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Bootstrapping GARCH Models Under Dependent Innovations

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Abstract

This study reflects on the inconsistency of the fixed-design residual bootstrap procedure for GARCH models under dependent innovations. We introduce a novel recursive-design residual block bootstrap procedure to accurately quantify the uncertainty around parameter estimates and volatility forecasts. A simulation study provides evidence for the validity of the recursive-design residual block bootstrap in the presence of dependent innovations. The resulting bootstrap confidence intervals are not only valid but also potentially narrower than the ones obtained from the inconsistent fixed design bootstrap, depending on the underlying data-generating process and the sample size. In an application to financial time series, we illustrate the empirical relevance of our proposed methods, showing evidence for the residual dependence and demonstrating notable differences between the confidence intervals obtained by the fixed- and the recursive-design bootstrap procedure.

Key words: GARCH; Dependent Innovations; Residual Block Bootstrap

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1 Introduction

Generalized autoregressive conditional heteroskedasticity (GARCH) models, first proposed by Engle (1982) and Bollerslev (1986), have attracted substantial attention in both practical applications and academic literature. For example, these models find utility in weather modelling, derivatives pricing and Value-at-Risk (VaR) estimation (Campbell and Diebold, 2005; Duan, 1995; Francq and Zakoïan, 2015; Hoga and Demetrescu, 2022; Li et al., 2023). The estimation of the model parameters is commonly conducted through maximum likelihood estimation (MLE), under the assumption that the innovations are independent and identically distributed (iid) random variables. The true distribution of innovations is, however, typically unknown. In such cases and under relatively mild assumptions, quasi-maximum likelihood estimation (QMLE) can be performed, where "quasi" indicates that the log-likelihood function is maximized "as if" the innovations are normally distributed. Although QMLE is not efficient when the innovations deviate from normality, the estimator remains consistent.

Since the introduction of GARCH models, a number of articles have addressed different aspects of their statistical properties. Weiss (1986) presents asymptotic theory for ARCH models, whereas Lumsdaine (1996) and Berkes et al. (2003) establish results for the local GARCH$(1, 1)$ and global GARCH(p, q) models, respectively. The term "local" indicates that the maximization of the log-likelihood function occurs within the proximity of the true parameter θ_0. Subsequently, Francq and Zakoïan (2004) establish the consistency and asymptotic normality (CAN) for GARCH models under even milder conditions. Hall and Yao (2003) provide results on ARCH and GARCH models in the heavy-tailed setting. See also Robinson and Zaffaroni (2006) for pseudo-maximum likelihood estimation. However, the aforementioned results all share the characteristic that they work under the assumption of iid innovations.

While the assumption of iid innovations is convenient and substantially eases asymptotic analysis, it is not fulfilled in all applications. Empirical evidence strongly suggests the presence of time-varying conditional higher moments within the innovations in several observed time series. Harvey and Siddique (1999) and Brooks et al. (2005), for example, find evidence of autoregressive conditional skewness, and autoregressive conditional kurtosis in financial time series, respectively. Jondeau and Rockinger (2003), León et al. (2005) and Dark (2010) allow for both autoregressive conditional skewness and kurtosis, and White Jr et al. (2008) propose a multi-quantile conditionally autoregressive Value-at-Risk (CAViaR) model to capture autoregressive conditional skewness and kurtosis.

D'Innocenzo et al. (2023) propose a score-driven model allowing for time-varying conditional mean, variance and kurtosis.

In the presence of dependence within the innovations, it becomes necessary to loosen the iid assumption. Lee and Hansen (1994) study the CAN of the local QMLE in GARCH$(1,1)$ models under the assumption of martingale difference innovations. More than a decade later, Escanciano (2009) proves CAN for the global QMLE of the semi-strong GARCH(p,q) model where the errors are assumed to be a conditionally homoscedastic martingale difference sequence. Francq and Zakoïan (2016) show similar results for equation-by-equation estimators for multivariate volatility models. We refer to Linton et al. (2010) for least absolute deviation estimation (LADE) for a nonstationary semi-strong GARCH$(1,1)$ model. Meitz and Saikkonen (2011) establish outcomes concerning the consistency and asymptotic normality of QMLE in nonlinear AR(p) models with GARCH$(1,1)$ errors, where the innovations are a strictly stationary and ergodic martingale difference sequence. Kouassi et al. (2017a) and Kouassi et al. (2017b) present results on pseudo-maximum likelihood estimation under dependent innovations for the univariate GARCH$(1,1)$ and GARCH$(2,2)$ model, respectively. See Lee and Kim (2022) for asympotics for the semi-strong augmented GARCH$(1,1)$ model and Francq and Zakoïan (2023) for quasi-likelihood estimation for weak location-scale dynamic models.

