Structural stability tests with unknown breakpoint for the efficient method of moments with application to stochastic volatility models

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Abstract

Tests for structural stability with unknown breakpoint are derived for and applied to the efficient method of moments. Three types of tests are discerned: Wald type tests, Predictive tests and Hansen type tests. The Hansen type test for structural stability with unknown breakpoint is a novelty for moment based techniques. Therefore for this test asymptotic and local power results are provided. It turns out this test has the same asymptotic distribution as the Hall and Sen test. All these tests are applied to an asymmetric stochastic volatility model for a series of daily observations of the S&P 500 index over the years 1963-1993. Our results indicate that the asymmetric stochastic volatility models fails all stability tests. The supremum type tests give an estimate of the breakpoint in first half of 1970. However the model seems to become more stable as time proceeds.

Keywords: efficient method of moments; (asymmetric) stochastic volatility; tests for structural stability with unknown breakpoint

JEL-classification: C15; C22; C52

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1 Introduction

In case the likelihood of a model is analytically intractable one must often resort to simulation-based estimation techniques such as *indirect inference* (Gouriéroux, Monfort & Renault (1993)) or the efficient method of moments (EMM, Gallant & Tauchen (1996)). Both EMM and indirect inference may be seen as a *minimum chi square* estimator or *M-estimator* as is GMM (Hansen (1982)). GMM based tests may therefore easily be used in EMM and indirect inference. Such has been done for the Hansen J*-test for overidentifying restrictions* (Gouriéroux, Monfort & Renault (1993)), Wald and score tests (Dhaene, Gouriéroux & Scaillet (1996)), structural stability tests (van der Sluis (1998*a*) and van der Sluis (1997*b*)). For a test statistic for EMM that has no counterpart in GMM see Liu & Zhang (1997). In the present paper tests for structural stability with unknown breakpoint for EMM are discussed. Most results for the tests hinge on results for Maximum Likelihood and GMM based tests by Andrews (1993), Andrews & Ploberger (1994), Hall & Sen (1996) Ghysels, Guay & Hall (1998) and Sowell (1996*b*). One test is a novelty in the field of moment-based inference.

As we shall see in section 3, for moment based techniques the source for the violation of the structural stability hypothesis can be split up in three parts: (i) parameter variation, (ii) violation of the moment conditions after the breakpoint only, and (iii) violation of the moment conditions both before and after the breakpoint. For each of these three alternatives, for known breakpoint three classes of most powerful tests has been derived in the literature. For unknown breakpoint we take these three classes as a starting point. For unknown breakpoint the testing procedures are much more involved since the unknown parameter representing the time of the breakpoint is only present under the alternative hypothesis and not under the null. This induces non standard asymptotic distributions of the test statistics. Path breaking work in Andrews (1993) and Andrews & Ploberger (1994) respectively treats taking the supremum of the test-statistics over all possible breakpoints and weighing the test-statistics over all possible breakpoints. The tests discussed in Andrews (1993) are supremum Wald type tests in an GMM context. For known breakpoint Wald type tests are known to have optimal power against parameter variation. The optimality properties of the supremum type tests are found to be very weak, but this type of tests has the advantage of providing an indication of the breakpoint. The tests discussed in Andrews & Ploberger (1994) are weighted Wald type tests in a Maximum Likelihood framework. Here it is shown that optimality depends on the nature of the alternative. So no optimal tests but rather a whole family can be derived, where the properties of each member class of this family depends on the nature of the alternative.

Sowell (1996*a*) discusses these types of tests in general in a GMM context. The weighted tests have better optimality properties conditional on the moments and on the weighing function that is employed. Since for EMM the moments are optimal with respect to the model we noted in van der Sluis (1998*a*) that the Wald test has not only optimal power against parameter variation but also its local power against the alternative of parameter variation uniformly dominates all other M-estimators based tests. For unknown breakpoint optimality in this respect is hard to obtain since optimality in this respect depends also on the weighing functions.

A next class of test statistics are the predictive tests. For unknown breakpoint Ghysels, Guay & Hall (1998) propose predictive tests in a GMM context. This test is based on the post-sample prediction test of Hoffman & Pagan (1989) and Ghysels & Hall (1990*a*). In this test the moment conditions are evaluated at the estimated parameters from the sample and the post-sample data. For known breakpoint it was shown by Ahn (1995) that the post-sample prediction tests has optimal power against the alternative of failure of the moment condition in the post-sample. For unknown breakpoint Ghysels, Guay & Hall (1998) consider supremum and weighted forms of the prediction statistic.

A next class of test for structural stability tests are the Hansen type tests for structural stability. In the known breakpoint case, Ghysels & Hall (1990*b*) consider tests which have optimal power against the alternative of moment failure in both ample and post-sample. In this paper this test statistic is generalized to the case of unknown breakpoint. Since this test was not considered for the unknown breakpoint in the literature, we derive its asymptotic distribution. Again weighted forms and supremum forms of this test are considered.

The last tests statistic we apply in this paper is based on work of Hall & Sen (1996)and Sowell (1996*b*). In these papers a different decomposition of the null hypothesis of structural stability is chosen. The tests in these papers are constructed each element of the test has only power against one of these alternatives. This has the advantage that each of the components reveals unique information of the nature of the instability of the model. It turns out that the asymptotic distribution of the Hansen test equals the asymptotic distribution of the Hall-Sen-Sowell test. It is argued in this paper that by using several tests the same information can be obtained from the Wald, Predictive type tests and Hansen type tests as from the Hall-Sen-Sowell tests.

In this paper we apply all the above tests statistics to an asymmetric stochastic volatility model (see Harvey & Shephard (1996)) for the S&P500 index over the range 1963-1993. We estimate this model by EMM with an EGARCH(1,1) score generator. Tables have been provided with asymptotic critical values for all of these test statistics. Also computer code is made available to calculate $P-$values from the non-standard limiting distributions of all of these test-statistics.

The setup of this paper will be as follows. In section 2 will give of review of EMM of Gallant & Tauchen (1996). This section is also intended to introduce some notation. Readers who are unfamiliar with EMM should consult the referenced papers. In section 3 structural stability tests with unknown breakpoint for EMM, are proposed. First, the decomposition of the structural stability hypothesis is given. After that, building on results from the GMM literature, subsequently, Wald type tests, Predictive tests, and Hansen type tests are discussed. For unknown breakpoint the Wald type tests statistics, the Prediction tests and the Hall-Sen test statistic are direct transcriptions from the GMM literature. The Hansen type test for unknown breakpoint is a novelty. Therefore for this test statistic the asymptotic distribution is derived in appendix A. In section 4 the application to asymmetric stochastic volatility is discussed. Section 5 concludes.

2 Efficient Method of Moments

In the original Indirect Inference setup of Smith (1993) and Gouriéroux, Monfort & Renault (1993) the connection between the auxiliary model and the structural model is established through the parameters of the auxiliary model. Gallant & Tauchen (1996) try to solve the efficiency problems Indirect Inference has by proposing the *efficient method of moments* (EMM) technique. Here the connection between the auxiliary model and the dynamic model is achieved by means of the scores of the auxiliary model, where strict guidelines are given for the choice of the auxiliary model. This method does not only yield asymptotically efficient estimators but *score calibration* makes more sense than *parameter calibration*. Namely, the score of the structural model provides the exactly identifying moment conditions for maximum likelihood estimation. Since this score is analytically intractable, we replace this score by the score of an auxiliary model. This score must be tractable and for efficiency this auxiliary score must span the score of the structural model (See Gallant & Long (1997) for details). Gallant & Tauchen (1996). In short the EMM method goes as follows: the sequence of densities for the data y_t for the structural model will be denoted

$$\{p_1(x_1 \mid \theta), \{p(y_t \mid x_t, \theta)\}_{t:=1}^{\infty}\}$$

Of course we will only observe a finite number of data $\{y_t\}_{t=1}^{n}$. The sequence of densities for the auxiliary process will be denoted as

$$\{f_1(w_1 \mid \beta), \{f(y_t \mid w_t, \beta)\}_{t:=1}^{\infty}\}$$

where x_t and w_t are observable endogenous variables[1]. We impose assumptions 1, 2 and 3 from Gallant & Long (1997) on the structural model. Assumption 1 is a technical assumption that implies strong mixing and ergodicity on (y_t, x_t) which are the standard regularity conditions for quasi maximum likelihood estimation. Assumption 2 is needed for the desirable properties of estimators based on *Hermite expansions* which will be explained below. Assumption 3 is technical but crucial since it imposes conditions on the structural model that are needed for the efficiency claim of EMM. It is needed that the score of the structural model can be approximated by polynomials. The stochastic volatility models that will be dealt with in this paper all obey these assumptions, provided they are stationary.

Let us define

$$m(\theta, \beta) := \int \int \frac{\partial}{\partial \beta} \ln f(y \mid w, \beta) p(y \mid x, \theta) dy p(x \mid \theta) dx$$

the expected score of the auxiliary model under the dynamic model. The expectation is written in integral form to anticipate on the fact the we will approximate this integral by standard Monte Carlo

[1] So x_t is a vector of lagged y_t and w_t is also a vector of lagged y_t. The laglength may differ however.

techniques. The simulation approach solely consists of calculating this function as

$$m_N(\theta,\beta) := \frac{1}{N}\sum_{\tau:=1}^{N} \frac{\partial}{\partial\beta} \ln f(y_\tau(\theta) \mid w_\tau(\theta),\beta) \tag{1}$$

Let n denote the sample size, the EMM estimator is defined as

$$\widehat{\theta}_n(\mathcal{I}_n) := \arg\min_{\theta\in\Theta} m_N^{'}(\theta,\widehat{\beta}_n)(\mathcal{I}_n)^{-1} m_N(\theta,\widehat{\beta}_n)$$

where $\mathcal{I}_n$ is a weighting matrix and $\widehat{\beta}_n$ denotes an estimator for the parameter of the auxiliary model. The optimal weighting matrix here is obviously

$$\mathcal{I}_0 = \lim_{n\to\infty} V_0[\frac{1}{\sqrt{n}}\sum_{t:=1}^{n}\{\frac{\partial}{\partial\beta}\ln f_t(y_t \mid w_t,\beta^*)\}]$$

where β^* is a (pseudo) true value.

With the theory of misspecified models (White (1994)) one can prove consistency for the parameters of the auxiliary model under several assumptions posed in Gallant & Tauchen (1996),

$$\lim_{n\to\infty}(\widehat{\beta}_n - \beta^*) = 0 \text{ a.s.}$$

and asymptotic normality

$$\sqrt{n}(\widehat{\beta}_n - \beta^*) \xrightarrow{d} N(0,(\mathcal{J}_0)^{-1}(\mathcal{I}_0)(\mathcal{J}_0)^{-1})$$

Here

$$\begin{aligned}\mathcal{I}_n &= V_0[\frac{1}{\sqrt{n}}\sum_{t:=1}^{n}\frac{\partial}{\partial\beta}\ln f_t(\widetilde{y}_t \mid \widetilde{w}_t,\widehat{\beta}_n)\\ \mathcal{J}_n &= -\frac{\partial}{\partial\beta} m_N^{'}(\theta_0,\widehat{\beta}_n)\end{aligned}$$

where θ_0 denotes the (pseudo) true value. Obviously,

$$\begin{aligned}\lim_{n\to\infty}\mathcal{I}_n &= \mathcal{I}_0\\ \lim_{n\to\infty}\mathcal{J}_n &= \mathcal{J}_0\end{aligned}$$

One can also prove for the scores

$$\sqrt{n} m_N(\theta_0,\widehat{\beta}_n) \xrightarrow{d} N(0,\mathcal{I}_0)$$

Hence consistency and asymptotic normality of the estimator of the structural parameters $\widetilde{\theta}_n$ follow:

$$\sqrt{n}(\widehat{\theta}_n(\mathcal{I}_0) - \theta_0) \xrightarrow{d} N(0,[\mathcal{M}_0^{'}(\mathcal{I}_0)^{-1}\mathcal{M}_0]^{-1})$$

where $\mathcal{M}_0 := \frac{\partial}{\partial\theta'} m(\theta_0,\beta^*)$.

In order to obtain *first order asymptotic efficiency* it is required that the auxiliary model in some sense embeds the structural model. The semi-nonparametric (SNP) density of Gallant & Nychka (1987)[2] may be a good choice, see Gallant & Tauchen (1996) and Gallant & Long (1997). The auxiliary model is built as follows. Let $y_t(\theta_0)$ denote the process under investigation, $\mu_t(\beta^*) := \mathsf{E}_{t-1}[y_t(\theta_0)]$, the conditional mean, $\sigma_t^2(\beta^*) := \mathsf{Var}_{t-1}[y_t(\theta_0) - \mu_t(\beta^*)]$ the conditional variance and $z_t(\beta^*) := [y_t(\theta_0) - \mu_t(\beta^*)] \cdot [\sigma_t^2(\beta^*)]^{-1/2}$ the standardized process. The SNP density now takes the following form

$$f(y_t; \beta) = \frac{1}{\sqrt{\sigma_t^2(\beta)}} \frac{[P_K(z_t(\beta), x_t)]^2 \phi(z_t(\beta))}{\int [P_K(u, x_t)]^2 \phi(u) du}$$

where ϕ denotes the standard normal density, $x := (y_{t-1}, ..., y_{t-L})$ and the polynomials

$$P_K(z, x_t) := \sum_{i:=0}^{K_z} a_i(x_t) z^i := \sum_{i:=0}^{K_z} [\sum_{j:=0}^{K_x} a_{ij} x_t^j] z^i \tag{2}$$

Note that the coefficients in the polynomial, a_{ij}, belong to the vector of auxiliary parameters β. A specific form for the polynomials is taken, namely orthogonal Hermite polynomials (see Gallant, Hsieh & Tauchen (1991) and Andersen & Lund (1997)). Relevant formulae for the derivatives can be found in Abramowitz & Stegun (1972) and Fenton & Gallant (1996). The model $\sigma_t^2(\beta)$ and $\mu_t(\beta)$ is chosen as a *leading term* in the Hermite expansion to relieve the expansion of some of its task, improving its small sample properties. We note that the SNP density offers a very convenient way of modelling. It omits the curse of dimensionality often encountered in the area of non-parametric models while sharing many of the optimality properties of the Kernel estimators of Silverman (1986).

In this paper *structural stability* is of interest, therefore the following notation is employed: n_i will denote the number of observations in the sample for $i = 1$ and in the post-sample for $i = 2$. The total number of observations $n = n_1 + n_2$. The auxiliary estimator that employs sample or post-sample data only, will be denoted $\widehat{\beta}_{n_i}$. The auxiliary estimator that employs both sample and post-sample data will be denoted $\widehat{\beta}_n$. Likewise the EMM estimators $\widehat{\theta}_{n_i}$ and $\widehat{\theta}_n$ are denoted.

In specification testing the following estimator for θ_0 will often be of (theoretical) interest, see e.g. Rothenberg (1973)

$$\widetilde{\theta}_n := \arg\min_{\theta \in \Theta} [\sum_{i:=1}^{2} n_i m_N^{'}(\theta, \widehat{\beta}_{n_i}) \mathcal{I}_i^{-1} m_N(\theta, \widehat{\beta}_{n_i})] \tag{3}$$

where $\mathcal{I}_i$ needs to be consistently estimated for $i = 1, 2$. From the appendix we have that $\widetilde{\theta}_n$ has the same asymptotic distribution as $\widehat{\theta}_n$.

