Unobserved heterogeneity;
process and parameter effects in life insurance

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ABSTRACT

In this paper life insurance contracts based on an urn-of-urns model, with age-at-death as observable variable, are analyzed. Premium payment functions based on the principles of "equivalence on an individual level" and "equivalence on a group level" are compared. Both the aggregate loss and its second moment for an individual contract are split in several components. Life insurance contracts are compared with non-life insurance contracts, also with respect to solidarity.

Keywords: **(Unobserved) heterogeneity; Urn-of-urns model; Life Insurance; Equivalence; Loss variance; Solidarity**

1. Introduction

Unobserved heterogeneity is a problem which involves both life and non-life insurance. In this context heterogeneity is defined as the existence of an *urn-of-urns* model (see e.g. Jewell (1980)) with a non-degenerate structure distribution. In an urn-of-urns model first the outcome of a random structure variable is drawn from an urn. Then the outcome of a certain random variable is drawn from the second urn which corresponds to the outcome of the first draw. In this paper we consider some aspects of unobserved heterogeneity in life insurance. We will restrict ourselves to single life policies.

In terms of non-life insurance, we assume that the distribution of an individual's total claim amount is known up to a risk parameter. The values of these risk parameters are not observable; the insurer only knows the parameters' distribution density (structure distribution). The company charges an average premium for each contract of the portfolio such that the expected loss for the whole portfolio is equal to zero, i.e. the expected value of the premium income is equal to the expected value of the claim amounts. This implies that, for each insured individual, the company is faced by both a *process effect* (the difference between claim amount and the expected claim amount for that individual) and a *parameter effect* (the difference between the expected claim amount for that individual and the average premium). An essential property of the parameter effect is that it does not depend on the draw from the second urn in the urn-of-urns model.

In the non-life case, the unconditional loss variance consists of two parts. In this paper we will investigate whether this is also the case for life contracts.

In Section 2 the main assumptions will be described. These involve an urn-of-urns model with respect to remaining lifetime and the definition of a loss function. The concepts *equivalence on an individual level* and *equivalence on a group level*, introduced in Spreeuw (1996), will be recalled here. If "equivalence on an individual level" was possible then each individual would have his/her own premium payment function, in this paper defined as *individual (discounted) premium payment function*. For two individuals with different risk parameters, different individual premium payment functions would result.

Since the heterogeneity is unobserved, only equivalence on a group level is possible. In this paper a premium payment function which satisfies this principle, and applies to all insured, is called *average (discounted) premium payment function*.

Section 3 will deal with the division of an aggregate loss into several components. The section will begin with the non-life case, mentioned in the second paragraph of this introduction. When in this paper we mention non-life insurance, we restrict ourselves to non-life policies where a premium is only paid at issue. Then a major difference between life insurance and non-life insurance is the period during which premiums have to be paid by the individual. Contrary to a non-life contract, this period is for a life contract often a time interval and not one single time point during the contract term. This is the reason why the expression which results from subtracting the *expected loss in case of equivalence on an individual level* (a more generalized definition of process effect) from the *aggregate loss*, defined as parameter effect in the non-life case, is in general not independent of the drawn outcome itself, in this case the age-at-death. In this paper we will split this difference further in a) the expected loss for the individual contract, which is similar to the parameter effect in the non-life case, and b) a remaining part. Hence the aggregate loss for an individual contract consists of three parts (the two parts mentioned in the previous sentence and the process effect). First and second moments of these three parts will be derived. It will be proved that the (unconditional) variance of the aggregate loss consists of the above mentioned second moments, besides a covariance term.

In Section 4 payment by single premium will be discussed. In the case of single premium payment, there is the most resemblance between a life contract and a non-life contract. The *average single premium* (based on the principle of equivalence on a group level) will be compared with *individual single premiums* (satisfying equivalence on an individual level). Similar to a non-life insurance contract, the parameter effect is equal to the expected loss for the individual contract. It will be shown that in this particular case two components of the loss variance, derived in Section 3, vanish. As a result, two terms remain, which are similar to the non-life case, namely the expectation of the variance in case of equivalence on an individual level and the variance of the expected loss.

To illustrate the results derived in previous sections, a numerical example, involving an endowment insurance, will be given in Section 5. We will compare the cases of single premium payment and continuous level premium payment.

Finally Section 6 deals with some solidarity aspects. In comparison with above all issues of this paper will now be considered from the insured's point of view. The concepts *risk solidarity* and *probabilistic solidarity*, as well as *ex ante transfer* and *ex post transfer*, the latter concepts being closely related to process effect and parameter effect, respectively, will be discussed.

2. Assumptions

Our assumptions mainly involve an urn-of-urns model which will be introduced next. Furthermore in this section, we will specify a loss function and define the principles *equivalence on an individual level* and *equivalence on a group level*.

