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Decomposing Portfolio Value-at-Risk:

A General Analysis

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Abstract

An intensive and still growing body of research focuses on estimating a portfolio's Value-at-Risk. Depending on both the degree of non-linearity of the instruments comprised in the portfolio and the willingness to make restrictive assumptions on the underlying statistical distributions, a variety of analytical methods and simulation-based methods are available. Aside from the total portfolio's VaR, there is a growing need for information about *(i)* the marginal contribution of the individual portfolio components to the diversified portfolio VaR, *(ii)* the proportion of the diversified portfolio VaR that can be attributed to each of the individual components consituting the portfolio, and *(iii)* the incremental effect on VaR of adding a new instrument to the existing portfolio.

Expressions for these marginal, component and incremental VaR metrics have been derived by Garman [1996a, 1997a] under the assumption that returns are drawn from a multivariate normal distribution. For many portfolios, however, the assumption of normally distributed returns is too stringent. Whenever these deviations from normality are expected to cause serious distortions in VaR calculations, one has to resort to either alternative distribution specifications or historical and Monte Carlo simulation methods. Although these approaches to overall VaR estimation have received substantial interest in the literature, there exist to the best of our knowledge no procedures for estimating marginal VaR, component VaR and incremental VaR in either a non-normal analytical setting or a Monte Carlo / historical simulation context.

This paper tries to fill this gap by investigating these VaR concepts in a general distribution-free setting. We derive a general expression for the marginal contribution of an instrument to the diversified portfolio VaR – whether this instrument is already included in the portfolio or not. We show how in a most general way, the total portfolio VaR can be decomposed in partial VaRs that can be attributed to the individual instruments comprised in the portfolio. These component VaRs have the appealing property that they aggregate linearly into the diversified portfolio VaR. We not only show how the standard results under normality can be generalized to non-normal analytical VaR approaches but also present an explicit procedure for estimating marginal VaRs in a simulation framework. Given the marginal VaR estimate, component VaR and incremental VaR readily follow. The proposed estimation approach pairs intuitive appeal with computational efficiency. We evaluate various alternative estimation methods in an application example and conclude that the proposed approach displays an astounding accuracy and a promising outperformance.

Keywords: Value-at-Risk, marginal VaR, component VaR, incremental VaR, non-normality, non-linearity, estimation, simulation

JEL classification: C13, C14, C15, G10, G11

1. Introduction

Value-at-Risk (VaR) is defined as a one-sided confidence interval on potential portfolio losses over a specific horizon. Interest in such a diagnostic metric can be traced back to Edgeworth [1888], but the developments in this field were really spurred by the release of *RiskMetrics*™ by J.P.Morgan in October 1994. An intensive and still growing body of research focuses on estimating a portfolio's VaR and *RiskMetrics* has become a classic framework by now. Depending on both the degree of non-linearity of the instruments comprised in the portfolio and the willingness to make restrictive assumptions on the underlying statistical distributions, a variety of analytical methods and simulation-based methods are currently available (see for example Duffie & Pan [1997] and Jorion [1997] for an overview). Aside from the total portfolio's VaR, there is a growing need for information about *(i)* the marginal contribution of the individual portfolio components to the diversified portfolio VaR, *(ii)* the proportion of the diversified portfolio VaR that can be attributed to each of the individual components consituting the portfolio, and *(iii)* the incremental effect on VaR of adding a new instrument to the existing portfolio.

Why should one be interested in marginal VaRs, component VaRs and incremental VaRs? Risking to answer a rhetoric question we can think of some reasons. Firstly, when presented a VaR figure, what positions constitute the big risks and what positions in contrast serve as hedges? Component VaR reveals pockets of risk concentrations as well as hedges. Secondly, when the overall VaR figure appears to be disturbingly high, how should one alter the portfolio mix (if possible) in order to mitigate the portfolio's overall risk exposure? The effects of changes in portfolio composition are measured by marginal VaRs. Thirdly, what is the change in portfolio VaR when a new instrument or trade is added to the portfolio? Does the VaR increase or decrease and how much? Incremental VaR measures this potential impact on overall VaR.

These three VaR metrics are closely related. Garman [1996a,b, 1997a,b] has derived expressions for these measures under the assumption that returns are drawn from a multivariate normal distribution. His results can readily be used in the so-called variance-covariance approach to VaR estimation (like the *RiskMetrics* framework). For many portfolios, however, the assumption of normally distributed returns does not apply. Fat tailed distributions are rule rather than exception for financial market factors and the inclusion of non-linear derivative instruments in the portfolio gives rise to distributional asymmetries. Whenever these deviations from normality are expected to cause serious biases in VaR calculations, one has to resort either to alternative distribution specifications (like the t-distribution) or to historical and Monte Carlo simulation methods. Although these approaches to overall VaR-estimation have received substantial interest in the literature, there exist to the best of our knowledge no procedures for estimating marginal VaR, component VaR and incremental VaR in a non-normal context. Estimating marginal VaR then would imply re-estimating overall VaR for a slightly changed portfolio composition. Since notably simulation methods are computationally intensive, this brute force approach is tedious, time-consuming and hence not feasible in a trading environment. For estimating component VaR even the recipe is missing.

This paper tries to fill this gap by investigating the concepts of marginal, component and incremental VaR in a general distribution-free setting. We show how the standard results under normality can be generalized to non-normal analytical VaR approaches and we present explicit procedures for estimating marginal VaR in simulation settings. Given the marginal VaR, component VaR and incremental VaR readily follow. Based on the intrinsic linearity of the portfolio operator, our approach pairs appealing simplicity with high accuracy and computational efficiency.

The structure of the paper is as follows. Section 2 introduces notation and gives some definitions. After summarizing marginal VaR and component VaR in a restrictive normal world, section 3 presents our theoretical key results. We derive a general expression for the marginal contribution of an instrument to the diversified portfolio VaR – whether this instrument is already included in the portfolio or not. We show how in a most general way, the total portfolio VaR can be decomposed in partial VaRs that can be attributed to the individual instruments comprised in the portfolio. These component VaRs have the appealing property that they aggregate into the diversified portfolio VaR. In section 4 we investigate the empirical issues connected to the practical application of our findings. We discuss the distributional assumptions under which the derived expressions hold, either exactly or as mean-square error approximations. In addition we provide guidelines to estimating the VaR measures. We propose various different procedures for estimating the common building block of marginal, component and incremental VaRs, viz. global and local regression models, asymmetric response models, rational approximants and the adjusted conditional mean model. We hereby do not limit ourselves to a "normal" world, but also cover non-normality, historical simulation methods and Monte Carlo simulation approaches. Section 5 evaluates the accuracy of the proposed estimators. On the basis of some prototypical examples we conclude that the preferred estimation procedure pairs computational simplicity with high accuracy. Finally, section 6 concludes the paper and presents some suggestions for further research.