In this paper we will take $p := \dim(\theta)$, the dimension of the parameters of the structural model, $q_i := \dim(\beta_i)$, the dimension of the parameters of the auxiliary model for the sample for $i = 1$ and for the post-sample for $i = 2$. The number of moment conditions for the sample q_1 and for the post-sample q_2 will be determined using several criteria. For the efficiency claim of EMM, it is necessary that q_i increases with n_i Note in this respect the conceptual difference with GMM. It will automatically happen

[2]Building on earlier work of Phillips (1983). See also Fenton & Gallant (1996a and b) for recent results on SNP densities.

that q_i increases with n_i when any of the model specification criteria such as the Akaike Information Criterion (AIC, Akaike (1973)), the Schwarz Criterion (BIC, Schwarz (1978)) or the Hannan-Quinn Criterion (HQC, Hannan & Quinn (1979) and Quinn (1980)) is used.[3]. The theory of model selection in the context of SNP models is not very well developed yet. Results in Eastwood (1991) may lead to believe AIC is optimal in this case. However, as for multivariate ARMA models, the AIC may overfit the model to noise in the data so we may be better off by following the BIC or the HQC. There is no general theory yet giving the rate at which terms must be added to the expansion to retain root$-n$ consistency, so one must be careful in expanding the SNP density. The same findings were reported in Andersen & Lund (1997). In their paper Gallant & Tauchen (1996) rely on the BIC in their applications. If testing by a battery of specification tests indicates that the BIC-optimal model is inadequate, see Bansal, Gallant, Hussey & Tauchen (1995) and Tauchen, Zhang & Liu (1996), the model is further expanded.

3 Structural Stability Tests with Unknown Breakpoint for EMM

In this section we will first give a decomposition of the structural stability hypothesis. After that, subsequently the Predictive tests, the Wald type tests and the Hansen type tests for structural stability with unknown breakpoint will be discussed.

Structural stability tests with unknown breakpoint does not fall into the regular class of tests because the breakpoint parameter π only appears under the alternative hypothesis and not under the null. Therefore treating π as a parameter does not give the standard asymptotic results. These type of tests were first considered by

Let $\pi \in \Pi \subset (0,1)$, the parameters that identifies the unknown breakpoint. Let $T_1(\pi) := \{t = 1, 2, ..., [\pi n]\}$ and $T_2(\pi) := \{t = [\pi n] + 1, ..., n\}$. Following the notation in previous section we will write $n_1 = [\pi n]$ and $n_2 = n - [\pi n]$, so we will surpress in our notation dependence of n_1 and n_2 on π. In the case of known breakpoint we could allow for different moment conditions before and after the breakpoint. We could also allow for a different number of moment conditions q_1 and q_2. However for unknown breakpoint the derivation of the asymptotic distributions of the test-statistics becomes very messy if not impossible if we allow for different score-generators before and after the break or more specifically if we let q_1 and q_2 be functions of π. Therefore in this paper we assume that for the case of unknown breakpoint the same score-generator is used before and after the break. To simplify notation we will use $q_1 = q_2 = q$ and we will write q. Since $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$ we will also write $\mathcal{M}$.

[3] AIC$= T \ln L - |\beta|$

BIC$= T \ln L - 0.5|\beta| \ln T$

HQC$= T \ln L - |\beta| \ln \ln T$

In this context, L denotes the loglikelihood of the auxiliary model in the optimum, T denotes the number of observations and $|\beta|$ the number of freely estimated auxiliary parameters. For some properties in the context of multivariate time-series see Lütkepohl (1990).

3.1 Decomposition of Structural Stability

Following, among others, Newey (1985), Ghysels & Hall (1990*a*) and Ahn (1995) in a GMM context, we identify the following individual null hypotheses for the EMM case:

$$\mathsf{H}_0^1 \quad : \quad m(\theta_0, \beta_1^*) = 0, \ \forall\, t \in T_1(\pi) \tag{4}$$

$$\mathsf{H}_0^2 \quad : \quad m(\overline{\theta}_0, \beta_2^*) = 0, \ \forall\, t \in T_2(\pi) \tag{5}$$

$$\mathsf{H}_0^3 \quad : \quad \theta_0 = \overline{\theta}_0 \tag{6}$$

where the function m is the expected score of the auxiliary model under the dynamic model as defined in the previous section. Here $\beta_i^* := \operatorname{plim}_{n_i \to \infty} \widehat{\beta}$. The *stability* hypothesis is defined as

$$\mathsf{H}_0^s : \mathsf{H}_0^1, \mathsf{H}_0^2 \text{ and } \mathsf{H}_0^3 \text{ hold}$$

The asymptotic power of the test depends on which of the individual null-hypotheses is violated. Define the following alternative hypotheses

$$\mathsf{H}_A^A \quad : \quad \text{At least one of } \mathsf{H}_0^1, \mathsf{H}_0^2 \text{ and } \mathsf{H}_0^3 \text{ does not hold}$$

$$\mathsf{H}_A^B \quad : \quad \mathsf{H}_0^1 \text{ holds, at least one of } \mathsf{H}_0^2 \text{ and } \mathsf{H}_0^3 \text{ does not hold}$$

$$\mathsf{H}_A^C \quad : \quad \mathsf{H}_0^1 \text{ and } \mathsf{H}_0^2 \text{ hold, } \mathsf{H}_0^3 \text{ does not hold}$$

This leads to the following local alternatives

$$\mathsf{H}_n^1 \quad : \quad \sqrt{n_1} m(\theta_0, \beta_1^*) = \delta_1 + o(1)$$

$$\mathsf{H}_n^2 \quad : \quad \sqrt{n_2} m(\theta_0, \beta_2^*) = \delta_2 + o(1)$$

$$\mathsf{H}_n^3 \quad : \quad m(\overline{\theta}_{0n}, \beta_2) = 0, \ \sqrt{n_2}(\overline{\theta}_{0n} - \theta_0) = \delta$$

Let $\mathsf{H}_l^A := \{(\mathsf{H}_n^1, \mathsf{H}_n^2)\}_{n:=1}^{\infty}$, $\mathsf{H}_l^B := \{(\mathsf{H}_0^1, \mathsf{H}_n^2)\}_{n:=1}^{\infty}$ and $\mathsf{H}_l^C := \{(\mathsf{H}_0^1, \mathsf{H}_n^3)\}_{n:=1}^{\infty}$.

We have the following interpretation of the individual components of the stability hypothesis.

3.2 Wald type tests for unknown breakpoint

Andrews (1993), Andrews & Ploberger (1994) and Sowell (1996*a*), derive a class of tests. Namely the Wald (W), Lagrange Multiplier (LM) and Likelihood Ratio (LR) tests for structural stability with unknown breakpoint. Let $\widehat{\mathcal{V}}_i(\theta) := \mathcal{M}'(\theta, \widehat{\beta}_{n_i})(\mathcal{I}_i)^{-1}\mathcal{M}(\theta, \widehat{\beta}_{n_i})$. In this paper we will use the following statistics

$$\mathsf{W}_n(\pi) \quad : \quad = n \cdot (\widehat{\theta}_{n_2} - \widehat{\theta}_{n_1})' [\pi^{-1} \widehat{\mathcal{V}}_1^{-1} + (1-\pi)^{-1} \widehat{\mathcal{V}}_2^{-1}]^{-1} (\widehat{\theta}_{n_2} - \widehat{\theta}_{n_1})$$

$$\mathsf{LR}_n(\pi) \; : \; = \sum_{i=1}^{2}[n_i m_N'(\widetilde{\theta}_n, \widehat{\beta}_{n_i})(\mathcal{I}_i)^{-1} m_N(\widetilde{\theta}_n, \widehat{\beta}_{n_i}) - n_i m_N'(\widehat{\theta}_{n_i}, \widehat{\beta}_{n_i})(\mathcal{I}_i)^{-1} m_N(\widehat{\theta}_{n_i}, \widehat{\beta}_{n_i})]$$

$$\mathsf{LM}_n(\pi) \; : \; = \sum_{i=1}^{2} n_i m_N'(\widetilde{\theta}_n, \widehat{\beta}_{n_i})(\mathcal{I}_i)^{-1} \mathcal{M}(\widetilde{\theta}_n, \widehat{\beta}_{n_i}) \widehat{\mathcal{V}}_i^{-1}(\widetilde{\theta}_n) \mathcal{M}'(\widetilde{\theta}_n, \widehat{\beta}_{n_i})(\mathcal{I}_i)^{-1} m_N(\widetilde{\theta}_n, \widehat{\beta}_{n_i})$$

Note the difference with Andrews (1993) of the LR tests. Andrews proposes to use the full-sample EMM estimator $\widehat{\theta}_n$ instead of the more natural restricted $\widetilde{\theta}_n$, as defined in (3), we employ here. Andrews (1993) proves in the GMM case $\mathsf{W}_n(\pi) = \mathsf{LR}_n(\pi) + o_p(1)$ and $\mathsf{W}_n(\pi) = \mathsf{LM}_n(\pi) + o_p(1)$. These findings carry over to the EMM case. We will refer to this equivalence class of tests as *Wald-type tests*. Andrews (1993) proposed $\sup_{\pi\in\Pi} \mathsf{W}_n(\pi)$, $\sup_{\pi\in\Pi} \mathsf{LR}_n(\pi)$ and $\sup_{\pi\in\Pi} \mathsf{LM}_n(\pi)$. As Andrews (1993) notes these statistics have very weak optimality properties. It has the advantage that we get an estimate of the time of breakpoint through $\widehat{\pi} = \arg\sup_{\pi\in\Pi} \mathsf{W}_n(\pi)$. Andrews & Ploberger (1994) and Sowell (1994) propose exponential weighted forms of $\mathsf{W}_n(\pi)$, $\mathsf{LR}_n(\pi)$ and $\mathsf{LM}_n(\pi)$. For example for the $\mathsf{LM}_n(\pi)$ statistic, the general form of the optimal tests is given by

$$(1+c)^{-p/2} \int \exp\{\frac{1}{2}\frac{c}{1+c}\mathsf{LM}_n(\pi)\} dJ(\pi) \tag{7}$$

The parameter c controls the weight given to different alternatives. Larger values of c give weight to alternatives further from the null. Smaller values of c give weight to alternatives close to the null. Taking $c \to 0$ in (7) yields $\int_\Pi \mathsf{LM}_n(\pi) dJ(\pi)$, which is thought to be powerful to alternatives that are close to the null and taking $c \to \infty$ in (7) yields $\ln[\int_\Pi \exp\{\frac{1}{2}\mathsf{LM}_n(\pi)\} dJ(\pi)]$, which is thought to be powerful against distant alternatives. Andrews, Lee & Ploberger (1993) deduce from a Monte Carlo study in a normal regression context that the power and size of their test changed very slowly with values of c. In this paper we therefore follow their conclusions and consider only extreme values for c. For LM we thus consider

$$\begin{aligned} \mathsf{SLM} &:= \sup_{\pi\in\Pi} \mathsf{LM}_n(\pi) \\ \mathsf{ALM} &:= \textstyle\int_\Pi \mathsf{LM}_n(\pi) dJ(\pi) \\ \mathsf{ELM} &:= \ln[\textstyle\int_\Pi \exp\{\frac{1}{2}\mathsf{LM}_n(\pi)\} dJ(\pi)] \end{aligned}$$

Likewise we denote LR and W. As shown by Sowell (1994) these statistics have optimality properties conditional on the weighting function. This weighting function can be interpreted in a Bayesian way as a prior on π. We may then interpreted the test statistic as a posterior odds ratio or more precisely the optimal tests are asymptotically equivalent to a Bayesian posterior odds ratio. By the *continuous mapping theorem* the asymptotic distributions of W, LR and LM test-statistics are easily derived form the asymptotic distribution of W, LR and LM itself. We have that $\mathsf{W}(\pi)$, $\mathsf{LR}(\pi)$ and $\mathsf{LM}(\pi)$ all converge in distribution to $\mathsf{BBH}_p(\pi)$. Where

$$\mathsf{BBH}_p(\pi) := \frac{(B_p(\pi) - \pi B_p(1))'(B_p(\pi) - \pi B_p(1))}{\pi(1-\pi)}$$

Sowell (1994) shows that for GMM the weighted Wald type tests have optimal asymptotic power towards alternative H_l^C. In van der Sluis (1998*a*) we showed that for Wald type tests we have extra optimality results for EMM as opposed to GMM. Since for EMM the moments are optimal with respect to the model we noted in van der Sluis (1998*a*) that the Wald test has not only optimal power against parameter variation but also its local power against the alternative of parameter variation uniformly dominates all other M-estimators based tests. For unknown breakpoint optimality in this respect is hard to obtain since optimality in this respect depends also on the weighing functions.

3.3 Predictive tests for unknown breakpoint

For EMM the following pendant of the PR quantity is proposed

$$\mathsf{PR}_n(\pi) := m_N(\widehat{\theta}_{n_1}, \widehat{\beta}_{n_2})' \mathcal{W}^{-1} m_N(\widehat{\theta}_{n_1}, \widehat{\beta}_{n_2}) \tag{8}$$

where $\mathcal{W} := \mathcal{I}_2 + \frac{1-\pi}{\pi}\mathcal{M}(\widehat{\theta}_{n_1}, \widehat{\beta}_{n_2})[\mathcal{M}(\widehat{\theta}_{n_1}, \widehat{\beta}_{n_1})'\mathcal{I}_1^{-1}\mathcal{M}(\widehat{\theta}_{n_1}, \widehat{\beta}_{n_1})]^{-1}\mathcal{M}(\widehat{\theta}_{n_1}, \widehat{\beta}_{n_2})'$.

Under several conditions one can prove that under the null

$$\mathsf{PR}_n(\pi) \xrightarrow{d} \mathsf{BBH}_p(\pi) + \mathsf{BMH}_{q-p}(\pi)$$

where

$$\mathsf{BMH}_{q-p}(\pi) := \frac{(B_{q-p}(1) - B_{q-p}(\pi))'(B_{q-p}(1) - B_{q-p}(\pi))}{1-\pi}$$

where the B_p are vectors of independent *Brownian motions* of dimension p. Here p is the dimension of θ, the number of structural parameters and q is the number of moment conditions, the number of auxiliary parameters. The asymptotic distribution is thus the sum of the Wald test for parameter variation and a test for overidentifying restrictions in $T_2(\pi)$. Continuous mappings of $\mathsf{PR}_n(\pi)$ yield familiar predictive test statistics.

$$\begin{aligned}
\mathsf{SPR} &:= \sup_{\pi\in\Pi} \mathsf{PR}_n(\pi) \\
\mathsf{APR} &:= \textstyle\int_\Pi \mathsf{PR}_n(\pi) dJ(\pi) \\
\mathsf{EPR} &:= \ln[\textstyle\int_\Pi \exp\{\frac{1}{2}\mathsf{PR}_n(\pi)\} dJ(\pi)]
\end{aligned}$$

Where J is a weighting function. The SPR is identified as the *sup-predictive test*. There are no formal optimality properties of this test known, see Andrews & Ploberger (1994). It is intuitively plausible and it provides an estimate of the unknown breakpoint by $\arg\sup_{\pi\in\Pi} \mathsf{PR}_T(\pi)$. The APR and EPR are motivated by Andrews & Ploberger (1994) and Sowell (1994). By the *continuous mapping theorem* the asymptotic distributions of these test-statistics are easily derived form the asymptotic distribution of $\mathsf{PR}_T(\pi)$. As for the Wald type tests in Andrews & Ploberger (1994) the general form of the optimal tests is given by

$$(1+c)^{-p/2}\int \exp\{\frac{1}{2}\frac{c}{1+c}\mathsf{PR}_n(\pi)\} dJ(\pi) \tag{9}$$

Taking $c \to 0$ in (9) yields APR, which is thought to be powerful to alternatives that are close to the null and taking $c \to \infty$ in (9) yields EPR, which is thought to be powerful against distant alternatives. The asymptotic distributions and critical values are discussed in subsection

When the breakpoint is known, we get that PR gives the (post-sample) prediction (PSP) test (discussed in van der Sluis (1997*b*) for the EMM case and earlier in Hoffman & Pagan (1989) and Ghysels & Hall (1990*a*) for the GMM case. Ahn (1995) shows, building on earlier work of Newey (1985) that the PSP test for GMM is an optimal GMM test that has maximum power toward H_l^B. This fact applies to the EMM formulation as well. Ghysels, Guay & Hall (1998) show that when the breakpoint is unknown the tests based on the quantity PR have the following properties: (i) when $p = q$ the Predictive test is asymptotically equivalent to the Wald type statistics. However for $q > p$ the predictive test has more degrees of freedom and hence less power than the Wald type tests against alternatives of parameter variation. (ii) if the correct breakpoint is chosen and only the moments conditions (4) do not hold, we have a noncentrality parameter, say $\text{ncp}_1(\pi_0)$ as a function of π. Vice versa if only the moments conditions (5) do not hold we have the noncentrality parameter $\text{ncp}_2(\pi_0)$. However if $\pi_0 = 0.5$, then $\text{ncp}_2 \geq \text{ncp}_1$, indicating that in this case the predictive test has more power against structural instability after the break. (iii) the SPR, APR and EPR have nontrivial power against alternatives for which the expectation of the score generator is not constant over the sample. The second property suggests it is desirable to perform the tests in two ways: using the parameters estimators of the first subsample to evaluate the moment conditions in the second subsample and using the parameters of the second subsample to evaluate the moment conditions in the first subsamples. We indicate in the application the PR statistic where the parameters estimates from the second subsamples are taken with PR^* :

$$\mathsf{PR}_n^*(\pi) := m_N(\widehat{\theta}_{n_2}, \widehat{\beta}_{n_1})' \mathcal{W}^{*-1} m_N(\widehat{\theta}_{n_2}, \widehat{\beta}_{n_1}) \tag{10}$$

where $\mathcal{W}^*$ is a straightforward modification of $\mathcal{W}$ in (8).

