# On the newsboy model with a cutoff transaction size

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**Abstract**

In this paper we analyse the effect of a cutoff transaction size in a simple newsboy setting. It is assumed that customers with an order larger than a prespecified size are satisfied in an alternative way, against additional cost. For compound Poisson demand with discrete order sizes, we show how to determine the average cost and an optimal cutoff transaction size. Because the computational effort to calculate the exact cost is quite large, we also consider an approximative model. By approximating the distribution of the total demand during a period by the normal distribution one can determine an expression for the average cost function that depends on the cutoff transaction size only. A main advantage of this approximation is that we can solve problems of any size. The quality of using the normal approximation is evaluated through a number of numerical experiments, which show that the approximative results are satisfactory.

*Keywords*: Inventory, newsboy model, erratic demand, cutoff transaction size.

## 1 Introduction

In practice, many inventory systems need to deal with erratic (or lumpy) demand patterns, which may be the result of occasionally occurring large transactions interspersed among

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a majority of small transactions (Silver [16], Hollier, Mak & Lam [5]). A well-recognised concept to prevent the large transactions from disturbing the inventory system is the use of a *cutoff transaction size*. With this concept, all demand smaller or equal than a prespecified cutoff transaction size (or maximum issue quantity, exceptional quantity, break quantity, weight limit, etc.) is routinely served, whereas large demand is alternatively delivered to the customer, e.g. by a direct delivery from a higher level stockpoint. Although this concept appears to gain popularity in practice (see e.g. Ballou [2], p. 43), it is not extensively analysed in the literature. The first theoretical contribution (to our knowledge) was made by Popp [14], who introduced the notion of a *combined inventory policy* where small demand was delivered from stock and large demand was delivered directly, thereby incurring a fixed setup cost. Using a rather approximative inventory model, he derived, for exponentially distributed order sizes and zero lead time, conditions under which the use of a cutoff transaction size was beneficial. Silver [16] discussed some factors contributing to an erratic demand pattern, and presented, for zero lead time and compound Poisson demand, a method to determine the average inventory cost and service levels in an $(s, S)$ model with a cutoff transaction size. More recently, Hollier, Mak & Lam [5, 6] and Mak & Lai [10, 11] presented results on the determination of the average cost in an $(s, S)$ model with compound Poisson demand and a positive lead time. They also considered a simple additional cost function for not satisfying customers with large transaction sizes. However, although their analysis is exact, the effect of the cutoff transaction size on the system performance is rather intransparent, i.e. the sensitivity of the cost function with respect to the cutoff transaction size is not obvious. Examples on the use of the concept in practice are given by Matz [12] and Nass, Dekker & Van Sonderen-Huisman [13]. Finally, for an overview on the effect of using a cutoff transaction size on the performance of a distribution system we refer to Kleijn & Dekker [7].

The objective of this paper is to analyse the effect of a cutoff transaction size on the average cost of a newsboy model, and determine a way of obtaining an optimal cutoff transaction size. We consider compound Poisson demand, and assume that the additional cost of alternatively delivering a large order (*overflow cost*) is known. In some practical situations this overflow cost function may represent the transportation cost of delivering a large order directly from the factory. Although in principle we are able to determine the optimal policy, we also analyse an approximative method in which the demand during a period is assumed to be normally distributed. This assumption enables us to derive an expression for the average cost function which only depends on the cutoff transaction size. Moreover, if the maximum size of a customer order and/or the arrival rate is large, the determination of the true optimal policy may fail due to computational problems. Using the normal approximation one can handle problems of any size. Finally, under the normal approximation it is possible to derive, for special cases of the overflow cost function, an upperbound on the optimal cutoff transaction size, which may also serve as a "quick and dirty" approximation of this optimal size.

Although the newsboy model in itself is of little (practical) importance, it is the building stone for a number of stochastic inventory models (Lee & Nahmias [9], Porteus [15]). For example, the newsboy model can easily be extended to multi-period, multi-echelon models with a positive lead time (see e.g. Clark & Scarf [3]). The results presented in this paper for normal distributed demand can also be used for these extended models (Dekker, Kleijn & De Kok [4]).

In Section 2 we derive exact expressions for the average cost of a newsboy model with a cutoff transaction size. Section 3 discusses an approximative analysis, where the demand during a period is assumed to be normally distributed. This allows for the derivation of an expression for the average cost as a function of only the cutoff transaction size. In Section 4 the quality of the normal approximation is tested through a number of computational experiments. The last section provides some concluding remarks.

## 2 Analysis of the newsboy model with a cutoff transaction size

In this section, the traditional newsboy model is extended with the notion of a cutoff transaction size. In this new model, demand from a customer is only satisfied from stock on hand if the size of the order does not exceed a prespecified cutoff transaction size, otherwise the customer is served in an alternative way, against additional cost. In order to distinguish customers by their order size, we assume that the demand is compound Poisson distributed. The notation that we use in this paper is listed in Table 1.