2. Some definitions

dollar-VaR and return-VaR

Given the current (t=0) marked-to-market portfolio value V_0, a confidence level of c and an evaluation horizon of Δt years, the overall portfolio VaR is defined by:

$$\text{(1)} \qquad \Pr\left\{V_0 - \tilde{V}_{\Delta t} < VaR\right\} = c$$

where the tilde marks a stochastic variable. Using continuous compounded returns, the stochastic end-of-horizon portfolio value is $\tilde{V}_{Dt} = V_0 \exp\left(\tilde{r}_p Dt\right)$. So the dollar-VaR can be expressed in terms of a continuously compounded **return-VaR** (or the critical **negative** return), denoted by r_p^* , defined over the VaR-horizon Δt:

$$\text{(2)} \qquad r_p^* = -\ln\left(\frac{V_0 - VaR}{V_0}\right) \approx VaR/V_0$$

satisfying $\Pr\left\{\tilde{r}_p \Delta t < -r_p^*\right\} = 1 - c$.

We assume that the portfolio p is composed of N individual instruments. Given the current dollar-positions $\{n_i\}_{i \in p}$ in these instruments and their time Δt market values $\{\tilde{P}_{i,\Delta t}\}_{i \in p}$, the end-of-horizon portfolio value is $\tilde{V}_{\Delta t} = \sum_{i \in p} n_i \tilde{P}_{i,\Delta t}$. The portfolio return over the holding period Δt then equals:[1]

$$\text{(3)} \qquad \tilde{r}_{p,Dt} = \sum_{i \in p} w_i \tilde{r}_{i,Dt}$$

where the portfolio weights $\{w_i\}_{i \in p}$ are defined by $w_i \equiv n_i P_{i,0} / V_0$, satisfying $\sum_{i \in p} w_i = 1$.[2] In fact, we allow eq.(3) to represent any partitioning of the portfolio as long as it is disjoint, exhaustive and linear in the portfolio weights. For example, w_i may indicate the proportion of the current portfolio value invested in instrument *i*, instrument

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[1] Note that eq.(3) is exact for percentage returns, but holds only approximately for logarithmic returns. We join in with conventional practice and assume that the cross-sectional aggregation property in eq.(3) holds with acceptable accuracy.

[2] The composition of the portfolio is assumed to be constant over the VaR-horizon, obviously.

type *i*, trade *i*, or geographical region *i*. For convenience, we'll henceforth refer to the portfolio components as instruments.

In the following, we focus on portfolio and instrument **returns** and on return-VaR. Without any loss of generality, our results can be transposed to dollar-VaR since $VaR = V_0\left[1-\exp\left(-r_p^*\right)\right]$.[3] For notational convenience, we set $\Delta t = 1$. Where it is obvious from the context, we'll suppress the time subscripts.

marginal VaR, component VaR and incremental VaR

On a return basis, the marginal VaR is the change in the return-VaR resulting from a marginal change in the relative position in instrument *i*. Hence, the marginal return-VaR of component *i*, M-VaR_i , equals:

(4) $$M\text{-}VaR_i \equiv \frac{\partial\ r_p^*}{\partial\ w_i} \qquad i \in p$$

In addition to the **marginal** contribution of individual instruments to the portfolio VaR, we consider the **total** contribution of each separate included instrument to the diversified portfolio VaR. In order to analyze the (return-) VaRs of the individual instruments within the context of the portfolio in which they are included, we must find out how to decompose the portfolio VaR into partial VaRs or component VaRs. We require that these component VaRs *(i)* can uniquely be attributed to each of the individual instruments comprised in that portfolio and *(ii)* aggregate linearly into the total diversified portfolio VaR. Denoting an instrument's *i* component return-VaR by C-VaR_i , we require that:

(5) $$r_p^* \equiv \sum_{i \in p} C\text{-}VaR_i$$

Of course, the relative magnitude of the individual component VaRs depends on the portfolio partitioning criterion involved. However, since we require the partitioning to be disjoint and exhaustive, the additivity property is independent of the particular partitioning criterion employed.

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[3] Whether focusing on returns or on dollar positions, transactions with zero initial value have to be decomposed into non-zero (long and short) positions. For example, a newly initiated long forward contract is represented by a long spot position and a short bond position. A swap contract consists of a long and a short position in two different bonds.

Note that because of interdependencies between instrument returns, the instruments' stand-alone VaRs do not add up to the diversified portfolio VaR. The break-down of VaR according to portfolio components or market risk factors as suggested by Fong & Vasicek [1997] suffers from the same shortcoming and is hence not useful.[4]

Note that eq.(4) also applies to an instrument not yet included in the portfolio. Suppose the initial portfolio p comprises $N\text{-}1$ instruments and consider this ($N\text{-}1$)-element portfolio as an N-element portfolio where $w_N = 0$. Now assume that instrument N is added to the portfolio with weight w_N . Then, to a first order approximation the change in portfolio VaR is:

$$(6) \qquad \Delta r_p^* \approx \left. \frac{\partial r_p^*}{\partial w_N} \right|_{w_N = 0} \cdot w_N$$

for w_N small. This is the **incremental** VaR of instrument N. Hence, the VaR of the augmented portfolio can be approximated by $r_p^* + \Delta r_p^*$. Although they have the same building block, the main difference between component VaR and incremental VaR is that the former applies exact while the latter is an approximation.

Since incremental VaR readily follows from marginal VaR, we concentrate on marginal VaR and component VaR. In the next section we first discuss marginal VaRs, component VaRs and their relationship in a normal world, whereafter we shift our attention to a general setting.

3. Generalizing marginal VaR and component VaR

The available formulas for estimating marginal VaR and component VaR (in dollar terms) rest on the multivariate normality assumption. We therefore first summarize the normal case. We then offer a completely general perspective on marginal VaR and component VaR. We analyze these metrics in a distribution-free setting and present our theoretical key results.

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[4] Hallerbach & Menkveld [1999] in contrast derive a disjoint, exhaustive and linear decomposition of overall VaR with respect to a multi-factor risk model. They also provide empirical evidence in a corporate VaR context.

multivariate normality

Analytical approaches to estimating VaR rest on assumptions with respect to the form of the portfolio return distribution. A popular (because tractable) assumption is that instrument returns are drawn from a multivariate normal distribution. Hence the portfolio return, continuously compounded over the horizon Δt , also follows a normal distribution, $\tilde{r}_p \sim N(m_p, s_p)$. For a 95%-VaR , for example, the return-VaR over the specified horizon is given by:

(7) $$r_p^* = -m_p + N^{-1}(c) \cdot s_p = -m_p + 1.65 \cdot s_p$$

From the definition of portfolio variance it readily follows that:

(8) $$\frac{\P s_p}{\P w_i} = \frac{Cov\left(\tilde{r}_i, \tilde{r}_p\right)}{s_p} = b_i \cdot s_p$$

where b_i is the slope coefficient in the OLS regression of $\tilde{r}_i$ on $\tilde{r}_p$,

(9) $$b_i = \frac{Cov\left(\tilde{r}_i, \tilde{r}_p\right)}{Var\left(\tilde{r}_p\right)}$$

and where $Cov(\cdot,\cdot)$ and $Var(\cdot)$ are the covariance and variance operators. Setting all expected returns equal to zero according to common practice[5], the marginal return-VaR follows from eqs.(4), (7) and (8):

(10) $$M\text{-}VaR_i = b_i \cdot r_p^*$$

This indeed corresponds to the expression for marginal dollar-VaR as derived by Garman [1996a,b].[6]

From the linearity of the covariance operator, it follows immediately from eq.(9) that:

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[5] Because of the short time horizon *Dt* involved in VaR estimations, the expected returns are often neglected. This practice is not always warranted, however, for example when there are substantial non-linearities in the instruments' pay-offs inducing convexity effects.