3.4 Hansen type tests for unknown breakpoint

We can also generalize the Hansen J test for structural stability for known breakpoint of Ghysels & Hall (1990*a*) in the GMM case, to unknown breakpoints. That is, to consider the value of the EMM criterion at the optimum of the restricted estimator $\widetilde{\theta}_n$

$$\mathsf{JSS}_n(\pi) := \sum_{i=1}^{2} n_i m_N'(\widetilde{\theta}_n, \widehat{\beta}_{n_i})(\mathcal{I}_i)^{-1} m_N(\widetilde{\theta}_n, \widehat{\beta}_{n_i})$$

For the asymptotic distribution of this quantity we obtain the following theorem.

Theorem 1 *Under standard assumptions we have that*

$$\mathsf{JSS}_n(\pi) \xrightarrow{d} \mathsf{BO}_{q-p}(\pi) + \mathsf{BMH}_{q-p}(\pi) + \mathsf{BBH}_p(\pi)$$

where

$$\mathsf{BO}_{q-p}(\pi) := \frac{B_{q-p}(\pi)' B_{q-p}(\pi)}{\pi}$$

For a proof see appendix A.

Asymptotically the JSS is the sum of the Wald test for parameter variation (corresponding to BBH), a test of the overidentifying restrictions in $T_2(\pi)$ (BMH) and a test for the overidentifying restrictions in $T_1(\pi)$ (corresponding to BO).

As for the Predictive and Wald type tests we can define

$$\begin{aligned}
\mathsf{SJSS} &:= \sup_{\pi\in\Pi} \mathsf{JSS}_n(\pi) \\
\mathsf{AJSS} &:= \int_\Pi \mathsf{JSS}_n(\pi) dJ(\pi) \\
\mathsf{EJSS} &:= \ln[\int_\Pi \exp\{\tfrac{1}{2}\mathsf{JSS}_n(\pi)\} dJ(\pi)]
\end{aligned}$$

Critical values of this asymptotic distributions of SJSS, AJSS and EJSS can be found in Tables 1,2 and 3 respectively. For known breakpoint Ghysels & Hall (1990*a*) show that JSS converges in distribution to $\chi^2_{q_1+q_2-p}$. For known breakpoint the JSS(π) test has optimal power towards local alternative H_l^A. We can consider $\sup_{\pi\in\Pi} \mathsf{JSS}(\pi)$. Again the asymptotic distributions of continuous mappings of $\mathsf{JSS}_n(\pi)$ depend on the distribution of $\mathsf{JSS}_n(\pi)$ itself.

3.5 Hall-Sen-Sowell type tests

In practice it may be useful to decompose the stability hypothesis into tests for parameter constancy and validity of the overidentifying restrictions both before and after the break.

Sowell (1996*b*) and Hall & Sen (1996) propose independently to use

$$\mathsf{HSS}(\pi) := \sum_{i=1}^{2} n_i m_N'(\widehat{\theta}_{n_i}, \widehat{\beta}_{n_i})(\mathcal{I}_i)^{-1} m_N(\widehat{\theta}_{n_i}, \widehat{\beta}_{n_i}) + \mathsf{W}(\pi)$$

obviously

$$\mathsf{HSS}(\pi) \xrightarrow{d} \mathsf{BO}_{q-p}(\pi) + \mathsf{BMH}_{q-p}(\pi) + \mathsf{BBH}_p(\pi)$$

This means that JSS and HSS have the same asymptotic distribution. This asymptotic distribution is the sum of the Wald test for parameter variation (corresponding to BBH), a test of the overidentifying restrictions in $T_2(\pi)$ (BMH) and a test for the overidentifying restrictions in $T_1(\pi)$ (corresponding to BO). This is not surprising since the HSS test is constructed in this way. The advantage of this test is that each of the component statistics reveals information about the nature of the instability. For known breakpoint each individual element of HSS has only power against violation of (4), (5) and (6). However the same conclusions can be drawn from using combinations of the Wald type, PR, PR* and the Hansen test for structural stability. Of course with such a strategy the right significance level for the individual tests much be set to ensure the desired significance level for the overall test. Similarly to the Wald, Predictive and JSS type tests the SHSS, AHSS and EHSS test statistics are defined.

3.6 Practical issues of the tests statistics and critical values

Note that for every choice of π, often θ must be estimated for $T_1(\pi)$, this may be very time-consuming. Without loss of asymptotic properties the LM and JSS may be used with $\widehat{\theta}_n$ instead of $\widetilde{\theta}_n$. This was not done in the application however. It should also be noted that all these tests are designed for one breakpoint. However we can split the sample up in several subsamples if the original sample is big enough for the asymptotic properties to hold.

The weighting function $J(\pi)$ is set equal to the density of the uniform distribution. In this for our problems quite natural choice we follow Andrews & Ploberger (1994) and Ghysels, Guay & Hall (1998). As a side effect doing so we can use the tables reported in these papers: Andrews & Ploberger (1994) for the Wald type statistics, Ghysels, Guay & Hall (1998) for the Prediction type statistics and Hall & Sen (1996) for the Hansen type statistics. Since in Hall & Sen (1996) only critical values for the interval [0.15,.85] are reported, we generated the distributions of SJSS, AJSS and EJSS using the same settings as Andrews (1993) Andrews & Ploberger (1994), Hall & Sen (1996) and Ghysels, Guay & Hall (1998). That is $10,000$ Monte Carlo replications and a grid of the interval $\Pi(N) = [\pi_0, 1-\pi_0] \cap \{\pi = j/N; j = 0, 1, \ldots, N\}$ was made setting $N = 3,600$ as argued in Andrews & Ploberger (1994). To obtain our results we used Ox under AIX. The Ox program *StabTest.ox* is available on from the author's homepage at

`http:\\www.fee.uva.nl\vak_groep\ake\vdsluis.htm`

http:\\www.fee.uva.nl\vak_groep\ake\vdsluis.htm. The results do not change substantially for a higher number of Monte Carlo replications. Since as a side result we obtained the distributions of the Prediction tests and Wald tests we could compare these to our own results. It was found that our results are comparable to the ones obtained by Andrews (1993), Table 1, Andrews & Ploberger (1994), Table 1 and Table 2 and Hall & Sen (1996). The small differences must be attributed to the different random number generators that were used. Since the random number generators have been improved over time we provide the critical values for the asymptotic results. The results we obtained for the asymptotic distribution of the PR based tests were rather different from the ones in Ghysels, Guay & Hall (1998). This is strange since JSS is the sum of the PR and $\mathsf{BO}_{q-p}(\pi)$ and the asymptotic distribution of the JSS based tests were in compliance with out results. Therefore we decided to report all the asymptotic critical values here and to base our inference on our own code. In applied economics one is often more interested in P-values than in critical values. This can be done by using numerical methods developed by e.g. Hansen (1997). We choose to do this by brute force because with the current state of computing technology we could obtain them in a couple of minutes on an IBM RS/6000 running under Ox[4]. The Ox program *Pvalues.ox* available from the author's homepage at the above internet address, calculates these P-values within a reasonable amount of time on an average computer. The P-values that are reported in the application are based on this program.

[4]From the author's experience about three times faster than a Pentium 166 Mhz with 32 MB.

4 Applications to stochastic volatility models

In this section we make an application to financial time-series in the context of stochastic volatility.

4.1 Description of the data and the structural model

The dataset consists of thirty years of daily data from the S&P500 index ranging from January 1st 1963 till January 26th 1993. The observations are not corrected for calendar effects, they are only converted into price movement series i.e. $100 \times [\ln(S\&P500_t) - \ln(S\&P500_{t-1})]$. It is reported that many financial time series, especially those from broad stock indices such as the S&P500 exhibit modest serial correlation in the mean process. The series have therefore been prewhitened for autocorrelation and mean, although the effect of the adjustment is modest. Figure 1 displays the unadjusted levels and the adjusted returns series. Salient features of the adjusted return series are provided in Table 15. In this table some features of sub-periods are also provided. In the last sub-period from 1983 till 1993 the influence of the 1987 October crash is substantial. This type of data sets may well be described by a so called stochastic volatility model (see Shephard (1996) and Ghysels, Harvey & Renault (1996) for a review). In the application we estimated an asymmetric stochastic volatility model of the following form

$$
\begin{aligned}
y_t &= \sigma_t \epsilon_t \qquad (11)\\
\ln \sigma_t^2 &= \omega + \gamma \ln \sigma_{t-1}^2 + \sigma_\eta \eta_t \\
\begin{bmatrix} \epsilon_t \\ \eta_{t+1} \end{bmatrix} &\sim NIID(0, \begin{bmatrix} 1 & \lambda \\ \lambda & 1 \end{bmatrix}), -1 \leq \lambda \leq 1
\end{aligned}
$$

This model was first proposed by Harvey & Shephard (1996). We labelled model in (11) as an ASARMAV(1,0)[5].

4.2 Implementation of the estimation and tests

Due to the computational burden of this types of tests in the application we set in (2) $K_z = 0$ and $K_x = 0$. Therefore we get the pure EGARCH model of Nelson (1991)

$$
\begin{aligned}
y_t &= \sigma_t z_t \qquad (12)\\
\sigma_t^2 &= \exp\{\omega + \sum_{i=1}^{p} \gamma_i \ln \sigma_{t-i}^2 + (1 + \sum_{j=1}^{q} \alpha_j L^j)[\kappa_1 z_{t-1} + \kappa_2 \langle |z_{t-1}| - \sqrt{2/\pi} \rangle]\}
\end{aligned}
$$

This can safely be done by relying on Monte Carlo results in van der Sluis (1998*b*). Here it was found that a pure EGARCH$(1,1)$ model in combination with an ASARMAV$(1,0)$ yields very efficient estimates. It is also very desirable for the test statistics, since we know from the same Monte Carlo

[5] Asymmetric Stochastic Autoregressive Moving Average Model

experiments that the small sample properties of the Hansen J-test are very good for the pure parametric model, whereas these properties slightly deteriorate for increasing order of the polynomial. This may have its reflection on the structural stability tests which have the same form as the Hansen J-test. Lastly, form the results in van der Sluis (1998*b*) and Andersen, Chung & Sørensen (1997) we know that convergence problems arise for high order Hermite polynomials. Since we need to perform about 1800 EMM estimations here we do not wish to take the risk of non-convergence at a certain point.

We also set $N = 20,000$ in (1) and use antithetic variables to decrease the Monte-Carlo variance in (1). For details on the implementation of EMM see van der Sluis (1997*a*). The grid was set $\Pi = [.20, .80]$. Also with respect to the computational burden, we decided to vary π with steps of .001 with is about 7 or 8 business days. This is enough to identify events with the breakpoint.

4.3 Discussion of results

Using the whole dataset we obtain the following EMM estimates for the ASARMAV(1,0) model with $t-$values below

$$
\begin{aligned}
y_t &= \sigma_t \epsilon_t \\
\ln \sigma_t^2 &= \underset{(-.009)}{-.001} + \underset{(11.1)}{.994} \ln \sigma_{t-1}^2 + \underset{(8.14)}{0.086} \eta_t \\
\mathsf{Cor}(\epsilon_t, \eta_{t+1}) &= \underset{(-315)}{-0.633}
\end{aligned}
$$

The Hansen J-test has a value of 2.43, which result in a $P-$value of 0.119. So on basis of the J-test this model is accepted. The individual $t-$values are all 1.56 in absolute value, which results in a P-value of 0.181. In figure 2 the Wald type statistics are plotted. In figure 3 the criterion value of the EMM criterion in the subsample are given. These corresponds to individual elements in the Hall-Sen-Sowell methodology. Figure 4 provides a plot of the predictive PR and PR* quantities. Figure 5 gives a plot of the Hansen test and of the Hall-Sen-Sowell test. They are plotted in the same figure since they have the same asymptotic distribution. After 1975 these quantities are remarkably similar.

In Table 16 the values of the test-statistics are given. All the tests given here reject the null of stability at all reasonable levels. As a side effect of the supremum tests we have an estimate of the time the break occurred at our disposal. Practitioners may use economic analysis to see what happened at the time of the break. The Wald and the HSS tests point at June 8th 1970, while the Hansen, Predictice, LR and LM tests point at March 3th 1970. We know that in the early 1970s, the U.S. economy was torn between guns and butter, struggling to pay for the Great Society's social programs while waging the war in Southeast Asia. Federal deficit spending rose, as did inflation and interest rates; economic growth slowed and employment fell. Confronted with *stagflation* – an extraordinary combination of rising prices and economic stagnation – Richard Nixon's administration cut taxes, raised interest rates and devalued the dollar in quick succession. Figure 6 gives the development of $\widehat{\theta}_{n_1}$. We observe a drop in the asymmetry parameter λ at the time of the breakpoint indicated by the supremum statistics. Note that the estimates become more stable as time proceeds because more data is used for estimation. In figure 7 there seems to be a sudden drop in the γ, σ_η and λ by the end of 1973. This may be attributed to the second oil crisis. Note that the parameters estimates become less stable as time proceeds because less data is used. The decrease in γ is remarkable. Figure 8 displays the $\widetilde{\theta}_n$. Also in this figure a sudden change in λ is observed. The other parameters also change but less heavy. Figure 9 and 10 shows the parameters estimates of the auxiliary EGARCH model as given in equation (12). More or less the same movements are observed here.

5 Conclusion

The theory for testing for structural stability with unknown breakpoint is taken from the GMM and maximum likelihood framework and put into the EMM framework. The Hansen tests for structural stability with unknown breakpoint that is proposed in this paper has no counterpart in the structural stability testing with unknown breakpoint. Therefore its asymptotic distribution was derived. It is found that its asymptotic distribution equals the test proposed byHall & Sen (1996). Since the tables in Ghysels, Guay & Hall (1998) were found to be inaccurate these are given here. Also computer code to generate P-values for all these tests have been supplied. We applied the theory to an asymmetric stochastic volatility model for thirty years of daily data of the S&P500 index ranging from 1963 to 1993. All the tests indicate rejection of the null hypothesis of structural stability. From the movements of the test-statistics we can see aspects of the data we could not see by eyeballing the levels or the returns. The supremum tests which give an estimate of the breakpoint point at the first half of 1970.

