Approximation by Penultimate Stable Laws

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Abstract

In certain cases partial sums of i.i.d. random variables with finite variance are better approximated by a sequence of stable distributions with indices $\alpha_n \to 2$ than by a normal distribution. We discuss when this happens and how much the convergence rate can be improved by using penultimate approximations. Similar results are valid for other stable distributions.

Key words & phrases: Partial sums, stable distribution, penultimate.

1 Introduction

Let $X_1, X_2, \cdots$ be independent random variables with common distribution function F. We assume that F is either in the domain of attraction of a stable law with index less than 2, that is

$$\begin{cases} \lim_{t\to\infty} \frac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} = x^{-\alpha}, \quad x > 0 \\ \lim_{t\to\infty} \frac{1-F(t)}{1-F(t)+F(-t)} = p \end{cases} \tag{1.1}$$

for some parameters $\alpha \in (0,2)$ and $p \in [0,1]$, or in the domain of attraction of a normal law, i.e.

$$S(x) := \int_0^x (1 - F(u) + F(-u))u\,du \in RV_0.$$

Then there exist $a_n > 0$ and $b_n \in R$ such that

$$\lim_{n\to\infty} P\{\sum_{i=1}^{n} X_i/a_n - b_n \leq x\} = G_\alpha(x) \tag{1.2}$$

for all x where G_α is a stable distribution function for $\alpha \in (0, 2)$ and

$$G_2(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-u^2/2}\, du.$$

Rate of convergence result in connection with (1.2) can be given under second order conditions. First let's concentrate on the case $\alpha < 2$.

Suppose there exists a function A with $\lim_{t\to\infty} A(t) = 0$ and not changing sign near infinity, such that

$$\begin{cases} \lim_{t\to\infty} \frac{\frac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} - x^{-\alpha}}{A(t)} = x^{-\alpha}\frac{x^\rho - 1}{\rho}, \quad x > 0 \\ \lim_{t\to\infty} \frac{\frac{1-F(t)}{1-F(t)+F(-t)} - p}{A(t)} = q. \end{cases} \tag{1.3}$$

Here q is a real constant. The function $|A|$ is then regularly varying with non-positive index ρ (notation: $|A(t)| \in RV_\rho$).

De Haan and Peng (1996) proved that under condition (1.3) for a suitable choice of the sequences a_n and b_n

$$\lim_{n\to\infty} \sup_{x\in R} |P\{\sum_{i=1}^{n} X_i/a_n - b_n \leq x\} - G_\alpha(x)|/|A(a_n)| \tag{1.4}$$

exists and is positive.

Now the question is: can we improve the convergence rate by using a sequence of stable distribution function G_{α_n} with $\alpha_n \to \alpha$ instead of G_α in relation (1.4)? In order to answer this question we note that an intermediate step in settling (1.4) is a second order relation for the characteristic function of F,

$$f(t) := \int_{-\infty}^{\infty} e^{itx}\, dF(x).$$

We take as an example the case $1 < \alpha < 2$ and $1 - F(x) = F(-x)$ for $x > 0$. In this case the relation for f is the following (see Lemma 1 of de Haan and Peng (1996)):

$$\lim_{n\to\infty} \frac{-n\log f(t/a_n) + \log g_\alpha(t)}{A(a_n)} = |t|^\alpha (s_\alpha + d_{\alpha-\rho}\frac{|t|^{-\rho} - 1}{\rho}), \tag{1.5}$$

where

$$g_\alpha(t) = \exp\{-|t|^\alpha , (1-\alpha)\cos\frac{\pi\alpha}{2}\}$$

is the characteristic function of $G_\alpha(x)$ and

$$\begin{aligned} d_\alpha &:= \textstyle\int_0^\infty x^{-\alpha}\sin x\,dx \\ &= , (1-\alpha)\sin\tfrac{\pi(1-\alpha)}{2} \quad (0<\alpha<2) \\ s_\alpha &:= \textstyle\int_0^\infty x^{-\alpha}\log x\sin x\,dx \\ &= , (1-\alpha)\sin\tfrac{\pi(1-\alpha)}{2}\{\tfrac{\Gamma'(1-\alpha)}{\Gamma(1-\alpha)} + \tfrac{\pi}{2}ctg\tfrac{\pi(1-\alpha)}{2}\} \quad (0<\alpha<2). \end{aligned}$$

We want to replace α by a sequence α_n for which

$$\lim_{n\to\infty}\frac{-n\log f(t/a_n)+\log g_{\alpha_n}(t)}{A(a_n)} = 0. \tag{1.6}$$

Note that for $n\to\infty$

$$\begin{aligned} & -\log g_{\alpha_n}(t) + \log g_\alpha(t) \\ = \;& |t|^{\alpha_n}, (1-\alpha_n)\cos\tfrac{\pi\alpha_n}{2} - |t|^\alpha, (1-\alpha)\cos\tfrac{\pi\alpha}{2} \\ = \;& (|t|^{\alpha_n} - |t|^\alpha), (1-\alpha_n)\cos\tfrac{\pi\alpha_n}{2} \\ & +|t|^\alpha(, (1-\alpha_n)\cos\tfrac{\pi\alpha_n}{2} - , (1-\alpha)\cos\tfrac{\pi\alpha}{2}) \\ \sim \;& (\alpha_n-\alpha)|t|^\alpha\log|t|, (1-\alpha)\cos\tfrac{\pi\alpha}{2} \\ & -(\alpha_n-\alpha)|t|^\alpha[, '(1-\alpha)\cos\tfrac{\pi\alpha}{2} + \tfrac{\pi}{2}, (1-\alpha)\sin\tfrac{\pi\alpha}{2}] \\ = \;& (\alpha_n-\alpha)|t|^\alpha\log|t|d_\alpha \\ & -(\alpha_n-\alpha)|t|^\alpha s_\alpha. \end{aligned}$$

This shows that if we take $\alpha_n := \alpha - A(a_n)$,

$$\lim_{n\to\infty}\frac{\log g_{\alpha_n}(t) - \log g_\alpha(t)}{A(a_n)} = \lim_{n\to\infty}\frac{-n\log f(t/a_n)+\log g_\alpha(t)}{A(a_n)}$$

when $\rho = 0$. So with that choice

$$\lim_{n\to\infty}\frac{-n\log f(t/a_n)+\log g_{\alpha_n}(t)}{A(a_n)} = 0,$$

i.e. the convergence rate can be improved.