(11) $\beta_p = \sum_{i \in p} w_i \beta_i = 1$

Hence,

(12) $r_p^* = \sum_{i \in p} w_i \left(\beta_i \cdot r_p^* \right) = \sum_{i \in p} w_i \cdot M\text{-}VaR_i$

where according to eq.(5) each term i identifies the component-VaR of instrument i :

(13) $C\text{-}VaR_i = w_i \cdot M\text{-}VaR_i \qquad i \in p$

So an instrument's i component return-VaR is simply given by its marginal return-VaR, multiplied by its investment weight in portfolio p . Eqs.(12) and (13) are fully consistent with the dollar-terms expressions as presented by Jorion [1997, p.154] and as derived by Garman [1997a,b].

It happens that this relationship between marginal and component VaR is much more general than is believed. Let us therefore escape from the restrictive multivariate world and analyze marginal- and component-VaRs in a general setting.

a general perspective

Eq.(3) identifies the portfolio return as a convex combination of the returns on the individual components. Therefore, the portfolio return $\tilde{r}_p$ and hence the return-VaR r_p^* are linearly homogeneous functions of the investment fractions $\{w_i\}_{i \in p}$. According to Euler's theorem it then immediately follows that:

(14) $r_p^* = \sum_{i \in p} \frac{\partial r_p^*}{\partial w_i} w_i = \sum_{i \in p} w_i \cdot M\text{-}VaR_i = \sum_{i \in p} C\text{-}VaR_i$ ■

So the relationship between marginal and component VaR is of a very general nature. It does not depend on any distributional assumptions but prevails since the portfolio return can be expressed as a linear combination of the individual component returns. Without loss of generalization, the components may be mapped in a non-linear fashion onto M

[6] Garman [1996a,b] terms this metric DelVaR and VaRdelta™, respectively.

underlying state variables (or market factors) or standardized positions (like cash flows on the maturity vertices in JPMorgan's [1996] *RiskMetrics*).

Maintaining the general nature of the setting, we can even obtain more insight in marginal and component VaRs . The only (and very weak) assumption we now make is that all relevant return distributions have finite first moments. From eq.(3), by the very definition of conditional expectations, we have:

$$(15) \qquad \tilde{r}_p = E\left\{\tilde{r}_p \middle| \tilde{r}_p\right\} = \sum_{i \in p} w_i E\left\{\tilde{r}_i \middle| \tilde{r}_p\right\}$$

Note that $E\left\{\tilde{r}_i \middle| \tilde{r}_p\right\}$ is to be interpreted as the expectation of $\tilde{r}_i$ conditional to the σ-field F relative to which $\tilde{r}_p$ is defined. Hence, this conditional expectation is a random variable. By taking iterated expectations, we get:

$$(16) \qquad r_p^* = -\sum_{i \in p} w_i E\left\{\tilde{r}_i \middle| \tilde{r}_p = -r_p^*\right\}$$

Since the portfolio return now takes the particular value $-r_p^*$, this conditional expectation becomes deterministic. Combining eqs.(4) & (16) yields:

$$(17) \qquad M\text{-}VaR_i = -E\left\{\tilde{r}_i \middle| -r_p^*\right\} \qquad\qquad i \in p \qquad \blacksquare$$

and hence, from eq.(14),

$$(18) \qquad C\text{-}VaR_i = -w_i \cdot E\left\{\tilde{r}_i \middle| -r_p^*\right\} \qquad\qquad i \in p \qquad \blacksquare$$

The intuition behind eqs(17-18) is clear. When there is a strong positive (negative) interdependence between $\tilde{r}_p$ and $\tilde{r}_i$, then large negative portfolio returns will be associated with large negative (positive) instrument returns. Increasing (decreasing) the weight w_i of the instrument will then lower the portfolio return even more, thus increasing the portfolio's VaR.

Once marginal VaR is known, component VaR (and incremental VaR when w_i is initially zero) readily follow. In the remainder we'll therefore focus on marginal VaR.

In the next section we investigate how to estimate this metric, in both analytic and simulation environments.

4. Estimating marginal VaRs (and component VaRs)

In analytical approaches to estimating VaR, either assumptions are made with respect to the specific form of the portfolio return distribution or this distribution is reconstructed in an approximate way by means of its first four or so estimated moments[7]. Simulation based methods come in two basic forms. Historical simulation approaches employ historical return "scenarios" to construct a sample frequency distribution from which the portfolio's return confidence interval can be determined. In Monte Carlo simulations, a very large number of drawings is made from predetermined (joint) distributions to construct the portfolio's return frequency distribution.

Simulation methods are much more flexible than analytical methods since they can effectively cope with non-linear return patterns from options and other non-linear derivative instruments. Given a particular sampling of market factors, the instruments can be marked-to-model and the end-of-horizon portfolio can be determined. Compared to Monte Carlo simulation, historical simulation has the advantage that no assumptions have to be made regarding the underlying return generating distributions and that it is relatively fast.[8] On the other hand, when there are (almost) no non-linear return patterns, analytical methods are much more tractable and display a higher computational efficiency.

The conventional formulas for estimating marginal VaR and component VaR rest on the multivariate normality assumption. To the best of our knowledge, there are no simple procedures available for estimating these metrics within simulation-based VaR approaches. In those cases one would have to re-estimate VaR for a slightly changed portfolio composition. Since simulation methods are computationally intensive, this brute force approach is tedious, time-consuming and hence not feasible in a trading environment. In this section, we provide guidelines to estimating marginal VaR and component VaR within non-normal analytical VaR methods and simulation methods.

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[7] This can be accomplished by means of moment expansions of the distribution function; see Kendall & Stuart [1969, pp.156-167] for a general discussion. Jarrow & Rudd [1982] apply the technique in option pricing and Hull [1998, pp.357-358] summarizes its use in a VaR context.

[8] However, historical simulation can suffer from a lack of sufficient observations to construct a comfortably reliable frequency distribution.

4.1 preamble

Estimating M-VaR entails two steps. First, $\tilde{r}_i$ is linked to the portfolio return $\tilde{r}_p$ in order to obtain $E\left\{\tilde{r}_i \middle| \tilde{r}_p\right\}$, and then the restriction that the portfolio return equals minus the return-VaR, $\tilde{r}_p = -r_p^*$ is imposed, in order to get $E\left\{\tilde{r}_i \middle| \tilde{r}_p = -r_p^*\right\}$.

