$ for all $k \in N$ satisfying $\eta_k(g - g_{i,j}) \geq \eta_j(g - g_{i,j})$.*

Lemma 4.3 provides a simple "marginal" check for stability against transfers by examining the incentive for two firms in a network $g$ to be jointly better off by forming a link. If $i$ and $j$ can jointly profit from a link, then this will lower the profits of all firms $k \neq i, j$ and may lead some other firms to sever their links. What is true, however, throughout this process of readjustment is that the original network $g$ could not have been stable against transfers.

We start by noting that an implication of increasing returns in gross profits is that in the class of *symmetric* networks there are only two possible stable networks: the empty and the complete. The proof follows as a corollary of the property of increasing returns (see Lemma 4.1) and is, therefore, omitted.

We now turn to the characterization of *asymmetric* networks. We start with a class of asymmetric *connected* networks that are commonly observed in empirical work and are stable only in the presence of transfers. These are the star and inter-linked star architectures. We first determine the conditions under which the star (or hub-spokes network) is stable against transfers.

**Proposition 4.2** *Let $n \geq 4$. Suppose that cost satisfies (1), demand satisfies (3) and that firms compete in quantities. Then there exist $F_H$ and $F_L$, where $0 < F_L < F_H$ such that the star architecture is stable against transfers if and only if $F_L < f < F_H$.*

The proof requires a verification of the three conditions of stability. First, we need to establish that no firm has an incentive to delete an existing link. This generates a upper bound on the value of $f$. Next we check that no pair of spoke firms has an incentive to form a link. This generates a lower bound on the value of $f$. Finally, we check that the transfers required to induce the spoke firms to form links with the center (or hub) are indeed feasible. This generates another upper bound on the value of $f$. Putting these bounds together yields the conditions in the result. The details of the computations are given in Appendix B.

We now elaborate on this result. *First*, in the star, each of the spoke firms derives a relatively low return from its link, due to the relative cost disadvantage with respect to the center. Hence, the stability of the star architecture is critically dependent on transfers from the center. If transfers were not permitted, each spoke firm would sever its link with the hub firm. This is also indicative of how market dominance can arise in a setting with ex-ante identical firms.

*Second*, inspecting the terms $F_L$ and $F_H$ in the proof of Proposition 4.2 shows that the star is stable for all $f \in [0, \infty)$ as $n \to \infty$, i.e over the entire parameter space. This result is once again a consequence of increasing returns in gross profits: for any fixed cost $f$ of link formation, however high, the center will be able to use transfers to induce spoke firms to form a link if its marginal profits from the links are high enough; the center's marginal profits in turn will be large enough if there are a sufficient number of spoke firms (i.e. a large enough value of $n(f)$) for the center to potentially form links with.

*Third*, Lemma 4.3 highlights an important relationship between a star and the connectedness of a stable network. In particular, it implies that if a network $g$ is stable against transfers, and the non-singleton component is a star, then $g$ must be connected. To see this, let $k$ be some firm which does not belong to the star component. Since there can be at most one non-singleton component in a stable network, $\eta_k(g) = 0$. Now consider a hub, $i$, and a spoke, $j$, in the star component. By definition, $\eta_j(g - g_{i,j}) = \eta_k(g - g_{i,j}) = 0$. Then, by Lemma 4.3, $i$ should have had a link with $k$ as well.

In addition to the star, the only other candidates for stability in the class of connected asymmetric networks are the inter-linked stars. In order to motivate the inter-linked star architectures, let $g$ be any connected network. Consider a partition of the set of firms, $\{h_1(g), h_2(g), \ldots, h_m(g)\}$, with $h_l(g) \cap h_k(g) = \emptyset$ for $l \neq k$, and $\cup_{l=1}^{m} h_l(g) = N$. Further, (i) if $i, j \in h_l(g)$, then $\eta_i(g) = \eta_j(g)$, (ii) if $i \in h_l(g)$ and $j \in h_k(g)$, $l < j$, then $\eta_i(g) < \eta_j(g)$, and (iii) if $i \in h_1(g)$, then $\eta_i(g) \geq 1$ (since $g$ is connected). We now characterize the number of links within an element $h_l(g)$ as well as across $h_l(g)$, $l = 1, 2, ..., m$ in a stable network. This characterization result shows that stable networks have an inter-linked star architecture.

**Proposition 4.3** *Suppose that marginal cost satisfies (1), demand satisfies (3) and that firms compete in quantities. Let $g \neq g^c$ be a connected network. If $g$ is stable against transfers then it satisfies the following properties. (i). If $i \in h_l(g)$, $j \in h_{l'}(g)$, $l \neq l'.|, then \eta_i(g) - \eta_j(g)| \geq 2$. (ii) If $i \in h_m(g)$, then $\eta_i(g) = n - 1$. (iii) If $g_{i,j} = 1$ for $i \in h_l(g)$ and $j \in h_{l'}(g)$, then $g_{i,k} = 1$ for any $k \in h_{l''}(g)$ where $l'' \geq l'$. Moreover, for any $i \in h_1(g)$, $g_{i,j} = 1$ if and only if $j \in h_m(g)$. (iv) If $i \in h_{m-l}(g)$, then $g_{i,j} = 1$ if and only if $j \in h_k(g)$, $k > l$.*

The proof of this result is given in Appendix B. Inter-linked stars are illustrated in Figure 5. For the case $n = 6$, these are the only asymmetric connected networks that are stable against transfers according to Proposition 4.3. Figure 5a presents a star, Figure 5b presents a network with two inter-linked stars with firms 1 and 2 forming two hubs, Figure 5c presents a network with three inter-linked stars with firms 1,2 and 3 forming the hubs. Finally, Figure 5d presents a network with four inter-linked asymmetric stars: the star with firm 1 as the hub is larger (in the sense of being connected to more spoke firms) than the three smaller stars with firms 2,3 and 4 as the hubs.