2a. The urn-of-urns model

We consider a large group of persons, all aged x_0. The distribution of remaining lifetime of an arbitrary member of this group, depends on the value of his/her individual, one-dimensional, risk parameter. The draw of an individual with risk parameter valued θ involves the outcome θ of a random variable Θ. The latter has structure density function $u(\theta)$. Given the outcome θ, the force-of-mortality function is equal to $\mu(x_0 + s; \theta)$ $(s \geq 0)$. Then the corresponding c.d.f. $F_{T|\Theta}(t, x_0; \theta)$, the probability of a person aged x_0 to die before age $x_0 + t$, is equal to

$$F_{T|\Theta}(t, x_0; \theta) = 1 - e^{-\int_0^t \mu(x_0 + s; \theta)\, \mathrm{d}s}. \tag{1}$$

The marginal c.d.f. of T, the latter being the random variable of the remaining lifetime of an arbitrary member of the population, is then given by

$$F_T(t, x_0) = 1 - \int_\theta u(\theta) e^{-\int_0^t \mu(x_0 + s; \theta)\, \mathrm{d}s}\, \mathrm{d}\theta. \tag{2}$$

So we have an urn-of-urns model, see e.g. Jewell (1980). Usually only the outcomes of the second draw (the selection of the outcome from the "small urn", in this case the age-at-death t) can be observed. However, we assume that both the conditional c.d.f.'s of remaining life-time $F_T(t, x_0)$ and the structure density $u(\theta)$ are known to the insurer. Only the outcome of the first draw (the draw of the "small urn" within the "large urn", in this case the outcome of Θ) remains unobserved.

2b. The loss function

We assume that there is a fixed infinitesimal interest rate, namely δ. Let $c(t)$ be the present value of the benefits due at t in case of dying at time t and $B(t)$ the total amount of benefit payments in $[0, t]$ while the insured is alive. Then the present value at time 0 of all benefits (life and death) (to be) paid if the insured dies at time t is equal to $\beta(t)$, where $\beta(t)$ is defined as follows:

$$\beta(t) = \begin{cases} e^{-\delta t} c(t) + \int_0^t e^{-\delta s}\, \mathrm{d}B(s), & (t > 0) \\ 0. & (t = 0) \end{cases} \tag{3}$$

In this article we will ignore expenses, although they can be defined such that they are part of $\beta(t)$. The present value of all premiums paid by the insured in $[0, t]$ in case of dying at time t is denoted by $\rho(t)$:

$$\rho(t) = \begin{cases} \int_0^t e^{-\delta s}\, d\Pi(s), & (t > 0) \\ 0, & (t = 0) \end{cases} \tag{4}$$

where $\Pi(s)$ is defined as the total amount of premium payments in $[0, s]$. We assume that the integrals, appearing in the right hand sides of formulas (3) and (4), exist for each t.

Then $l(t)$, the loss incurred in case of dying at time t, is equal to

$$l(t) = \beta(t) - \rho(t). \qquad (t \geq 0) \tag{5}$$

By writing this loss as a function of T we obtain the random loss:

$$l(T) = \beta(T) - \rho(T). \qquad (T \geq 0) \tag{6}$$

These loss functions are formulated in a very general way. In Section 3 we will use the same notation for loss functions, where the term $\rho(t)$ is subject to "equivalence on a group level". The latter principle will be defined in the next subsection.

2c. Two equivalence concepts

We first consider equivalence on an individual level. An *individual (discounted) premium payment function* for an individual with risk parameter θ is defined as a function $\rho(t)$ which is a solution of the following equation:

$$E[l(T)|\Theta=\theta] = 0$$

$$\Leftrightarrow \quad E[\rho(T)|\Theta=\theta] = E[\beta(T)|\Theta=\theta]$$

$$\Leftrightarrow \quad \int_0^\infty \rho(t)\, dF_{T|\Theta}(t, x_0; \theta) = \int_0^\infty \beta(t)\, dF_{T|\Theta}(t, x_0; \theta). \tag{7}$$

In the remainder of this paper we will denote such a solution by $\rho(t;\theta)$.

In our case of "equivalence on a group level", the *average (discounted) premium payment function*, denotes a function $\rho(t)$ which is applicable to all insured and satisfies an equation similar to (7), though now with respect to the marginal distribution of T (see equation (2)):

$$E\big[E[l(T)|\Theta]\big] = 0$$

$$\Leftrightarrow \quad E\big[E[\rho(T)|\Theta]\big] = E\big[E[\beta(T)|\Theta]\big]$$

$$\Leftrightarrow \quad \int_\theta \left(\int_0^\infty \rho(t)\, dF_{T|\Theta}(t, x_0; \theta) \right) u(\theta)\, d\theta = \int_\theta \left(\int_0^\infty \beta(t)\, dF_{T|\Theta}(t, x_0; \theta) \right) u(\theta)\, d\theta$$

$$\Rightarrow \quad \int_0^\infty \rho(t)\, dF_T(t, x_0) = \int_0^\infty \beta(t)\, dF_T(t, x_0). \tag{8}$$

We define $\bar{\rho}(t)$ as a function $\rho(t)$ which satisfies equality (8).

Note that neither $\rho(t;\theta)$, nor $\bar{\rho}(t)$, has to be unique unless sufficient restrictions are imposed on the premium payment function. The latter is the case in the following example, involving continuous level premium payment.

Example 1 (level premium payment)

If a contract with duration n years is purchased by a level premium paid continuously during the first m years ($m \leq n$) while the insured is alive, we have that $\rho(t)$ in (4) is defined as

$$\rho(t) = \pi \int_0^{t\wedge m} e^{-\delta s}\, ds, \qquad (t \geq 0) \tag{9}$$

where $t\wedge m$ is defined as the minimum of t and m. In this case, a unique value for π will result, since there are sufficient constraints imposed on $\rho(t)$, using one of the equivalence principles in subsection 2c. Hence $\rho(t)$ is then uniquely determined as well.