If ρ is less than zero, then for no choice of α_n cancellation is possible, so we can not improve the convergence rate in this case.

Next let's consider the case $\alpha = 2$.

Since $x^{-2}S(x) \downarrow 0$ as $x \to \infty$, the function

$$a(x) := \sup\{a : 2a^{-2}S(a) \geq x^{-1}\}$$

is well defined for $x > 1/2$. We have

$$2x(a(x))^{-2}S(a(x)) = 1. \tag{1.7}$$

De Haan and Peng (1997) proved that

$$\lim_{n\to\infty} sup_{x\in R} \frac{|P(\sum_{i=1}^{n} X_i/a(n) \leq x) - G_2(x)|}{n(1-F(a(n))+F(-a(n)))} \tag{1.8}$$

exists and is positive under the condition

$$\begin{cases} 1-F(x)+F(-x) \in RV_{\rho-2} \quad (-1<\rho\leq 0) \\ \lim_{x\to\infty} \frac{1-F(x)}{1-F(x)+F(-x)} = p^* \in [0,1]. \end{cases} \tag{1.9}$$

Using the same arguments as in the case $\alpha < 2$, we find that the rate of (1.8) can be improved only in the case $\rho = 0$ of (1.9).

The result in section 2 shows that for $1 < \alpha < 2$ the convergence rate can be improved 'a little' if the condition (1.3) holds for $\rho = 0$, that is, if the convergence rate is slow. In that case the convergence rate $A(a_n)$ is replaced by $\{A(a_n)\}^2$. See also Remark 2.2 about the case $0 < \alpha \leq 1$.

In section 3 we consider the normal limit distribution. We shall show that if (1.9) holds for $\rho = 0$ the convergence rate can be improved ' a little' when one approximates by a sequence of stable distributions with $\alpha_n \to 2$ instead of by the normal distribution. In that case the rate $n(1-F(a(n))+F(-a(n)))$ is replaced by $[n(1-F(a(n))+F(-a(n)))]^2$. The phenomenon has been observed before in Iglesias Pereira, Oliveira and Pestana (1996) and Oliveira (1996).

2 Main result for $\alpha \in (1, 2)$

Throughout this section we assume that $\alpha \in (1,2)$ (but see Remark 2.2 for $0 < \alpha \leq 1$) and $EX_1 = 0$. We now need an even more stringent condition than the second order condition (1.3). In fact we need a third order condition:

suppose there exists a function $A_0(t)$ with $\lim_{t\to\infty} A_0(t) = 0$ and not changing sign near infinity such that

$$\lim_{t\to\infty} \frac{\frac{\frac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} - x^{-\alpha}}{A(t)} - x^{-\alpha}\log x}{A_0(t)} = H(x) \tag{2.1}$$

where $H(x)$ is not a multiple of $x^{-\alpha}\log x$ and suppose

$$\lim_{t\to\infty} \frac{\frac{1-F(t)}{1-F(t)+F(-t)} - p}{A^2(t)} = q_0 \in (-\infty, \infty). \tag{2.2}$$

Note that (2.1) is equivalent to

$$\lim_{t\to\infty} \frac{(tx)^\alpha K(tx) - t^\alpha K(t) - A(t)t^\alpha K(t)\log x}{A(t)t^\alpha K(t)A_0(t)} = x^\alpha H(x) \tag{2.3}$$

where $K(x) := 1 - F(x) + F(-x)$. From Theorem 1 of de Haan and Stadtmüller (1996) we can assume

$$H(x) = x^{-\alpha}\frac{1}{\rho'}[\frac{x^{\rho'} - 1}{\rho'} - \log x] \quad (\rho' \le 0). \tag{2.4}$$

Let $U(t)$ denote the generalized inverse of the function $1/(1 - F(t) + F(-t))$. If (1.1) holds, $1 < \alpha < 2$ and $EX_1 = 0$, the sequence $\sum_{i=1}^n X_i/U(n)$ converges in distribution to G_α whose characteristic function is

$$g_\alpha(t) := \exp\{-|t|^\alpha, (1-\alpha)[\cos\frac{\pi\alpha}{2} - i\mathbf{sgn}(t)(2p-1)\sin\frac{\pi\alpha}{2}]\}$$

where $\mathbf{sgn}(t) = \begin{cases} 1, & t \ge 0 \\ -1, & t < 0. \end{cases}$

Now we can state our main results.