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A Appendix

Proof of theorem 1: The observed score is defined as

$$s_n(\beta) := \frac{1}{n}\sum_{t=1}^{n}\frac{\partial}{\partial\beta}\ln f(y_t|w_t;\beta)$$

Let B_q^0 denote a q-dimensional Brownian bridge, $B_q^0(t) := B_q(t) - tB_q(1)$.. By the delta-method we have for every $t \in \{L+1, \ldots, n\}$

$$s_i(\widehat{\beta}_n) = s_i(\beta^*) + \frac{\partial s_i(\beta^{*i})}{\partial\beta'}(\widehat{\beta}_n - \beta^*) + o_p(1)$$

$$\frac{1}{\sqrt{n}}\mathcal{I}_0^{-1/2}\sum_{i=1}^{[nt]} s_i(\beta^*) \to B_q(t) \tag{13}$$

$$\text{plim}\,\beta^{*i} = \beta^*$$

$$\text{plim}\,\frac{1}{nt}\sum_{i=1}^{[nt]}\frac{\partial s_i(\beta^{*i})}{\partial\beta} = -\mathcal{I}_0$$

Equation (13) holds because the score $s_i(\beta^*)$ are martingale difference sequence series and using Billingsley (1963) theorem 23.1 and using

$$\sqrt{n}(\widehat{\beta}_n - \beta^*) \to (\mathcal{I}_0)^{-1/2}B_q(1)$$

we have

$$\frac{1}{\sqrt{n}}\mathcal{I}_0^{-1/2}\sum_{i=1}^{[nt]} s_i(\widehat{\beta}_n) \to B_q^0(t)$$

Likewise for $\pi \in (0,1)$ we define

$$\begin{aligned} s_{1\pi n}(\beta) &= \frac{1}{\pi n}\sum_{t=1}^{[\pi n]}\frac{\partial}{\partial\beta}\ln f(y_t|w_t;\beta) \\ s_{2\pi n}(\beta) &= \frac{1}{(1-\pi)n}\sum_{t=[\pi n]+1}^{n}\frac{\partial}{\partial\beta}\ln f(y_t|w_t;\beta) = \frac{s_n(\beta) - \pi s_{1\pi n}(\beta)}{1-\pi} \end{aligned}$$

Similarly to equation (13) we have

$$\mathcal{I}_0^{-1/2}\sqrt{n}s_{1\pi n}(\beta^*) \xrightarrow{d} \frac{B_q(\pi)}{\pi}$$

and

$$\mathcal{I}_0^{-1/2}\sqrt{n}s_{2\pi n}(\beta^*) \xrightarrow{d} \frac{B_q(1) - B_q(\pi)}{1-\pi}$$

Now define $\widehat{\beta}_{1\pi n}$ by the property $s_{i\pi n}(\widehat{\beta}_{1\pi n}) = 0$. A standard Taylor series argument gives us

$$\sqrt{n}(\widehat{\beta}_{1\pi n} - \beta^*) \xrightarrow{d} \mathcal{J}_0^{-1}\mathcal{I}_0^{1/2}\frac{B_q(\pi)}{\pi}$$
$$\sqrt{n}(\widehat{\beta}_{1\pi n} - \beta^*) \xrightarrow{d} \mathcal{J}_0^{-1}\mathcal{I}_0^{1/2}\frac{B_q(1) - B_q(\pi)}{1-\pi}$$

By the delta method

$$\sqrt{n}m_N(\theta_0, \widehat{\beta}_{1\pi n}) \xrightarrow{d} \mathcal{I}_0^{1/2}\frac{B_q(\pi)}{\pi}$$
$$\sqrt{n}m_N(\theta_0, \widehat{\beta}_{2\pi n}) \xrightarrow{d} \mathcal{I}_0^{1/2}\frac{B_q(1) - B_q(\pi)}{1-\pi}$$

Now let

$$\begin{aligned} Q_{1\pi n}(\theta) \quad &: \quad = [\pi n] m_N(\theta_0, \widehat{\beta}_{1\pi n})' \mathcal{I}_0^{-1} m_N(\theta_0, \widehat{\beta}_{1\pi n}) \\ Q_{2\pi n}(\theta) \quad &: \quad = [(1-\pi)n] m_N(\theta_0, \widehat{\beta}_{2\pi n})' \mathcal{I}_0^{-1} m_N(\theta_0, \widehat{\beta}_{2\pi n}) \end{aligned}$$

We have that

$$Q_{1\pi n}(\theta_0) \xrightarrow{d} \frac{B_q(\pi)' B_q(\pi)}{\pi}$$
$$Q_{2\pi n}(\theta_0) \xrightarrow{d} \frac{[1 - B_q(\pi)]'[1 - B_q(\pi)]}{1-\pi}$$

For

$$\widehat{\theta}_{i\pi n} = \arg\min_{\theta\in\Theta} Q_{i\pi n}(\theta)$$

the delta method gives

$$\sqrt{n}(\widehat{\theta}_{1\pi n} - \theta_0) \xrightarrow{d} -[\mathcal{M}_0'\mathcal{I}_0^{-1}\mathcal{M}_0]^{-1}\mathcal{M}_0'\mathcal{I}_0^{-1/2}\frac{B_q(\pi)}{\pi}$$
$$\sqrt{n}(\widehat{\theta}_{2\pi n} - \theta_0) \xrightarrow{d} -[\mathcal{M}_0'\mathcal{I}_0^{-1}\mathcal{M}_0]^{-1}\mathcal{M}_0'\mathcal{I}_0^{-1/2}\frac{B_q(1) - B_q(\pi)}{1-\pi}$$

and for the restricted estimator

$$\widetilde{\theta}_{\pi n} = \arg\min_{\theta\in\Theta}\sum_{i=1}^{2} Q_{i\pi n}(\theta)$$

we find

$$\sqrt{n}(\widetilde{\theta}_n - \theta_0) \xrightarrow{d} -[\mathcal{M}_0'\mathcal{I}_0^{-1}\mathcal{M}_0]^{-1}\mathcal{M}_0'\mathcal{I}_0^{-1/2}B_p(1)$$

Another application of the delta method gives us

$$\begin{aligned} \sqrt{n}m_N(\widehat{\theta}_{1\pi n}, \widehat{\beta}_{1\pi n}) \quad &= \quad \sqrt{n}m_N(\theta_0, \widehat{\beta}_{1\pi n}) - \sqrt{n}\mathcal{M}_0(\widehat{\theta}_{1\pi n} - \theta_0) + o_p(1) \\ &\xrightarrow{d} (\mathcal{I}_0^{1/2} - \mathcal{M}_0[\mathcal{M}_0'\mathcal{I}_0^{-1}\mathcal{M}_0]^{-1}\mathcal{M}_0'\mathcal{I}_0^{-1/2})\frac{B_q(\pi)}{\pi} \\ &= \quad \mathcal{I}_0^{1/2}[I_q - SS']\frac{B_q(\pi)}{\pi} \end{aligned}$$

where $S := \mathcal{I}_0^{-1/2}\mathcal{M}_0[\mathcal{M}_0'\mathcal{I}_0^{-1}\mathcal{M}_0]^{-1/2},$ a semi-orthogonal $q \times p$ matrix (hence $S'S = I_p)$ of rank p. Similarly

$$\begin{aligned}
\sqrt{n}m_N(\widehat{\theta}_{2\pi n}, \widehat{\beta}_{2\pi n}) &= \sqrt{n}m_N(\theta_0, \widehat{\beta}_{2\pi n}) - \sqrt{n}\mathcal{M}_0(\widehat{\theta}_{2\pi n} - \theta_0) + o_p(1) \\
&\xrightarrow{d} (\mathcal{I}_0^{1/2} - \mathcal{M}_0[\mathcal{M}_0'\mathcal{I}_0^{-1}\mathcal{M}_0]^{-1}\mathcal{M}_0'\mathcal{I}_0^{-1/2})\frac{B_q(1) - B_q(\pi)}{1-\pi} \\
&= \mathcal{I}_0^{1/2}[I_q - SS']\frac{B_q(1) - B_q(\pi)}{1-\pi}
\end{aligned}$$

and

$$\begin{aligned}
\sqrt{n}m_N(\widetilde{\theta}_{\pi n}, \widehat{\beta}_{1\pi n}) &\xrightarrow{d} \mathcal{I}_0^{1/2}[I_q - SS']\frac{B_q(\pi)}{\pi} + \mathcal{I}_0^{1/2}SS'\frac{B_q^0(\pi)}{\pi} \\
\sqrt{n}m_N(\widetilde{\theta}_{\pi n}, \widehat{\beta}_{2\pi n}) &\xrightarrow{d} \mathcal{I}_0^{1/2}[I_q - SS']\frac{B_q(1) - B_q(\pi)}{1-\pi} - \mathcal{I}_0^{1/2}SS'\frac{B_q^0(\pi)}{1-\pi}
\end{aligned}$$

Let $S_\perp$ be a $q \times (q-p)$ matrix such that $[S, S_\perp]$ is orthogonal; hence $S_\perp'S_\perp = I_{q-p}$ and $S_\perp'S = 0$. Then $B_p = S'B_q$ and $B_{q-p} = S_\perp'B_q$ are two independent standard vector Brownian motions. Furthermore,

$$[I_q - SS']B_q = [I_q - SS'][SS' + S_\perp'S_\perp]B_q = S_\perp B_q$$

and the $p-$vector Brownian bridge B_p^0 is independent of B_{q-p}. Therefore

$$\begin{aligned}
&Q_{1\pi n}(\widehat{\theta}_{1\pi n}) \to \frac{B_{q-p}'(\pi)B_{q-p}(\pi)}{\pi} \\
&Q_{2\pi n}(\widehat{\theta}_{2\pi n}) \xrightarrow{d} \frac{[B_q(1) - B_q(\pi)]'[B_q(1) - B_q(\pi)]}{1-\pi} \\
&\sum_{i=1}^{2} Q_{i\pi n}(\widetilde{\theta}_{\pi n}) \xrightarrow{d} \frac{B_{q-p}'(\pi)B_{q-p}(\pi)}{\pi} + \frac{[B_q(1) - B_q(\pi)]'[B_q(1) - B_q(\pi)]}{1-\pi} + \frac{B_p^0(\pi)'B_p^0(\pi)}{\pi(1-\pi)}
\end{aligned}$$

		$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$q-p=1$	1%	17.22	19.82	22.02	24.55	25.78	28.41	30.01	31.63
	5%	13.19	15.56	17.77	19.85	21.07	23.46	25.27	26.86
	10%	11.33	13.63	15.63	17.75	18.86	21.20	22.99	24.34
$q-p=2$	1%	21.76	23.93	25.76	28.03	29.89	31.68	33.46	34.97
	5%	17.33	19.21	21.12	23.18	24.78	26.45	28.44	29.92
	10%	15.12	17.13	18.81	20.83	22.40	24.08	25.84	27.33
$q-p=3$	1%	25.51	27.93	29.26	31.57	32.98	35.00	37.48	38.09
	5%	20.57	22.61	24.36	26.26	27.93	29.30	31.36	32.90
	10%	18.34	20.36	22.01	23.88	25.44	26.75	28.70	30.13
$q-p=4$	1%	28.60	31.26	32.32	34.55	36.33	38.20	39.31	41.72
	5%	23.75	25.90	27.43	29.30	30.88	32.23	34.01	35.69
	10%	21.38	23.36	24.98	26.64	28.21	29.66	31.37	32.97
$q-p=5$	1%	32.35	34.56	36.34	37.93	39.61	40.94	42.89	44.29
	5%	27.20	29.04	30.86	32.19	33.95	35.33	37.22	38.36
	10%	24.65	26.46	28.09	29.50	31.28	32.44	34.36	35.54
$q-p=6$	1%	35.48	36.88	39.28	40.41	42.54	44.17	44.92	47.39
	5%	30.17	31.96	33.44	35.01	36.67	38.04	39.74	41.31
	10%	27.53	29.11	30.75	32.29	33.92	35.26	37.01	38.27
$q-p=7$	1%	38.79	40.69	42.82	44.70	46.23	47.65	48.95	50.83
	5%	33.20	35.04	36.84	38.27	39.87	41.21	42.67	44.28
	10%	30.52	32.09	33.68	35.31	36.68	38.11	39.58	41.05
$q-p=8$	1%	41.79	43.70	45.17	47.42	48.91	50.50	51.90	53.52
	5%	36.07	37.50	39.47	40.96	42.38	43.89	45.25	46.76
	10%	33.00	34.57	36.34	37.96	39.21	40.81	42.14	43.62

Table 1: Asymptotic critical values for $\text{SJSS}_n(\pi)$ with $\pi \in \Pi = [.15, .85]$

		$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$q-p=1$	1%	16.73	17.10	21.66	23.65	25.78	27.71	29.22	30.99
	5%	12.71	13.21	17.26	19.28	21.07	22.87	24.75	26.15
	10%	10.88	11.37	15.13	17.15	18.86	20.53	22.30	23.72
$q-p=2$	1%	21.50	23.35	25.21	27.44	29.32	31.27	32.62	34.71
	5%	16.91	18.81	20.56	22.53	24.19	25.87	27.72	29.10
	10%	14.66	16.54	18.31	20.21	21.75	23.38	25.16	26.50
$q-p=3$	1%	25.03	27.93	28.83	30.93	32.55	33.91	36.63	37.45
	5%	20.10	22.61	23.78	25.54	27.38	28.65	30.86	32.25
	10%	17.88	20.36	21.51	23.22	24.92	26.10	28.17	29.50
$q-p=4$	1%	28.21	30.82	31.70	33.99	35.80	37.10	38.67	41.09
	5%	23.13	25.31	26.95	28.76	30.28	31.79	33.39	34.95
	10%	20.90	22.80	24.43	23.89	27.54	29.00	30.53	32.21
$q-p=5$	1%	31.94	33.99	35.99	37.27	38.93	40.23	42.36	43.56
	5%	26.57	28.36	30.22	31.54	33.28	34.73	36.45	37.79
	10%	24.05	25.77	27.35	28.99	30.64	31.85	33.69	34.90
$q-p=6$	1%	34.94	36.16	38.76	39.95	41.62	43.46	44.51	46.67
	5%	29.45	31.38	32.80	34.28	36.04	37.27	39.26	40.44
	10%	26.87	28.48	30.08	31.51	33.19	34.55	36.28	37.47
$q-p=7$	1%	38.23	40.04	42.30	43.94	45.55	47.13	48.52	50.39
	5%	32.57	34.28	36.01	37.70	39.19	40.61	42.02	43.54
	10%	29.85	31.48	33.04	34.59	36.00	37.46	38.85	40.26
$q-p=8$	1%	41.35	43.13	44.28	46.61	48.12	49.77	51.22	52.93
	5%	35.52	36.94	38.69	40.19	41.79	43.24	44.59	46.04
	10%	32.39	33.88	35.59	37.33	38.54	40.08	41.47	42.94

Table 2: Asymptotic critical values for $\text{SJSS}_n(\pi)$ with $\pi \in \Pi = [.2, .8]$