The level premium for an individual with parameter θ in case of equivalence on an individual level, in the remainder called the *individual level premium* and denoted by $\pi(\theta)$, is

$$\pi(\theta) = \frac{E[\beta(T)\,|\,\Theta=\theta]}{E[\bar{a}_{\overline{T\wedge m|}}\,|\,\Theta=\theta]}. \tag{10}$$

On the other hand, applying the principle of equivalence on an individual level will result in the following *average level premium* $\bar{\pi}$:

$$\bar{\pi} = \frac{E[\beta(T)]}{E[\bar{a}_{\overline{T\wedge m|}}]}. \tag{11}$$

■

3. Separation of the loss; process effect and parameter effect; first and second moments

In this section we will separate the aggregate loss, first for non-life and then for life contracts. In subsection 3a the aggregate loss of a non-life policy will be split in components. Division of the aggregate loss of a life contract is the subject of subsection 3b, the approach being based on subsection 3a. Subsections 3c and 3d will deal with first and second moments, respectively. It will be proved at the end of subsection 3d that these second moments are part of the loss variance for an arbitrary contract.

3a. Separation of aggregate loss for non-life insurance contracts

Generally speaking, an "urn-of-urns model", as described in e.g. Jewell (1980), involves a two stage draw. At the first stage, which is unobserved, an outcome of a random variable Θ, having density function $u(\theta)$, is drawn. Then, given $\Theta = \theta$, the outcome of the second stage

h involves the realization of a random variable H (e.g. claim amount), with corresponding density function being a function of θ. Then in non-life insurance, "equivalence on a group level" implies that each individual pays a premium equal to the unconditional expectation of H, namely

$$\mathrm{E}[H] = \mathrm{E}\big[\mathrm{E}[H|\Theta]\big]. \tag{12}$$

For an individual contract with outcome h and risk parameter θ, the loss is

$$l(h) = h - \mathrm{E}[H]. \tag{13}$$

For an individual contract with risk parameter θ, this loss can be separated as follows:

$$l(h) = l_1(h;\theta) + \big(l(h) - l_1(h;\theta)\big), \tag{14}$$

where the *process effect* is defined as

$$l_1(h;\theta) = h - \mathrm{E}[H|\Theta{=}\theta], \tag{15}$$

being equal to the actual loss in case of equivalence on an individual level. The remaining term is defined as the *parameter effect*, which is independent of h:

$$l(h) - l_1(h;\theta) = \mathrm{E}[H|\Theta{=}\theta] - \mathrm{E}[H]. \tag{16}$$

The *parameter effect* is equal to the expected loss for the individual contract. (The concepts in italics are related to "process risk" and "parameter risk", respectively, which are introduced and verbally described in Cummins et al. (1983, p. 68).)

3b. Separation of aggregate loss for life insurance contracts

With the urn-of-urns model described in Section 2 as basis, we will now separate the aggregate loss for the life contract into the process effect and the difference between the aggregate loss and the process effect. Similar to above subsection, by the process effect the loss corresponding to equivalence on an individual level is denoted. The other term will be again separated, namely into the expected loss for the individual contract and a remaining component.

The drawn outcomes of the urn-of-urns model in the previous section are times-of-death, rather than claim amounts. In the life case the relevant variables are functions of these times-of-death, namely those defined in formulas (3) and (4), instead of the draws themselves. Besides, interest has to be taken into account.

In this paper we suppose that the principle of equivalence on a group level is used by the insurer. Therefore, each insured pays premiums according to the average (discounted) premium payment function $\overline{\rho}(\cdot)$, which satisfies equality (8):

$$\mathrm{E}\big[\mathrm{E}[\overline{\rho}(T)|\Theta]\big] = \mathrm{E}\big[\mathrm{E}[\beta(T)|\Theta]\big]. \tag{17}$$

Then at time-until-death t the loss for the insurer is equal to

$$l(t) = \beta(t) - \bar{\rho}(t). \tag{18}$$

(Note that the form of $l(t)$ is now more specific than in definition (5).) For an individual with risk parameter θ this loss can be separated in two parts:

$$l(t) = l_1(t;\theta) + \big(l(t) - l_1(t;\theta)\big). \tag{19}$$

In (19) the *process effect* is defined as

$$l_1(t;\theta) = \beta(t) - \rho(t;\theta), \tag{20}$$

being equal to the loss in case of equivalence on an individual level. The process effect has expectation zero on the level of an individual contract. The remaining part of (19) is given by

$$l(t) - l_1(t;\theta) = \rho(t;\theta) - \bar{\rho}(t). \tag{21}$$

Recall from subsection 3a that in the non-life case the difference between the aggregate loss and the process effect is equal to the expected loss for the individual contract.

We will now investigate whether this also holds for the life case, by deriving the expected loss for the individual contract and subtracting it from the right hand side of (21). The expectation of the latter is denoted by $EL(\theta)$ (EL is short for "Expected Loss"):

$$EL(\theta) = \mathrm{E}[l(T) - l_1(T;\theta)\,|\,\Theta = \theta] = \mathrm{E}[l(T)\,|\,\Theta = \theta], \tag{22}$$

since

$$\mathrm{E}[l_1(T;\theta)\,|\,\Theta = \theta] = 0 \text{ for each } \theta. \tag{23}$$

So $EL(\theta)$ is equal to the expected loss for the individual contract.