Proposition 4.3 also implies that in the class of *unconnected* asymmetric networks, if the non-singleton component is incomplete, then it will be an inter-linked star. Note, however, that the non-singleton component cannot be a star by virtue of part (i) of the proposition. In fact, as we argued earlier, if the non-singleton component is a star, then the network must be connected. If the non-singleton component in an asymmetric unconnected network is complete, then we have the familiar dominant group network, $g^k$. With regard to this architecture, we can now prove:

**Proposition 4.4** *Suppose that cost satisfies (1), demand satisfies (3) and that firms compete in quantities. If $g^k$ is stable against transfers then it is also stable. The converse, however, is not true.*

**Proof** Suppose $g^k$ is stable against transfers. Then, for any $i, j$ such that $g_{i,j} = 0$, net profits must satisfy:

$$\Pi_i(g) + \Pi_j(g) < \Pi_i(g + g_{i,j}) + \Pi_j(g + g_{i,j}) \tag{13}$$

This implies that if $\Pi_i(g + g_{i,j}) > \Pi_i(g)$, then $\Pi_j(g + g_{i,j}) < \Pi_j(g)$. Therefore, condition (2) in the definition of stability is also satisfied.

Since $g^k$ is stable against transfers, for any $i, j$ such that $g_{i,j} = 1$:

$$\Pi_i(g) + \Pi_j(g) > \Pi_i(g - g_{i,j}) + \Pi_j(g - g_{i,j}) \tag{14}$$

However, in $g^k$, the only firms who are linked are those which belong to the non-singleton complete component. Therefore, these firms have the same net profits, i.e. $\Pi_i(g) = \Pi_j(g)$ and $\Pi_i(g - g_{i,j}) = \Pi_j(g - g_{i,j})$. It follows from (14) that $\Pi_i(g) > \Pi_i(g - g_{i,j})$ and $\Pi_j(g) > \Pi_j(g - g_{i,j})$. Therefore, condition (1) in the definition of stability is also satisfied.

Since all firms in the non-singleton complete component have identical profits, it follows that any net transfers between these firms must be zero. Therefore, if $g^k$ satisfies condition (3) in the definition of stability against transfers, then it also satisfies condition (3) of stability.

The converse of the above implication is not true: it is possible for $g^k$ to be stable for some range of values of $f$ but not be stable against transfers. Consider the case where $n = 4$ and the network $g^2$ where firms 1 and 2 are linked while firms 3 and 4 are singletons with no links. It can be verified that $g^2$ is pairwise stable if $f \in (\frac{3\gamma}{25}[2(\alpha-\gamma_0)-\gamma], \frac{3\gamma}{25}[2(\alpha-\gamma_0)+2\gamma])$. However, by virtue of Lemma 4.3, $g^2$ is not stable against transfers.

$\square$

This result shows that in the class of dominant group networks, allowing for transfers refines the set of stable networks.

# 5 Conclusion

Empirical research suggests that collaboration among firms is common; there is also some evidence to suggest that this collaboration has been increasing in recent years. Collaboration between firms seems to display two striking patterns: one, the overall structure of collaboration in an industry is often asymmetric and two, the relations are intransitive. In this paper, we have developed a simple model of network formation to examine the incentives of firms to form collaboration links with other firms. In particular, we have been concerned with the interaction between market competition, on the one hand, and the networks of collaboration, on the other hand.

Our analysis has clarified the nature of collaboration structures that are strategically stable under different market conditions. An important finding is that even in settings where firms are (ex-ante) symmetric, strategically stable networks are often asymmetric, with some firms having many collaboration links, while other firms are poorly linked. We characterized such structures, finding that the star, inter-linked stars and the dominant group architecture are strategically stable. These asymmetries translate into different levels of competitiveness for firms and hence have a serious influence on market performance. The model of links between firms which we have used is quite simple and should be seen as a first step in a more systematic analysis of the interaction between firms' collaboration networks and markets. We now briefly discuss some issues that should be explored in future work.

First, we take up the issue of spillovers. Our analysis does not accommodate spillovers across the collaborative links of firms. In the received literature, spillovers from the R&D activity of firm $i$ is assumed to affect firms $j \neq i$ uniformly.[19] In our framework, one plausible definition of 'distance' between firms in a network, is the number of links in the shortest path between the firms. This would allows us to implement the more realistic idea that firms that are 'far apart' receive lower spillovers as compared to firms that are 'close' in the network.

The second issue is social welfare. The potential conflict between stable and efficient outcomes is an important one which has figured prominently in the coalition formation literature. We have been able to obtain a characterization of efficiency in the case of a homogeneous product oligopoly with small fixed costs of link formation. In the case of price competition, a network $g$ is efficient if and only if there are two or more firms with $(n-1)$ links. Since the empty network is uniquely stable, we see a conflict between stability and efficiency. In the case of quantity competition, the complete network is uniquely efficient over a large range of parameter values ($\alpha > 3\gamma_0$ and $\gamma_0 > (n-1)\gamma$). Over this range, there is no conflict between stable and efficient networks.[20] Clearly, the architecture of efficient networks, particularly in the presence of large fixed costs of link formation, is an important one which needs to be examined in future research.