		$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$q-p=1$	1%	9.62	10.98	12.55	14.01	15.18	16.45	17.90	19.06
	5%	6.74	8.25	9.51	10.79	11.98	13.32	14.65	15.84
	10%	5,46	6.95	8.05	9.42	10.52	11.86	13.00	14.06
$q-p=2$	1%	13.63	14.68	16.00	17.32	18.64	19.76	21.17	22.45
	5%	10.00	11.23	12.49	13.70	14.81	16.01	17.54	18.56
	10%	8.37	9.73	10.85	12.00	13.28	14.36	15.75	16.71
$q-p=3$	1%	16.37	18.12	19.02	20.41	21.83	22.58	24.14	25.25
	5%	12.86	14.13	15.18	16.47	17.57	18.72	20.29	21.23
	10%	11.14	12.35	13.47	14.73	15.89	16.88	18.21	19.28
$q-p=4$	1%	19.16	21.03	21.86	22.95	24.78	25.61	27.20	27.80
	5%	15.64	16.80	17.84	18.93	20.20	21.28	22.71	23.76
	10%	13.73	14.70	15.96	17.10	18.19	19.26	20.55	21.58
$q-p=5$	1%	22.50	23.72	25.58	26.16	27.36	28.84	29.76	30.60
	5%	18.34	19.58	20.82	21.79	23.04	23.95	25.49	26.44
	10%	16.44	17.54	18.63	19.70	20.99	21.85	23.25	24.09
$q-p=6$	1%	25.44	26.45	27.99	28.77	30.34	31.56	32.22	34.07
	5%	20.86	22.12	23.16	24.37	25.48	26.47	27.81	28.93
	10%	18.71	19.89	20.88	22.04	23.31	24.24	25.50	26.56
$q-p=7$	1%	28.10	29.79	30.78	31.81	33.04	34.54	35.51	36.96
	5%	23.49	24.65	26.01	26.86	28.25	29.42	30.43	31.62
	10%	21.19	22.29	23.62	24.63	25.73	26.88	27.84	29.02
$q-p=8$	1%	30.91	32.04	32.76	34.64	35.82	36.62	37.81	39.27
	5%	25.93	27.05	28.05	29.53	30.42	31.67	32.75	33.91
	10%	23.46	24.50	25.74	27.02	27.87	29.03	30.23	31.28

Table 3: Asymptotic critical values for $\text{AJSS}_n(\pi)$ with $\pi \in \Pi = [.15, .85]$

		$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$q-p=1$	1%	9.79	10.98	12.81	14.37	15.59	16.81	18.48	19.50
	5%	6.82	8.25	9.68	11.06	12.22	13.56	14.88	16.09
	10%	5.55	6.95	8.19	9.57	10.69	12.07	13.19	14.30
$q-p=2$	1%	13.76	14.68	16.06	17.68	19.02	20.33	21.58	22.45
	5%	10.09	11.23	12.65	13.88	15.11	16.24	17.54	18.56
	10%	8.45	9.73	10.98	12.20	13.43	14.49	15.75	16.71
$q-p=3$	1%	16.60	18.35	19.40	20.67	22.21	23.01	24.54	25.63
	5%	12.95	14.28	15.47	16.66	17.81	18.95	20.59	21.47
	10%	11.24	12.48	13.56	14.82	16.00	17.02	18.41	19.47
$q-p=4$	1%	19.35	21.18	22.02	23.28	25.32	26.07	27.36	28.41
	5%	15.74	16.94	18.04	19.18	20.47	21.59	22.87	24.03
	10%	13.84	14.83	16.11	17.21	18.38	19.47	20.69	21.79
$q-p=5$	1%	22.60	23.94	25.86	26.53	27.62	29.26	29.80	30.92
	5%	18.49	19.76	21.01	22.03	23.22	24.20	25.67	26.65
	10%	16.53	17.64	18.79	19.87	21.15	21.98	23.46	24.35
$q-p=6$	1%	25.60	26.63	28.26	29.13	30.64	31.65	32.58	34.36
	5%	20.98	22.34	23.41	24.52	25.72	26.70	28.05	29.17
	10%	18.84	20.09	21.13	22.17	23.50	24.38	25.70	26.75
$q-p=7$	1%	28.35	29.95	31.08	32.30	33.42	25.09	35.94	37.29
	5%	23.72	24.79	26.24	27.13	28.56	29.55	30.66	31.89
	10%	21.35	22.40	23.75	24.77	25.96	27.02	28.08	29.24
$q-p=8$	1%	31.17	32.41	32.99	34.97	36.11	36.97	38.25	39.45
	5%	26.04	27.26	28.20	29.73	30.67	31.89	32.92	34.20
	10%	23.66	24.67	25.88	27.15	28.08	29.33	30.45	31.51

Table 4: Asymptotic critical values for $\text{AJSS}_n(\pi)$ with $\pi \in \Pi = [.2, .8]$

		$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$q-p=1$	1%	5.82	6.77	7.92	8.98	9.71	10.72	11.50	12.19
	5%	4.09	5.04	5.97	6.92	7.66	8.61	9.34	10.06
	10%	3.34	4.28	5.10	5.97	6.75	7.56	8.32	8.94
$q-p=2$	1%	8.08	8.88	9.82	10.71	11.58	12.49	13.14	13.87
	5%	5.97	6.81	7.61	8.50	9.26	9.97	10.84	11.56
	10%	5.03	5.90	6.60	7.39	8.13	8.90	9.70	10.36
$q-p=3$	1%	9.72	10.72	11.40	12.46	13.15	13.88	15.06	15.59
	5%	7.53	8.39	9.21	9.97	10.75	11.40	12.35	12.97
	10%	6.57	7.35	8.07	8.88	9.61	10.22	11.10	11.73
$q-p=4$	1%	11.20	12.40	12.97	13.90	14.76	15.41	16.12	17.08
	5%	9.01	9.93	10.60	11.44	12.17	12.82	13.59	14.40
	10%	7.97	8.74	9.51	10.24	10.89	11.62	12.37	13.13
$q-p=5$	1%	13.03	13.95	14.83	15.58	16.16	16.93	17.90	18.44
	5%	10.62	11.40	12.22	12.88	13.65	14.25	15.15	15.70
	10%	9.47	10.23	10.97	11.64	12.43	12.95	13.88	14.39
$q-p=6$	1%	14.43	15.18	16.29	16.83	17.71	18.46	18.91	20.01
	5%	11.98	12.80	13.46	14.21	15.00	15.64	16.44	17.09
	10%	10.80	11.51	12.19	12.93	13.72	14.28	15.10	15.70
$q-p=7$	1%	16.10	17.04	17.89	18.64	19.50	20.25	20.84	21.72
	5%	13.45	14.29	15.09	15.75	16.52	17.21	17.87	18.56
	10%	12.20	12.90	13.65	14.35	14.99	15.64	16.33	17.04
$q-p=8$	1%	17.64	18.39	19.00	20.14	20.90	21.55	22.25	23.00
	5%	14.81	15.54	16.28	17.07	17.68	18.46	19.03	19.75
	10%	13.42	14.09	14.87	15.64	16.24	16.98	17.60	18.32

Table 5: Asymptotic critical values for $\text{EJSS}_n(\pi)$ with $\pi \in \Pi = [.15, .85]$

		$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$q-p=1$	1%	5.81	6.77	7.89	8.96	9.73	10.58	11.34	12.18
	5%	4.09	5.01	5.92	6.89	7.58	8.48	9.29	10.01
	10%	3.34	4.23	5.04	5.91	6.69	7.45	8.19	8.82
$q-p=2$	1%	8.11	8.79	9.76	10.69	11.48	12.42	13.11	13.91
	5%	5.92	6.77	7.55	8.38	9.13	9.90	10.77	11.40
	10%	4.97	5.82	6.57	7.31	8.06	8.77	9.55	10.20
$q-p=3$	1%	9.70	10.63	11.32	12.36	13.15	13.75	15.00	15.43
	5%	7.43	8.32	9.12	9.93	10.65	11.28	12.23	12.90
	10%	6.49	7.28	8.01	8.78	9.52	10.11	11.01	11.60
$q-p=4$	1%	11.16	12.36	12.84	13.82	14.70	15.28	16.08	16.96
	5%	8.96	9.83	10.50	11.33	12.09	12.76	13.48	14.28
	10%	7.88	8.66	9.37	10.11	10.81	11.49	12.23	12.95
$q-p=5$	1%	13.00	13.86	14.83	15.49	16.16	16.70	17.78	18.21
	5%	10.51	11.33	12.11	12.80	13.55	14.20	14.99	15.64
	10%	9.36	10.13	10.83	11.56	12.28	12.82	13.69	14.26
$q-p=6$	1%	14.38	15.03	16.12	16.73	17.59	18.44	18.97	19.85
	5%	11.89	12.73	13.36	14.09	14.86	15.42	16.30	16.95
	10%	10.72	11.42	12.07	12.78	13.58	14.15	14.98	15.55
$q-p=7$	1%	15.96	16.94	17.84	18.60	19.31	20.23	20.79	21.71
	5%	13.38	14.19	14.96	15.63	16.42	17.14	17.75	18.47
	10%	12.10	12.76	13.53	14.25	14.86	15.52	16.23	16.89
$q-p=8$	1%	17.53	18.24	18.86	19.93	20.84	21.50	22.14	22.84
	5%	14.77	15.43	16.12	16.92	17.52	18.31	18.94	19.61
	10%	13.32	13.99	14.72	15.48	16.09	16.81	17.49	18.18

Table 6: Asymptotic critical values for $\text{EJSS}_n(\pi)$ with $\pi \in \Pi = [.2, .8]$

		$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$q-p=1$	1%	14.76	17.47	20.24	22.68	24.61	26.82	27.97	29.98
	5%	11.04	13.62	15.97	18.16	19.91	22.05	23.78	25.45
	10%	9.41	11.85	13.96	16.09	17.89	19.71	21.45	23.01
$q-p=2$	1%	17.85	19.94	22.02	24.77	26.05	28.35	30.18	31.98
	5%	13.35	15.71	17.68	19.84	21.84	23.44	25.34	27.02
	10%	11.51	13.73	15.73	17.62	19.48	21.11	22.97	24.48
$q-p=3$	1%	19.39	22.15	24.15	26.29	28.45	29.83	32.47	33.23
	5%	15.20	17.67	19.56	21.66	23.38	24.93	27.05	28.47
	10%	13.26	15.63	17.48	19.32	21.12	22.42	24.38	25.95
$q-p=4$	1%	21.13	24.16	25.67	27.78	29.92	31.48	33.37	35.00
	5%	17.19	19.23	21.15	22.12	24.73	26.28	28.28	29.89
	10%	15.10	17.03	18.86	20.83	22.45	23.96	25.78	27.33
$q-p=5$	1%	23.45	25.71	28.09	29.56	31.79	33.01	35.07	37.07
	5%	19.06	20.95	23.00	24.53	26.55	27.76	29.77	31.40
	10%	16.84	18.85	20.62	22.22	24.04	25.42	27.28	28.80
$q-p=6$	1%	25.48	26.99	29.19	31.26	33.13	34.45	36.32	38.27
	5%	20.56	22.58	24.46	26.17	28.09	29.32	31.35	32.92
	10%	18.29	20.23	22.15	23.72	25.58	26.89	28.70	30.22
$q-p=7$	1%	27.31	28.98	30.93	33.02	34.88	37.02	38.22	40.33
	5%	22.47	24.27	26.33	27.86	29.69	31.33	32.74	34.65
	10%	20.11	21.82	23.86	25.14	26.92	28.68	30.19	31.84
$q-p=8$	1%	28.79	31.03	32.74	34.75	37.24	37.70	40.19	42.20
	5%	23.77	25.81	27.68	29.49	31.24	32.66	34.11	35.89
	10%	21.46	23.32	25.08	26.87	28.48	29.95	31.24	33.02

Table 7: Asymptotic critical values for $\text{SPR}_n(\pi)$ with $\pi \in \Pi = [.15, .85]$

		$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$q-p=1$	1%	14.37	17.10	19.64	21.92	24.12	26.08	27.47	29.72
	5%	10.58	13.21	15.44	17.67	19.50	21.44	23.20	24.69
	10%	8.96	11.37	13.56	15.58	17.29	19.15	20.79	22.28
$q-p=2$	1%	17.49	19.79	21.58	24.17	25.43	27.97	29.46	31.37
	5%	12.88	15.28	17.22	19.24	21.20	22.68	24.79	26.17
	10%	11.03	13.22	15.23	17.09	18.85	20.55	22.31	23.72
$q-p=3$	1%	18.97	21.71	23.53	26.29	27.91	29.13	32.07	32.73
	5%	14.73	17.13	19.18	21.66	22.88	24.34	26.53	27.92
	10%	12.89	15.08	16.90	19.32	20.64	21.83	23.85	25.27
$q-p=4$	1%	20.72	23.79	25.09	27.20	29.31	30.60	32.75	34.58
	5%	16.77	18.70	20.54	22.52	24.10	25.75	27.57	29.15
	10%	14.52	16.47	18.28	20.24	21.79	23.39	25.03	26.60
$q-p=5$	1%	22.96	25.25	27.53	28.84	31.41	32.55	34.28	36.70
	5%	18.55	20.38	22.36	24.01	25.97	27.27	29.11	30.81
	10%	16.32	18.21	20.05	21.64	23.51	24.92	26.60	28.15
$q-p=6$	1%	24.79	26.36	28.50	30.45	32.48	33.80	35.69	37.32
	5%	19.98	22.05	23.78	25.47	27.44	28.62	30.75	32.25
	10%	17.80	19.58	21.62	23.08	24.95	26.29	28.19	29.56
$q-p=7$	1%	26.69	28.35	30.53	32.48	34.45	36.46	37.55	39.82
	5%	21.80	23.64	25.74	27.10	28.94	30.75	32.15	33.79
	10%	19.50	21.28	23.13	24.53	26.34	28.09	29.52	31.01
$q-p=8$	1%	28.32	30.65	32.19	34.17	36.48	36.88	39.63	41.57
	5%	23.29	25.26	27.07	28.81	30.61	31.99	33.47	35.20
	10%	20.95	22.67	24.45	26.19	27.80	29.25	30.54	32.31

Table 8: Asymptotic critical values for $\text{SPR}_n(\pi)$ with $\pi \in \Pi = [.2, .8]$

		$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$q-p=1$	1%	7.09	8.72	10.40	12.03	13.35	14.62	16.34	17.39
	5%	4.89	6.45	7.73	9.18	10.37	11.87	13.06	14.36
	10%	3.85	5.32	6.62	7.97	9.16	10.39	11.73	12.76
$q-p=2$	1%	9.86	10.80	12.27	13.90	15.36	16.45	17.82	19.18
	5%	6.63	8.05	9.27	10.68	11.91	13.19	14.53	15.74
	10%	5.41	6.82	8.05	9.33	10.47	11.69	12.97	14.11
$q-p=3$	1%	11.76	12.61	13.95	15.35	16.73	17.86	19.45	20.57
	5%	8.41	9.49	10.88	12.03	13.41	14.54	16.01	17.03
	10%	6.80	8.19	9.44	10.64	11.86	12.93	14.24	15.42
$q-p=4$	1%	12.75	14.31	15.09	16.44	17.88	19.23	20.92	21.84
	5%	9.64	10.88	12.17	13.36	14.65	15.83	17.17	18.30
	10%	8.20	9.42	10.58	11.89	13.11	14.24	15.41	16.58
$q-p=5$	1%	14.30	15.58	17.19	18.16	19.89	21.17	21.93	23.51
	5%	11.11	12.45	13.71	14,71	16.18	17.05	18.50	19.66
	10%	9.60	10.98	12.05	13.21	14.51	15.49	16.70	17.89
$q-p=6$	1%	15.73	17.03	18.21	19.69	21.08	21.94	23.25	24.90
	5%	12.37	13.76	15.09	15.98	17.47	18.40	19.79	21.05
	10%	10.85	12.07	13.27	14.41	15.57	16.75	17.96	19.02
$q-p=7$	1%	17.36	18.69	19.95	21.08	22.79	23.62	24.87	26.39
	5%	13.86	14.99	16.51	17.40	18.81	19.98	21.21	22.48
	10%	12.15	13.31	14.76	15.75	16.86	18.12	19.27	20.39
$q-p=8$	1%	18.79	20.23	21.28	22.97	24.03	24.89	26.72	28.10
	5%	15.21	16.44	17.58	18.94	20.04	21.26	22.14	23.75
	10%	13.36	14.52	15.74	17.03	18.19	19.35	20.26	21.59

Table 9: Asymptotic critical values for $\text{APR}_n(\pi)$ with $\pi \in \Pi = [.15, .85]$