Theorem 2.1. *Let F be a non-lattice distribution function. Suppose (2.1), (2.2) and (2.4) hold for some $1 < \alpha < 2$ and $\rho' < 0$. Then (recall $EX_1 = 0$)*

$$\begin{aligned} &\lim_{n\to\infty}\{A(U(n))\}^{-2}[P(\textstyle\sum_{i=1}^n X_i/U(n) \le x) - G_{\alpha - A(U(n))}(x)] \\ = \;& -\tfrac{1}{2\pi}\int_{-\infty}^{\infty} \tfrac{e^{-itx}}{-it} g_\alpha(t)(C_1(t) + i\mathbf{sgn}(t)C_2(t))\,dt \end{aligned} \tag{2.5}$$

uniformly for all x, where

$$C_1(t) = \int_0^\infty [-(x/|t|)^{-\alpha}(\log(x/|t|))^2/2]\sin x\,dx$$

and

$$C_2(t) = \int_0^\infty [-(2p-1)(x/|t|)^{-\alpha}(\log(x/|t|))^2/2) + 2q_0(x/|t|)^{-\alpha}](1-\cos x)\,dx.$$

Remark 2.1. *Suppose (2.1), (2.2) and (2.4) hold for $\rho' = 0$. Assume*

$$\lim_{t\to\infty} A_0(t)/A(t) = c \in (-\infty, \infty).$$

Then the left hand side of (2.5) still exists, but the limit function is different.

Remark 2.2. *For $\alpha \le 1$ we also have a version of Theorem 2.1 when F is assumed to be symmetric (cf. Remark 5 of de Haan and Peng (1996)).*

3 Main result for $\alpha = 2$

We assume throughout this section that $EX_1 = 0$ and that G^*_α is the stable law with characteristic function

$$g^*_\alpha = \exp\{-|t|^\alpha(1-\alpha/2),\ (1-\alpha)[\cos\frac{\pi\alpha}{2} - i\mathbf{sgn}(t)(2p^*-1)\sin\frac{\pi\alpha}{2}]\}.$$

Define $\alpha^*_n := 2 - 2n(1 - F(a(n)) + F(-a(n)))$. We now need a condition stronger than (1.8). Suppose there exists a function $A^*(t)$ with $\lim_{t\to\infty} A^*(t) = 0$ and not changing sign near infinity such that

$$\left\{\begin{array}{ll} \lim_{t\to\infty} \frac{\frac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} - x^{-2}}{A^*(t)} = x^{-2}\frac{x^{\rho^*}-1}{\rho^*}, & x > 0 \\ \lim_{t\to\infty} \frac{\frac{1-F(t)}{1-F(t)+F(-t)} - p^*}{t(1-F(a(t))+F(a(t)))} = q^*, & \end{array}\right. \tag{3.1}$$

where $\rho^* \le 0$ and q^* is a real constant.

Now we state our main theorem.

Theorem 3.1. *Let F be a non-lattice distribution function. Suppose (3.1) holds for $\rho^* < 0$. Then*

$$\begin{array}{rl} & \lim_{n\to\infty} \frac{P(\sum_{i=1}^n X_i/a(n) \le x) - G^*_{\alpha^*_n}(x)}{[n(1-F(a(n))+F(-a(n)))]^2} \\ = & -\frac{1}{2\pi}\int_{-\infty}^\infty \frac{e^{-itx}}{-it} e^{-t^2/2}[C^*_1(t) + i\mathbf{sgn}(t)C^*_2(t)]\,dt \end{array} \tag{3.2}$$

uniformly for all x, where

$$\begin{array}{rl} C^*_1(t) & = -2\int_{|t|}^\infty (x/|t|)^{-2}\log(x/|t|)\sin x\,dx \\ & -2\int_0^{|t|}(x/|t|)^{-2}\log(x/|t|)(\sin x - x)\,dx \end{array}$$

and

$$C_2^*(t) = \int_0^\infty [-2(2p^*-1)(x/|t|)^{-2}\log(x/|t|) + 2q^*(x/|t|)^{-2}](1-\cos x)\,dx.$$

Remark 3.1. *Suppose (3.1) holds for $\rho^* = 0$. Assume that*

$$\lim_{t\to\infty} A_0^*(t)/[n(1-F(a(n))+F(-a(n)))] = c_0 \in (-\infty,\infty).$$

Then the left hand of (3.2) still exists, but the limit function is different.

4 Proofs

For the proof of Theorem 2.1 we need a number of auxiliary results.

Lemma 4.1. *Suppose that (2.1) and (2.4) hold for $\rho' < 0$. Then for $x > 0$*

$$\lim_{t\to\infty} \frac{\frac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} - x^{-\alpha+A(t)}}{A^2(t)} = -x^{-\alpha}(\log x)^2/2.$$

Proof. Relation (2.1) implies that $A(t)$ is slowly varying and $A_0(t)$ is ρ'-varying, hence

$$\lim_{t\to\infty} A_0(t)/A(t) = 0. \tag{4.1}$$

Note that

$$\begin{aligned} & \frac{\frac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} - x^{-\alpha+A(t)}}{A^2(t)} \\ = & \frac{\frac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} - x^{-\alpha} - A(t)x^{-\alpha}\log x}{A(t)A_0(t)} \times \frac{A_0(t)}{A(t)} \\ & - \frac{x^{-\alpha+A(t)} - x^{-\alpha} - A(t)x^{-\alpha}\log x}{A^2(t)}. \end{aligned} \tag{4.2}$$

We now use (2.1), (4.1), (4.2) and

$$\begin{aligned} & x^y - x^{y_0} - (y-y_0)x^{y_0}\log x \\ = & \tfrac{1}{2}(y-y_0)^2 x^{y_0+\theta(y-y_0)}(\log x)^2 \\ \le & \tfrac{1}{2}(y-y_0)^2 x^{y_0}(\log x)^2 e^{|(y-y_0)\log x|} \end{aligned} \tag{4.3}$$

for all $x > 0$, where $\theta \in [0,1]$. Lemma 4.1 follows easily. $\square$

Lemma 4.2. *Suppose (2.1), (2.2) and (2.4) hold for $\rho' < 0$. Then*

$$\frac{n(1 - F(U(n)x) + F(-U(n)x)) - x^{-\alpha + A(U(n))}}{A^2(U(n))} \to -x^{-\alpha}(\log x)^2/2 \qquad (4.4)$$

and

$$\begin{aligned} &\frac{n(1-F(U(n)x)-F(-U(n)x))-(2p-1)x^{-\alpha+A(U(n))}}{A^2(U(n))} \\ \to\ & -(2p-1)x^{-\alpha}(\log x)^2/2 + 2q_0 x^{-\alpha}. \end{aligned} \qquad (4.5)$$

Proof. Similar to the proof of Proposition 2 of de Haan and Peng (1996) using Lemma 4.1 and (2.2). □

The following lemma is an extension of a result of Drees (1995).