Thirdly, we discuss the role of ex-ante asymmetries between firms. In our analysis, we have assumed that all firms are ex-ante symmetric with respect to initial costs and have the same cost reduction function. This seems to us to be the natural starting point, and our results illustrate how significant network asymmetries can emerge even in such a symmetric setting. In some important cases, however, it is natural to start with asymmetric firms. While we expect that asymmetric networks will become more prominent, further work on this subject is needed to clarify the precise structure of such networks.

[19] There is a very large literature on the subject of R& D spillovers. See e.g., d'Aspremont, and Jacquemin (1988), Katz (1986), Suzumura (1992) and Vonortas (1994).

[20] We thank Sang-Seung Yi for pointing out that over some parameter range, a dominant group network with $k = n - 1$ may be efficient.

# 6 Appendix A

**Proof of Proposition 3.1** We first show that $g^c$ is stable. In $g^c, \eta_i(g^c) = n-1$, $\forall i \in N$. Therefore, firm $i$ has a marginal cost of $\gamma_0 - \gamma(n-1)$ and payoff of:

$$T_i(g^c) \;=\; (\alpha - \gamma_0) + \gamma(n-1) \tag{15}$$

There are no links to add so condition (ii) of stability is automatically satisfied. We check condition (i) next. Suppose we set $g_{i,j} = 0$ for some pair $i$ and $j$. In the ensuing network, $g^c - g_{i,j}$, the payoff to $i$ is given by:

$$T_i(g^c - g_{i,j}) \;=\; \alpha - (n-1)[\gamma_0 - \gamma(n-2)] + (n-2)[\gamma_0 - \gamma(n-1)] \;=\; (\alpha - \gamma_0) \tag{16}$$

The payoff to firm $j$ is identical. There is no incentive to delete link $g_{i,j} = 1$ since $T(g^c) - T_i(g^c - g_{i,j}) = \gamma(n-1) > 0$.

We now show that $g^c$ is the unique stable network. Consider a stable network $g \neq g^c$. Then, there exists a pair of firms $i, j \in N$ with $g_{i,j} = 0$. We show that both $i$ and $j$ are strictly better off by forming a link. In the network, $g + g_{i,j}$, the payoff to firm $i$ is given by:

$$T_i(g + g_{i,j}) \;=\; (\alpha - \gamma_0) + n\gamma\eta_i(g + g_{i,j}) - \gamma\eta_j(g + g_{i,j}) - \gamma \sum_{k \neq i,j} \eta_k(g + g_{i,j})) \tag{17}$$

Note that $\eta_l(g + g_{i,j}) = \eta_l(g) + 1$ for $l = i, j$ and $\eta_k(g + g_{i,j}) = \eta_k(g)$ for $k \neq i, j$. Therefore, $T_i(g + g_{i,j}) - T_i(g) = \gamma(n-1) > 0$. An identical argument establishes that for firm $j$, $T_j(g + g_{i,j}) - T_j(g) = \gamma(n-1) > 0$. Thus, condition (ii) is violated and $g$ is not stable, a contradiction. $\square$

**Proof of Proposition 3.2** Consider some non-empty network $g$. There are two possibilities. First, there is some firm $i \in N$ which is the *unique* lowest cost firm. But this implies that firm $i$ must have at least two links since all firms are ex-ante identical. However, since firm $i$ is the unique lowest cost firm, all other firms make zero profits. In particular, consider $j \neq i$ such that $g_{i,j} = 1$. For this firm, condition (i) of stability is violated since $\pi_j(g) = \pi_j(g - g_{i,j}) = 0$. Hence, firm $i$ cannot be uniquely minimal cost in a stable network. The second possibility, given that links are bilateral, is that one or more pairs of firms have minimal cost. Let $i, j \in N$ be two firms with minimal costs. Under price competition both firms make zero profits. If these firms would delete their links they would still make zero profits. Thus $\pi_i(g) = \pi_i(g - g_{i,j}) = 0$. This once again violates condition (i) of stability.

Thus the only candidate for a stable network is $g^e$. Condition (i) is trivially satisfied since there are no links to sever. In the network $g^e + g_{i,j}$, there are two lowest cost firms, $i$ and $j$. From (7), it follows that both firms will get a payoff of zero. Thus condition (ii) is satisfied. This completes the proof. $\square$

**The Differentiated Product Oligopoly:** Consider first the case of quantity competition in a differentiated oligopoly. The inverse demand for firm $i \in N$ is given by:

$$p_i \;=\; \alpha - q_i - \beta \sum_{j \neq i} q_j \;, \quad i \in N \tag{18}$$

where $0 < \beta < 1$. The Cournot output of firm $i \in N$, given a network $g$, is:

$$q_i(g) \;=\; \frac{(\alpha - \gamma_0)(2-\beta) + [2 + (n-1)\beta]\gamma\eta_i(g) - \beta\gamma \sum_k \eta_k(g)}{[2 + (n-1)\beta][2-\beta]} \tag{19}$$

The Cournot equilibrium profits are given by $\Pi_i(g) = q_i^2(g)$. Under a positive monotonic transform, the payoff to firm $i \in N$ can be written as:

$$\pi_i(g) \;=\; (\alpha - \gamma_0)(2-\beta) + [2 + (n-1)\beta]\gamma\eta_i(g) - \beta\gamma \sum_k \eta_k(g) \tag{20}$$

Under (1), it can be easily verified that for $n \geq 3$:

$$\pi_i(g + g_{i,j}) - \pi_i(g) \;=\; \gamma[2 + (n-3)\beta] \;>\; 0 \tag{21}$$

Therefore, the additional monotonicity condition of Theorem 3.1 is met, (which also implies that (SY1) is satisfied). Hence, the complete network is the unique stable network in the case of quantity competition. This result can be contrasted with Bloch (1995, Proposition 3) where, under the restriction $0 < \beta < 1$, the equilibrium coalition structure consists of two asymmetrically-sized coalitions with the size of the larger coalition being the integer closest to $(3n-1)/4 + 1/(2\beta)$.

