Approximation by penultimate extreme value distributions

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1 Introduction and statement of results.

In their basic paper on extreme value distributions R.A. Fisher and L.H.C. Tippett [1928] remark that if one approximates the distribution of the successive maxima of normal samples not by the limit distribution $\exp -e^{-x}$ but by a sequence of other extreme value distributions converging to the limit distribution, the approximation is asymptotically improved. This sequence of approximating extreme value distributions they called penultimate distributions.

The subject has been taken up by a number of authors. We mention C.W. Anderson in his thesis [1971], J.P. Cohen [1982], M.I. Gomes [1984], Gomes and Pestana [1987] and R.-D. Reiss [1989].

The present paper aims at answering the following questions:

1. In what circumstances can the convergence rate be improved by the use of penultimate approximations?

2. What are the precise conditions for improvement and what is the new convergence rate?

It is by now well understood that the proper framework for convergence rate results is second order theory. This means in short that while for convergence in distribution for maxima a first order expansion suffices, for a rate of convergence result an expansion of second order is needed. The second order theory has been worked out in full generality (de Haan and

Stadtmüller (1996)). However this leads to the need to consider many different cases and to calculate complicated integrals. This would be rather tedious particularly in the case of penultimate approximations since in that case even third order theory is needed. So rather than to proceed in full generality we shall assume sufficient differentiability (i.e. von Mises type conditions) and this allows us to proceed in a unified and relatively elegant way.

Suppose $X_1, X_2, \ldots$ are i.i.d. random variables with common distribution function F. Assume that the sequence of sample maxima $\max(X_1, X_2, \ldots, X_n)$, suitably normalized, converges to one of the extreme-value distributions

$$G_\gamma(x) := \exp -(1+\gamma x)^{-1/\gamma} \qquad (\gamma \in \mathbb{R},\ 1+\gamma x > 0).$$

We assume that von Mises' sufficient conditions for the convergence of $F^n(a_n x + b_n)$ to $G_\gamma(x)$ are fulfilled i.e. with

$$u(x) := -\log - \log F$$

and

$$v(t) := u^{\leftarrow}(t)$$

(the arrow denoting the inverse function) we assume

$$\lim_{t \uparrow x^*} \left(\frac{1}{u'}\right)'(t) = \gamma \tag{1.1}$$

where $x^* := \sup\{x \mid F(x) < 1\}$ or equivalently

$$\lim_{t\to\infty} \frac{v''(t)}{v'(t)} = \gamma. \tag{1.2}$$

Relations (1.1) or (1.2) imply

$$\lim_{t\uparrow x^*} u(t + \frac{x}{u'(t)}) - u(t) = \gamma^{-1}\log(1+\gamma x) \tag{1.3}$$

for $1+\gamma x > 0$ or equivalently

$$\lim_{t\to\infty} \frac{v(t+x)-v(t)}{v'(t)} = \frac{e^{\gamma x}-1}{\gamma} \tag{1.4}$$

for all x. This is first order theory. Next we discuss second order conditions.

Write for $t < x^*$

$$\varphi(t) := \left(\frac{1}{u'}\right)'(t) - \gamma.$$

Suppose that moreover for some $\rho \le 0$

$$\lim_{t\uparrow x^*} \frac{\varphi'(t)}{u'(t)\varphi(t)} = \rho \tag{1.5}$$

or equivalently

$$\lim_{t\to\infty} \frac{v'''(t)-\gamma v''(t)}{v''(t)-\gamma v'(t)} = \gamma + \rho. \tag{1.6}$$

Then

$$\lim_{t\to\infty} \frac{\frac{v''(t+x)}{v'(t+x)} - \gamma}{\frac{v''(t)}{v'(t)} - \gamma} = e^{\rho x} \tag{1.7}$$

for all x and second order versions of (1.3) and (1.4) hold:

$$\lim_{t\uparrow x^*} \frac{u\left(t+\frac{x}{u'(t)}\right) - u(t) - \gamma^{-1}\log(1+\gamma x)}{\varphi(t)} = -\int_0^x (1+\gamma s)^{-2}\int_0^s (1+\gamma p)^{\rho/\gamma}dpds. \tag{1.8}$$

locally uniformly for $1+\gamma x>0$ or equivalently

$$\lim_{t\to\infty}\frac{\frac{v(t+x)-v(t)}{v'(t)}-\frac{e^{\gamma x}-1}{\gamma}}{\frac{v''(t)}{v'(t)}-\gamma}=\int_0^x e^{\gamma s}\int_0^s e^{\rho y}dyds=:H_{\gamma,\rho}(x) \tag{1.9}$$

(see de Haan and Resnick, 1996b).