Lemma 4.3. *Let l be a measurable function. Suppose there exist a real parameter γ and functions $a_1(t) > 0$ and $a_2(t) \to 0$ with constant sign near infinity such that for all $x > 0$*

$$\lim_{t\to\infty} \frac{\frac{l(tx)-l(t)}{a_1(t)} - \frac{x^\gamma - 1}{\gamma}}{a_2(t)} = \tilde{h}(x)$$

exists as a finite limit and $\tilde{h}(x)$ is not a multiple of $\frac{x^\gamma-1}{\gamma}$. The function a_1 is the regularly varying of index γ and $|a_2(t)|$ regularly varying of index $\beta \le 0$.

Then there exist functions $a_3(t) > 0$ and $a_4(t)$ (with $|a_4(t)| > 0$) with the property that for all $\epsilon, \epsilon' > 0$ there exists $t_0 > 0$ such that for all $t \ge t_0$, $tx \ge t_0$

$$x^{-\gamma-\beta} e^{-\epsilon'|\log x|} \left| \frac{\frac{l(tx)-l(t)}{a_3(t)} - \frac{x^\gamma-1}{\gamma}}{a_4(t)} - h(x) \right| \le \epsilon$$

where

$$h(x) = \begin{cases} (\log x)^2/2 & \text{for } \beta = 0, \gamma = 0 \\ x^\gamma \log x & \text{for } \beta = 0, \gamma \neq 0 \\ \frac{x^{\gamma+\beta}-1}{\gamma+\beta} & \text{for } \beta < 0. \end{cases}$$

Proof. Suppose $\beta = 0$ and $\gamma = 0$. We proceed as in Omey and Willekens (1988) and Drees (1995). Write

$$l_1(t) := l(t) - \frac{1}{t}\int_0^t l(s)\, ds. \qquad (4.6)$$

Then l_1 is in the class Π. Hence by de Haan and Pereira (1997, Appendix) there exists a slowly varying function L with the property that for all $\epsilon^{(1)}, \epsilon^{(2)} > 0$ there exists $t_0 > 0$ such that for $t \geq t_0$, $tx \geq t_0$

$$e^{-\epsilon^{(2)}|\log x|}\left|\frac{l_1(tx) - l_1(t)}{L(t)} - \log x\right| \leq \epsilon^{(1)}. \tag{4.7}$$

Next note that (4.6) implies

$$l(t) = l_1(t) + \int_0^t \frac{l_1(s)}{s}\, ds.$$

Hence

$$\begin{aligned} & \tfrac{l(tx)-l(t)-l_1(t)\log x - L(t)\log x}{L(t)} - (\log x)^2/2 \\ = \quad & \tfrac{l_1(tx)-l_1(t)}{L(t)} - \log x + \int_1^x \left(\tfrac{l_1(ts)-l_1(t)}{L(t)} - \log s\right) ds. \end{aligned}$$

Choose $\epsilon > 0$. By (4.7) for $t \geq t_0$, $tx \geq t_0$

$$\begin{aligned} & \left|\tfrac{l(tx)-l(t)-l_1(t)\log x - L(t)\log x}{L(t)} - (\log x)^2/2\right| \\ \leq \quad & \epsilon^{(1)} e^{\epsilon^{(2)}|\log x|} + \epsilon^{(1)}\left|\int_1^x e^{\epsilon^{(2)}|\log x|}\tfrac{ds}{s}\right| \\ = \quad & \epsilon^{(1)} e^{\epsilon^{(2)}|\log x|} + \tfrac{\epsilon^{(1)}}{\epsilon^{(2)}}\left|e^{\epsilon^{(2)}|\log x|} - 1\right|. \end{aligned}$$

Let $\epsilon^{(1)}/\epsilon^{(2)} \leq \epsilon$, $\epsilon^{(2)} \leq \epsilon \wedge \epsilon'$. Then the expression is at most $\epsilon e^{\epsilon'|\log x|}$.

For $\beta = 0$ and $\gamma > 0$. By Theorem 2 of de Haan and Stadtmüller (1996) the function $t^{-\gamma} l(t)$ is in the class Π. Hence by (4.7) for each $\epsilon, \epsilon' > 0$ there exists $t_0 > 0$ such that for $t \geq t_0$, $tx \geq t_0$

$$\begin{aligned} & x^{-\gamma}\left|\tfrac{l(tx)-l(t)-\gamma l(t)\frac{x^\gamma - 1}{\gamma}}{t^\gamma L(t)} - x^\gamma \log x\right| \\ = \quad & \left|\tfrac{(tx)^{-\gamma} l(tx) - t^{-\gamma} l(t)}{L(t)} - \log x\right| \\ \leq \quad & \epsilon e^{\epsilon|\log x|}. \end{aligned}$$