		$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$q-p=1$	1%	7.31	9.12	10.71	12.44	12.80	15.08	16.69	17.85
	5%	5.02	6.63	7.94	9.42	10.59	12.12	13.39	14.62
	10%	3.96	5.43	6.76	8.07	9.34	10.61	11.91	12.99
$q-p=2$	1%	10.14	11.13	12.57	14.27	15.67	16.95	18.28	19.71
	5%	6.77	8.25	9.52	10.82	12.18	13.42	14.80	16.03
	10%	5.50	6.94	8.21	9.47	10.67	11.89	13.17	14.28
$q-p=3$	1%	11.40	12.94	14.35	15.68	17.19	18.20	19.93	21.12
	5%	8.37	9.70	11.08	12.32	13.71	14.84	16.26	17.35
	10%	7.08	8.32	9.62	10.80	12.05	13.14	14.48	15.60
$q-p=4$	1%	13.05	14.63	15.38	16.91	18.47	19.61	21.45	22.30
	5%	9.79	11.04	12.38	13.62	14.90	16.15	17.55	18.64
	10%	8.30	9.54	10.77	12.09	13.25	14.44	15.65	16.83
$q-p=5$	1%	14.53	15.92	17.52	18.75	20.30	21.56	22.40	24.07
	5%	11.34	12.71	13.90	14.92	16.41	17.34	18.81	20.04
	10%	9.74	11.10	12.22	13.35	14.70	15.73	16.91	18.15
$q-p=6$	1%	16.15	17.39	18.61	20.19	21.57	22.34	23.89	25.17
	5%	12.57	13.98	15.31	16.28	17.79	18.71	20.11	21.39
	10%	11.02	12.21	13.47	14.61	15.82	16.99	18.19	19.24
$q-p=7$	1%	17.62	19.06	20.33	21.44	23.25	24.15	25.46	27.10
	5%	14.09	15.25	16.71	17.64	19.10	20.33	21.52	22.76
	10%	12.32	13.44	14.92	15.88	17.12	18.36	19.45	20.62
$q-p=8$	1%	19.16	20.65	21.69	23.28	24.44	25.47	27.01	28.59
	5%	15.44	16.63	17.82	19.26	20.43	21.57	22.46	23.98
	10%	13.49	14.67	15.88	17.21	18.38	19.55	20.51	21.79

Table 10: Asymptotic critical values for $\text{APR}_n(\pi)$ with $\pi \in \Pi = [.2, .8]$

		$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$q-p=1$	1%	4.76	5.75	6.99	8.05	8.92	9.85	10.62	11.45
	5%	3.19	4.20	5.14	6.12	6.87	7.84	8.59	9.34
	10%	2.48	3.47	4.34	5.21	5.99	6.87	7.58	8.26
$q-p=2$	1%	6.17	7.05	7.98	9.16	9.76	10.75	11.60	12.47
	5%	4.22	5.19	5.95	6.89	7.74	8.48	9.40	10.12
	10%	3.44	4.31	5.13	5.94	6.76	7.51	8.30	9.00
$q-p=3$	1%	6.93	8.04	8.80	9.82	10.80	11.49	12.68	13.13
	5%	5.09	6.05	6.91	7.72	8.55	9.25	10.21	10.91
	10%	4.25	5.14	5.96	6.75	7.55	8.15	9.05	9.71
$q-p=4$	1%	7.83	9.02	9.62	10.58	11.50	12.23	13.05	13.86
	5%	5.99	6.82	7.62	8.46	9.24	10.00	10.82	11.52
	10%	5.02	5.83	6.62	7.47	8.20	8.87	9.68	10.36
$q-p=5$	1%	8.85	9.70	10.88	11.42	12.55	13.06	13.99	14.89
	5%	6.86	7.66	8.43	9.18	10.12	10.71	11.59	12.31
	10%	5.89	6.71	7.44	8.15	8.96	9.61	10.42	11.15
$q-p=6$	1%	9.67	10.40	11.38	12.26	13.09	13.81	14.66	15.41
	5%	7.54	8.41	9.19	9.96	10.83	11.35	12.32	13.03
	10%	6.53	7.32	8.21	8.89	9.67	10.27	11.16	11.81
$q-p=7$	1%	10.61	11.47	12.28	13.11	14.08	15.00	15.63	16.59
	5%	8.38	9.15	10.12	10.74	11.59	12.34	12.98	13.84
	10%	7.31	8.11	8.97	9.54	10.34	11.10	11.84	12.52
$q-p=8$	1%	11.40	12.35	13.10	14.01	14.95	15.32	16.44	17.40
	5%	9.08	9.89	10.74	11.53	12.31	13.02	13.60	14.47
	10%	8.02	8.75	9.59	10.34	11.05	11.76	12.32	13.14

Table 11: Asymptotic critical values for $\text{EPR}_n(\pi)$ with $\pi \in \Pi = [.15, .85]$

		$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
$q-p=1$	1%	4.77	5.77	6.95	8.00	8.89	9.75	10.57	11.51
	5%	3.15	4.20	5.11	6.06	6.82	7.75	8.52	9.26
	10%	2.47	3.42	4.31	5.14	5.94	6.78	7.50	8.17
$q-p=2$	1%	6.14	7.06	7.94	9.07	9.79	10.63	11.52	12.41
	5%	4.17	5.14	5.94	6.82	7.63	8.37	9.36	10.00
	10%	3.39	4.27	5.11	5.90	6.67	7.43	8.22	8.87
$q-p=3$	1%	6.93	8.00	8.71	9.80	10.82	11.45	12.67	13.15
	5%	5.03	6.00	6.84	7.61	8.49	9.15	10.14	10.80
	10%	4.22	5.09	5.91	6.66	7.47	8.09	8.92	9.60
$q-p=4$	1%	7.72	8.99	9.64	10.48	11.38	12.00	13.01	13.76
	5%	5.95	6.76	7.52	8.37	9.19	9.93	10.69	11.43
	10%	4.96	5.76	6.52	7.36	8.10	8.79	9.54	10.26
$q-p=5$	1%	8.79	9.64	10.79	11.33	12.48	13.01	13.83	14.83
	5%	6.80	7.59	8.38	9.13	10.03	10.61	11.49	12.25
	10%	5.83	6.63	7.37	8.06	8.87	9.50	10.30	11.01
$q-p=6$	1%	9.59	10.41	11.20	12.05	13.04	13.73	14.53	15.27
	5%	7.51	8.32	9.13	9.86	10.72	11.23	12.24	12.89
	10%	6.47	7.24	8.12	8.74	9.55	10.17	11.06	11.68
$q-p=7$	1%	10.50	11.38	12.20	12.96	14.11	14.94	15.46	16.51
	5%	8.29	9.04	10.01	10.60	11.41	12.28	12.93	13.76
	10%	7.26	8.00	8.87	9.44	10.29	11.02	11.73	12.38
$q-p=8$	1%	11.40	12.28	12.95	13.97	14.82	15.23	16.46	17.27
	5%	9.04	9.75	10.62	11.44	12.23	12.91	13.53	14.32
	10%	7.97	8.69	9.45	10.21	10.95	11.65	12.21	13.01

Table 12: Asymptotic critical values for $\text{EPR}_n(\pi)$ with $\pi \in \Pi = [.2, .8]$

		$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
SW	1%	12.41	15.55	18.01	20.52	22.66	25.02	26.51	28.35
	5%	8.75	11.67	14.10	16.48	18.43	20.35	22.16	23.84
	10%	7.14	9.92	12.33	14.44	16.40	18.01	20.05	21.41
EW	1%	3.35	4.79	5.82	7.08	8.00	9.05	9.84	10.72
	5%	2.01	3.24	4.28	5.26	6.05	6.98	7.83	8.56
	10%	1.47	2.54	3.51	4.39	5.29	6.00	6.90	7.51
AW	1%	4.53	6.62	8.44	10.09	11.54	13.29	14.61	15.86
	5%	2.81	4.65	6.16	7.64	9.02	10.28	11.79	12.99
	10%	2.13	3.72	5.19	6.56	7.82	9.00	10.45	11.40

Table 13: Asymptotic critical values for $\mathrm{SW}_n(\pi)$, $\mathrm{EW}_n(\pi)$ and $\mathrm{AW}_n(\pi)$ with $\pi \in \Pi = [.15, .85]$

		$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$
SW	1%	11.59	15.19	17.50	20.11	22.08	24.17	25.96	27.98
	5%	8.37	11.20	13.71	16.06	17.76	19.74	21.67	23.17
	10%	6.75	9.49	11.90	13.93	15.87	17.46	19.40	20.69
EW	1%	3.31	4.76	5.85	7.05	7.86	8.91	9.82	10.69
	5%	1.99	3.22	4.24	5.23	6.03	6.86	7.80	8.51
	10%	1.46	2.53	3.49	4.34	5.21	5.90	6.82	7.37
AW	1%	4.82	6.96	8.81	10.52	11.93	13.66	15.05	16.44
	5%	2.91	4.83	6.36	7.85	9.25	10.51	12.06	13.31
	10%	2.20	3.83	5.33	6.71	8.00	9.12	10.67	11.62

Table 14: Asymptotic critical values for $\mathrm{SW}_n(\pi)$, $\mathrm{EW}_n(\pi)$ and $\mathrm{AW}_n(\pi)$ with $\pi \in \Pi = [.2, 8]$

	1.1.63-26.1.93	1.1.63-31.12.72	1.1.73-31.12.82	1.1.83-26.1.93
mean	.0255	.0252	.007	.044
std.dev	.890	.610	.943	1.05
skewness	-2.29	.227	.227	-4.34
excess kurtosis	63.0	4.31	1.53	94.3
maximum	8.71	-2.85	-4.08	-22.9
minimum	-22.9	4.90	4.65	8.71
Normality χ^2	36159	794	159	6583
# observations	7565	2493	2527	2545

Table 15: Salient Features of the adjusted returns

	Test statistic	$p-$value	$\widehat{\pi}$
SPR	25.2765	0.000	March 3th 1970 (34)
EPR	31.992	0.000	
APR	27.272	0.000	
SPR*	94.2785	0.000	March 3th 1970 (34)
EPR*	42.334	0.000	
APR*	40.448	0.000	
SW	25.2765	0.002	June 8th 1970 (43)
EW	8.8009	0.002	
AW	12.147	0.003	
SLR	68.6677	0.000	March 3th 1970 (34)
ELR	29.374	0.000	
ALR	22.718	0.000	
SLM	70.7615	0.000	March 3th 1970 (34)
ELM	30.665	0.000	
ALM	22.718	0.000	
SJSS	87.7955	0.000	March 3th 1970 (34)
EJSS	38.907	0.000	
AJSS	36.893	0.000	
SHSS	45.6112	0.000	June 8th 1970 (43)
EHSS	20.204	0.000	
AHSS	31.250	0.000	

Table 16: Test statistics for unknown breakpoint $\pi \in \Pi = [.2, .8]$

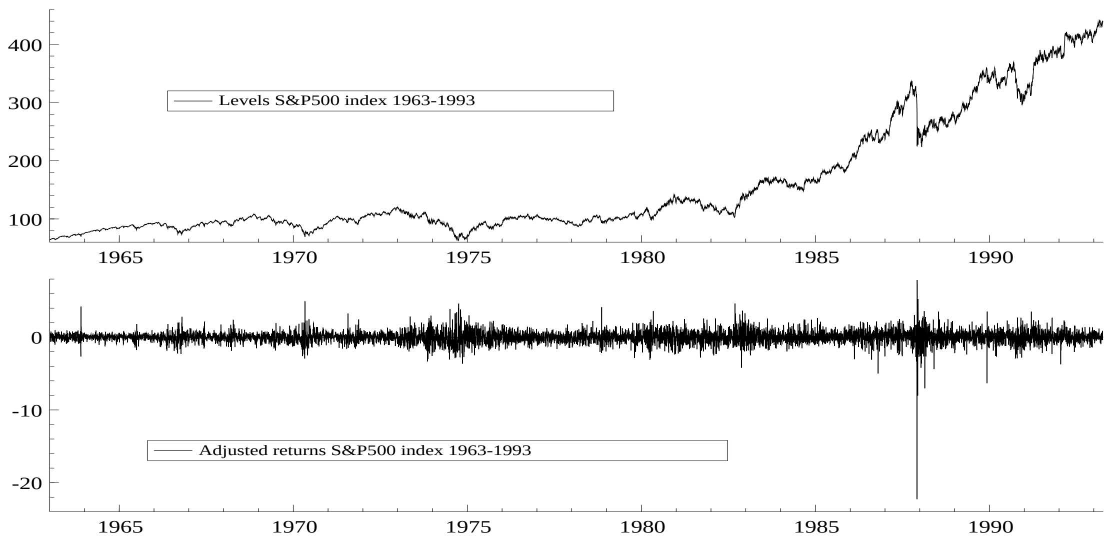

Figure 1: Levels and returns of S&P500, 1963-1993

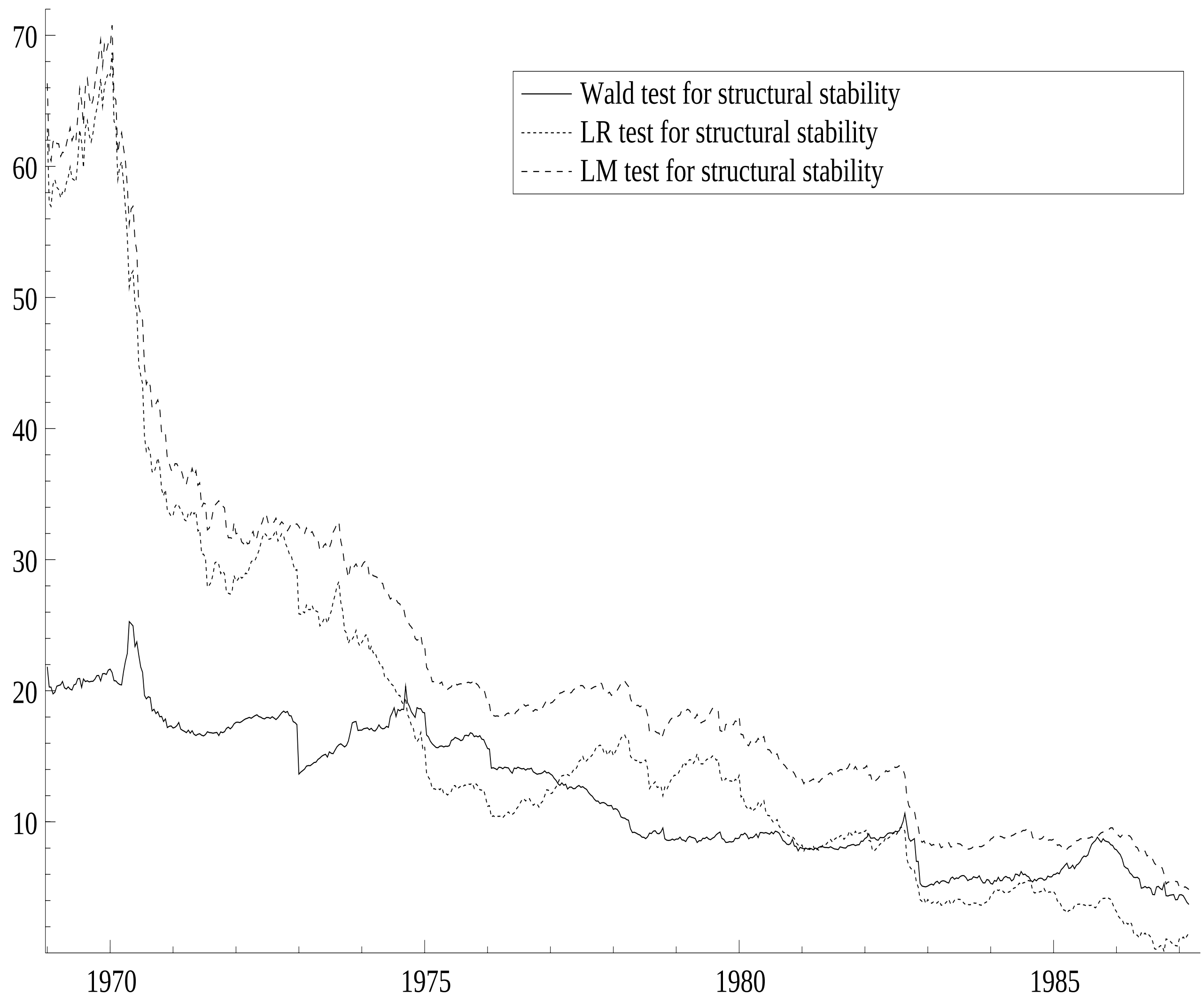

Figure 2: Plot of $\mathsf{W}_n(\pi)$, $\mathsf{LR}_n(\pi)$ and $\mathsf{LM}_n(\pi)$