Suppose that we model the relationship between $\tilde{r}_i$ and $\tilde{r}_p$ by some function $f_i(\cdot)$:

(19) $$\tilde{r}_i = f_i\left(\tilde{r}_p\right) + \tilde{x}_i$$

where without loss of generality, we let the additive disturbance term satisfy $E\left\{\tilde{x}_i\right\} = 0$. Assuming that the relevant first and second moments exist, the choice $f_i(\cdot) = E\left\{\tilde{r}_i \middle| \tilde{r}_p\right\}$ minimizes the mean-squared error of the fit $E\left\{\tilde{x}_i^{\,2}\right\}$.[9] Hence, the least-squares representation of eq.(19) is:

(20) $$\tilde{r}_i = E\left\{\tilde{r}_i \middle| \tilde{r}_p\right\} + \tilde{x}_i$$

which in turn implies the semi-independence of the disturbance term:[10]

(21) $$E\left\{\tilde{x}_i \middle| \tilde{r}_p\right\} = 0$$

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[9] This follows from applying iterated expectations: $E\left\{\left[\tilde{r}_i - f_i(\cdot)\right]^2\right\} = E\left\{E\left\{\left[\tilde{r}_i - f_i(\cdot)\right]^2 \middle| \tilde{r}_p\right\}\right\}$.

[10] At first sight, the assumption of semi-independence between $\tilde{x}_i$ and $\tilde{r}_p$ may seem unwarranted. After all, since the portfolio p is a convex combination of all instruments, the disturbance term $\tilde{x}_i$ is part of the return on p. However, multiplying the LHS and RHS of eq.(20) with w_i and summing over $i \in p$ implies that $\tilde{x}_p = \sum_{i \in p} w_i \tilde{x}_i = 0$.

orthogonal projection

In order to obtain a general insight in $E\left\{\tilde{r}_i \middle| \tilde{r}_p\right\}$, we consider the orthogonal projection of $\tilde{r}_i$ into the subspace spanned by the portfolio return $\tilde{r}_p$:

$$\text{(22)} \qquad \tilde{r}_i = a_i + b_i \tilde{r}_p + \tilde{e}_i$$

This construction is **always** possible. In statistical terms, eq.(22) represents an ordinary linear least-squares approximation to the relationship between $\tilde{r}_i$ and $\tilde{r}_p$, satisfying $E\left\{\tilde{e}_i\right\} = 0$ and the orthogonality condition $E\left\{\tilde{e}_i \tilde{r}_p\right\} = 0$ of the disturbances. Assuming that second moments exist, it follows from the latter that $Cov\left(\tilde{r}_i, \tilde{r}_p\right) = b_i \cdot Var\left(\tilde{r}_p\right)$, which defines the slope coefficient b_i in this general setting as in eq.(9) above. Also, for the intercept we have $a_i = E\left\{\tilde{r}_i\right\} - b_i \cdot E\left\{\tilde{r}_p\right\}$, which conforms to well-known results from standard OLS regression theory.[11]

From eq.(22) we thus have:

$$\text{(23)} \qquad E\left\{\tilde{r}_i \middle| \tilde{r}_p\right\} = E\left\{\tilde{r}_i\right\} + b_i\left[\tilde{r}_p - E\left\{\tilde{r}_p\right\}\right] + E\left\{\tilde{e}_i \middle| \tilde{r}_p\right\}$$

where $E\left\{\tilde{r}_i \middle| \tilde{r}_p\right\}$ is a random variable. When the actual relationship between $\tilde{r}_i$ and $\tilde{r}_p$ is linear, the conditional expectation eq.(23) is linear in $\tilde{r}_p$. This in turn implies the semi-independence $E\left\{\tilde{e}_i \middle| \tilde{r}_p\right\} = 0$. Together with the restriction that $\tilde{r}_p = -r_p^*$, eq.(23) then transforms to the deterministic expression:

$$\text{(24)} \qquad E\left\{\tilde{r}_i \middle| \tilde{r}_p = -r_p^*\right\} = E\left\{\tilde{r}_i\right\} + b_i\left[-r_p^* - E\left\{\tilde{r}_p\right\}\right]$$

Hence, from eq.(17) we obtain:

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[11] Likewise, it is easy to see that on the portfolio level we have a slope of unity (see eq.(11)) and a zero intercept, $a_p = \sum_{i \in p} w_i a_i = E\left\{\tilde{r}_p\right\} - b_p \cdot E\left\{\tilde{r}_p\right\} = 0$. When *p* is a market index or "the market portfolio" of financial assets, eq.(22) is known as the market model; see Stapleton & Subrahmanyam [1983], e.g. We explicitly do not label the slope coefficients as "betas" since this term is contaminated with diverse interpretations in finance.

(25) $$M\text{-}VaR_i = -E\{\tilde{r}_i\} + b_i\left[r_p^* + E\{\tilde{r}_p\}\right]$$ ■

which reduces to eq.(10) when expected returns are set to zero.

Note that eq.(25) also applies to an instrument not (yet) included in the portfolio. Suppose the portfolio *p* comprises $N\text{-}1$ instruments and we want to know whether a marginal addition of instrument N lowers or increases the portfolio VaR. Consider the initial ($N\text{-}1$)-element portfolio as a N-element portfolio where $w_N = 0$. The coefficient b_N measured with respect to the ($N\text{-}1$)-element portfolio is equal to the slope coefficient measured with respect to the N-element portfolio with $w_N = 0$. So when that the slope is greater (smaller) than unity, then the marginal inclusion of instrument N increases (lowers) the portfolio VaR.

4.2 analytical approaches

Expression (25) applies exactly when the linearity condition $E\left\{\tilde{e}_i \middle| \tilde{r}_p\right\} = 0$ is satisfied, whence $E\left\{\tilde{r}_i \middle| \tilde{r}_p\right\}$ is linear in $\tilde{r}_p$. This condition is satisfied whenever $\tilde{r}_i$ and $\tilde{r}_p$ follow a bivariate elliptical (or spherical) distribution. [12] The class of elliptical distributions – also known as location-scale or "two-parameter" distributions – includes the Student t distribution, the exponential distribution and symmetric stable (or Pareto-Lévy) distributions (see Press [1972, p.455]), of which the normal distribution is a special case. Also non-normal variance mixtures of multivariate normal distributions belong to the elliptical class. Together with the t-distribution, variance mixtures are especially relevant in a VaR context, since these distributions possess fat tails.

Elliptical distributions possess density functions depending on only quadratic functions of the variates, ensuring a symmetrical shape. Any linear combination of elliptically distributed variates is still elliptical. Hence, when the component returns follow a multivariate elliptical distribution, $\tilde{\boldsymbol{r}} \sim EL(\boldsymbol{\Delta},\boldsymbol{\Omega})$, also the portfolio return is elliptically distributed. When first moments exists, the location parameters $\boldsymbol{\Delta}$ are the

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[12] The linearity of the conditional mean for elliptical distributions is proven by Kelker [1970, p.424]; see also Chmielewski [1981, p.72]. Fang, Kotz & Ng [1990] provide an extensive overview and Owen & Rabinovitch [1983] discuss the application of elliptical distributions in portfolio theory. Although for all elliptical distributions the conditional mean is linear (when it exists), except for the normal case the distribution of the conditional expectation depends on the conditioning variable – here $\tilde{r}_p$. So the higher conditional moments of $\tilde{r}_i$, and hence of the

means of the distribution. When also the second moment exists, the covariance matrix **Σ** exists and **Σ**=c·**Ω**, where c is a nonnegative scalar independent of **Δ** and **Ω**.[13] So not only applies the expression for marginal VaR in eq.(25) exact, also the slope coefficient retains it conventional definition eq.(9).