It is assumed that the starting inventory level is zero. Now the problem is to determine how much to order ($S$) and how to set the cutoff transaction size $q$, such that the expected cost over a single period is minimised. This problem reduces to the traditional newsboy

| | |
|---|---|
| $N$ | (Poisson distributed) number of customers arriving during a period |
| $\lambda$ | arrival rate of customers, i.e. $\lambda := \mathbb{E}(N)$ |
| $Y_i$ | (random) order size of $i$th customer |
| $a$ | order size distribution, i.e. $a(j) := \Pr\{Y_i = j\}$ |
| $M$ | maximum order size, i.e. $M := \max\{j \geq 0 : a(j) > 0\}$ |
| $q$ | cutoff transaction size |
| $1_{\{A\}}$ | indicator function of the event $A$ |
| $a_q$ | order size distribution for cutoff transaction size $q$, |
| | i.e. $a_q(j) := \Pr\{Y_i 1_{\{Y_i \leq q\}} = j\}$ |
| $D_q$ | (random) demand during a period for cutoff transaction size $q$, |
| | i.e. $D_q = \sum_{i=1}^{N} Y_i 1_{\{Y_i \leq q\}}$ |
| $f_q$ | pdf of demand during a period for cutoff transaction size $q$, |
| | i.e. $f_q(j) = \Pr\{D_q = j\}$ |
| $F_q$ | cdf of demand during a period for cutoff transaction size $q$, |
| | i.e. $F_q(j) = \Pr\{D_q \leq j\}$ |
| $S$ | order-up-to level |
| $S(q)$ | optimal order-up-to level for cutoff transaction size $q$ |
| $C(S,q)$ | expected total cost during a period for order-up-to level $S$ |
| | and cutoff transaction size $q$ |
| $C(q)$ | minimum expected cost during a period for cutoff transaction size $q$, |
| | i.e. $C(q) := C(S(q), q)$ |
| $c$ | unit ordering cost |
| $h$ | unit holding cost |
| $p$ | unit penalty cost |
| $\pi$ | *overflow cost function*, |
| | i.e. $\pi(j)$ denotes the cost of alternatively satisfying an order of size $j$ |

Table 1: Notation.

problem if the cutoff transaction size is set equal to infinity (or the maximum size $M$ of a customer order), since then all demand is handled on a routine basis. The total cost consists of

- ordering or production cost for realising the initial stock level
- holding cost for units in stock at the end of the period
- penalty cost for unsatisfied demand during the period
- overflow cost for alternatively delivering large orders during the period

It can be verified that the expected total cost for a period is given by

$$C(S,q) = IC(S,q) + OC(q)$$

with $IC(S,q)$ the expected inventory (ordering, holding and penalty) cost, given by

$$IC(S,q) = cS + h\sum_{j=0}^{S}(S-j)f_q(j) + p\sum_{j=S+1}^{\infty}(j-S)f_q(j)$$

and $OC(q)$ the expected overflow cost which equals

$$OC(q) = \lambda \mathbb{E}(\pi(Y_i)1_{\{Y_i>q\}}) = \lambda\sum_{j=q+1}^{M}\pi(j)a(j)$$

Our objective is to find the solution of the optimisation problem

$$\begin{aligned} \inf\{C(S,q) : 0 \leq S < \infty, 0 \leq q \leq M\} &= \\ \inf\{OC(q) + \inf\{IC(S,q) : 0 \leq S < \infty\} : 0 \leq q \leq M\} \end{aligned}$$

Observe that the optimisation problem $\inf\{IC(S,q) : 0 \leq S < \infty\}$ is just a standard newsboy problem with demand distribution $F_q$, and its solution is given by (see e.g. Porteus [15])

$$S(q) = \min\{j \in \mathbb{N}_+ : F_q(j) \geq \frac{p-c}{p+h}\} \tag{1}$$

Observe that if $p \leq c$, the optimal order-up-to level will be zero. Hence, we will henceforth assume that $p > c$. In order to calculate $S(q)$ we observe that the distribution function $f_q$ can be computed using Adelson's recursion scheme (Adelson [1]). With

$$a_q(j) = \begin{cases} a(0) + \sum_{i=q+1}^{M} a(i) & \text{if } j = 0 \\ a(j) & \text{if } j = 1, \ldots, q \\ 0 & \text{otherwise} \end{cases}$$

it follows that $f_q(j)$ satisfies the recursive relations (see e.g. Tijms [18])

$$f_q(j) = \begin{cases} e^{-\lambda(1-a_q(0))} & \text{if } j = 0 \\ (\lambda/j)\sum_{i=0}^{j-1}(j-i)a_q(j-i)f_q(i) & \text{if } j = 1, 2, \ldots \end{cases}$$

Hence, an efficient way to determine $S(q)$ would be to recursively calculate $f_q(0), \ldots, f_q(j)$ until $F_q(j) := \sum_{i=0}^{j} f_q(i) \geq (p-c)/(p+h)$. Substituting the optimal value $S(q)$ into the cost function $C(S,q)$ we get the one-dimensional minimum expected cost function

$$C(q) := \inf\{C(S,q) : 0 \leq S < \infty\} = C(S(q), q)$$

Since the order sizes are discrete and bounded by $M$, we can use enumeration over $q = 0, \ldots, M$ to find the optimal cutoff transaction size, i.e. the optimal solution of $\min\{C(q) : 0 \leq q \leq M\}$, and the associated expected cost. In fact, it can be verified that $C(S,x) \equiv C(S,q)$ with $q = \max\{j \leq x : a(j) > 0\}$, and hence one only needs to consider cutoff transaction sizes $q$ for which $a(q) > 0$. Observe that this will considerably reduce the computational effort needed to determine the optimal cutoff transaction size.