Since (1.9) is considerably easier to handle than (1.8), we shall mainly use (1.9) as in de Haan and Resnick (1996b).

Also (1.9) has a simple interpretation. Let W be a random variable with distribution function $\exp -e^{-x}$. Then M_n has the same distribution as $v(W+\log n)$. Hence

$$\frac{M_n-v(\log n)}{v'(\log n)}-\frac{e^{\gamma W}-1}{\gamma}\overset{d}{=}\frac{v(W+\log n)-v(\log n)}{v'(\log n)}-\frac{e^{\gamma W}-1}{\gamma},$$

so (1.9) shows that the rate of convergence of $(M_n-v(\log n))/v'(\log n)$ towards its limit $(e^{\gamma W}-1)/\gamma$ is of order $v''(\log n)/\{v'(\log n)\}-\gamma$.

Now we return to our first question: in what circumstances (i.e. for which combination of γ and ρ in (1.8)) can the convergence rate be improved by the use of penultimate approximations? Following loosely de Haan and Resnick (1996b, pages 112 and 113) we write with the transformation $a_n x+b_n=v(u+\log n)$

$$F^n(a_n x+b_n)-G_\gamma(x)$$

$$=G_\gamma\left(\frac{e^{u\gamma}-1}{\gamma}\right)-G_\gamma\left(\frac{v(u+\log n)-b_n}{a_n}\right)$$

$$=\left(\frac{v(u+\log n)-b_n}{a_n}-\frac{e^{u\gamma}-1}{\gamma}\right)\quad\times$$

$$G_\gamma'\left(\frac{e^{u\gamma}-1}{\gamma}+\theta\left(\frac{v(u+\log n)-b_n}{a_n}-\frac{e^{u\gamma}-1}{\gamma}\right)\right)$$

for some $\theta\in[0,1]$, depending on u,γ and n.

So the uniform rate of convergence of $F^n(a_nx+b_n)$ to $G_\gamma(x)$ is controlled by the uniform rate of convergence of $\frac{v(u+\log n)-b_n}{a_n}$ to $\frac{e^{\gamma u}-1}{\gamma}$. Hence, if we want to improve the rate for the former, we have to improve the rate for the latter, i.e. we have to improve the rate in (1.9) by using penultimate approximations. So we look at the convergence rate of

$$\frac{v(t+x)-v(t)}{v'(t)}-\frac{e^{\gamma(t)x}-1}{\gamma(t)}$$

to zero as $t\to\infty$ where $\gamma(t)$ is a suitably chosen function converging to γ as $t\to\infty$. We have

$$\frac{v(t+x)-v(t)}{v'(t)}-\frac{e^{\gamma x}-1}{\gamma}=\left(\frac{v''(t)}{v'(t)}-\gamma\right)H_{\gamma,\rho}(x)+o\left(\frac{v''(t)}{v'(t)}-\gamma\right) \tag{1.10}$$

by (1.9) and

$$\begin{aligned}\frac{e^{\gamma(t)x}-1}{\gamma(t)}-\frac{e^{\gamma x}-1}{\gamma}&=\int_0^x(e^{\gamma(t)s}-e^{\gamma s})ds=\\ &=(\gamma(t)-\gamma)\int_0^x se^{\gamma s}ds+\\ &\qquad+\frac{1}{2}(\gamma(t)-\gamma)^2\int_0^x s^2e^{\gamma s}ds+o\left((\gamma(t)-\gamma)^2\right)\end{aligned} \tag{1.11}$$

as $t\to\infty$. Now the first order terms in (1.10) and (1.11) should cancel in order to get an improvement. Hence improvement is possible only if $\rho=0$ and one chooses $\gamma(t):=v''(t)/v'(t)$. This answers the first question.