Any distribution assumption within analytical VaR approaches is likely to be covered by the elliptical class. When the portfolio contains non-linear derivative instruments, the fist four or so statistical moments of the portfolio return distribution can be derived under the assumption that the returns on the underlying values are multivariate normal. Using series expansions (like Cornish-Fisher) one can then reconstruct the portfolio return distribution in an approximate sense from its moments (or cumulants) in order to derive confidence intervals (see footnote 7). Eq.(7) is then replaced by:

(26) $$r_p^* \approx -m_p + k(c) \cdot s_p$$

where $k(\cdot)$ is a function of the cumulants of $\tilde{r}_p$. For return distributions displaying negative skewness or fat tails we have $k(c) > N^{-1}(c)$. However, since $k(\cdot)$ is a function of the moments of the portfolio's return distribution, and hence of w_i ,

(27) $$\frac{\P\ r_p^*}{\P\ w_i} \neq b_i \cdot r_p^*$$

so the conventional formulas for M-Var and C-VaR under normality no longer apply. This is adequately illustrated in section 5 (see also footnote 21).

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disturbance term $\tilde{e}_i$, (insofar they exist) are heteroskedastic, implying time-varying residual (co-) variances.

[13] We can be more specific for symmetric stable distributions. For the (conditional) expectation to exist, their characteristic exponent must exceed one. When the characteristic exponent is smaller than two, the variance of the distribution does not exist. Hence, the slope coefficients b_i are no longer defined by eq.(9). In that case, the mean-absolute deviation estimator is a sensible candidate for the slope coefficient. Minimizing the mean-absolute deviations of the errors instead of minimizing the mean-square errors can be done in a conventional OLS regression framework by using an iterative WLS technique. See Maddala [1977, pp.309ff], e.g.

4.3 simulation-based approaches

When it is no longer possible to specify return distributions that are both realistic and tractable, one has to resort to simulation methods. In this section we discuss the estimation of marginal VaR and component VaR in a simulation context. As a preliminary we first investigate the effects of non-linearity and distributional asymmetry on marginal VaR estimates. We then suggest some estimation methods: asymmetric response approximation, rational approximation and linear local approximation. These methods will be evaluated in section 5.

non-linearity .....

Distributional asymmetries and/or non-linearities in the relationship between $\tilde{r}_i$ and $\tilde{r}_p$ destroy the linearity of $E\left\{\tilde{r}_i \middle| \tilde{r}_p\right\}$. In that case, a linear relationship between $\tilde{r}_i$ and $\tilde{r}_p$ is no longer implied, but linearity could be **imposed** by forcing a linear least-squares regression of the form of eq.(22). Now assume for a moment that the portfolio return does follow a normal distribution (by invoking the central limit theorem, e.g.) but that **no** assumptions are made with respect to the distributions of the individual components' returns (aside from some mild regularity restrictions). When $f_i(\cdot)$ is at least once differentiable with derivative $f_i'(\cdot)$, then from eq.(19) we have:

$$(28) \qquad Cov(\tilde{r}_i, \tilde{r}_p) = Cov\left(f_i(\tilde{r}_p), \tilde{r}_p\right) + Cov(\tilde{x}_i, \tilde{r}_p) = E\left\{f_i'(\tilde{r}_p)\right\} \cdot Var(\tilde{r}_p)$$

where the last equality follows from applying Stein's lemma and eq.(21). Hence, the slope from the forced regression in eq.(22) equals:

$$(29) \qquad b_i = E\left\{ f_i'(\tilde{r}_p)\right\}$$

In words: the linear OLS slope coefficient linking $\tilde{r}_i$ and $\tilde{r}_p$ is given by the expected value of the gradient of the (unknown) function $f_i(\cdot)$.[14] The expectation is taken with

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[14] The assumption of normality of $\tilde{r}_p$ is only a matter of convenience and does in no way affect our conclusion. To see this, we note a remarkable parallel between the interpretation of the regression coefficient under normality in eq.(29) on the one hand and some robust regression estimators that are designed to cope with **deviations** from normality (especially outliers) on the other. Theil [1950] and Sen [1968], for example, developed estimators for the slope coefficient b_i

respect to the distribution of $\tilde{r}_p$, so the OLS slope provides a **global** approximation to $f_i(\cdot)$. But when $f_i(\cdot)$ is non-linear, this gives a problem. For estimating the return-VaR we are interested in the relationship between $\tilde{r}_i$ and $\tilde{r}_p$ in the **left tail** of the distribution, i.e. where $\tilde{r}_p = -r_p^*$. So what we need is not a global linear approximation to $f_i(\cdot)$ but instead a **local** approximation in the neighbourhood of where $\tilde{r}_p = -r_p^*$.

….. and asymmetry

What happens when the true relationship between $\tilde{r}_i$ and $\tilde{r}_p$ is non-linear and the underlying distributions are no longer symmetric? White [1980, p.155] proves that linear least squares estimates provide consistent (i.e. asymptotically correct) estimates of a well-defined weighted linear least squares approximation to the true but unknown function, $f_i(\cdot)$ in eq.(20) in our case. The distribution function of the independent variable ($\tilde{r}_p$) hereby serves as weighting function.[15] In this case, the global mean-square approximation provided by the OLS slope is even more inappropriate. The slope estimate may be tilted towards the slope of the function $f_i(\cdot)$ in the right tail of $\tilde{r}_p$'s distribution, instead of towards the left tail which is relevant in the VaR context.

The problem of functional non-linearity and distributional asymmetry is by no means insurmountable. In effect, the discussion above clearly suggests one obvious solution to the estimation problem, i.e. applying a **local** approximation to the relationship between $\tilde{r}_i$ and $\tilde{r}_p$. Given the portfolio VaR and estimates of b_i, marginal VaR and component VaR readily follow. In the next section we discuss local linear approximations and present alternatives.

estimation by linear local approximation

Figure 1 shows why functional non-linearity and distributional asymmetry render a global linear regression approximation inappropriate. Combined with the assumed convexity of the return relationship, the left-skewed portfolio return distribution pushes

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which are the median of the set of slopes Dr_i / Dr_p joining pairs of points (r_i, r_p). The slopes Dr_i / Dr_p can be imagined as gradients, so these estimators would resemble the **median** of gradients. Another example are Hinich & Talwar [1975] who propose the (trimmed) mean or median of slope coefficients from a large number of non-overlapping sub-sample OLS regressions. As these sub-samples contain very few observations, the coefficient from each of these regressions may be considered as a local linear approximation to the gradient, evaluated in the sub-sample mean. Hence, the (trimmed) mean or median of these coefficients can then be considered as the mean (or median) of the gradients.