# 3 Approximative analysis of the newsboy model with a cutoff transaction size

A main problem of the exact analysis is the fact that the computation time increases exponentially when the arrival rate and/or the maximum size $M$ of a customer order increases. This problem does not occur if the total demand during a period $D_q$ is normally distributed. Justified by the central limit theorem, the normal distribution is often used as an approximation of the real demand distribution. In a recent paper, Tyworth & O'Neill [19] reported that although this approximation in many cases leads to a misspecification of the optimal policy parameters, the sensitivity of the expected optimal cost appears to be much less. We will now show that by approximating the distribution of the demand during a period by the normal distribution it is possible to obtain an easy expression for the minimum expected cost $C(q)$. First, we introduce some additional notation in Table 2.

For a given cutoff transaction size $q$, it follows (see e.g. Tijms [18]) that the first two moments $\mu_q$ and $\sigma_q^2$ of the variable $D_q$ are given by

$$\mu_q = \lambda \sum_{j=0}^{\infty} j a_q(j) = \lambda \sum_{j=1}^{q} j a(j)$$

| | |
|---|---|
| $\mu_q$ | mean demand during a period for cutoff transaction size $q$, i.e. $\mu_q = \mathbb{E}(D_q)$ |
| $\sigma_q^2$ | variance of demand during a period for cutoff transaction size $q$, i.e. $\sigma_q^2 = \mathrm{Var}(D_q)$ |
| $S_N(q)$ | optimal order-up-to level for cutoff transaction size $q$ and normal demand |
| $C_N(S,q)$ | expected total cost during a period for order-up-to level $S$, cutoff transaction size $q$ and normal demand |
| $C_N(q)$ | minimum expected cost during a period for cutoff transaction size $q$ and normal demand, i.e. $C_N(q) := C_N(S_N(q), q)$ |
| $\varphi, \Phi$ | pdf and cdf of the standard normal distribution |
| $G_q$ | cdf of normal distribution with mean $\mu_q$ and variance $\sigma_q^2$ |

Table 2: Additional notation.

and

$$\sigma_q^2 = \lambda \sum_{j=0}^{\infty} j^2 a_q(j) = \lambda \sum_{j=1}^{q} j^2 a(j)$$

Approximating $F_q$ by a normal distribution $G_q$ with mean $\mu_q$ and variance $\sigma_q^2$ it follows from (1) that the optimal order-up-to level is given by

$$S_N(q) = \min\{j \geq 0 : j \geq \mu_q + z\sigma_q\}$$

with $z := \Phi^{-1}((p-c)/(p+h))$ the safety stock multiplier. Moreover, the expected total cost during a period is approximated by

$$C_N(S,q) = cS + h\int_0^S (S-y)dG_q(y) + p\int_S^\infty (y-S)dG_q(y) + \lambda \sum_{j=q+1}^{M} \pi(j)a(j)$$

Although the optimal order-up-to level needs to be an integer, we substitute $S = \mu_q + z\sigma_q$ into the expected cost funtion $C_N(S,q)$ to obtain that the minimum expected cost during a period is approximately equal to (see e.g. Porteus [15])

$$\begin{aligned} C_N(q) &= c\mu_q + \sigma_q\left[(c+h)z + (p+h)I(z)\right] + \lambda \sum_{j=q+1}^{M} \pi(j)a(j) \\ &= \lambda \mathbb{E}(\pi(Y_i)) + k\sigma_q - \lambda \sum_{j=0}^{q} (\pi(j) - cj)a(j) \qquad (2) \end{aligned}$$

where $I(\cdot)$ denotes the normal loss function (Tijms [18], Silver & Peterson [17]) and $k := (c+h)z + (p+h)I(z)$. From Tijms [18] we learn that $I(z) = \varphi(z) - z(1 - \Phi(z))$, and since $z = \Phi^{-1}((p-c)/(p+h))$ this implies that $k = \varphi(z)$. Again, we can use enumeration over all cutoff transaction sizes $q$ for which $a(q) > 0$ to determine the optimal cutoff transaction size. However, for the approximative model it is possible to characterise an upperbound on the optimal cutoff transaction size.