Similarly for $\beta = 0$ and $\gamma < 0$. For $\beta < 0$, from Theorem 2 of de Haan and Stadtmüller (1996) we have for some positive $\tilde{a}_1$ and all $x > 0$

$$\lim_{t\to\infty} \frac{l_2(tx) - l_2(t)}{\tilde{a}_1(t)} = \frac{x^{\gamma+\beta} - 1}{\gamma + \beta}$$

with $l_2(t) = l(t) - c\frac{t^\gamma - 1}{\gamma}(c > 0)$. Hence by de Haan and Pereira (1997, Appendix) there exist $a_5(t) > 0$ with the property that for all $\epsilon, \epsilon' > 0$ there exists $t_0 > 0$ such that for $t \geq t_0$, $tx \geq t_0$

$$\begin{aligned} & x^{-\gamma-\beta}\left|\frac{l(tx)-l(t)-ct^\gamma\frac{x^\gamma-1}{\gamma}}{a_5(t)} - \frac{x^{\gamma+\beta}-1}{\gamma+\beta}\right| \\ = \ & x^{-\gamma-\beta}\left|\frac{l_2(tx)-l(t)}{a_5(t)} - \frac{x^{\gamma+\beta}-1}{\gamma+\beta}\right| \\ \leq \ & \epsilon e^{-\epsilon'|\log x|}. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 4.4. *Suppose the conditions of Lemma 4.1 hold. Then for any $\epsilon > 0$ there exists $t_0 > 0$ such that for all $t \geq t_0$, $tx \geq t_0$*

$$\begin{aligned} & \left|\frac{\frac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} - x^{-\alpha+A(t)}}{A^2(t)}\right| \\ \leq \ & \epsilon x^{-\alpha}(|\log x| + e^{\epsilon|\log x|}) + \frac{1}{2}x^{-\alpha}(\log x)^2 e^{\epsilon|\log x|}. \end{aligned}$$

Proof. Note that (2.3) implies

$$\lim_{t\to\infty}\frac{(tx)^\alpha K(tx) - t^\alpha K(t) - A(t)t^\alpha K(t)(1+\frac{A_0(t)}{\rho'})\log x}{A(t)t^\alpha K(t)A_0(t)} = \frac{1}{\rho'}\frac{x^{\rho'}-1}{\rho'}.$$

By Lemma 4.3 for any $\epsilon > 0$ there exist functions $a_1(t)$, $a_2(t)$ and $t_0 > 0$ such that for all $t \geq t_0$, $tx \geq t_0$

$$\left|\frac{(tx)^\alpha K(tx) - t^\alpha K(t) - a_1(t)\log x}{a_2(t)} - \frac{1}{\rho'}\frac{x^{\rho'}-1}{\rho'}\right| \leq \epsilon x^{\rho'} e^{\epsilon|\log x|}. \qquad (4.8)$$

It is easy to see that

$$\begin{cases} \frac{a_1(t)}{A(t)t^\alpha K(t)} \to 1 \\ \frac{a_2(t)}{A(t)t^\alpha K(t)A_0(t)} \to 1 \\ \frac{a_1(t)-A(t)t^\alpha K(t)}{A(t)t^\alpha K(t)A_0(t)} \to 1/\rho'. \end{cases}$$

Note that

$$
\begin{aligned}
& \frac{\frac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} - x^{-\alpha+A(t)}}{A^2(t)} \\
= \; & x^{-\alpha}\frac{(tx)^{\alpha}K(tx) - t^{\alpha}K(t) - A(t)t^{\alpha}K(t)\log x}{A(t)t^{\alpha}K(t)A_0} \times \frac{A_0(t)}{A(t)} \\
& - \frac{x^{-\alpha+A(t)} - x^{-\alpha} - A(t)x^{-\alpha}\log x}{A^2(t)} \\
= \; & x^{-\alpha}\Big[\frac{(tx)^{\alpha}K(tx) - t^{\alpha}K(t) - a_1(t)\log x}{a_2(t)} - \frac{1}{\rho'}\frac{x^{\rho'}-1}{\rho'}\Big] \times \\
& \frac{a_2(t)}{A(t)t^{\alpha}K(t)A_0(t)} \times \frac{A_0(t)}{A_0(tx)} \times \frac{A_0(tx)}{A(tx)} \times \frac{A(tx)}{A(t)} \\
& + x^{-\alpha}(\log x)\frac{a_1(t) - A(t)t^{\alpha}K(t)}{A(t)t^{\alpha}K(t)A_0(t)} \times \frac{A_0(t)}{A(t)} \\
& + x^{-\alpha}\frac{x^{\rho'}}{(\rho')^2}\frac{a_2(t)}{A(t)t^{\alpha}K(t)A_0(t)} \times \frac{A_0(t)}{A_0(tx)} \times \frac{A_0(tx)}{A(tx)} \times \frac{A(tx)}{A(t)} \\
& - x^{-\alpha}(\rho')^{-2}\frac{a_2(t)}{A(t)t^{\alpha}K(t)A_0(t)} \times \frac{A_0(t)}{A(t)} \\
& - \frac{x^{-\alpha+A(t)} - x^{-\alpha} - A(t)x^{-\alpha}\log x}{A^2(t)}
\end{aligned}
$$

and $A_0(t)/A(t) \to 0$. Using (4.3), (4.8) and Potter bounds,

$$
\begin{aligned}
& \Big|\frac{\frac{1-F(tx)+F(-tx)}{1-F(t)+F(-t)} - x^{-\alpha+A(t)}}{A^2(t)}\Big| \\
\le \; & x^{-\alpha}\epsilon(e^{\epsilon|\log x|} + |\log x|) + \tfrac{1}{2}x^{-\alpha}(\log x)^2 e^{\epsilon|\log x|}.
\end{aligned}
$$