In order to answer the second question (precise conditions and ensuing rate) we start again and proceed in a slightly different manner. Write $\gamma(t) := \frac{v''(t)}{v'(t)}$.

Note that

$$\frac{v(t+x)-v(t)}{v'(t)} - \frac{e^{\gamma(t)x}-1}{\gamma(t)}$$

$$= \int_0^x \left[\exp\{\log v'(t+s) - \log v'(t)\} - \exp\{s\frac{v''(t)}{v'(t)}\}\right] ds$$

$$= \int_0^x e^{\gamma(t)s}\left[\exp\left\{\int_0^s \left(\frac{v''(t+u)}{v'(t+u)} - \frac{v''(t)}{v'(t)}\right) du\right\} - 1\right] ds$$

$$= \int_0^x e^{\gamma(t)s}\left[\exp\left\{\int_0^s \int_0^u \gamma'(t+v) dv du\right\} - 1\right] ds.$$

Hence

$$\frac{\frac{v(t+x)-v(t)}{v'(t)} - \frac{e^{\gamma(t)x}-1}{\gamma(t)}}{\gamma'(t)}$$

$$= \int_0^x e^{\gamma(t)s} \frac{\exp\left\{\gamma'(t)\int_0^s \int_0^u \frac{\gamma'(t+u)}{\gamma'(t)} dv du\right\} - 1}{\gamma'(t)} ds$$

It is clear that in order to get convergence for $t \to \infty$, we need to have $\lim_{t\to\infty} \gamma'(t) = 0$ and $\lim_{t\to\infty} \gamma'(t+u)/\gamma'(t)$ exists for all u, locally uniformly. Given (1.2) and (1.6) this is ensured if we assume

$$\lim_{t\to\infty} \frac{\gamma''(t)}{\gamma'(t)} = 0. \tag{1.12}$$

Then $\lim_{t\to\infty} \gamma'(t+u)/\gamma'(t) = 1$ for all u, locally uniformly, $\lim_{t\to\infty} \gamma'(t) = 0$ and

$$\lim_{t\to\infty} \frac{\frac{v(t+x)-v(t)}{v'(t)} - \frac{e^{\gamma(t)x}-1}{\gamma(t)}}{\gamma'(t)} = \int_0^x e^{\gamma s} \int_0^s \int_0^u 1 dv du ds = \frac{1}{2}\int_0^x s^2 e^{\gamma s} ds \tag{1.13}$$

for all x (see Lemma 2.1). We remark that the limit in (1.12) has to be zero, otherwise (1.9) would not hold with $\rho = 0$.

In the next section it will be shown that (1.12) is the precise condition for improvement of the convergence rate. As relation (1.13) suggests, the new rate is $\gamma'(\log n)$. This provides an answer to the second question.

Two issues remain to be considered. Firstly, the condition (1.12) is phrased in terms of v, which is the inverse function of $-\log - \log F$. It is much more convenient to have a condition phrased in terms of the distribution function itself and its derivatives. It will be proved (Lemma 2.2) that (1.12) is equivalent to

$$\lim_{t \uparrow x^*} \frac{\varphi''(t)}{u'(t)\varphi'(t)} = -\gamma \tag{1.14}$$

with as before $\varphi(t) = \left(\frac{1}{u'}\right)(t) - \gamma$. We now formulate the resulting statement.

Theorem 1 Let $u := -\log - \log F$. Suppose

$$\lim_{t \uparrow x^*} \left(\frac{1}{u'}\right)'(t) = \gamma \tag{1.15}$$

(von Mises' first order condition),

$$\lim_{t \uparrow x^*} \frac{\varphi'(t)}{u'(t)\varphi(t)} = 0 \tag{1.16}$$

with $\varphi(t) := (\frac{1}{u'})'(t) - \gamma$ (von Mises type second order condition) and

$$\lim_{t \uparrow x^*} \frac{\varphi''(t)}{u'(t)\varphi'(t)} = -\gamma \tag{1.17}$$