Rewriting the aggregate loss as a function of both the process effect and $EL(\theta)$ we get:

$$\beta(t) - \bar{\rho}(t) = l_1(t;\theta) + l_2(t;\theta) + l_3(t;\theta), \tag{24}$$

where

$$l_2(t;\theta) = EL(\theta) \text{ for each } t, \text{ and} \tag{25}$$

$$l_3(t;\theta) = \rho(t;\theta) - \bar{\rho}(t) - EL(\theta). \tag{26}$$

The latter term is to be interpreted as the difference between the actual parameter effect and its expectation. Since (26) in general depends on t, (21) is not equal to the expected loss for the individual contract.

3c. Analysis of first moments

We have that $l_1(t;\theta)$ and $l_3(t;\theta)$ have expectation zero on an individual level and that $l_2(t;\theta)$ is equal to its expectation on the level of an individual contract, since it does not depend on t:

$$\mathrm{E}[l_1(T;\theta)\,|\,\Theta=\theta] = 0 \text{ for each } \theta, \text{ so } \mathrm{E}[l_1(T;\Theta)\,|\,\Theta] \equiv 0,$$

$$\mathrm{E}[l_2(T;\theta)\,|\,\Theta=\theta]] = EL(\theta) = \mathrm{E}[l(T)\,|\,\Theta=\theta], \tag{27}$$

$$\mathrm{E}[l_3(T;\theta)\,|\,\Theta=\theta] = 0 \text{ for each } \theta, \text{ so } \mathrm{E}[l_3(T;\Theta)\,|\,\Theta] \equiv 0.$$

All three parts have average expectation zero:

$$\mathrm{E}_\Theta\big[\mathrm{E}_T[l_i(T;\Theta)\,|\,\Theta]\big] = 0 \text{ for each } i \in \{1,2,3\}. \tag{28}$$

3d. Analysis of second moments

In this subsection we will derive the second moments of all three components, as well as the covariances between these terms.

Since

$$\mathrm{E}[l(T)] = \mathrm{E}[\beta(T) - \bar{\rho}(T)] = 0, \tag{29}$$

we have that the variance of the aggregate loss is equal to

$$\begin{aligned} & \mathrm{Var}[l(T)] = \mathrm{E}[l^2(T)] = \mathrm{E}\big[\mathrm{E}[l^2(T)\,|\,\Theta]\big] \\ = \quad & \mathrm{E}\big[\mathrm{E}[\{l_1(T;\Theta) + l_2(T;\Theta) + l_3(T;\Theta)\}^2\,|\,\Theta]\big] \\ = \quad & \sum_{i,j\in\{1,2,3\}} \mathrm{E}\big[\mathrm{E}[l_i(T;\Theta)\, l_j(T;\Theta)\,|\,\Theta]\big]. \end{aligned} \tag{30}$$

Theorem 1 (Composition of the loss variance)

$$\begin{aligned} & \mathrm{Var}[l(T)] \\ = \quad & \mathrm{E}\big[\mathrm{Var}[\beta(T) - \rho(T;\Theta)\,|\,\Theta]\big] \\ & + \mathrm{Var}\big[\mathrm{E}[\beta(T) - \bar{\rho}(T)\,|\,\Theta]\big] \\ & + \mathrm{E}\big[\mathrm{Var}[\rho(T;\Theta) - \bar{\rho}(T)\,|\,\Theta]\big] \\ & + 2\,\mathrm{E}\big[\mathrm{Cov}[\beta(T) - \rho(T;\Theta), \rho(T;\Theta) - \bar{\rho}(T)\,|\,\Theta]\big]. \end{aligned} \tag{31}$$

Verbal interpretation:

Equation (31) shows that the variance of the loss for an arbitrary individual contract consists of:

- the *expected loss variance in case of equivalence on an individual level*;
- the *variance of the expected loss*;
- the *expected variance, due to the deviation of the actual difference between the aggregate loss and the process effect from the expected difference* and
- *two times* the *expected covariance between*, on the one hand, *the process effect and*, on the other hand, *the difference between the aggregate loss and the process effect.*

Proof

The theorem is proved by calculating $\mathrm{E}\big[\mathrm{E}[l_i(T;\Theta)\, l_j(T;\Theta)\,|\,\Theta]\big]$ for each possible combination of i and j. We will first consider the cases with $i = j$:

$$\mathrm{E}\big[\mathrm{E}_T[l_1^2(T;\Theta)\,|\,\Theta]\big] = \mathrm{E}\big[\mathrm{Var}[l_1(T;\Theta)\,|\,\Theta]\big]$$

$$= \mathrm{E}\big[\mathrm{Var}[\beta(T;\Theta) - \rho(T;\Theta)\,|\,\Theta]\big]; \tag{32}$$

$$\mathrm{E}\big[\mathrm{E}_T[l_2^2(T;\Theta)\,|\,\Theta]\big] = \mathrm{E}\big[\mathrm{E}_T[EL^2(\Theta)\,|\,\Theta]\big]$$

$$= \mathrm{E}\big[\mathrm{E}^2[\rho(T;\Theta) - \bar{\rho}(T)\,|\,\Theta]\big]$$

$$= \mathrm{E}\big[\mathrm{E}^2[\beta(T)\,|\,\Theta] - \mathrm{E}^2[\,\bar{\rho}(T)\,|\,\Theta]\big]$$

$$= \mathrm{Var}\big[\mathrm{E}[\beta(T) - \bar{\rho}(T)\,|\,\Theta]\big]; \tag{33}$$

$$\mathrm{E}\big[\mathrm{E}_T[l_3^2(T;\Theta)\,|\,\Theta]\big]$$

$$= \mathrm{E}\big[\mathrm{E}_T[\{\rho(T;\Theta) - \bar{\rho}(T) - EL(\Theta)\}^2\,|\,\Theta]\big]$$

$$= \mathrm{E}\big[\mathrm{Var}[\rho(T;\Theta) - \bar{\rho}(T)\,|\,\Theta]\big]. \tag{34}$$