This completes the proof of the lemma. □

Lemma 4.5. *Suppose the conditions of Lemma 4.2 hold. Then for any* $\epsilon > 0$ *there exists* $N_0 > 0$ *such that for all* $n \ge N_0$, $U(n) \ge N_0$, $U(n)x \ge N_0$

$$
\begin{aligned}
& \Big|\frac{n[1-F(U(n)x)+F(-U(n)x)] - x^{-\alpha+A(U(n))}}{A^2(U(n))}\Big| \\
\le \; & \epsilon x^{-\alpha}(|\log x| + e^{\epsilon|\log x|}) \\
& + \tfrac{1}{2}x^{-\alpha}(\log x)^2 e^{\epsilon|\log x|}
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
& \Big|\frac{n[1-F(U(n)x)-F(-U(n)x)] - (2p-1)x^{-\alpha+A(U(n))}}{A^2(U(n))}\Big| \\
\le \; & \epsilon x^{-\alpha}(|\log x| + e^{\epsilon|\log x|}) \\
& + \frac{|2p-1|}{2}x^{-\alpha}(\log x)^2 e^{\epsilon|\log x|} \\
& + 2q_0 x^{-\alpha}e^{\epsilon|\log x|}.
\end{aligned}
\tag{4.10}
$$

Proof. Note that

$$
\begin{aligned}
& \frac{n[1-F(U(n)x)+F(-U(n)x)] - x^{-\alpha+A(U(n))}}{A^2(U(n))} \\
= \; & \frac{\frac{1-F(U(n)x)+F(-U(n)x)}{1-F(U(n))+F(-U(n))} - x^{-\alpha+A(U(n))}}{A^2(U(n))} \\
& + \frac{1-F(U(n)x)+F(-U(n)x)}{1-F(U(n))+F(U(n))} \times \frac{n[1-F(U(n))+F(-U(n))]-1}{A^2(U(n))}
\end{aligned}
$$

and

$$\begin{cases} \frac{1-F(U(n)x)+F(-U(n)x)}{1-F(U(n))+F(-U(n))} \in RV_{-\alpha} \\ \frac{n[1-F(U(n))+F(-U(n))]-1}{A^2(U(n))} \to 0. \end{cases}$$

Thus (4.9) follows from Lemma 4.4 and Potter bounds.

Note that

$$\begin{aligned} & \frac{n[1-F(U(n)x)-F(-U(n)x)]-(2p-1)x^{-\alpha+A(U(n))}}{A^2(U(n))} \\ = \quad & (2p-1)\frac{n[1-F(U(n)x)+F(-U(n)x)]-x^{-\alpha+A(U(n))}}{A^2(U(n))} \\ & +n[1-F(U(n))+F(-U(n))] \times \frac{1-F(U(n)x)+F(-U(n)x)}{1-F(U(n))+F(-U(n))} \times \\ & \frac{\frac{1-F(U(n)x)-F(-U(n)x)}{1-F(U(n)x)+F(-U(n)x)}-(2p-1)}{A^2(U(n)x)} \times \frac{A^2(U(n)x)}{A^2(U(n))}. \end{aligned}$$

Hence (4.10) follows easily. This completes the proof of the lemma. □

Lemma 4.6. *Suppose the conditions of Lemma 4.2 hold. Let f denote the characteristic function of F. Define $\alpha_n = \alpha - A(U(n))$. Then*

$$\begin{aligned} & \lim_{n\to\infty} \frac{-n\log f(t/U(n))+\log g_{\alpha_n}(t)}{A^2(U(n))} \\ = \quad & \int_0^\infty [-(x/|t|)^{-\alpha}(\log(x/|t|))^2/2] \sin x \, dx \\ & +i\mathbf{sgn}(t)\int_0^\infty [-(2p-1)(x/|t|)^{-\alpha}(\log(x/|t|))^2/2 \\ & +2q_0(x/|t|)^{-\alpha}](1-\cos x)\, dx \\ =: \quad & C_1(t) + i\mathbf{sgn}(t)C_2(t). \end{aligned}$$

Proof. Note that for $|t| \neq 0$

$$\begin{aligned} & n(1-f(t/U(n))) - \log g_{\alpha_n}(t) \\ = \quad & n\int_0^\infty t\sin(tx)[1-F(U(n)x)+F(-U(n)x)]\, dx \\ & +in\int_0^\infty t(1-\cos(tx))[1-F(U(n)x)-F(-U(n)x)]\, dx \\ & -|t|^{\alpha_n}, (1-\alpha_n)\cos\tfrac{\pi\alpha_n}{2} \\ & +i\mathbf{sgn}(t)|t|^{\alpha_n}, (1-\alpha_n)(2p-1)\sin\tfrac{\pi\alpha_n}{2} \\ = \quad & \int_0^\infty [n(1-F(U(n)x/|t|)+F(-U(n)x/|t|)) - (x/|t|)^{-\alpha_n}] \sin x\, dx \\ & +i\mathbf{sgn}(t)\int_0^\infty [n(1-F(U(n)x/|t|)-F(-U(n)x/|t|)) \\ & -(2p-1)(x/|t|)^{-\alpha_n}](1-\cos x)\, dx. \end{aligned}$$

By Lemma 4.5, $\alpha > 1$ and Lebesgue's dominated convergence theorem we have

$$\begin{aligned} & \frac{1}{A^2(U(n))}\int_1^\infty [n(1-F(U(n)x/|t|)+F(-U(n)x/|t|)) - (x/|t|)^{-\alpha_n}] \sin x\, dx \\ & \to \int_1^\infty [-(x/|t|)^{-\alpha}(\log(x/|t|))^2/2] \sin x\, dx. \end{aligned}$$