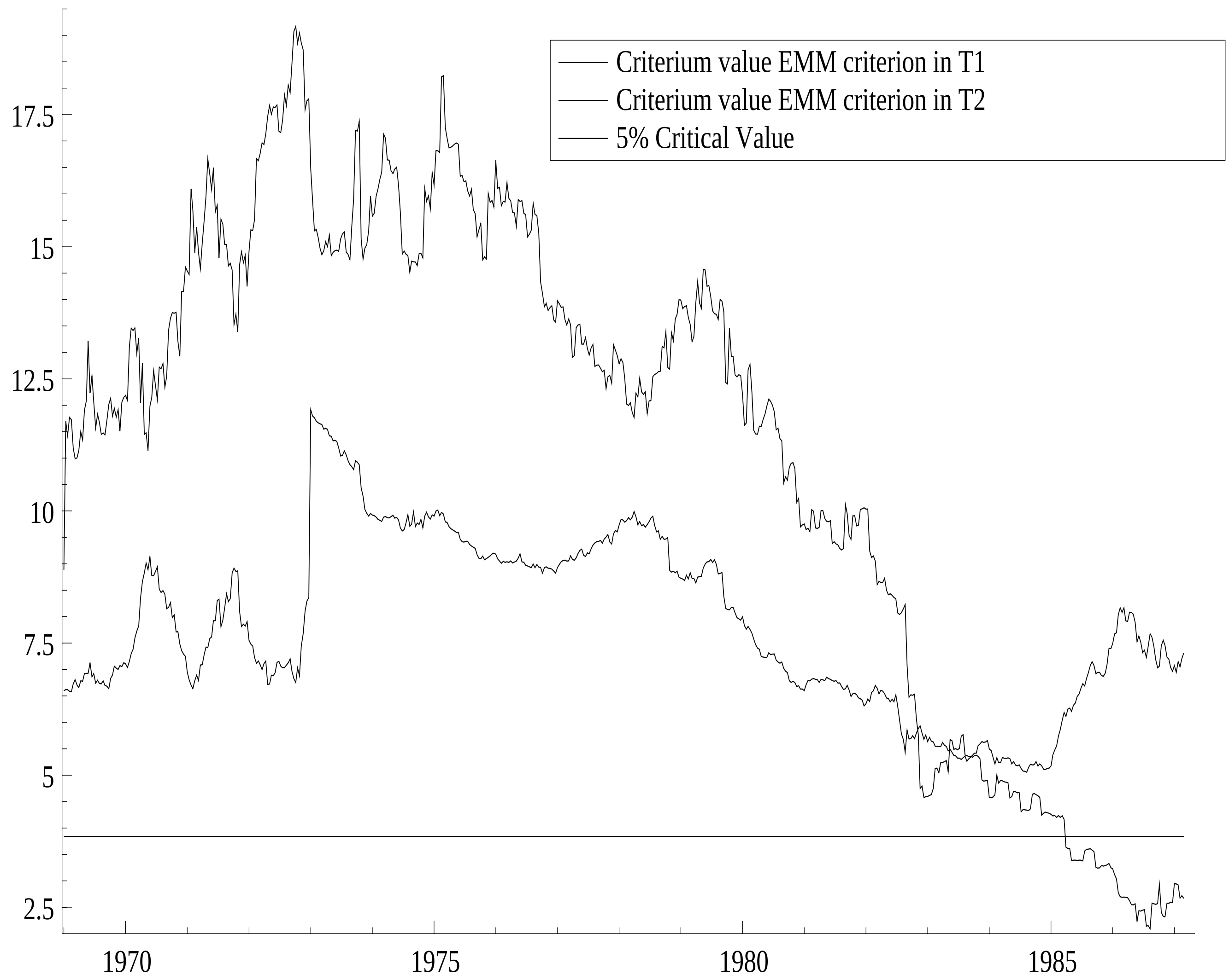

Figure 3: Plot of criterion values in $T_1(\pi)$ and $T_2(\pi)$

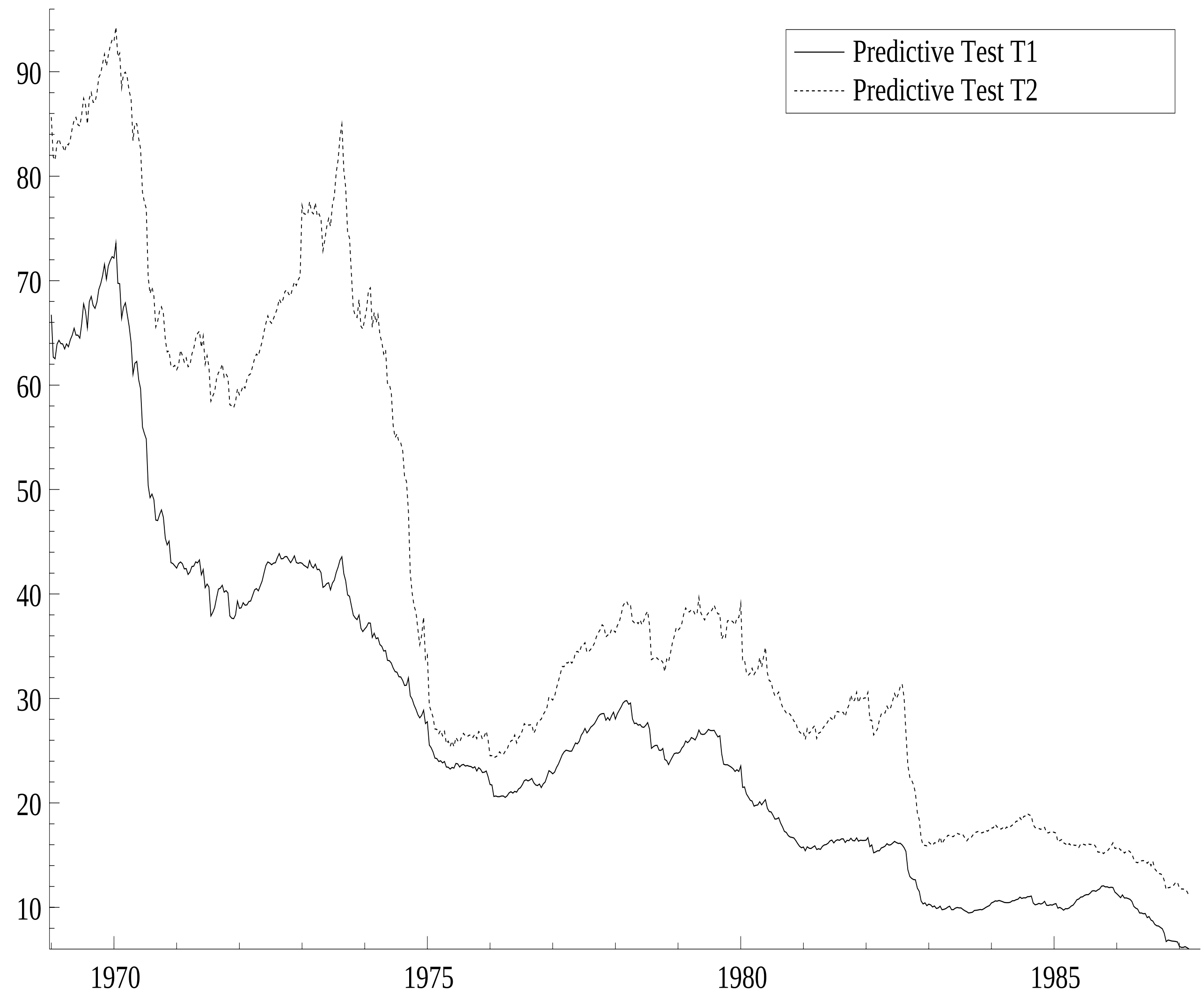

Figure 4: Plot of $\mathsf{PR}_n(\pi)$ and $\mathsf{PR}^*_n(\pi)$

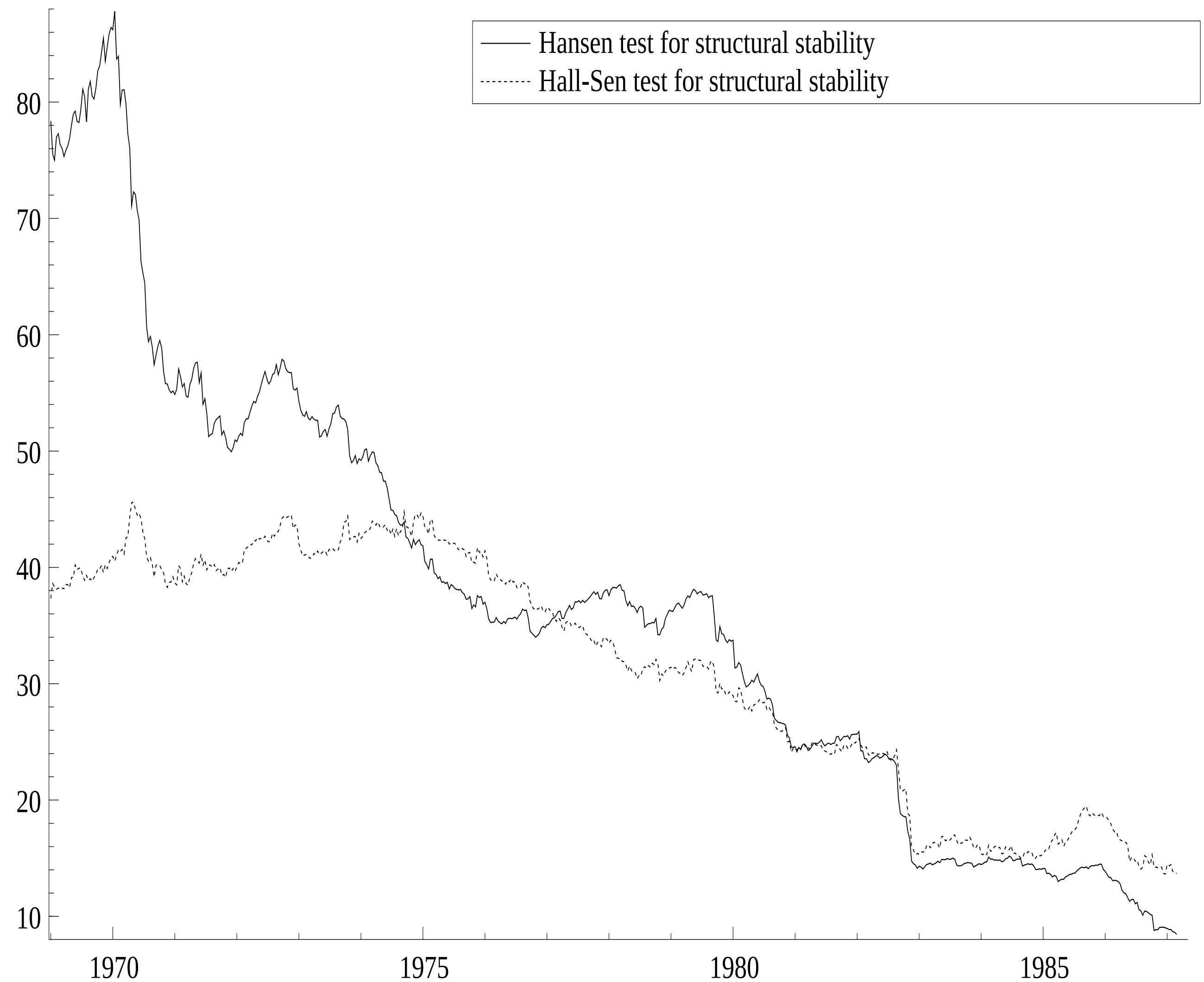

Figure 5: Plot of $\mathsf{JSS}_n(\pi)$ and $\mathsf{HSS}_n(\pi)$

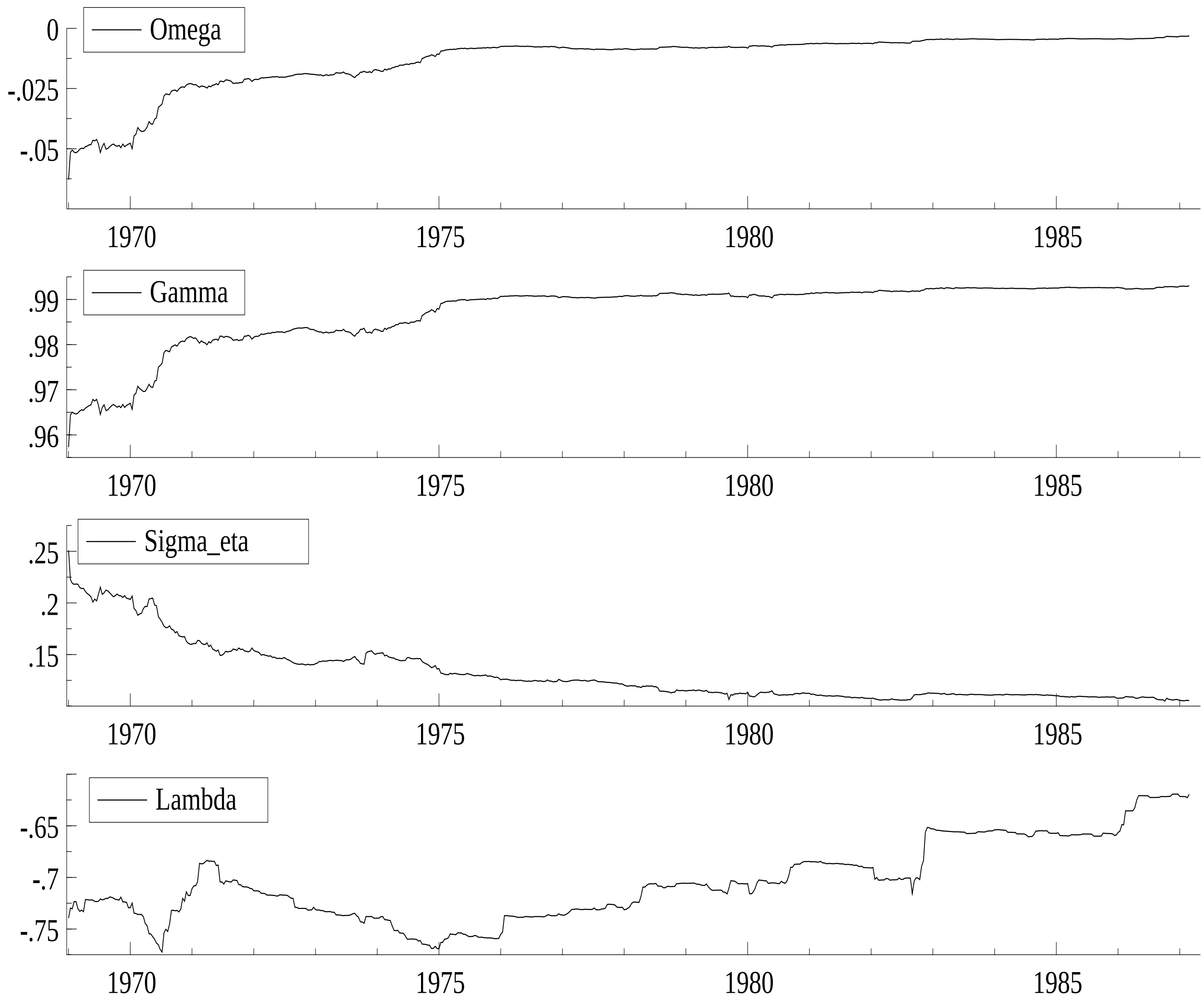

Figure 6: Time-varying estimates of SV model for data in $T_1(\pi)$