For the remaining cases we can restrict ourselves to all combinations with $i < j$ ($j < i$ gives the same results):

$$\mathrm{E}\big[\mathrm{E}_T[l_1(T;\Theta)\; l_2(T;\Theta)\,|\,\Theta]\big] = \mathrm{E}\big[EL(\Theta)\,\mathrm{E}_T[l_1(T;\Theta)\,|\,\Theta]\big] = 0; \tag{35}$$

$$\mathrm{E}\big[\mathrm{E}_T[l_2(T;\Theta)\; l_3(T;\Theta)\,|\,\Theta]\big] = \mathrm{E}\big[EL(\Theta)\mathrm{E}_T[l_3(T;\Theta)\,|\,\Theta]\big] = 0; \tag{36}$$

$$\mathrm{E}\big[\mathrm{E}_T[l_1(T;\Theta)\; l_3(T;\Theta)\,|\,\Theta]\big]$$

$$= \mathrm{E}\big[\mathrm{E}_T[(l_1(T;\Theta) - \mathrm{E}[l_1(T;\Theta)\,|\,\Theta])\; l_3(T;\Theta)\,|\,\Theta]\big]$$

$$= \mathrm{E}\big[\mathrm{Cov}[l_1(T;\Theta), l(T) - l_1(T;\Theta)\,|\,\Theta]\big]$$

$$= \mathrm{E}\big[\mathrm{Cov}[\beta(T) - \rho(T;\Theta), \rho(T;\Theta) - \bar{\rho}(T)\,|\,\Theta]\big]. \qquad (37)$$

By substituting all last sides of formulas (32) to (37) into (30) we get the desired result. ■

4. Single premium payment

Single premiums in life insurance and premiums in non-life insurance have one thing in common: the amount is paid at one single time point and intended to cover the expected amount of benefits during the contract period. Therefore we will consider this special case of premium payment below in order to be able to compare the urn-of-urns model concerning time-until-death with the traditional one, mentioned at the beginning of Section 3. We will give formulas for the single premiums and analyze both the loss components and their second moments.

4a. Special form of premium payment function

If a contract is paid by single premium, the formulas for $\rho(t;\theta)$ reduce as follows:

$$\rho(t;\theta) = \begin{cases} SP(\theta), & (t > 0) \\ 0, & (t = 0) \end{cases} \qquad (38)$$

where the *individual single premium*, is

$$SP(\theta) = \mathrm{E}[\beta(T)\,|\,\Theta{=}\theta]. \qquad (39)$$

For $\bar{\rho}(t)$ we get

$$\bar{\rho}(t) = \begin{cases} \overline{SP}, & (t > 0) \\ 0. & (t = 0) \end{cases} \qquad (40)$$

In this equation $\overline{SP}$, the *average single premium*, equals

$$\overline{SP} = \mathrm{E}[\beta(T)] = \mathrm{E}\big[\mathrm{E}[\beta(T)\,|\,\Theta]\big] = \mathrm{E}[SP(\Theta)]. \qquad (41)$$

4b. Separation in different parts; second moments

The three parts of formula (24) are in this case

$$l_1(t;\theta) = \beta(t) - SP(\theta), \text{ and} \tag{42}$$

$$l_2(t;\theta) = EL(\theta) = \mathrm{E}[\rho(T;\theta) - \bar{\rho}(T)|\Theta = \theta]$$

$$= \quad SP(\theta) - \overline{\overline{SP}}. \tag{43}$$

$$l_3(t;\theta) = SP(\theta) - \overline{\overline{SP}} - \mathrm{E}[\rho(T;\theta) - \bar{\rho}(T)|\Theta = \theta] = 0. \tag{44}$$

So we have that in the case of single premium payment the difference between the aggregate loss and the process effect is not dependent on time-until-death *t* and hence is equal to the expected loss.

The first, third and fourth term of the right hand side of (31) are in this case

$$\mathrm{E}\big[\mathrm{Var}[\beta(T) - \rho(T;\Theta)|\Theta]\big] = \mathrm{E}\big[\mathrm{Var}[\beta(T) - SP(\Theta)|\Theta]\big]$$

$$= \quad \mathrm{E}\big[\mathrm{Var}[\beta(T)|\Theta]\big]; \tag{45}$$

$$\mathrm{E}\big[\mathrm{Var}[\rho(T;\Theta) - \bar{\rho}(T)|\Theta]\big] = \mathrm{E}\big[\mathrm{Var}[SP(\Theta) - \overline{\overline{SP}}|\Theta]\big] = 0; \tag{46}$$

$$\mathrm{E}\big[\mathrm{Cov}[\beta(T) - \rho(T;\Theta), \rho(T;\Theta) - \bar{\rho}(T)|\Theta]\big]$$

$$= \quad \mathrm{E}\big[\mathrm{Cov}[\beta(T) - \rho(T;\Theta), SP(\Theta) - \overline{\overline{SP}}|\Theta]\big] = 0. \tag{47}$$

So the unconditional variance reduces to

$$\mathrm{Var}[l(T)] = \mathrm{E}\big[\mathrm{Var}[l(T)|\Theta]\big] + \mathrm{Var}\big[\mathrm{E}[l(T)|\Theta]\big] \tag{48}$$

$$= \quad \mathrm{E}\big[\mathrm{Var}[\beta(T)|\Theta]\big] + \mathrm{Var}\big[\mathrm{E}[\beta(T)|\Theta]\big], \tag{48$'$}$$

since adding a constant term, in this case the average single premium, to the argument of a variance does not influence the value of the variance itself. As the right hand side of formula (48) demonstrates, the loss variance in the case of single premium payment consists solely of the expected variance of the loss in case of equivalence of an individual level and the variance of the conditional expected loss.