By Lemma 4.5, $|\frac{\sin x}{x}| \leq 1$ as $0 \leq x \leq 1$, $\alpha < 2$ and Lebesgue's dominated convergence theorem we have

$$\frac{1}{A^2(U(n))} \int_{|t|N_0/U(n)}^1 [n(1 - F(U(n)x/|t|) + F(-U(n)x/|t|)) - (x/|t|)^{-\alpha_n}] \sin x \, dx$$
$$\to \int_0^1 [-(x/|t|)^{-\alpha} (\log(x/|t|))^2/2] \sin x \, dx.$$

Combining

$$\begin{aligned}
& \frac{1}{A^2(U(n))} | \int_0^{|t|N_0/U(n)} n[1 - F(U(n)x/|t|) + F(-U(n)x/|t|)] \sin x \, dx| \\
= & \frac{1}{A^2(U(n))} | \int_0^1 n(1 - F(N_0 y) + F(-N_0 y)) \frac{|t|N_0}{U(n)} \sin(|t|N_0 y/U(n)) \, dy| \\
= & O(\frac{n}{U^2(n)A^2(U(n))}) \to 0 \quad (\text{ since } U \in RV_{1/\alpha})
\end{aligned}$$

and

$$\frac{1}{A^2(U(n))} | \int_0^{|t|N_0/U(n)} (x/|t|)^{-\alpha_n} \sin x \, dx| \to 0$$

(similar to the proof of the above relation), we get

$$\begin{aligned}
& \frac{1}{A^2(U(n))} \int_0^\infty [n(1 - F(U(n)x/|t|) + F(-U(n)x/|t|)) \\
& -(x/|t|)^{-\alpha_n}] \sin x \, dx \\
\to & C_1(t).
\end{aligned}$$

Similarly

$$\begin{aligned}
& \frac{1}{A^2(U(n))} \int_0^\infty [n(1 - F(U(n)x/|t|) - F(-U(n)x/|t|)) \\
& -(2p-1)(x/|t|)^{-\alpha_n}](1 - \cos x) \, dx \\
\to & C_2(t).
\end{aligned}$$

When expanding $-\log f = -\log(1 - (1 - f))$, we find that the second (and higher) order term is of lower order, hence the result of the lemma. □

Lemma 4.7. *Suppose the conditions of Lemma 4.2 hold. Then for any $\epsilon > 0$ there exists $N_0 > 0$ such that for all $n \geq N_0$, $U(n) \geq N_0$, $U(n)/|t| \geq N_0$*

$$\begin{aligned}
& |\frac{-n \log f(t/U(n)) + \log g_{\alpha_n}(t)}{A^2(U(n))}| \\
\leq & C(|t|^\alpha (1 + |\log|t|| + (\log|t|)^2)(1 + e^{\epsilon|\log t|}))
\end{aligned}$$

where C is a positive constant.

Proof. It follows by using the same arguments as in the proofs of Lemma 4.4 and Lemma 4.5. □

Proof of Theorem 2.1. The proof is quite similar to the proof of Theorem 1 of de Haan and Peng (1996) by using Lemma 4.6 and Lemma 4.7. □

For the proof of Theorem 3.1 we need also some lemmas.

Lemma 4.8. *Suppose (3.1) holds for $\rho^* < 0$. Then for $x > 0$*

$$\lim_{n\to\infty} \frac{\frac{1-F(a(n)x)+F(-a(n)x)}{1-F(a(n))+F(-a(n))} - x^{-\alpha_n^*}}{n(1-F(a(n))+F(-a(n)))} = -2x^{-2}\log x \tag{4.11}$$

and

$$\lim_{n\to\infty} \frac{\frac{1-F(a(n)x)-F(-a(n)x)}{1-F(a(n))+F(-a(n))} - (2p^*-1)x^{-\alpha_n^*}}{n(1-F(a(n))+F(-a(n)))} = -2(2p^*-1)x^{-2}\log x + 2q^*x^{-2}. \tag{4.12}$$

Proof. From $S(x) \in RV_0$, (1.7) and $\rho^* < 0$ we have

$$\lim_{n\to\infty} \frac{A^*(n)}{n(1-F(a(n))+F(-a(n)))} = 0. \tag{4.13}$$

Combining (4.13) with

$$\begin{aligned} & \frac{\frac{1-F(a(n)x)+F(-a(n)x)}{1-F(a(n))+F(-a(n))} - x^{-\alpha_n^*}}{n(1-F(a(n))+F(-a(n)))} \\ = \quad & \frac{\frac{1-F(a(n)x)+F(-a(n)x)}{1-F(a(n))+F(-a(n))} - x^{-2}}{A^*(n)} \times \frac{A^*(n)}{n(1-F(a(n))+F(-a(n)))} \\ & + \frac{x^{-2}-x^{-\alpha_n^*}}{n(1-F(a(n))+F(-a(n)))} \end{aligned} \tag{4.14}$$

and

$$x^y - x^{y_0} = (y-y_0)x^{y_0+\theta(y-y_0)}\log x \quad \theta \in [0,1], \tag{4.15}$$

we have (4.11). Note that

$$\begin{aligned} & \frac{1-F(a(n)x)-F(-a(n)x)}{1-F(a(n))+F(-a(n))} - (2p^*-1)x^{-\alpha_n^*} \\ = \quad & (2p^*-1)\left[\frac{1-F(a(n)x)+F(-a(n)x)}{1-F(a(n))+F(-a(n))} - x^{-\alpha_n^*}\right] \\ & + \frac{1-F(a(n)x)+F(-a(n)x)}{1-F(a(n))+F(-a(n))} \times \left[\frac{1-F(a(n)x)-F(-a(n)x)}{1-F(a(n)x)+F(-a(n)x)} - (2p^*-1)\right] \end{aligned}$$