[15] See also Spanos [1986, p.459] for a clear exposition.

the global OLS-b_i towards a lower value. The local linear approximation in the neighborhood of where $\tilde{r}_p = -r_p^*$ would yield the slope estimate that is relevant for computing marginal-VaR and component-VaR.

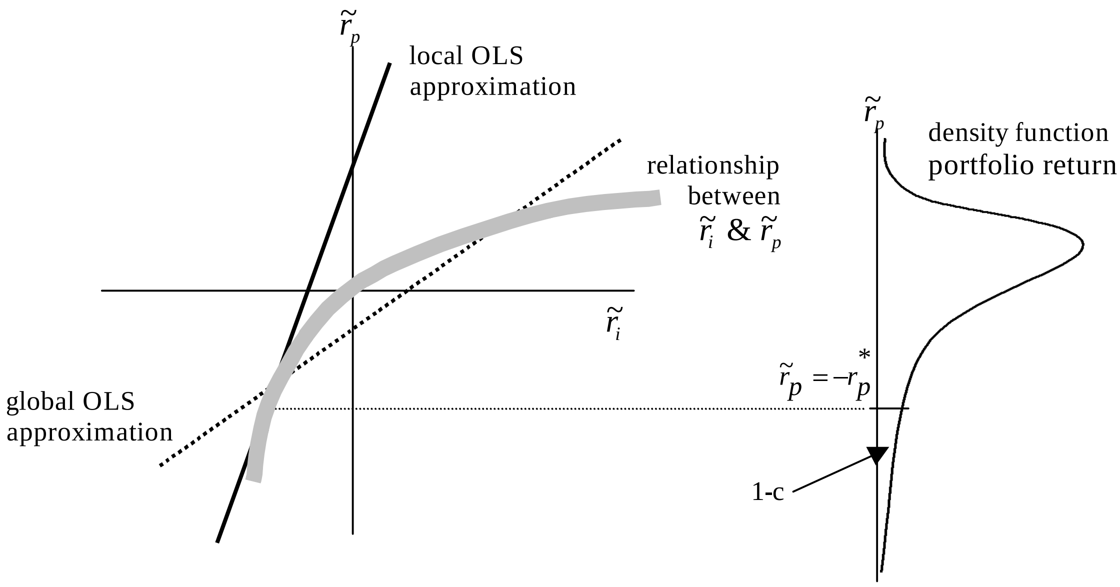

Figure 1: Global and local OLS slope estimates in case of functional non-linearity and distributional asymmetry. The fat gray curve indicates the set of portfolio and instrument return combinations.

The local estimation of the slope coefficients is required in analytical non-parametric approaches as well as in simulation-based approaches. The procedure is the same. Starting from the raw sample return data matrix or from the Monte Carlo generated return data matrix, first select the complete data rows for which:

$$r_p \in \left[-r_p^* - q, -r_p^* + q\right] \qquad (30)$$

for some small $q > 0$. Using only the selected data window, perform next OLS regressions of eq.(22) to obtain estimates $\hat{b}_i$ of the slope coefficients of the individual instruments. Finally, use the estimated slopes in eq.(25).

The number T* of selected data rows depends on the width $2q$ of the return window. On the one hand, the interval must be narrow enough to yield a truly local approximation. On the other hand, the interval must contain a sufficient number of return observations in order to provide reliable slope estimates. To be on the safe side, let T*≥15. The consequences for the total sample size can be put in perspective by assuming normality. For a VaR confidence level of c=95% (and centering around the 5% tail probability), we expect 2% of the observations to lie between $-0.94 \cdot r_p^*$ and $-1.06 \cdot r_p^*$. Requiring a minimum of T*=15 observations in this interval implies a total sample size of 750. Likewise, we expect 6% of the observations to lie between $-0.64 \cdot r_p^*$ and $-1.25 \cdot r_p^*$, indicating a minimum total sample size of 250 (i.e. a full year of daily data). The latter interval would still be sufficiently narrow to provide a suitable local approximation.

For Monte Carlo simulations, the number of drawings is sufficiently high to ensure that a fairly large number of generated portfolio returns will fall within a small interval. Especially here, because of this high number of drawings, brute force investigations into the effect of marginally changed portfolio compositions on the portfolio VaR are prohibitive. The local regression approximation then signifies an efficient method of VaR-analysis.

But is it also a robust method? Intuitively, this depends on the range of observed instrument returns r_i that belong to the interval of r_p as defined by eq.(30). When the implied window of r_i is too small, the local regression line will be tilted towards an infinitely large positive or negative value – irrespective of the actual functional relationship. The degree of robustness would profit from an increase in the number of observations available for the local approximation. This leads us to two alternative estimation methods.

estimation by an asymmetric response model

Increasing the domain of the local linear approximation leads us to an alternative estimation model. Assume that the relationship between $\tilde{r}_i$ and $\tilde{r}_p$ can be approximated by a two-piece linear function. Define a cut-off return f (which may be the mean or median of the portfolio return $\tilde{r}_p$, or zero) and set:

$$\tilde{r}_p^+ \equiv \max\left[f, \tilde{r}_p\right] \qquad (31)$$
$$\tilde{r}_p^- \equiv \min\left[f, \tilde{r}_p\right]$$

The regression model eq.(22) is now extended to:

$$(32) \qquad \tilde{r}_i = a_i + b_i^+ \tilde{r}_p^+ + b_i^- \tilde{r}_p^- + \tilde{e}_i$$

The estimate $\hat{b}_i^-$ of the relevant slope is then used in lieu of b_i. Setting the cut-off point f equal to the median of the portfolio return $\tilde{r}_p$ implies that effectively half of the total sample size is used to estimate the slope. This is a significant improvement over the local linear approximation method, but the linearity imposed over a wider range of $\tilde{r}_p$ is the price paid. Since by construction, the “positive” and “negative” parts of $\tilde{r}_p$ are orthogonal, omitting the positive part from the regression has no consequences for the estimate of b_i^-.[16] However, when estimating the **full** equation, time-varying volatility effects can be accounted for, either directly (by using (G)ARCH) or indirectly (for example via the general iterative weighted least-squares approach proposed by Davidian & Carroll [1987]).

estimation by rational approximation

The third approximation method allows use of all available sample data to estimate $E\left\{\tilde{r}_i \middle| \tilde{r}_p = -r_p^*\right\}$. Assuming that the function $f_i(\cdot)$ in eq.(19) is continuous and analytic, it can be approximated by a n-order Taylor series expansion (around the mean portfolio return, e.g.). Since that expansion is linear in the coefficients, their estimates can be obtained via OLS regression. However, polynomial approximations usually become unsatisfactory when it is necessary to approximate a function over a wide interval. Moreover, for tractable order n, they lack the capacity to adjust to complex non-linearities. In many cases, for a given amount of computational effort, a function can be approximated with greater accuracy by the use of rational functions, rather than by the use of polynomials. A rational function approximation, like the Padé approximation, can be seen as a kind of generalization of a Taylor series approximation.[17] A (m,k)-order rational approximant takes the form of the ratio of an m-order and a k-order polynomial. In our experience, m=k=2 usually suffices to capture complex non-linear empirical relationships.[18]

[16] Of course, the R^2 of the regression and hence the significance of the slope estimate will then decrease.