**Lemma 3.1** *An optimal solution $q_N^*$ of the optimisation problem* $\inf\{C_N(q) : 0 \le q \le M\}$ *satisfies*

$$q_N^* \le q_u := \max\{j \ge 0 : \frac{1}{2}k\sigma_M^{-1}j^2 - (\pi(j) - cj) < 0\}$$

**Proof:** By (2) we obtain that

$$C_N(q) - C_N(q-1) = k(\sigma_q - \sigma_{q-1}) - \lambda(\pi(q) - cq)a(q)$$

for $0 < q \le M$. Since due to the concavity of the function $x \to \sqrt{x}$ it follows for $0 \le x < \infty$ and $0 < y < \infty$ that $\sqrt{y} - \sqrt{x} \ge \frac{1}{2}y^{-1/2}(y - x)$, we obtain (by taking $y = \sigma_q^2$ and $x = \sigma_{q-1}^2$) that

$$\sigma_q - \sigma_{q-1} \ge \frac{1}{2}\sigma_q^{-1}\lambda q^2 a(q)$$

Hence, since $\sigma_q^{-1} \ge \sigma_M^{-1}$ this yields

$$C_N(q) - C_N(q-1) \ge \lambda a(q)\left(\frac{1}{2}k\sigma_M^{-1}q^2 - (\pi(q) - cq)\right)$$

Since for any $q > q_u$ it follows that $C_N(q) \ge C_N(q_u)$ the desired result follows. □

An immediate consequence of the above result is that for an affine overflow cost function $\pi(j) = \pi_0 + \pi_1 j$ an upperbound on the optimal cutoff transaction order size is given by

$$q_u = \frac{(\pi_1 - c)\sigma_M}{k} + \sqrt{\frac{(\pi_1 - c)^2\sigma_M^2}{k^2} + \frac{2\pi_0\sigma_M}{k}} \qquad (3)$$

Since this upperbound is very easy to compute, it may be used as a "quick and dirty" approximation for the optimal cutoff transaction size.

This concludes our analysis of the newsboy model with a cutoff transaction size.

# 4 Computational results

The main objective of this section is to test the quality of the normal approximation, in particular with respect to the optimal cutoff transaction size and the maximum cost reduction that can be obtained by introducing a cutoff transaction size. We evaluated 4 different order size distributions and considered affine overflow cost. For each distribution we generated examples by choosing the parameter values from the following sets: $h \in \{1\}$, $p \in \{10, 50, 100, 500\}$, $c \in \{5, 10, 25, 50\}$, $\pi_0 \in \{0, 10, 25, 100\}$, $\pi_1 = c + \alpha(p - c)$ with $\alpha \in \{0.2, 0.4, 0.6, 0.8\}$ and $\lambda \in \{1, 2, 5, 10\}$. Since we require $p > c$, this leads to 768 different data sets for each distribution.

The first two order size distributions we used are based on examples given by Hollier, Mak & Lam [6] and Silver [16]. The third distribution is created using a geometric distribution with parameter 0.2 ($a(j) := 0.2 \times 0.8^{j-1}$) for $j = 1, \ldots, 15$, and setting the tail of the distribution at $j = 25$. The last order size distribution was based on real-life demand data from the CERN laboratory (Krever [8]), scaled with a factor 100 to allow for the calculation of the exact cost. In Table 3 the order size distributions are presented. For each order size distribution, we calculated for all 768 cases the cost reduction obtained by using a cutoff transaction size. The values in column *exact* report the relative cost reductions obtained using the exact cost analysis, i.e.

$$\frac{C(M) - \inf\{0 \leq q \leq M : C(q)\}}{C(M)}$$

whereas in column *approximation* the relative cost reductions are presented which were obtained using the approximative cost function, i.e.

$$\frac{C_N(M) - \inf\{0 \leq q \leq M : C_N(q)\}}{C_N(M)}$$

We also calculated the relative cost reduction that one gets when the cutoff transaction size equals the cutoff transaction size that minimises the approximative cost function, i.e.

$$\frac{C(M) - C(q_N^*)}{C(M)}$$

with $q_N^*$ the optimal solution of $\inf\{0 \leq q \leq M : C_N(q)\}$. This reduction is presented in column *optimal normal*. Finally, column *upperbound* reports the relative cost reduction related to using the upperbound $q_u$ given in (3) as the cutoff transaction size, i.e.

$$\frac{C(M) - C(q_u)}{C(M)}$$

In Table 4 the minimum, average and maximum relative cost reduction ($\times$ 100%) are presented.