(von Mises type penultimate condition). Then

$$\lim_{n\to\infty} \frac{F^n(a_n x + b_n) - G_{\gamma_n}(x)}{\gamma'(\log n)} =$$

$$(-\log G_\gamma(x))^{1+\gamma} G_\gamma(x) M_\gamma(-\log - \log G_\gamma(x)) \quad (1.18)$$

uniformly for $x \in \mathbb{R}$ with $a_n := v'(\log n)$,

$$b_n := \begin{cases} v(\log n) & \text{for } \gamma \geq 0 \\ v(\infty) - \gamma^{-1} v'(\log n) & \text{for } \gamma < 0; \end{cases}$$

here $v := u^{\leftarrow}$ (the inverse of the function u). Further

$$\gamma_n := \gamma(\log n)$$

with

$$\gamma(t) := \frac{v''(t)}{v'(t)}$$

and

$$M_\gamma(x) := \begin{cases} \int_0^x u^2 e^{\gamma u} du & \text{for } \gamma \geq 0 \\ -\int_x^\infty u^2 e^{\gamma u} du & \text{for } \gamma < 0. \end{cases}$$

Moreover

$$\lim_{n\to\infty} \frac{\int_{-\infty}^{\infty} \left| \frac{d}{dx} F^n(a_n x + b_n) - G'_{\gamma_n}(x) \right| dx}{\gamma'(\log n)} =$$

$$= \int_{-\infty}^{\infty} e^{-(\gamma+1)u} \exp\left[e^{-u}\right] \left| M'_\gamma(u) - (\gamma+1) M_\gamma(u) + e^{-u} M_\gamma(u) \right| du. \quad (1.19)$$

Remark. The results should be compared with Theorems 3.1 and 4.1 of de Haan and Resnick (1994b). Under the given conditions $\gamma'(t)$ is of lower order than the original rate function $\gamma(t) - \gamma$, but the improvement is not great since γ' satisfies $\lim_{t\to\infty} \gamma'(t+x)/\gamma'(t) = 1$, the same relation as for $|\gamma(t) - \gamma|$.

The result of Theorem 1 can be simplified somewhat with some loss of generality. In order to do so let us return to the expansions (1.10) and (1.11). Since the second order terms (the first to the right) cancel by choosing $\rho = 0$ and $\gamma(t) = v''(t)/v'(t)$, the third order terms take over. So we need a third order expansion in (1.10). This is done in Lemma 2.3 below. The third order rate function in Lemma 2.3 is $\{v'''(t) - \gamma v''(t) - \gamma(v''(t) - \gamma v'(t))\}/v'(t)$. We want this rate function to be smaller than the one in (1.11) which is $(\gamma(t) - \gamma)^2$, so that the resulting convergence rate is $(\gamma(t) - \gamma)^2$. Hence we assume

$$\lim_{t\to\infty} \frac{\{v'''(t) - \gamma v''(t) - \gamma(v''(t) - \gamma v'(t))\}}{v'(t) \cdot \left(\frac{v''(t)}{v'(t)} - \gamma\right)^2} = 0. \tag{1.20}$$

We shall show (proof of Theorem 2) that this is equivalent to

$$\lim_{t\to\infty} \frac{\varphi'(t)}{u'(t)\{\varphi(t)\}^2} = -1. \tag{1.21}$$

It turns out that this condition alone (with the first and second order conditions (1.15) and (1.16) but without condition (1.17)) is sufficient for the desired convergence rate:

Theorem 2. Suppose (1.15), (1.16) and

$$\lim_{t\uparrow x*} \frac{\varphi'(t)}{u'(t)\{\varphi(t)\}^2} = -1. \tag{1.22}$$

Then (1.18) and (1.19) hold with $\gamma'(\log n)$ replaced by $-(\gamma_n - \gamma)^2$.

The conclusion is that improvement by penultimate approximations is possible only in case the approximation by the limiting extreme-value distribution is very slow (i.e. if $\rho = 0$). In that case the improvement is not spectacular: the rate becomes $(\gamma_n - \gamma)^2$ instead of $\gamma_n - \gamma$ (both are slowly varying functions).