[17] See for example Young & Gregory [1973, Ch.6.12] for an exposition.

[18] See Hallerbach & Kremer [1993] on this point. Hallerbach [1994] provides detials and another empirical application.

The (2,2)-order rational approximation to eq.(19) is:

$$(33) \qquad \tilde{r}_i = \frac{a_i + b_{i1}\tilde{r}_p + b_{i2}\tilde{r}_p^2}{1 + c_{i1}\tilde{r}_p + c_{i2}\tilde{r}_p^2} + \tilde{e}_i$$

The parameters of this equation can be estimated using various algorithms[19], some of which are standard in most commercial statistical software packages.[20] Given these estimates and using eqs.(4) and (17), an instrument's i marginal return-VaR is given by:

$$(34) \qquad \frac{\partial r_p^*}{\partial w_i} = \frac{\hat{a}_i + r_p^*\left(\hat{b}_{i1} + \hat{b}_{i2} r_p^*\right)}{1 + r_p^*\left(\hat{c}_{i1} + \hat{c}_{i2} r_p^*\right)}$$

Given the appropriateness of the approximant, the advantage of this approach is that all available sample data can be used to estimate marginal VaR and component VaR . In addition, the relationship between $\tilde{r}_i$ and $\tilde{r}_p$ is truly evaluated at the point where $\tilde{r}_p = -r_p^*$. The relative complexity of the procedure is an obvious disadvantage.

estimation by conditional mean

Considering computational complexity and intuitive appeal, the last estimation procedure presented here is on the other extreme of the spectrum. Instead of first modelling the relationship between portfolio and instrument return, the conditional expectation is estimated directly. The procedure is as follows. Choose a window of portfolio returns according to eq.(30) and select the corresponding T* complete data rows from the sample(d) data matrix. T* should be large enough to establish a sufficiently reliable estimate, but not too large. Then, for each instrument *i*, compute minus the (conditional) mean return over the T* observations in the data window:

$$(35) \qquad \bar{r}_i^* = -\frac{1}{T*}\sum_{t \in T*} r_{i,t} \qquad\qquad i \in p$$

At first sight, this would give us an estimate of instrument *i* 's marginal VaR according to eq.(17). However, since the tail of the portfolio's return distribution tapers off, the

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[19] Cf. Beck & Arnold [1976, Ch.7] or Draper & Smith [1981, Ch.10].

[20] Press et.al [1989, Ch.14.4], for example, provide program source codes. Also, the solver from Excel can be used.

median of the T* return observations will be higher than the mean. Hence we expect that the portfolio-weighted average of (minus) the conditional mean returns is below the portfolio VaR:

(36) $$\bar{r}_p^* \equiv \sum_{i \in p} w_i \bar{r}_i^* < r_p^*$$

Therefore we introduce a correction factor j :

(37) $$j \equiv \frac{r_p^*}{\bar{r}_p^*}$$

The adjusted estimate of instrument i 's marginal VaR then becomes:

(38) $$M\text{-}VaR_i = j \cdot \bar{r}_i^* \qquad i \in p$$

Note that the adjusted conditional mean represents the probability-weighted least squares aproximation to $E\left\{\tilde{r}_i \middle| \tilde{r}_p = -r_p^*\right\}$. The adjustment implies linear interpolation of the tail of the return distribution in the q -window. The corresponding estimate for the component VaR according to eq.(18) is $C\text{-}VaR_i = w_i \cdot M\text{-}VaR_i$. Due to the adjustment factor in eq.(38) this estimate for component VaR indeed satisfies eq.(14). Given its simplicity and intuitive appeal, this is our *a priori* preferred procedure. In section 5 we investigate whether this method can actually stand the test.

In the spirit of eqs.(37) and (38) we can also adjust the marginal VaRs (and hence the component VaRs) estimated through other procedures by:

(39) $$adjusted\ M\text{-}VaR_i = M\text{-}VaR_i \cdot \frac{r_p^*}{\sum_{j \in p} w_j \cdot M\text{-}VaR_j}$$

This ensures that the portfolio-weighted average of the adjusted estimated marginal VaRs (i.e. the sum of the component VaRs) equals the initially estimated overall portfolio VaR.

5. Evaluation of the estimators

In order to evaluate the proposed estimation procedures, we present a prototypical example in a simulation context. After describing the data we first discuss estimated marginal and component VaRs. Then we evaluate the use of marginal VaR estimates for approximating augmented portfolio VaR through incremental VaR.

data

The VaR horizon is one trading day and we generate 1,000 samplings. This could be a four year historical simulation or a small scale Monte Carlo simulation. We consider a single stochastic market factor: a stock on which various European options are written. The current value of the stock is set to *S*=100. Its continuous compounded return is drawn from a normal distribution with mean $m = 15\%$ p.a. and standard deviation $s = 40\%$ p.a. For each sampling, the portfolio is marked-to-model with the sampled stock price and the maturity of the options decreased with one day. The options are valued using the Black & Scholes model. The risk free interest rate is 5% p.a.

The initial portfolio we consider is a strap with a remaining maturity of one month. It consists of 2 long calls *C1* and one long put *P1* written on the stock. The strap is at-the-money in the forward sense, i.e. the exercise price X equals the forward price of the stock, $X = S \cdot \exp(.08 \cdot .05) = 100.40$. Hence *C(X)* = *P(X)* = 4.51. The current value of the strap is 13.53. We choose an at-the-money strap because of the high gamma (which induces non-linearity) and the resulting negatively skewed return distribution. In order to evaluate the estimated marginal VaRs, we consider three candidate trades:

Table 1: Composition of initial and augmented portfolios

			augmented portfolios					
	initial portfolio		A		B		C	
	#	w_i	#	w_i	#	w_i	#	w_i
C1	2	.67	2	.38	2	.61	2	.87
P1	1	.33	1	.19	1	.30	1	.44
S	-	-	.1	.43	-	.09	-	-
P2	-	-	-	-	25	-	-	-
C2	-	-	-	-	-	-	-.5	-.31
value :	13.53		23.53		14.88		10.34	

A. a position in the underlying stock *S*,

B. a position in a deep out-of-the-money put option *P2* with exercise price 85, remaining maturity of two weeks and current value 0.05, and

C. a position in a two-month call option *C2* with exercise price 100.8 (at-the-money forward) and current value 6.38.

Table 1 summarizes the portfolios.

marginal and component VaRs

The overall VaR of the initial strap portfolio with confidence level 95% is $r_p^* = 24.73\%$. This corresponds to $k(c) = 2.24$ in eq.(26) whereas under normality $k(c) = 1.65$. We use some of the methods outlined in section 4 to estimate marginal VaRs and component VaRs.

There are three OLS regression procedures. Global OLS uses all 1,000 samplings in eq.(22) to estimate the slope coefficient b_i in eq.(10).[21] This is the conventional procedure under normality. The asymmetric response model is eqs.(31-32) with $f = median$, so 50% of the data is used to estimate the relevant slope b_i^-. The local regression procedure uses 25 observations centered around $-r_p^*$ to estimate the slope coefficient b_i in eq.(10).