Next, we consider price competition in a differentiated oligopoly. Inverting the inverse demand given by (18) yields the demand functions:

$$q_i \;=\; a - bp_i + c\sum_{j\neq i} p_j\,, \quad i \in N \tag{22}$$

where the parameters of the demand function satisfy:

$$a = \frac{\alpha}{1+(n-1)\beta}, \quad b = \frac{1+(n-2)\beta}{(1-\beta)(1+(n-1)\beta)}, \quad c = \frac{\beta}{(1-\beta)(1+(n-1)\beta)}$$

The Bertrand equilibrium prices, given the network $g$, are:

$$p_i(g) \;=\; = \;\frac{a+b\gamma_0}{2b+(1-n)c} - \frac{b\gamma\eta_i(g)}{2b+c} - \frac{bc\gamma\sum_{k\in N}\eta_k(g)}{[2b+(1-n)c][2b+c]} \tag{23}$$

The Bertrand equilibrium profits are:

$$\Pi_i(g) \;=\; b\left[p_i(g) - \gamma_0 + \gamma\eta_i(g)\right]^2, \quad i \in N \tag{24}$$

Under a positive monotonic transform, the Bertrand payoffs can be written as:

$$\pi_i(g) \;=\; p_i(g) - \gamma_0 + \gamma\eta_i(g) \tag{25}$$

Under (1), it can be verified that the second condition of Theorem 3.1 holds:

$$\pi_i(g+g_{i,j}) - \pi_i(g) \;=\; \frac{\gamma\,[2b^2+(1-n)(b+c)c]}{[2b+(1-n)c][2b+c]} \tag{26}$$

Note, however, that $2b^2+(1-n)(b+c)c > 0$ is equivalent to $2+\beta[(3n-3)-\beta(2n-3)] > 0$, and the latter is clearly true since $0 < \beta < 1$. Thus the additional monotonicity condition of Theorem 3.1 is satisfied. Therefore, the complete network is the unique stable network in the case of price competition also. In contrast, Bloch (1995, Proposition 4) shows that the unique equilibrium coalition structure in this case is identical to the one derived for the Cournot model.

**Proof of Theorem 3.2:** The proof builds on two lemmas. We state and prove them first.

**Lemma A:** *Let $g$ be a stable network. Then every non-singleton component in $g$ is complete.*

**Proof** Suppose that $g$ is a stable network and $g' \subset g$ is a non-singleton component of $g$. We show that $g'$ must be complete. We know that no unique firm can have the lowest cost in $g'$; this follows from an argument as in the first part of Proposition 3.2. Thus, there must exist at least a pair of firms $i, j \in N$ such that $c_i(g) = c_j(g) = \min_{k \in N} c_k(g)$. Consider any other firm $l \in g'$, $l \neq i, j$. If such a firm has $c_l(g) > c_i(g)$, then under (AC), clearly this cannot be uniquely optimal for the firm. For instance, firm $l$ can delete a link $g_{l,k} = 1$ and retain zero profits. Hence, all firms in $g'$ must have the same costs, and these costs must be minimum. Thus, $c_j(g) = \min_{k \in g} c_k(g)$ $\forall j \in g'$. Finally, if $i, j \in g'$ are not connected, then under Assumption SY2(i), they can do strictly better by forming a link. Thus $g'$ must be complete. □

**Lemma B:** *In a stable network $g$, there can be at most one non-singleton component.*

**Proof** Suppose there are two non-singleton components, $g'$ and $g''$ and let firm $i \in g'$ and that firm $j \in g''$. From the proof of Lemma A we know that firms $i$ and $j$ are minimum cost firms. It now follows from Assumption SY2(i), that these firms can do strictly better by forming a link. This violates condition (ii) in the definition of stability. Thus $g$ is not a stable, a contradiction. This shows that a stable network cannot have more than one non-singleton component. □

We have shown that in a market with four or more firms there can be at most one non-singleton component, and that it is complete. This means that the only candidates for stable networks are networks of the following form: there is a complete component with $k \geq 1$ firms and there are $n - k$ singleton components. The proof of the theorem shows that networks with $k = 1$ and $k = 2$ are not stable, while the networks with $k \geq 3$ are stable.

We are now ready to complete the proof of Theorem 3.2.

**Proof of Theorem 3.2:** The candidates for stable networks can be parameterized in terms of the size of the non-singleton component, $k$. Given the ex-ante symmetry of firms, Assumption SY2(i) immediately implies that a network with $k = 1$ cannot be stable. Next consider $k = 2$. This is a network with one component with 2 firms and (since $n \geq 4$) at least 2 singleton components. Given specification (2), it follows that if the two singleton firms form a link then they have will have the same costs as the two firms already in the 2 firm component. Under Assumption (AC) this yields them positive payoffs, violating requirement (ii) in the definition of stability. Thus any network $g$ with $k = 2$ is not stable. We are left with networks where $k \geq 3$. In such a network every firm $i$ in the non-singleton component is a minimum cost firm, with (say) marginal cost $c_i(g)$. Under specification (2), it follows that $c_i(g) < c_j(g)$, for all firms $j$ which are singleton components. Thus under assumption (AC), $\pi_i(g) > 0$ and $\pi_j(g) = 0$. Now suppose a firm $j$ forms a link with another firm $i$. Then the marginal cost of the former firm will fall still further and under (2) will remain below the marginal cost of firm $j$. Thus firm $j$ has no incentive to form such a link. Since $k \geq 3$, and competition is specified by assumption (AC), it is also clear that two singleton component firms $j$ and $k$ do not have an incentive to form a link either. Finally, using assumption (SY2(ii)), it follows that firms in the non-singleton component have no incentive to delete a link. We have thus shown that both requirements (i) and (ii) are satisfied for any network with the structure: a non-singleton complete component with $k \geq 3$ firms and $n-k$ singleton firms. This completes the proof. □