The similarity with the urn-of-urns model, described at the beginning of subsection 3a, with random variable *H* is entirely clear, since

$$\mathrm{Var}[l(H)] = \mathrm{Var}[H] = \mathrm{E}\big[\mathrm{Var}[H|\Theta]\big] + \mathrm{Var}\big[\mathrm{E}[H|\Theta]\big]. \tag{49}$$

5. An example

In this section we illustrate the results derived in Sections 3 and 4 for an example, involving an n-year endowment insurance. We assume that the amount insured is equal to 1.

The function $\beta(t)$, derived in (3), now reads

$$\beta(t) = \begin{cases} e^{-\delta(t\wedge n)}, & (t > 0) \\ 0. & (t = 0) \end{cases} \tag{50}$$

Payment of this insurance is by either single premium or n year continuous level premium. We will treat both cases separately.

The individual single premium for a person with risk parameter θ is equal to (see equation (3)):

$$SP(\theta) = \mathrm{E}[e^{-\delta(T\wedge n)} | \Theta = \theta], \tag{51}$$

while for the average single premium we have (cf. equation (4)):

$$\overline{SP} = \mathrm{E}[e^{-\delta(T\wedge n)}] = \mathrm{E}\big[\mathrm{E}[e^{-\delta(T\wedge n)} | \Theta]\big]. \tag{52}$$

To be able to compare the quantities computed in this subsection with the respective results to be derived in subsection 5b we will frequently make use of the following well known formula:

$$e^{-\delta(t\wedge n)} = 1 - \delta\, \bar{a}_{\overline{t\wedge n}|}. \tag{53}$$

Using the formulas (10), (11) and (53), we get for the individual level premium and average level premium

$$\pi(\theta) = \frac{\mathrm{E}[e^{-\delta(T\wedge n)} | \Theta{=}\theta]}{\mathrm{E}[\bar{a}_{\overline{T\wedge n}|} | \Theta{=}\theta]} = \frac{1}{\mathrm{E}[\bar{a}_{\overline{T\wedge n}|} | \Theta{=}\theta]} - \delta, \tag{54}$$

and

$$\bar{\pi} = \frac{\mathrm{E}[e^{-\delta(T\wedge n)}]}{\mathrm{E}[\bar{a}_{\overline{T\wedge n}|}]} = \frac{1}{\mathrm{E}[\bar{a}_{\overline{T\wedge n}|}]} - \delta, \tag{55}$$

respectively.

Table 1 displays the formulas for the components of the aggregate loss itself and components, while expressions for the variance parts of the aggregate loss are given in Table 2 (single premium payment) and Table 3 (level premium payment).

Table 1

The aggregate loss and its components for an n-year continuous level endowment insurance.

Component	Single premium payment	Level premium payment
$l_1(t;\theta)$	$e^{-\delta(t\wedge n)} - \mathrm{E}[e^{-\delta(T\wedge n)} \mid \Theta=\theta]$	$\dfrac{e^{-\delta(t\wedge n)} - \mathrm{E}[e^{-\delta(T\wedge n)} \mid \Theta=\theta]}{1-\mathrm{E}[e^{-\delta(T\wedge n)} \mid \Theta=\theta]}$
$l_2(t;\theta)$	$\mathrm{E}[e^{-\delta(T\wedge n)} \mid \Theta=\theta] - \mathrm{E}[e^{-\delta(T\wedge n)}]$	$\dfrac{\mathrm{E}[e^{-\delta(T\wedge n)} \mid \Theta=\theta] - \mathrm{E}[e^{-\delta(T\wedge n)}]}{1-\mathrm{E}[e^{-\delta(T\wedge n)}]}$
$l_3(t;\theta)$	0	$\dfrac{(e^{-\delta(t\wedge n)} - \mathrm{E}[e^{-\delta(T\wedge n} \mid \Theta=\theta])\,(\mathrm{E}[e^{-\delta(T\wedge n)}] - \mathrm{E}[e^{-\delta(T\wedge n)} \mid \Theta=\theta])}{(1-\mathrm{E}[e^{-\delta(T\wedge n)}])(1-\mathrm{E}[e^{-\delta(T\wedge n)} \mid \Theta=\theta])}$
$l(t)$	$e^{-\delta(t\wedge n)} - \mathrm{E}[e^{-\delta(T\wedge n)}]$	$\dfrac{e^{-\delta(t\wedge n)} - \mathrm{E}[e^{-\delta(T\wedge n)}]}{1-\mathrm{E}[e^{-\delta(T\wedge n)}]}$

Table 2

The loss variance and its components for an n-year level continuous endowment insurance in case of single premium payment.