The fourth procedure is the (2,2)-Padé approximation according to the rational approximant eqs.(33-34). Finally we have the conditional mean procedure according to eq.(35) with T*=21 observations centered around $-r_p^*$ and adjusted according to eq.(37).

Table 2 contains the estimates. All methods recognize that the call options contribute to the risk exposure as measured by the VaR whereas the put option serves as a hedge. However, the estimates for the marginal VaRs and hence the component VaRs show considerable differences. Global regression generates estimates that are most pronounced – but also incorrect given the non-normality at hand. The estimates from the rational approximant and the conditional mean are mid-range and almost indistinguishable from one another while local regression estimates are smallest. A priori

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[21] Of course, the value of $k(c)$ implied by s_p and r_p^* in eq.(26) could be used to estimate marginal VaRs. However, from eq.(8) it follows that $\partial r_p^* / \partial w_i = \hat{k} \cdot \partial s_p / \partial w_i = \hat{k} \cdot b_i \cdot s_p = b_i \cdot r_p^*$ so this decomposition of overall VaR corresponds exactly to the global OLS procedure!

Table 2: Marginal- and component-VaR estimates for the initial portfolio.

The 95% return-VaR is $r_p^* = 24.73\%$. Global regression uses all 1,000 scenarios. The asymmetric response model is eqs.(31-32) with $f = median$. Local regression uses 25 observations centered around $-r_p^*$. (2,2)-Padé is the rational approximant eq.(34). The adjusted conditional mean is according to eqs.(35-37) with T*=21 observations centered around $-r_p^*$. Component VaRs are adjusted according to eq.(39).

$r_p^* = 24.73\%$	OLS			(2,2)-Padé	conditional
	global	asymmetric	local	approx.	mean
M-VaR(C1)	.6776	.5830	.5269	.5674	.5630
M-VaR(P1)	-.6142	-.4242	-.3417	-.3939	-.3892
M-VaR(S)	.0566	.0441	.0378	.0415	.0415
M-VaR(P2)	-2.0650	-1.4415	-1.1627	-1.0560	-1.0464
M-VaR(C2)	.4803	.4011	.3566	.3880	.3850
C-VaR(C1)	45.22%	38.87%	36.59%	37.87%	37.79%
C-VaR(P1)	-20.49%	-14.14%	-11.86%	-13.14%	-13.06%

we expect global OLS estimates to be biased by the non- linear relationship between $\tilde{r}_i$ and $\tilde{r}_p$ and it seems that the local regression estimates overshoot.

incremental VaRs

In order to evaluate the estimated marginal VaRs, we next estimate incremental VaRs. We add another instrument to the strap portfolio and use marginal VaR to estimate the overall VaR of the augmented portfolio. Note however, that expression (6) for incremental VaR only applies for a small weight w_N of the instrument added. But we are here considering a small portfolio and we want to evaluate the effect of larger changes in composition. For larger changes in the portfolio, an additional adjustment must be made. Suppose the initial portfolio p with current value $V_{p,0}$ is composed of $N-1$ instruments. Next we add instrument N with current value $V_{N,0}$ to the portfolio. Then the portfolio weight of this instrument in the augmented portfolio p' becomes:

$$(40) \qquad w_N = \frac{V_{N,0}}{V_{p0} + V_{N,0}}$$

and the weights of the initial instruments in p' change according to:

(41) $w_i \to (1 - w_N) w_i \qquad i = 1, \ldots, N-1$

Now, by definition, the VaR of the initial portfolio p is:

(42) $$r_p^* = \sum_{i=1}^{N-1} w_i \cdot M\text{-}VaR_i$$

Give the changed portfolio weights, we would have for the augmented portfolio p':

(43) $$r_{p'}^* = (1 - w_N) \sum_{i=1}^{N-1} w_i \cdot M\text{-}VaR_i' + w_N \cdot M\text{-}VaR_N'$$

where the prime indicates that the marginal VaRs are measured with respect to p'. Then, to a first order approximation:

(44) $$\begin{aligned} r_{p'}^* &\approx (1 - w_N) \sum_{i=1}^{N-1} w_i \cdot M\text{-}VaR_i + w_N \cdot M\text{-}VaR_N \\ &\equiv (1 - w_N) r_p^* + w_N \cdot M\text{-}VaR_N \end{aligned}$$

We use eq.(44) and the information in Table 2 to estimate the VaR of the augmented portfolios A, B and C. We confine ourselves to the global regression method and the proposed adjusted conditional mean method. Table 3 confronts these estimates with the portfolio VaRs obtained by brute force full-fledged re-estimation ("actual" VaR).

Table 3: **Actual VaR and estimated VaR (via incremental VaR) for the augmented portfolios A, B and C.**
Global regression uses all 1,000 scenarios. The adjusted conditional mean is according to eqs.(35-37) with T*=21 observations centered around $-r_p^*$. Confidence level: 95%.

	actual	OLS global	conditional mean
A	16.00%	2.40%	15.99%
B	13.18%	3.87%	13.06%
C	20.41%	17.54%	20.48%

As expected the global regression method (which embraces the very quick but even more dirty variant of the conventional variance-covariance method, as described in footnote 21) fails miserably, whereas the conditional mean method yields almost exact results. Given the non-normality and the highly non-linear portfolio at hand we expected the former but could only dream of the latter. Although this is a highly stylized example we conclude that the adjusted conditional mean method is not only intuitively appealing and embarrassingly simple but also promises to be highly accurate for non-linear portfolios in general. We performed some additional simulations on the basis of random portfolios and these results strengthen our confidence. We expect the adjusted conditional mean method to live up to its promises in a more scrutinous empirical investigation.

6. Summary and conclusions

This paper investigates the concepts of marginal, component and incremental VaR in a general setting. We derive a distribution-free expression for the marginal contribution of an instrument to the diversified portfolio VaR – whether this instrument is already included in the portfolio or not. We show how in a most general way, the diversified portfolio VaR can be decomposed in component VaRs that can be attributed to the individual instruments comprised in the portfolio. We show how the standard results under normality can be generalized to non-normal analytical VaR approaches and we discuss the distributional assumptions under which the derived expressions hold, either exactly or as mean-square error approximations. In addition we present explicit procedures for estimating marginal VaR and component VaR in simulation settings, viz. global and local regression models, asymmetric response models, rational approximants and the adjusted conditional mean model. Based on the intrinsic linearity of the portfolio operator, the last approach pairs appealing simplicity with high computational efficiency.

We evaluated the performance of various estimation methods applied to a prototypical non-linear portfolio in a simulation setting. The conventional variance-covariance method – albeit adapted for non-normality – fails miserably, whereas the conditional mean method yields almost exact results. The latter method is not only intuitively appealing and embarrassingly simple but also pairs high accuracy with computational efficiency. We expect that this outperformance can be maintained for non-linear portfolios in general. Additional simulations strengthen our confidence and a more detailed empirical investigation will be the subject of a next paper.

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