# 7 Appendix B

**Proof of Lemma 4.3:** Since $g$ is a stable network, $i$ and $j$ should have no incentive to sever their link. Letting $\Delta\pi_i(g - g_{i,j}) \equiv \pi_i(g) - \pi_i(g - g_{ij})$:

$$\Delta\pi_i(g - g_{i,j}) + \Delta\pi_j(g - g_{i,j}) \ > \ 2f \tag{27}$$

Let $T_i(g - g_{i,j}) \equiv (\alpha - \gamma_0) + n\gamma\eta_i(g - g_{i,j}) - \gamma \sum_{l \neq i} \eta_l(g - g_{i,j})$. The above inequality can be written as:

$$\frac{(n-1)\gamma}{(n+1)^2} \left[T_i(g - g_{i,j}) + T_j(g - g_{ij}) + (n-1)\gamma\right] \ > \ f \tag{28}$$

Now consider $k \neq i, j$ such that $\eta_k(g - g_{i,j}) \geq \eta_j(g - g_{i,j})$ but $g_{i,k} = 0$. Consider the network $g + g_{i,k}$ and let $\pi_i(g) \equiv \pi_i(g + g_{i,k}) - \pi_i(g)$. Then:

$$\Delta\pi_i(g) + \Delta\pi_k(g) \ = \ \frac{2(n-1)\gamma}{(n+1)^2} \left[T_i(g) + T_k(g) + (n-1)\gamma\right] \tag{29}$$

Note that:

$$
\begin{aligned}
\eta_l(g) &= \eta_l(g - g_{i,j}) + 1, \qquad l = i, j \\
\eta_k(g) &= \eta_k(g - g_{i,j}) \;\; \geq \;\; \eta_j(g - g_{i,j}) \\
\eta_l(g) &= \eta_l(g - g_{i,j}), \qquad l \neq i, j, k
\end{aligned} \tag{30}
$$

Therefore, $T_i(g) = T_i(g - g_{i,j}) + (n-1)\gamma$ and $T_k(g) \geq T_j(g - g_{ij}) - 2\gamma$. Substituting in (29) and recalling (28) it follows that:

$$
\begin{aligned}
\Delta\pi_i(g) + \Delta\pi_k(g) &= \frac{2(n-1)\gamma}{(n+1)^2}\left[T_i(g - g_{i,j}) + (n-1)\gamma + T_j(g - g_{i,j}) - 2\gamma + (n-1)\gamma\right] \\
&= \Delta\pi_i(g - g_{i,j}) + \Delta\pi_j(g - g_{i,j}) + \frac{2(n-1)(n-3)\gamma^2}{(n+1)^2} \\
&> 2f
\end{aligned} \tag{31}
$$

Therefore, $i$ and $k$ have a profitable deviation from $g$ by forming a link. This contradicts the stability of $g$ against transfers. □

**Proof of Proposition 4.2:** Suppose that $g^s$ is a star network with 1 firm at the center and $(n-1)$ firms at the spokes. Denote the center firm by $n$ and typical firms at the spokes by $i$ and $j$. If firm $n$ deletes all its links then we arrive at the empty network, denoted by $g^e$. If firm $i$ or firm $n$ deletes a link, then we arrive the network $g^s - g_{n,i}$. We now write down the three incentive requirements. The requirement that firm $n$ and firm $i$ wish to maintain their link may be written as:

$$
[\pi_n(g^s) - \pi_n(g^s - g_{n,i})] + [\pi_i(g^s) - \pi_j(g^s - g_{n,i})] > 2f \tag{32}
$$

The requirement that firms $i$ and $j$ do not have an incentive to form a link may be written as follows:

$$
[\pi_i(g^s + g_{i,j}) - \pi_i(g^s)] + [\pi_j(g^s + g_{i,j}) - \pi_j(g^s)] < 2f \tag{33}
$$

The requirement that there exists a set of transfers such that firms have no incentives to isolate themselves by deleting all their links is written as follows. For some $t_i$, for $i = 1, 2, ..n$, it is true that

$$\pi_n(g^s) - (n-)f + \sum_{j \in N_n(g)} (t_j^n - t_n^j) \quad \geq \quad \pi_n(g^e)$$

$$\pi_i(g^s) - f + (t_n^i - t_i^n) \geq \pi_i(g^s - g_{n,i}), \forall i \in N \backslash \{n\}. \tag{34}$$