| **order size distribution 1** (mean 4.49, variance 35.48, coef. of var. 1.32) | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 12 |
| $a(i)$ | 0.35 | 0.20 | 0.07 | 0.15 | 0.01 | 0.02 | 0.02 | 0.05 | 0.01 | 0.02 |
| $i$ | 14 | 15 | 18 | 20 | 24 | 25 | 28 | 30 | 36 | 40 |
| $a(i)$ | 0.02 | 0.05 | 0.005 | 0.004 | 0.003 | 0.002 | 0.005 | 0.002 | 0.003 | 0.001 |
| $i$ | 42 | 45 | 50 | | | | | | | |
| $a(i)$ | 0.002 | 0.002 | 0.001 | | | | | | | |
| **order size distribution 2** (mean 2.1, variance 54.99, coef. of var. 3.53) | | | | | | | | | | |
| $i$ | 1 | 5 | 75 | | | | | | | |
| $a(i)$ | 0.90 | 0.09 | 0.01 | | | | | | | |
| **order size distribution 3** (mean 5.16, variance 25.01, coef. of var. 0.96) | | | | | | | | | | |
| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $a(i)$ | 0.20 | 0.16 | 0.128 | 0.102 | 0.082 | 0.066 | 0.052 | 0.042 | 0.034 | 0.027 |
| $i$ | 11 | 12 | 13 | 14 | 15 | 25 | | | | |
| $a(i)$ | 0.022 | 0.017 | 0.014 | 0.011 | 0.009 | 0.034 | | | | |
| **order size distribution 4** (mean 11.16, variance 127.73, coef. of var. 1.01) | | | | | | | | | | |
| $i$ | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 10 | 11 | 12 |
| $a(i)$ | 0.18 | 0.02 | 0.08 | 0.04 | 0.14 | 0.02 | 0.02 | 0.10 | 0.14 | 0.02 |
| $i$ | 13 | 16 | 18 | 20 | 21 | 22 | 30 | 35 | 38 | 46 |
| $a(i)$ | 0.02 | 0.02 | 0.02 | 0.04 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 | 0.02 |
| $i$ | 50 | | | | | | | | | |
| $a(i)$ | 0.02 | | | | | | | | | |

Table 3: Order size distributions.

If the maximum order size or the arrival rate of customers is rather large, then it is computationally impossible to use the exact cost function $C(S, q)$ in order to determine the optimal cutoff transaction size and the corresponding relative cost reduction. In this case, one may use the approximative cost function $C_N(S, q)$. Comparing the columns *exact* and *approximation*, we can see how well the relative cost reduction is estimated when using the approximative cost function. From Table 4 we see that using the normal approximation

| order size distribution | | exact | approximation | optimal normal | upperbound |
|---|---|---|---|---|---|
| 1 | min | 0.00 | 0.00 | -6.00 | -3.00 |
| | av. | 5.85 | 6.76 | 4.99 | 3.35 |
| | max | 52.00 | 55.00 | 52.00 | 34.00 |
| 2 | min | 0.00 | 0.00 | -9.00 | -3.00 |
| | av. | 20.51 | 29.41 | 17.49 | 12.93 |
| | max | 67.00 | 74.00 | 67.00 | 61.00 |
| 3 | min | 0.00 | 0.00 | -7.00 | -6.00 |
| | av. | 3.75 | 4.31 | 3.34 | 2.01 |
| | max | 47.00 | 50.00 | 47.00 | 32.00 |
| 4 | min | 0.00 | 0.00 | -6.00 | -2.00 |
| | av. | 4.37 | 5.06 | 3.99 | 2.40 |
| | max | 48.00 | 50.00 | 48.00 | 34.00 |

Table 4: Minimum (min), average (av.) and maximum (max) relative cost reduction ($\times$ 100%) obtained by cutoff transaction size for 768 data sets.

of the demand generates results which are close to the exact results. However, the relative cost reduction tends to be overestimated when the approximative cost function is used. In particular for order size distribution 2, the difference is about 9%, although it should be noted that this order size distribution is highly lumpy. Since the order size can only attain 3 different values (1, 5 or 75), the use of the normal approximation of the total demand can easily lead to relatively large errors. The other distributions are smoother, and the results of using the approximative cost function are much better.

It is also interesting to determine the quality of the optimal cutoff transaction size obtained by minimising the approximative cost function. Comparing columns *exact* and *optimal normal*, we see that on average the relative cost reduction when using the approximative optimal cutoff transaction size is only slightly smaller than optimal. However, in some cases, it leads to a bad performance, i.e. an increase in cost. The worst case, for order size distribution 2, led to an increase of 9%. An analysis of the worst cases revealed that they occurred for $\lambda = 1$, i.e. the lowest arrival rate. Since the normal approximation is justified using the central limit theorem, it is clear that its quality will increase if the arrival rate of customers increases.

Using the upperbound $q_u$ as a "quick and dirty" approximation of the optimal cutoff transaction size gives satisfactory results. Although the relative cost reductions are less than optimal, the worst case behaviour is good compared to the worst case behaviour of using the cutoff transaction size that minimises the approximative cost function. Finally, one can observe that the relative cost reduction appears to increase with the coefficient of variation of the order size distribution, which is defined as the ratio of the standard deviation and the mean of the order size distribution. This can be explained by the fact that the inventory holding cost is increasing with the variability of the demand. If this variability is relatively large, then rejecting the demand from a small fraction of customers with large order sizes will cause a significant reduction in the demand variability and thus the inventory holding and shortage cost.