and $n(1-F(a(n))+F(-a(n))) \in RV_0$. (4.12) follows easily. □

Lemma 4.9. *Suppose (3.1) holds for $\rho^* < 0$. Then for any $\epsilon > 0$ there exists $N_0 > 0$ such that for all $n \geq N_0$, $a(n) \geq N_0$, $a(n)x \geq N_0$*

$$\begin{aligned} & \frac{|\frac{1-F(a(n)x)+F(-a(n)x)}{1-F(a(n))+F(-a(n))} - x^{-\alpha_n^*}|}{n(1-F(a(n))+F(-a(n)))} \\ \leq \quad & \epsilon x^{-2} e^{\epsilon|\log x|} + 2x^{-2}\log x| e^{\epsilon|\log x|} \end{aligned}$$

and

$$\begin{aligned} & \frac{|\frac{1-F(a(n)x)-F(-a(n)x)}{1-F(a(n))+F(-a(n))} - (2p^*-1)x^{-\alpha_n^*}|}{n(1-F(a(n))+F(-a(n)))} \\ \leq \quad & \epsilon x^{-2} e^{\epsilon||logx|} + 2|2p^*-1|x^{-2}|\log x| e^{\epsilon|\log x|} + 2q^* x^{-2} e^{\epsilon|\log x|}. \end{aligned}$$

Proof. Similar to the proofs of Lemma 2.4 and Lemma 2.5. □

Lemma 4.10. *Suppose (3.1) holds for $\rho^* < 0$. Then*

$$\begin{aligned} & \lim_{n\to\infty} \frac{-n\log f(t/a(n)) + \log g^*_{\alpha_n^*}(t)}{[n(1-F(a(n))+F(-a(n)))]^2} \\ = \quad & -2\int_{|t|}^{\infty} (x/|t|)^{-2}\log(x/|t|)\sin x\, dx \\ & -2\int_0^{|t|} (x/|t|)^{-2}\log(x/|t|)(\sin x - x)\, dx \\ & +i\mathbf{sgn}(t)\int_0^{\infty} [-2(2p^*-1)(x/|t|)^{-2}\log(x/|t|) + 2q^*(x/|t|)^{-2}](1-\cos x)\, dx. \end{aligned}$$

Proof. Note that for $t \neq 0$

$$\begin{aligned} & n(1-f(t/a(n)) - \log g^*_{\alpha_n^*}(t) \\ = \quad & \int_0^{\infty} \{n(1-F(a(n)x/|t|) + F(-a(n)x/|t|)) - (1-\alpha_n^*/2)(x/|t|)^{-\alpha_n^*}\}\sin x\, dx \\ & +i\mathbf{sgn}(t)\int_0^{\infty} \{n(1-F(a(n)x/|t|) - F(-a(n)x/|t|)) \\ & -(2p^*-1)(1-\alpha_n^*/2)(x/|t|)^{-\alpha_n^*}\}(1-\cos x)\, dx \\ = \quad & \int_{|t|}^{\infty} \{n(1-F(a(n)x/|t|) + F(-a(n)x/|t|)) - (1-\alpha_n^*/2)(x/|t|)^{-\alpha_n^*}\}\sin x\, dx \\ & +\int_0^{|t|} \{n(1-F(a(n)x/|t|) + F(-a(n)x/|t|)) \\ & -(1-\alpha_n^*/2)(x/|t|)^{-\alpha_n^*}\}(\sin x - x)\, dx \\ & +\int_0^{|t|} \{n(1-F(a(n)x/|t|) + F(-a(n)x/|t|)) - (1-\alpha_n^*/2)(x/|t|)^{-\alpha_n^*}\}x\, dx \\ & +i\mathbf{sgn}(t)\int_0^{\infty} \{n(1-F(a(n)x/|t|) - F(-a(n)x/|t|)) \\ & -(2p^*-1)(1-\alpha_n^*/2)(x/|t|)^{-\alpha_n^*}\}(1-\cos x)\, dx \end{aligned}$$

and

$$\begin{aligned} & \int_0^{|t|} \{n(1-F(a(n)x/|t|) + F(-a(n)x/|t|)) - (1-\alpha_n^*/2)(x/|t|)^{-\alpha_n^*}\}x\, dx \\ = \quad & |t|^2\{\int_0^1 n(1-F(a(n)x) + F(-a(n)x))x\, dx - (1-\alpha_n^*/2)\tfrac{1}{2-\alpha_n^*}\} \\ = \quad & 0 \quad (\text{ by } (1.7)). \end{aligned}$$

The rest of the proof is similar to that of Lemma 4.6. □

Lemma 4.11. *Suppose (3.1) holds for* $\rho^* < 0$*. Then for any* $\epsilon > 0$ *there exists* $N_0 > 0$ *such that for all* $n \geq N_0$*,* $a(n) \geq N_0$*,* $a(n)/|t| \geq N_0$

$$\frac{|-n\log f(t/a(n)) + \log g^*_{\alpha^*_n}(t)|}{[n(1-F(a(n))+F(-a(n)))]^2} \leq C^*|t|^2(1+|\log|t||)(1+e^{\epsilon|\log|t||})$$

where C^* *is a positive constant.*

Proof. Similar to the proof of Lemma 4.7 by using Lemma 4.10. □

Proof of Theorem 3.1. The proof is quite similar to the proof of Theorem 1 of de Haan and Peng (1997) by using Lemma 4.10 and Lemma 4.11. □

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