We note that the gross profits for different firms can be written as follows:

$$\pi_n(g^s) = \frac{[\alpha - \gamma_0 + (n-1)^2\gamma]^2}{(n+1)^2} \tag{35}$$

$$\pi_n(g^s - g_{n,i}) = \frac{[\alpha - \gamma_0 + (n-2)(n-1)\gamma]^2}{(n+1)^2} \tag{36}$$

$$\pi_n(g^e) = \frac{[\alpha - \gamma_0]^2}{(n+1)^2} \tag{37}$$

$$\pi_i(g^s) = \frac{[\alpha - \gamma_0 + (3-n)\gamma]^2}{(n+1)^2} \tag{38}$$

$$\pi_i(g^s - g_{n,i}) = \frac{[\alpha - \gamma_0 - 2(n-2)\gamma]^2}{(n+1)^2} \tag{39}$$

$$\pi_i(g^s + g_{i,j}) = \frac{[\alpha - \gamma_0 + 2\gamma]^2}{(n+1)^2} \tag{40}$$

We now substitute the above payoff terms in the incentive conditions(32)-(34). We start with (32). After substitution and rearrangement, we get the following term.

$$\frac{\gamma(n-1)[4(\alpha - \gamma_0) + (n-1)\gamma(2n-3) - \gamma(3n-7)]}{(n+1)^2} > 2f \tag{41}$$

Similarly, (33) can be rewritten as follows:

$$\frac{2\gamma(n-1)[2(\alpha - \gamma_0) + \gamma(2 - n + 3)]}{(n+1)^2} < 2f \tag{42}$$

Define:

$$F' = \frac{\gamma(n-1)[4(\alpha - \gamma_0) + (n-1)\gamma(2n-3) - \gamma(3n-7)]}{2(n+1)^2} \tag{43}$$

$$F_L = \frac{2\gamma(n-1)[2(\alpha - \gamma_0) + \gamma(2 - n + 3)]}{2(n+1)^2} \tag{44}$$

Conditions (32) and (33) are satisfied if and only if the fixed costs are such that $F_L < f < F'$. It is easily verified that $F_L < F'$ if $n > 3$.

Finally, we construct the set of transfers. Recall that we only require that there exists a set of transfers which makes firms want to retain their links in $g^s$ rather than delete all their links. Note that for the star to be stable it must be the case that the spokes do not have an incentive to form a link with each other. Given the symmetry in their situation, it follows that their marginal payoffs from the additional link are the same. This requirement taken along with increasing returns implies that if the star is to be stable then it must be the case that each of the spoke firms also do not have an incentive to form a link with the central firm. Thus transfers have to made by the central firm to each of the spokes. The minimum value of this transfer is given by:

$$t_n^i = \pi_i(g^s - g_{n,i}) - \pi_i(g^s) + f \tag{45}$$

Given the above expressions we can rewrite this minimum transfer as:

$$t_n^i = f - \frac{(n-1)\gamma[2(\alpha - \gamma_0) - \gamma(3n-7)]}{(n+1)^2} \tag{46}$$

We wish to show that the central firm has an incentive to make such transfers to each of the spoke firms rather than delete all links. This incentive is satisfied if and only if:

$$\pi_n(g^s) - (n-1)(f + t_n^i) \geq \pi_n(g^e) \tag{47}$$

After some rearrangement this requirement can be expressed as:

$$\frac{(n-1)\gamma[4(\alpha - \gamma_0) + (n-1)^2\gamma - \gamma(3n-7)]}{(n+1)^2} \geq 2f \tag{48}$$

Define:

$$F'' = \frac{(n-1)\gamma[4(\alpha - \gamma_0) + (n-1)^2\gamma - \gamma(3n-7)]}{2(n+1)^2} \tag{49}$$

It can be checked that $F'' > F_L$, for all $n > 3$. Define $F_H = \min\{F', F''\}$. The proof now follows. □

**Proof of Proposition 4.3** We prove the parts in sequence.

(i). Suppose $g$ is connected and asymmetric. It follows then that $g$ induces a partition with at least two elements. The claim is proved if we show that $\eta_i(g) - \eta_j(g) \geq 2$ for any pair $i \in h_{l+1}(g)$ and $j \in h_l(g)$ with $1 \leq l \leq m-1$. Suppose $\eta_i(g) - \eta_j(g) = 1$. Then there exists some player $k \neq i, j$ such that $g_{i,k} = 1$ but $g_{j,k} = 0$. However, note that $\eta_i(g - g_{i,k}) = \eta_j(g - g_{i,k}) = \eta_j(g)$. Hence, from Lemma 4.3 we infer that $g$ is not stable against transfers, a contradiction.

(ii). Suppose to the contrary that $\eta_i(g) < n-1$ for $i \in h_m(g)$. Consider any $j \neq i$ such that $g_{i,j} = 0$. Since $g$ is connected, there exists some $k \neq i, j$ such that $g_{j,k} = 1$. However, by definition of $h_m(g)$, $\eta_i(g - g_{j,k}) > \eta_k(g - g_{j,k})$. Lemma 4.3 now implies that since $g_{i,j} = 0$, $g$ is not stable against transfers.

(iii). Suppose not. Then there exists some $k \in h_{l''}(g)$, $l'' \geq l'$, such that $g_{i,k} = 0$. However, by assumption, $\eta_k(g - g_{i,j}) > \eta_j(g - g_{i,j})$. Therefore, from Lemma 4.3 $g$ is not stable against transfers, a contradiction. This completes the proof of the first part of the statement.

We now show that if $i, j \in h_1(g)$, then $g_{i,j} = 0$. Suppose not. From the previous argument, $g_{i,k} = 1\ \forall k \neq i$. This implies $\eta_i(g) = n-1$, a contradiction to the hypothesis that $i \in h_1(g)$.