Intuitively, one could imagine that there is a relation between the optimal cutoff transaction size and the order size distribution. In particular, it is expected that an optimal cutoff transaction size will coincide with a peak in the distribution. However, this does not seem to be the case in general. For all 4 order size distributions we determined the percentage of cases that a certain cutoff transaction size was optimal, and plotted these results against the order size distribution in Figures 1, 2, 3 and 4.

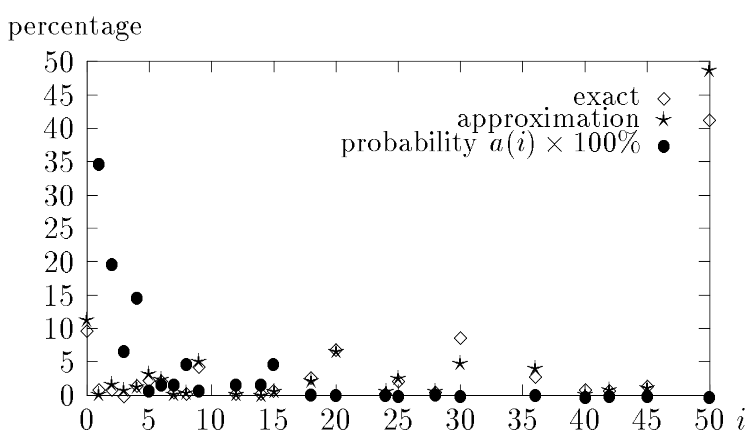


Figure 1: Percentage of cases that cutoff transaction size $i$ was optimal for distribution 1.

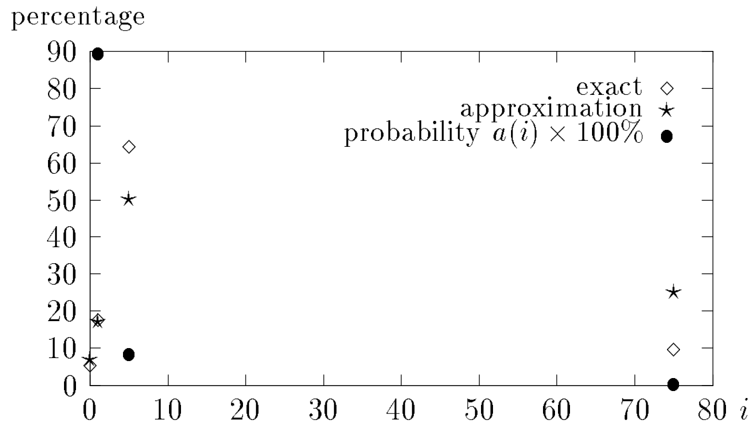


Figure 2: Percentage of cases that cutoff transaction size $i$ was optimal for distribution 2.

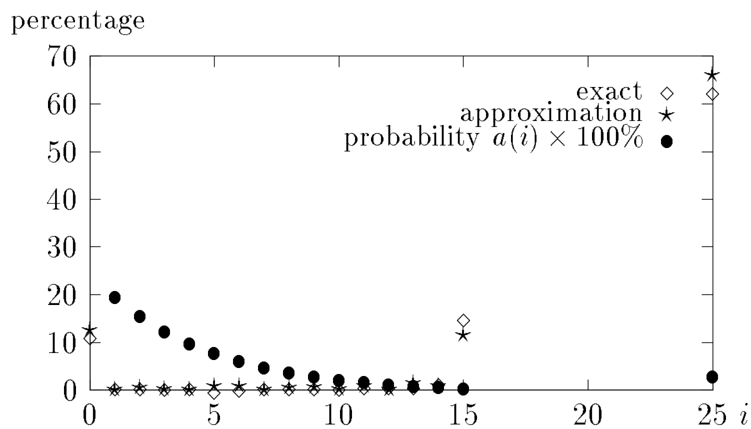


Figure 3: Percentage of cases that cutoff transaction size $i$ was optimal for distribution 3.

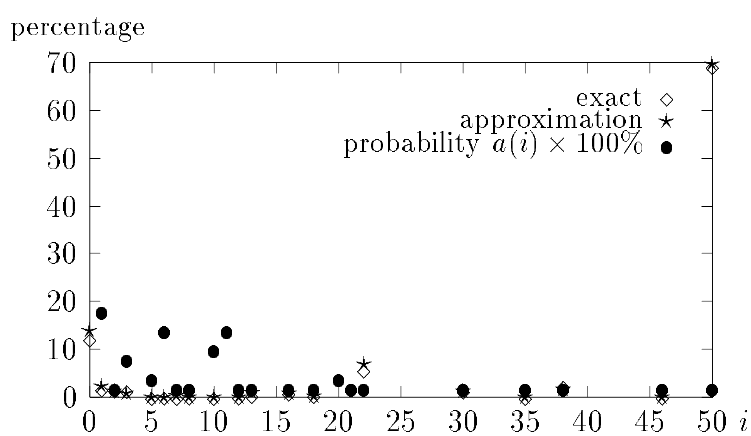


Figure 4: Percentage of cases that cutoff transaction size $i$ was optimal for distribution 4.