Component	Formula
$\mathrm{E}\big[\mathrm{Var}[\beta(T) - \rho(T;\Theta) \mid \Theta]\big]$	$\mathrm{E}\big[\mathrm{Var}[e^{-\delta(T\wedge n)} \mid \Theta]\big]$
$\mathrm{Var}\big[\mathrm{E}[\beta(T) - \bar{\rho}(T) \mid \Theta]\big]$	$\mathrm{Var}\big[\mathrm{E}[e^{-\delta(T\wedge n)} \mid \Theta]\big]$
$\mathrm{E}\big[\mathrm{Var}[\rho(T;\Theta) - \bar{\rho}(T) \mid \Theta]\big]$	0
$2\,\mathrm{E}\big[\mathrm{Cov}[\beta(T) - \rho(T;\Theta),\ \rho(T;\Theta) - \bar{\rho}(T) \mid \Theta]\big]$	0
$\mathrm{Var}[l(T)]$	$\mathrm{Var}[e^{-\delta(T\wedge n)}]$

Table 3	
The loss variance and its components for an n-year level continuous endowment insurance in case of continuous level premium payment.	
Component	Formula
$\mathrm{E}\big[\mathrm{Var}[\beta(T) - \rho(T;\Theta)\,|\,\Theta]\big]$	$\mathrm{E}\left[\dfrac{\mathrm{Var}[e^{-\delta(T\wedge n)}\,|\,\Theta]}{(1-\mathrm{E}[e^{-\delta(T\wedge n)}\,|\,\Theta])^2}\right]$
$\mathrm{Var}\big[\mathrm{E}[\beta(T) - \bar{\rho}(T)\,|\,\Theta]\big]$	$\dfrac{\mathrm{Var}\big[\mathrm{E}[e^{-\delta(T\wedge n)}\,|\,\Theta]\big]}{(1-\mathrm{E}[e^{-\delta(T\wedge n)}])^2}$
$\mathrm{E}\big[\mathrm{Var}[\rho(T;\Theta) - \bar{\rho}(T)\,|\,\Theta]\big]$	$\dfrac{\mathrm{E}\left[\left(\dfrac{\mathrm{E}[e^{-\delta(T\wedge n)}\,|\,\Theta]-\mathrm{E}[e^{-\delta(T\wedge n)}]}{1-\mathrm{E}[e^{-\delta(T\wedge n)}\,|\,\Theta]}\right)^2 \mathrm{Var}[e^{-\delta(T\wedge n)}\,|\,\Theta]\right]}{(1-\mathrm{E}[e^{-\delta(T\wedge n)}])^2}$
$2\,\mathrm{E}\big[\mathrm{Cov}[\beta(T) - \rho(T;\Theta),$ $\rho(T;\Theta) - \bar{\rho}(T)\,|\,\Theta]\big]$	$2\,\dfrac{\mathrm{E}\left[\dfrac{\mathrm{E}[e^{-\delta(T\wedge n)}]-\mathrm{E}[e^{-\delta(T\wedge n)}\,|\,\Theta]}{(1-\mathrm{E}[e^{-\delta(T\wedge n)}\,|\,\Theta])^2}\mathrm{Var}[e^{-\delta(T\wedge n)}\,|\,\Theta]\right]}{1-\mathrm{E}[e^{-\delta(T\wedge n)}]}$
$\mathrm{Var}[l(T)]$	$\dfrac{\mathrm{Var}[e^{-\delta(T\wedge n)}]}{(1-\mathrm{E}[e^{-\delta(T\wedge n)}])^2}$

The next numerical example is based on the following proportional model:

$$\mu(x_0+s;\ \theta) = \mu(x_0+s)\ \theta, \tag{56}$$

where $\mu(x_0 + s)$, denoting the force of mortality of a "standard" individual at age $x_0 + s$, corresponds to the graduated Dutch life table for males (GBM 1985-1990) and θ is usually defined as the *frailty* variable, see, for instance, Vaupel et al. (1979).

We assume the structure distribution $u(\theta)$ to be of the form:

$$u(\theta) = 0.8\,I_{\{0.5\}}(\theta) + 0.2\,I_{\{4\}}(\theta), \tag{57}$$

using the discrete equivalent of (2) to calculate $F_{T|\Theta}(t, x_0; \theta)$. In (57), $I_{\{a\}}(\theta)$ is the indicator function, giving value 1 if $\Theta = a$ and 0 otherwise.

The age-at-issue, x_0, is equal to 35, while for n the value 30 is taken. The variance components in Tables 2 and 3 are based on an insured amount of 100 and calculated at annual interest rates of 4% and 8% (corresponding with values for δ equal to 0.03922 and 0.07696, respectively). The results are shown in Table 4.

Table 4

Illustration of the formulas in Tables 2 and 3, calculated at annual interest rates of 4% and 8%.

Component	Single premium payment		Level premium payment	
	4%	8%	4%	8%
$\mathrm{E}\big[\mathrm{Var}[\beta(T) - \rho(T;\Theta) \mid \Theta]\big]$	65.86	74.24	165.44	104.28
$\mathrm{Var}\big[\mathrm{E}[\beta(T) - \bar{\rho}(T) \mid \Theta]\big]$	9.45	8.16	21.49	10.68
$\mathrm{E}\big[\mathrm{Var}[\rho(T;\Theta) - \bar{\rho}(T) \mid \Theta]\big]$	0	0	0.93	0.29
$2\,\mathrm{E}\big[\mathrm{Cov}[\beta(T) - \rho(T;\Theta), \rho(T;\Theta) - \bar{\rho}(T) \mid \Theta]\big]$	0	0	−16.62	−7.48
$\mathrm{Var}[l(T)]$	75.31	82.40	171.24	107.77

The table above illustrates a result derived in Wolthuis & Kling (1992), namely that for insurances with non-increasing $\beta(t)$, payment by single premium gives the lowest value for the loss variance $\mathrm{Var}[l(T)]$. Note furthermore that, assuming level premium payment, the third component, $\mathrm{E}\big[\mathrm{Var}[\rho(T;\Theta) - \bar{\rho}(T) \mid \Theta]\big]$, is in both cases small, but that the covariance part is not negligible.