We next show that if $i \in h_1(g)$ and $j \in h_m(g)$, then $g_{i,j} = 1$. Suppose to the contrary that $g_{i,j} = 0$ for some $i \in h_1(g)$ and $j \in h_m(g)$. This implies from the previous argument that $g_{i,k} = 0\ \forall k \in N \backslash \{i\}$, thereby contradicting the connectedness of $g$.

Finally we show that if $i \in h_1(g)$ and $j \in h_l(g)$, for $l < m$ then $g_{i,j} = 0$. Suppose to the contrary that $g_{i,j} = 1$ for some $i \in h_1(g)$ and $j \in h_l(g)$ for $1 < l < m$. By the argument above, this implies that firm $j$ has a link with all firms $k \neq j$, i.e. $\eta_j(g) = n-1$. Hence, $j \in h_m(g)$, a contradiction. We note that this result also implies that $h_1(g)$ is the unique element of the partition with this property.

(iv). We prove this part by induction. Fix $l = 1$. We first show that if $i \in h_{m-1}(g)$ then $g_{i,j} = 0$, for $j \in h_1(g)$. Suppose this claim is false. Then $g_{i,j} = 1$ for some $j \in h_1(g)$. From part (iii), this implies that $g_{i,k} = 1\ \forall k \in h_l(g)$, $l \geq 1$, which implies $\eta_i(g) = n-1$. This contradicts the hypothesis that $i \in h_{m-1}(g)$. We next show that if $j \in h_k(g)$, $k > 1$, then $g_{i,j} = 1$. Suppose not and let $g_{i,j} = 0$. Then from part (iii) it follows that $g_{j,k} = 0$ $\forall k \notin h_m(g)$. Then from part (iii) this implies $j \in h_1(g)$, a contradiction.

Now suppose that the hypothesis is true for $\hat{l} \geq 1$, i.e. if $i \in h_{m-\hat{l}}(g)$, then $g_{i,j} = 1$ if and only if $j \in h_k(g)$, $k > \hat{l}$. We now show that the hypothesis is also true for $\hat{l} + 1$.

We first prove that if $i \in h_{m-\hat{l}-1}(g)$ and $j \in h_k(g)$, $k \leq \hat{l}+1$, then $g_{i,j} = 0$. Suppose the claim is false. Then $g_{i,j} = 1$ for some $j \in h_l(g)$, $l \leq \hat{l}+1$. From part (iii), this implies $g_{i,r} = 1$, $\forall r \in h_l(g)$, $l \geq \hat{l}+1$. Using the induction hypothesis, this means $\eta_i(g) \geq \eta_r(g)$ for $r \in h_{m-\hat{l}}(g)$. This contradicts the hypothesis that $i \in h_{m-\hat{l}-1}$.

To prove the converse, we need to show that for any $j \in h_k(g)$, $k > \hat{l}+1$, implies $g_{i,j} = 1$. Suppose not. Then $g_{i,j} = 0$ for some $j \in h_l(g)$ for $l > \hat{l}+1$. Then $g_{j,k} = 0$, $\forall k \in h_{l'}(g)$, $l' \leq m - \hat{l} - 1$. However, this implies $j \in h_{\hat{l}+1}(g)$, a contradiction. □

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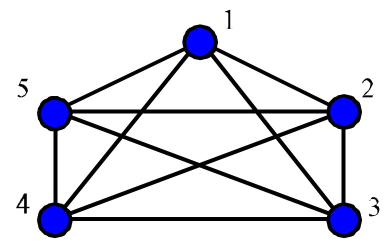