From Table 3 we read that order size distribution 1 has a large peak at $i = 15$ and a small one at $i = 28$. However, from Figure 1 it follows that $q = 20$ and $q = 30$ were popular choices for the optimal cutoff transaction size. For distributions 2 and 3 the relation between the order size distributions and the optimal cutoff transaction sizes was significant. In 65% of the cases for distribution 2 the optimal cutoff transaction size was equal to 5, whereas for distribution 3 in 15% of the cases $q = 15$ was optimal. Hence, in both situations a good policy seemed to be to satisfy all order sizes, except the largest one. For order size distribution 4 the results were similar to distribution 1. The popular optimal cutoff transaction size 22 did not correspond to a peak in the order size distribution. We also mention that, for all distributions, the percentage of cases with $q = 0$ optimal was about 10%. In these cases the additional cost of not satisfying demand from stock on hand was less than the cost of holding inventory. The optimal cutoff transaction size was equal to the maximum size of a customer order in respectively 42%, 10%, 62% and 69% of the cases. Since a cutoff transaction size equal to the maximum order size is equivalent with a traditional policy without the concept of a cutoff transaction size, the cost reduction in these cases was zero. Finally, again the results of the approximative model are closely related to the exact results, as can be observed from Figures 1, 2, 3 and 4.

We now consider two arbitrary examples for order size distribution 4. For both examples, we plotted in Figures 5 and 6 the exact cost function $C(q)$ and the approximative cost function $C_N(q)$ for all values of the cutoff transaction size $q$.

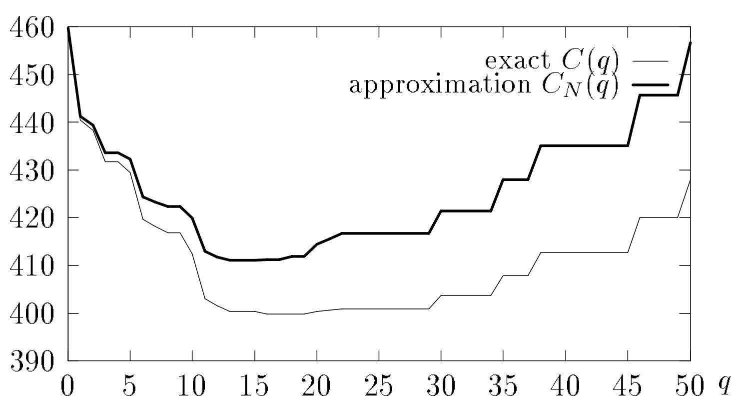


Figure 5: A plot of $C(q)$ and $C_N(q)$, for distribution 4 and $c = 5$, $p = 10$, $\pi_0 = 25$, $\pi_1 = 6$ and $\lambda = 5$.

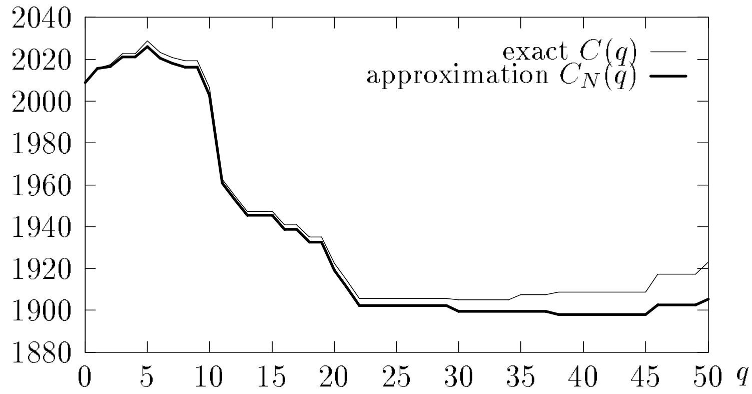


Figure 6: A plot of $C(q)$ and $C_N(q)$, for distribution 4 and $c = 10$, $p = 50$, $\pi_0 = 0$, $\pi_1 = 18$ and $\lambda = 10$.

As can be seen, the normal approximation is closer to the exact cost when the arrival rate is larger. However, even though there is a significant difference between the exact and approximated cost in Figure 5, the shapes of the function are similar and the optimal cutoff transaction sizes are close (18 (*exact*) vs. 13 (*approximation*)), and also the relative cost reductions are similar (7% vs. 10%). The value of the upperbound $q_u$ in Figure 5 was equal to 27, which led to a reduction of 6% in average cost. The difference between the true cost function and the approximative cost function is very small in Figure 6. Also the optimal cutoff transaction sizes and relative cost reductions (30 and 0.07% (*exact*) vs. 38 and 0.04% (*approximation*)) are comparable. The value of the upperbound was equal to 50 (the maximum size of a customer order).