We conclude with three remarks.

1. With some effort it can be proved that for $\gamma > 0$ Theorems 1 and 2 also hold with $G_{\gamma_n^*}(x)$ instead of $G_{\gamma_n}(x)$ where

   $$\gamma_n^* := \log v(\log n) - \frac{1}{\log n} \int_0^{\log n} \log v(\log s) ds.$$

   This can be connected with estimation theory as follows. The well-known Hill estimator for γ

   $$H_{k,n} := \frac{1}{k} \sum_{i=1}^{k} \log \frac{X_{(n-i+1,n)}}{X_{(n-k,n)}}$$

   is really designed to estimate γ_n^* rather than γ (see e.g. de Haan and Resnick (1996a), Proposition 2.2). So approximation of the distribution of the maximum by penultimate distributions using the estimated extreme value index $H_{k,n}$ rather than γ, may turn out to be quite good in the $\rho = 0$ case. We do not elaborate on this.

2. No further rate improvement is possible by changing the attraction coefficients a_n and b_n.

3. I have tried to find a third order relation for u equivalent to (2.12)

below but failed. Only under the extra condition (1.22) such equivalent third order relation has been found.

2 Proofs.

We need a number of auxiliary results.

Lemma 2.1 Suppose (1.2) and (1.6) hold. Set $\gamma(t) := v''(t)/v'(t)$ and $\varphi(t) := \left(\frac{1}{u'}\right)'(t) - \gamma$. Assume

$$\lim_{t\to\infty} \frac{\gamma''(t)}{\gamma'(t)} = 0. \tag{2.1}$$

Then

$$\lim_{t\to\infty} \gamma'(t) = 0 \text{ and } \lim_{t\to\infty} \frac{\gamma'(t+x)}{\gamma'(t)} = 1 \tag{2.2}$$

locally uniformly. Moreover we have the following inequalities:

Given $\varepsilon > 0$, there exists $t_\varepsilon = t_0(\varepsilon)$ such that for $t \geq t_0$ and $t + x \geq t_0$

$$(1-\varepsilon)e^{-\varepsilon|x|}\frac{x^2}{2}e^{\gamma x} < \frac{\frac{v'(t+x)}{v'(t)} - e^{\gamma(t)x}}{\gamma'(t)} < (1+\varepsilon)e^{\varepsilon|x|}\frac{x^2}{2}e^{\gamma x}. \tag{2.3}$$

Moreover, for $\gamma \geq 0$,

$$(1-\varepsilon)e^{-\varepsilon|x|} < \frac{\frac{v(t+x)-v(t)}{v'(t)} - \frac{e^{\gamma(t)x}-1}{\gamma(t)}}{\gamma'(t)M_\gamma(x)} < (1+\varepsilon)e^{\varepsilon|x|} \tag{2.4}$$

and for $\gamma < 0$ (note that then $v(\infty) < \infty$)

$$(1-\varepsilon)e^{-\varepsilon|x|} < \frac{\frac{v(t+x)-v(\infty)-\gamma^{-1}v'(t)}{v'(t)} - \frac{e^{\gamma(t)x}-1}{\gamma(t)}}{\gamma'(t)M_\gamma(x)} < (1+\varepsilon)e^{\varepsilon|x|}, \tag{2.5}$$

with

$$M_\gamma(x) := \begin{cases} \int_0^x u^2 e^{\gamma u} du & \text{for } \gamma \geq 0 \\ -\int_x^\infty u^2 e^{\gamma u} du & \text{for } \gamma < 0. \end{cases} \tag{2.6}$$

Proof. First note that by (2.1)

$$\log \gamma'(t+x) - \log \gamma'(t) = \int_0^x \frac{\gamma''(t+u)}{\gamma'(t+u)} du \to 0$$

locally uniformly as $t \to \infty$. Hence,

$$\lim_{t\to\infty} \frac{\gamma'(t+x)}{\gamma'(t)} = 1 \tag{2.7}$$

locally uniformly.

Relation (2.7) implies

$$\lim_{t\to\infty} \frac{\gamma(t+x)-\gamma(t)}{\gamma'(t)} = \lim_{t\to\infty} \int_0^x \frac{\gamma'(t+u)}{\gamma(t)} du = x$$

locally uniformly.