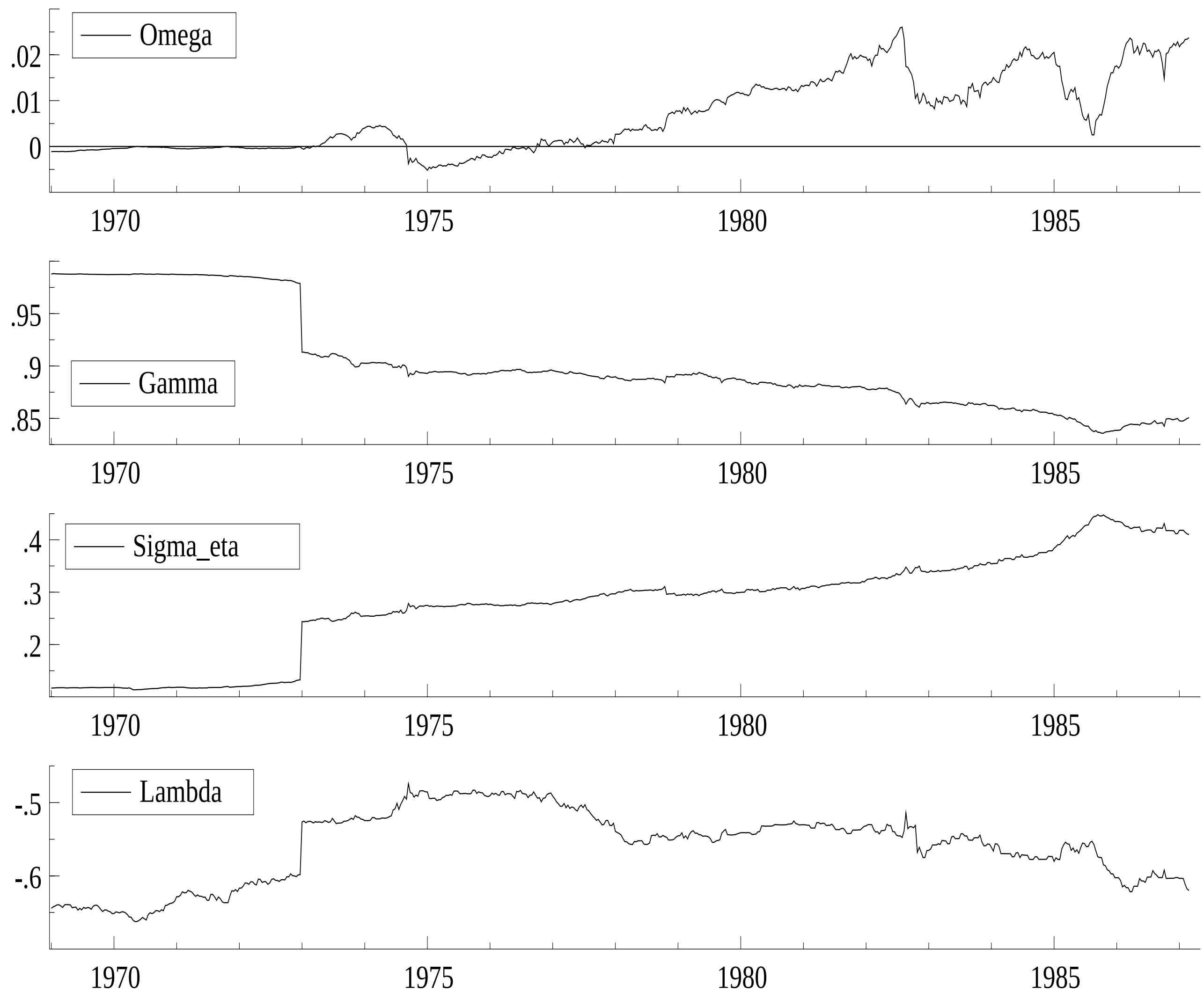

Figure 7: Time-varying estimates of SV model for data in $T_2(\pi)$

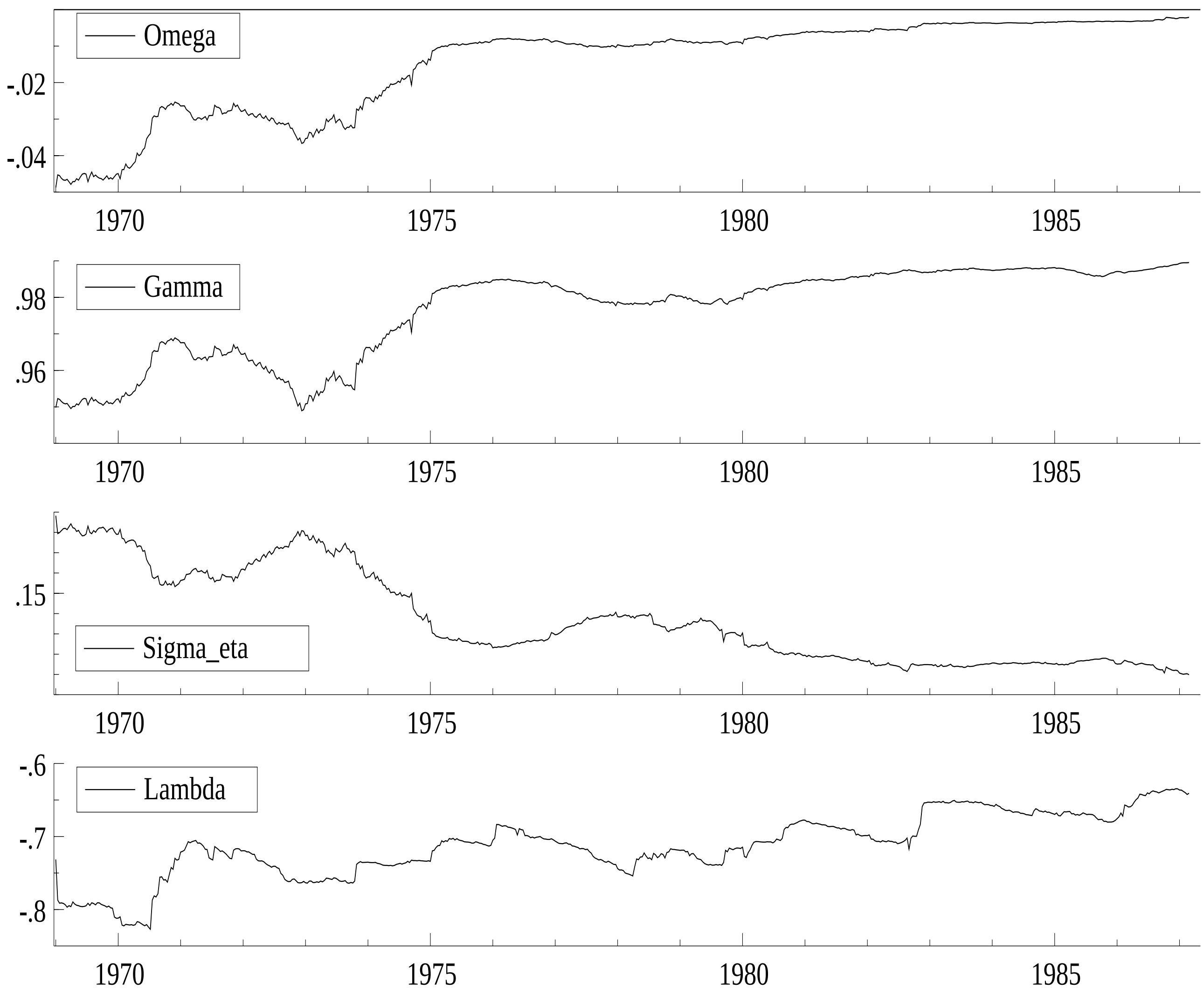

Figure 8: Time-varying restricted estimates of SV model

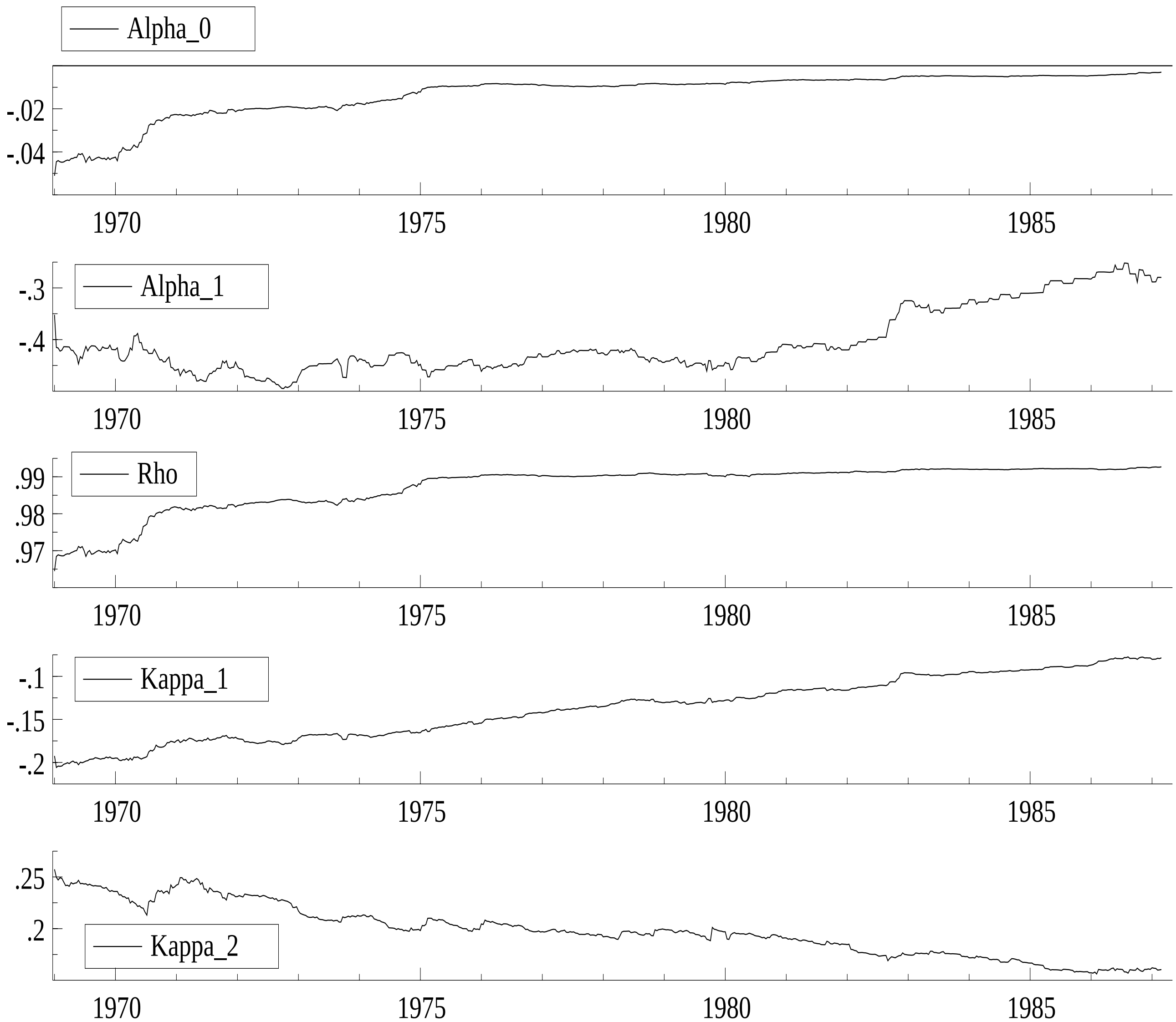

Figure 9: Time-varying estimates of EGARCH model for $T_1(\pi)$

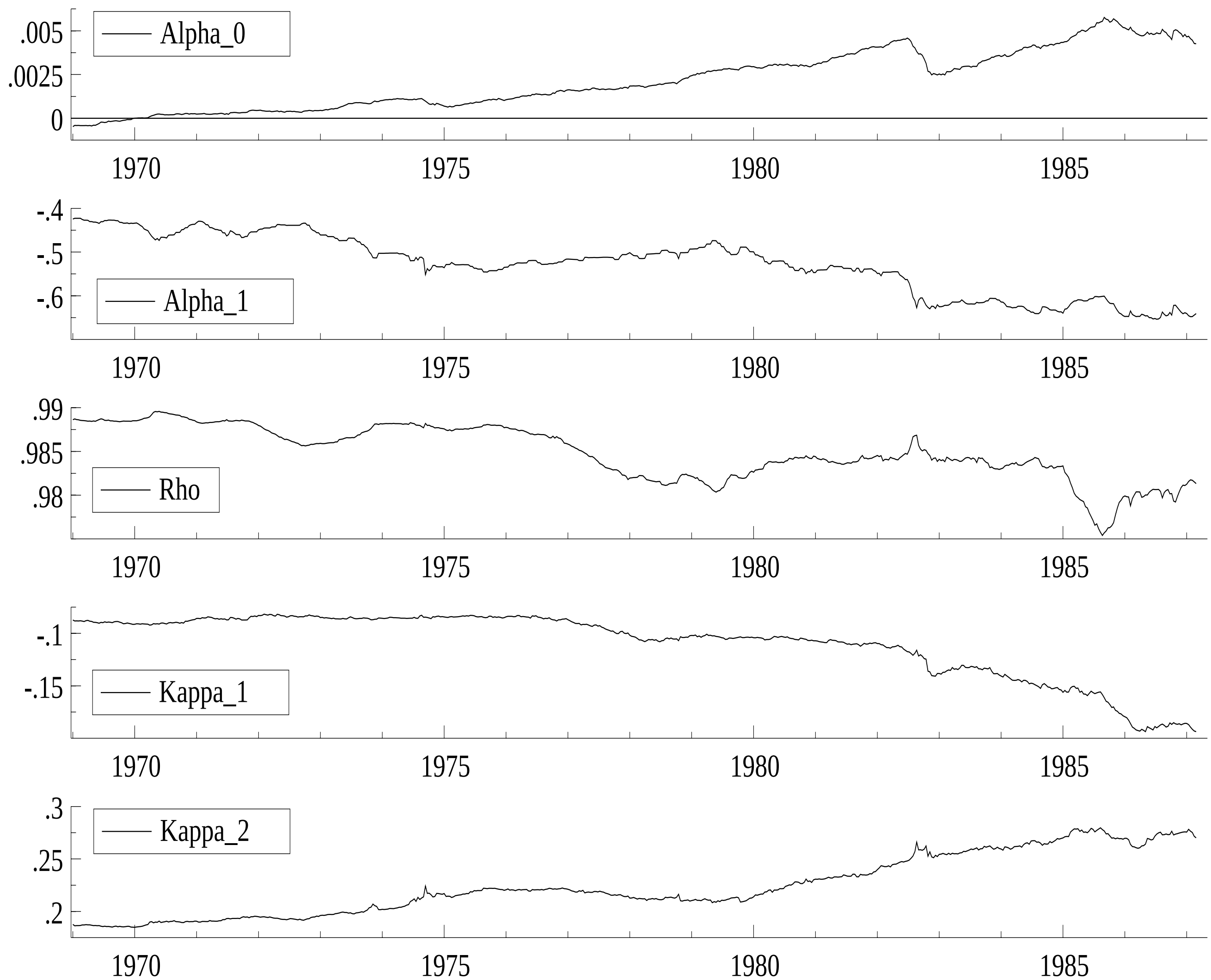

Figure 10: Time-varying estimates of EGARCH model for $T_2(\pi)$