To conclude our section on computation results, we observe from Figures 5 and 6 that in general the cost functions $C(q)$ and $C_N(q)$ do not have a shape that could allow us to design a straight algorithm to find the optimal $q$. Together with the observation that the optimal cutoff transaction size does not have a relation with the order size distribution, this justifies the use of enumeration to find the optimal cutoff transaction size. The remark at the end of Section 2, that only cutoff transaction sizes $q$ for which $a(q) > 0$ need to be evaluated, can also be verified from Figures 5 and 6. It can be observed that $C(q + 1) = C(q)$ and $C_N(q + 1) = C_N(q)$ if $a(q + 1) = 0$.

# 5 Concluding remarks

In this paper an analysis of the newsboy model, extended with the notion of a cutoff transaction size, was presented. This extension allows the delivery of large demands in an alternative way, thus preventing the large orders from disrupting the inventory system. The main contributions are the derivation of the exact cost and an approximative expression of the cost as a function of only the cutoff transaction size. The approximation originates from fitting a normal distribution on the distribution of the total demand during a period. From the computational experiments it follows that this approximative analysis gives satisfactory results. A major advantage of using the normal approximation is the fact that it requires much less computational effort. Therefore, it can handle order size distributions with a wide range of possible order sizes, whereas the computational effort needed to calculate the exact cost grows exponentionally with the range of the order size distribution.

The results presented in Section 4 indicate that the optimisation problem associated with finding the optimal cutoff transaction size is in general not an easy problem due to the nonconvexity of the average cost function. Since only a relatively small number of cutoff transaction sizes need to be evaluated, the use of enumeration to find the optimal policy is justified.

Finally, we mention that it is possible to extend the approximative results to more general systems, since the newsboy type equations appear in many inventory models. For example, the results can be extended to a multi-period, multi-echelon inventory system with positive lead times (see Dekker, Kleijn & De Kok [4]).

# References

[1] R.M. Adelson. Compound Poisson distributions. *Operations Research Quarterly*, 17:73–75, 1966.

[2] R.H. Ballou. *Business Logistics Management, 3rd edition*. Prentice Hall, Englewood Cliffs, NJ, 1992.

[3] A.J. Clark and H. Scarf. Optimal policies for a multi–echelon inventory problem. *Management Science*, 6:475–490, 1960.

[4] R. Dekker, M.J. Kleijn, and A.G. De Kok. The break quantity rule's effect on inventory costs in a 1–warehouse, $N$–retailers distribution system. *To appear in the International Journal of Production Economics*, 1997.

[5] R.H. Hollier, K.L. Mak, and C.L. Lam. Continuous review $(s, S)$ policies for inventory systems incorporating a cutoff transaction size. *International Journal of Production Research*, 33:2855–2865, 1995.

[6] R.H. Hollier, K.L. Mak, and C.L. Lam. An inventory model for items with demands satisfied from stock or by special deliveries. *International Journal of Production Economics*, 42:229–236, 1995.

[7] M.J. Kleijn and R. Dekker. Using break quantities for tactical optimisation in multi-stage distribution systems. Technical Report 9737/A, Econometric Institute, Erasmus University Rotterdam, The Netherlands, 1997.

[8] M. Krever. L.I.M.S.: Logistics Inventory Management System. Technical Report v96.1, CERN Laboratory, Geneva, Switzerland, 1996.

[9] H.L. Lee and S. Nahmias. Single–product, single–location models. In S.C. Graves, A.H.G. Rinnooy Kan, and P. Zipkin, editors, *Handbooks in OR & MS, Vol. 4*. Elsevier Science Publishers B.V., North–Holland, 1993.

[10] K.L. Mak and K.K. Lai. The determination of optimal partial fill policy for an inventory system with lumpy demand items. *Applied Mathematical Modelling*, 19:724–737, 1995.

[11] K.L. Mak and K.K. Lai. Optimal $(s, S)$ policy for an inventory system when the demand can be partially satisfied. *International Journal of Systems Science*, 26:213–231, 1995.

[12] A.W. Matz. The shop counter maximum issue problem. *OMEGA*, 11:75–79, 1983.

[13] R. Nass, R. Dekker, and W. van Sonderen-Huisman. Distribution optimization by means of break quantities: A case study. *To appear in the Journal of the Operational Research Society*, 1997.

[14] W. Popp. Simple and combined inventory policies, production to stock or to order? *Management Science*, 11:868–873, 1965.

[15] E.L. Porteus. Stochastic inventory theory. In D.P. Heyman and M.J. Sobel, editors, *Handbooks in OR & MS, vol. 2*. Elsevier Science Publishers B.V., North–Holland, 1990.

[16] E.A. Silver. Some ideas related to the inventory control of items having erratic demand patterns. *Journal of the Canadian Operations Research Society*, 8:87–100, 1970.

[17] E.A. Silver and R. Peterson. *Decision Systems for Inventory Management and Production Planning*. John Wiley & Sons, New York, 1985.

[18] H.C. Tijms. *Stochastic Models: An Algorithmic Approach*. Wiley, New York, 1994.

[19] J.E. Tyworth and L. O'Neill. Robustness of the normal approximation of lead-time demand in a distribution setting. *Naval Research Logistics*, 44:165–186, 1997.