This implies by the theory of Π-variation, since $\lim_{t\to\infty}\gamma(t) = \gamma$,

$$\lim_{t\to\infty} \frac{\gamma'(t)}{\gamma(t)-\gamma} = 0.$$

and hence $\lim\limits_{t\to\infty} \gamma'(t) = 0$.

We have Potter type bounds: given $\varepsilon' > 0$ there exists $t_0' = t_0'(\varepsilon')$ such that

for $t \geq t_0', t+x \geq t_0'$.

$$(1-\varepsilon')e^{-\varepsilon'|x|} < \frac{\gamma'(t+x)}{\gamma'(t)} < (1+\varepsilon')e^{\varepsilon'|x|}.$$

This implies

$$(1-\varepsilon')e^{-\varepsilon'|x|} \cdot \frac{x^2}{2} < \int_0^x \int_0^u \frac{\gamma'(t+v)}{\gamma'(t)} dv du < (1+\varepsilon')e^{\varepsilon'|x|} \cdot \frac{x^2}{2} \tag{2.8}$$

Further for $|w| < w_0(\varepsilon')$

$$1-\varepsilon' < \frac{e^w - 1}{w} < 1+\varepsilon'. \tag{2.9}$$

Next note that

$$\frac{v'(t+x)}{v'(t)} - e^{\gamma(t)x} = e^{\gamma(t)x}.\frac{\exp\{\gamma'(t)\int_0^x\int_0^u \frac{\gamma'(t+v)}{\gamma'(t)}dudu\} - 1}{\gamma'(t)}.$$

Application of the inequalities (2.7) and (2.8) yields (2.2). Relations (2.3) and (2.4) are obtained by straightforward integration from (2.2). □

Remark. Since a limit $\tau < 0$ in (2.1) would result in $\lim_{t\to\infty} \gamma'(t)/\{\gamma(t) - \gamma\} = \tau$, the rate of convergence can not be improved in that case.

Lemma 2.2. Suppose (1.2) and (1.6) hold. The statements (1.12) and (1.14) are equivalent.

Proof. Since

$$\gamma(t) - \gamma = \frac{v''(t)}{v'(t)} - \gamma = (\frac{1}{u'})'(v(t) - \gamma = \varphi(v(t)) \quad \text{(recall that } v = u^{\leftarrow}\text{)},$$

we have

$$\frac{\gamma''(t)}{\gamma'(t)} = \frac{v''(t)}{v'(t)} + v'(t)\frac{\varphi''(v(t))}{\varphi'(v(t))} = \frac{v''(t)}{v'(t)} + \frac{\varphi''(v(t))}{u'(v(t))\varphi'(v(t))}.$$

We know that $v''(t)/v'(t) \to \gamma \quad (t \to \infty)$ (cf.(1.2)). Hence (1.12) and (1.14) are equivalent.

Proof of Theorem 1. The line of proof is exactly the same as the proof of Theorem 3.1 in De Haan and Resnick (1996b). Corollary 2.4 of that paper should be replaced by our Lemma 2.1. Lemma 3.3 from De Haan and Resnick (1996b) holds without change in our situation. The statement of Lemma 3.2 from De Haan and Resnick (1996b) on M_γ for $\rho = 0$ holds for our M_γ without change (except for the fact that the "$\wedge$" in the statement of Lemma 3.2 should be a "$\vee$" sign. With these alterations the proof of Theorem 3.1 in De Haan and Resnick applies in our context. $\square$

Lemma 2.3. Suppose (1.2) and (1.6) so that (1.10) holds. Set

$$Q(t) := \frac{v'''(t) - \gamma v''(t)}{v''(t) - \gamma v'(t)} - \gamma - \rho.$$

Assume

$$\lim_{t\to\infty} \frac{Q'(t)}{Q(t)} = \tau \leq 0, \tag{2.10}$$

then

$$v(t+x) - v(t) - v'(t)\frac{e^{\gamma x} - 1}{\gamma} - \{v''(t) - \gamma v'(t)\} H_{\gamma,\rho}(x) =$$

$$= \left[v'''(t) - \gamma v''(t) - (\rho + \gamma)\{v''(t) - \gamma v'(t)\}\right]$$
$$\left[\int_0^x e^{\gamma s} \int_0^s e^{\rho y} \int_0^y e^{\tau u} du dy ds + o(t)\right] \tag{2.11}$$

for all $x (t \to \infty)$.