**Figure 1a: Complete Network**

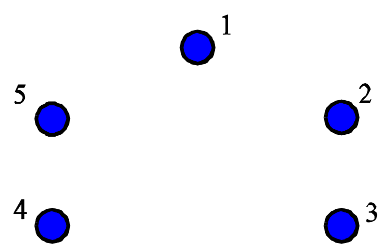


**Figure 1b: Empty Network**

**Figure 1: Symmetric Pair-Wise Stable Networks for n=5**

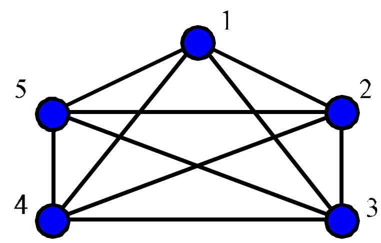


**Figure 2a: Dominant Group with No Fringe Firms**

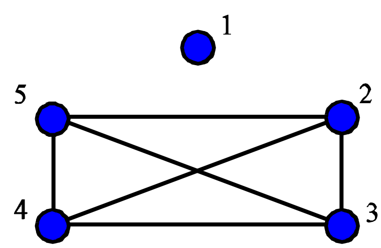


**Figure 2b: Dominant Group with One Fringe Firm**

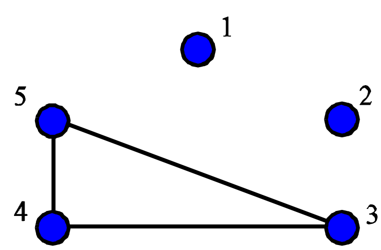


**Figure 2c: Dominant Group with Two Fringe Firms**

**Figure 2: Dominant Group Architecture for n=5**

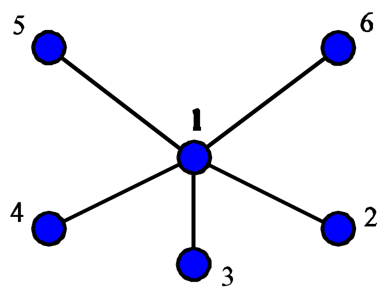


**Figure 5a: Star Network with Firm 1 as the Center**

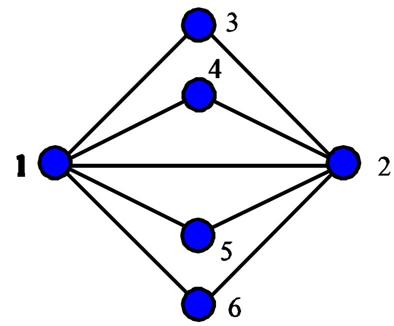


**Figure 5b: Two Inter-Linked Stars with Firms 1 and 2 as the Centers**

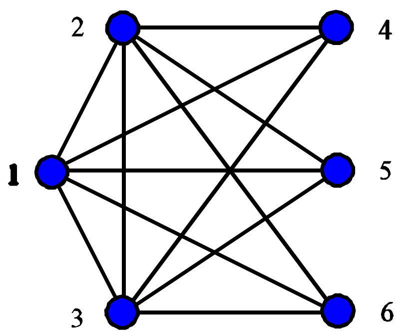


**Figure 5c: Three Inter-Linked Stars with Firms 1,2 and 3 as the Centers**

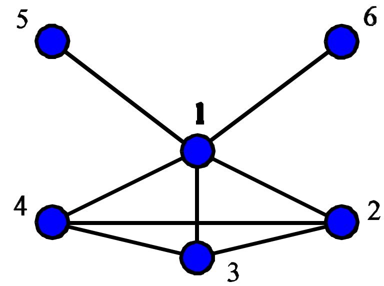


**Figure 5d: Four Asymmetrically-Sized Inter-Linked Stars**

**Figure 5: Asymmetric Connected Networks that are Pair-Wise Stable Against Transfers for n=6**

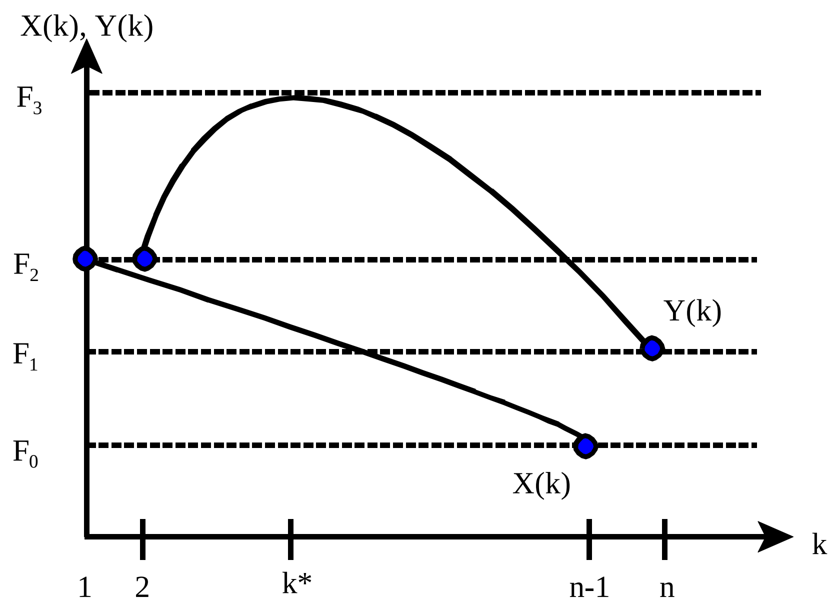


**Figure 3: Pair-Wise Stability of the Dominant Group Architecture**

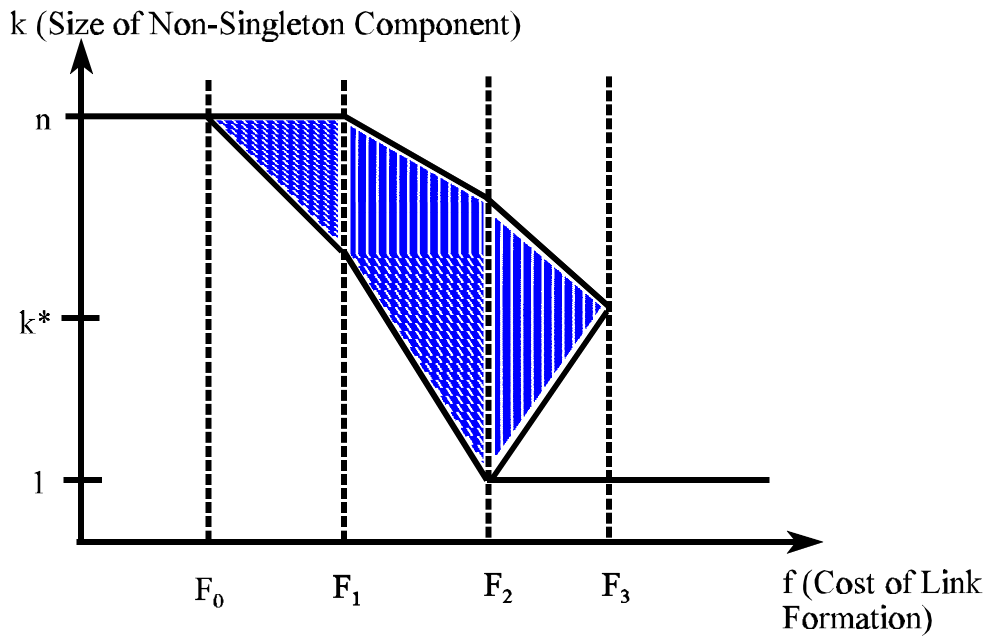


**Figure 4: Non-Monotonicity in Size of Dominant Group with respect to Cost of Link Formation**