6. Solidarity aspects

In the middle of the 1980's some papers have been published on solidarity in non-life insurance, the main paper being De Wit & Van Eeghen (1984). In the latter the concepts *probabilistic solidarity* and *risk solidarity* are defined. As will be seen below each of them indicates a variance part of the right hand side of (49). In this section we will also discuss the paper of Posthuma (1985), which is based on non-life insurance contracts and has an approach similar to the one in Section 3, though from the insured's point of view.

In this section we will use the notation of subsection 3a. For an individual with parameter θ, the quantity $\mathrm{E}[H \mid \Theta = \theta]$ is called the *individual premium*.

De Wit & Van Eeghen (1984) define the *probabilistic solidarity* as

$$\mathrm{E}\big[\mathrm{Var}[H \mid \Theta]\big]. \tag{58}$$

In (31) the term reflecting the probabilistic solidarity is given by

$$\mathrm{E}\big[\mathrm{Var}[\beta(T) - \rho(T;\Theta)\,|\,\Theta]\big]. \tag{59}$$

Besides, De Wit & Van Eeghen denote the *risk solidarity* by

$$\mathrm{Var}\big[\mathrm{E}[H\,|\,\Theta]\big]. \tag{60}$$

The term in (31) corresponding to this concept is

$$\mathrm{Var}\big[\mathrm{E}[\beta(T) - \bar{\rho}(T)\,|\,\Theta]\big]. \tag{61}$$

Posthuma (1985) considers two quantities, having the same absolute value as $l_1(h;\theta)$ and $l_2(h;\theta)$ in (14), respectively, but opposite sign. The first component, the *ex post transfer*, denoted by $POST(h;\theta)$, is equal to minus the process effect:

$$POST(h;\theta) = -\,l_1(h;\theta) = \mathrm{E}[H\,|\,\Theta{=}\theta] - h. \tag{62}$$

If there is equivalence on an individual level, considering all insured with the same value θ, an individual with a positive ex post transfer (having a claim amount smaller than the individual premium) subsidizes a person with a negative ex post transfer (having a claim amount larger than the individual premium).

Since sign differences are eliminated by taking squares, the expectation of the second moment of all ex post transfers is also equal to the probabilistic solidarity in De Wit & Van Eeghen (1984).

$$\mathrm{E}\big[\mathrm{E}_T[(\mathrm{E}[H\,|\,\Theta] - H)^2\,|\,\Theta]\big] = \mathrm{E}\big[\mathrm{Var}[H\,|\,\Theta]\big]. \tag{63}$$

On the other hand, also in Posthuma (1985), the *ex ante transfer* has been introduced, being equal to minus the parameter effect:

$$ANTE(h;\theta) = -\big(l(h) - l_1(h;\theta)\big) = \mathrm{E}[H\,|\,\Theta{=}\theta] - \mathrm{E}[H],\ \text{for each } h. \tag{64}$$

The interpretation is that, if there is equivalence on a group level, a person with a positive ex ante transfer (a person who pays a net premium which is higher than the individual premium) subsidizes one with a negative ex ante transfer (someone whose net premium is lower than the individual premium).

Since the parameter effect and the ex ante transfer only differ with respect to sign, the risk solidarity in De Wit & Van Eeghen (1984) can also be obtained by taking the expectation of the square of the ex ante transfers:

$$\mathrm{E}\big[(\mathrm{E}[H\,|\,\Theta] - \mathrm{E}[H])^2\,|\,\Theta\big] = \mathrm{Var}\big[\mathrm{E}[H\,|\,\Theta]\big]. \tag{65}$$

As indicated in the previous subsection, life contracts paid by single premium are in this respect very similar to non-life contracts. To define the ex post transfer and the ex ante transfer, we only have to replace H by $\beta(T)$ and h by $\beta(t)$ in both formulas (62) and (64), respectively.

Problems arise when the premium payment is not restricted to time-at-issue. Similar to Spreeuw (1996), where the particular case of continuous level premium payment has been considered, we can define the ex ante transfer more generally as minus the expected loss for the individual contract:

$$ANTE(t;\theta) = -EL(\theta). \tag{66}$$

It also seems reasonable to define the ex post transfer as the loss in case of equivalence on an individual level, though with reversed sign:

$$POST(h;\theta) = -l_1(t;\theta) = \rho(t;\theta) - \beta(t). \tag{67}$$

Contrary to single premium payment, however, with these definitions we can not separate the total transfer (being minus the aggregate loss) in only an ex ante transfer and an ex post transfer. The reason is that one term remains, being equal to

$$-l_3(t;\theta) = -\left(\rho(t;\theta) - \bar{\rho}(t) - EL(\theta)\right), \tag{68}$$

indicating the deviation of the actual value of the premium differences, from the expected value.

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