Remark. The function in square brackets is

$$e^{(\gamma+\rho)t} \frac{d}{dt} e^{-\rho t} \frac{d}{dt} e^{\gamma t} \frac{d}{dt} v(t).$$

This reveals the structure of the expansion and enables one to extend it at will.

Proof. Relation (2.10) implies

$$\lim_{t\to\infty} \frac{Q(t+x)}{Q(t)} = e^{\tau x} \tag{2.12}$$

for all x.

Hence by combining with (1.2) (implying $v'(t+x)/v'(t) \to e^{\gamma x}, t \to \infty$) and (1.7) we get

$$\lim_{t\to\infty} \frac{v'''(t+x) - \gamma v''(t+x) - (\rho+\gamma)\left[v''(t+x) - \gamma v'(t+x)\right]}{v'''(t) - \gamma v''(t) - (\rho+\gamma)\left[v''(t) - \gamma v'(t)\right]} = e^{(\gamma+\rho+\tau)x} \tag{2.13}$$

for all x. Now

$$v(t+x) - v(t) - v'(t)\frac{e^{\gamma x}-1}{\gamma} - \{v''(t) - \gamma v'(t)\}H_{\gamma,\rho}(x) = \tag{2.14}$$

$$\int_0^y e^{\gamma s} \int_0^s e^{\rho y} \int_0^y e^{-(\rho+\gamma)u} \big\{v'''(\log n + u) - \gamma v''(\log n + u) - (\rho+\gamma)$$

$$\big[v''(\log n + u) - \gamma v'(\log n + u)\big]\big\}\, dudyds.$$

The statement follows by the local uniformity in (2.13).

Proof of Theorem 2.

Note that

$$\gamma'(t) = (\gamma(t) - \gamma)\left\{\frac{v'''(t) - \gamma v''(t)}{v''(t) - \gamma v'(t)} - \gamma\right\} - (\gamma(t) - \gamma)^2.$$

Hence (1.20) holds if and only if

$$\lim_{t\to\infty} \frac{\gamma'(t)}{(\gamma(t) - \gamma)^2} = -1. \tag{2.15}$$

Moreover one sees by working out the derivatives that

$$\frac{v'''(u(t)) - \gamma v''(u(t))}{v''(u(t)) - \gamma v'(u(t))} - \frac{v''(u(t))}{v'(u(t))} = \frac{\varphi'(t)}{u'(t)\cdot\varphi(t)}.$$

Hence (1.20) is equivalent to (1.22).

Next note that (2.16) is equivalent to

$$\lim_{t\to\infty} \frac{d}{dt}\left(\frac{1}{\gamma(t)-\gamma}\right) = 1.$$

As a consequence

$$\lim_{t\to\infty} \frac{\gamma(t+x)-\gamma(t)}{-(\gamma(t)-\gamma)^2} = \lim_{t\to\infty} \frac{1}{\gamma(t+x)-\gamma} - \frac{1}{\gamma(t)-\gamma} = x$$

for all x, locally uniformly. It follows that the function $(\gamma(t)-\gamma)^2$ satisfies

$$\lim_{t\to\infty} \frac{(\gamma(t+x)-\gamma)^2}{(\gamma(t)-\gamma)^2} = 1$$

for all x and hence by (2.16)

$$\lim_{t\to\infty} \frac{\gamma'(t+x)}{\gamma'(t)} = 1 \tag{2.16}$$

for all x. The proof of Lemma 2.1 reveals that (2.17) is sufficient for the results of Lemma 2.1. Hence Theorem 1 holds. But by (2.16) one can replace the rate function $\gamma'(\log n)$ in Theorem1 by the rate function $-(\gamma(\log n)-\gamma)^2$. The proof is complete. □

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