The aforementioned studies, while contributing valuable theoretical insights, predominantly focus on the estimation of a range of GARCH models within the context of dependent innovations. However, to the best of our knowledge, a gap remains in the literature regarding bootstrap procedures to enhance the accuracy of approximating the finite sample distribution of parameter estimates and volatility forecasts in such non-iid settings. This paper aims to partially fill this gap. We present a novel solution: a recursive-design residual block bootstrap technique tailored for GARCH processes under dependent innovations. Our findings provide evidence that the characteristics of a GARCH process under dependent innovations can be effectively maintained through a bootstrap approach, particularly when the bootstrap GARCH processes follow a recursive-design scheme and the bootstrap residuals are generated by sampling random blocks of the model residuals. In our simulation study, innovation trajectories are generated from processes characterized by unchanging conditional mean and variance, yet displaying time-varying conditional higher moments. In doing so, we gain insights into how time-varying conditional higher moments affect the variance of parameter estimates and, consequently, volatility forecasts.

Numerous studies have explored bootstrapping techniques for GARCH processes. Among the widely used methods is the residual bootstrap, with two notable variations: recursive-design (Hidalgo and Zaffaroni, 2007; Jeong, 2017; Pascual et al., 2006) and fixed-design (Beutner et al., 2024; Cavaliere et al., 2018; Shimizu, 2010). Pascual et al. (2006) propose a recursive-design residual bootstrap for GARCH processes with iid innovations and therefore ignore potential dependence in the higher moments. Cavaliere et al. (2018) present a fixed volatility bootstrap for a class of ARCH(q) models. We will argue that within the context of dependent innovations, the fixed-design bootstrap method falls short in accurately quantifying the distribution of parameter estimates and volatility forecasts, even when a block or stationary bootstrap procedure is performed instead of sampling individual draws from the residuals' distribution. Corradi and Iglesias (2008) reconsider a block bootstrap procedure for GARCH processes with iid innovations (as proposed by Gonçalves and White (2004)), in which one resamples blocks from the likelihood function instead of the observables. The methods introduced above are suitable for GARCH models featuring iid innovations, yet they prove inadequate when the assumed iid nature of the innovations becomes overly restrictive. Gonçalves and Kilian (2004), for example, show that basic residual-based bootstrap procedures are invalid for autoregressions with conditional heteroskedasticity.

We demonstrate the practical significance of our approach by applying it to financial time series. To showcase the effectiveness of our proposed method, we compare it with the inconsistent fixed-design procedure. In this application, we calculate the next period's volatility and bootstrap the confidence intervals using log-returns data from the EU Emission Trading System and USD/EUR exchange rate. The findings reveal residual dependence, leading to noticeable distinctions between the confidence intervals generated by our recursive-design procedure and those produced by the fixed-design bootstrap.

In the subsequent section, we discuss the estimation of the parameters and the volatility forecasts. Moving forward, Section 3 addresses the inadequacy of a fixed-design block bootstrap approach, followed by a demonstration of how the recursive-design block bootstrap procedure effectively quantifies uncertainty surrounding parameter estimates. Section 4 presents a comprehensive simulation study to numerically validate our assertions. Our empirical analysis is conducted in Section 5, and ultimately, Section 6 concludes.

2 Estimation

Throughout this paper, we assume the following updating equation:

$$\epsilon_t = \sigma_t \eta_t, \tag{1}$$

where $t \in \mathbb{Z}$ and $\{\epsilon_t\}$ represents a sequence of observables, e.g. log-returns, $\{\sigma_t\}$ denotes the volatility process, and $\{\eta_t\}$ is a conditionally homoscedastic martingale difference sequence of innovations, with cumulative distribution function (cdf) F, satisfying $\mathbb{E}\left[\eta_t|\mathcal{F}_{t-1}\right] = 0$ and $\mathbb{E}\left[\eta_t^2|\mathcal{F}_{t-1}\right] = 1$. Here, $\mathcal{F}_u$ is the sigma-algebra generated by $\{\epsilon_t, t \leq u\}$. The volatility process is a measurable function of the observables, i.e.,

$$\sigma_t(\theta_0) = \sigma(\epsilon_{t-1}, \epsilon_{t-2}, ..., \epsilon_1, \epsilon_0, \epsilon_{-1}, ...; \theta_0). \tag{2}$$

We have $\sigma : \mathbb{R}^\infty \times \Theta \to (0, \infty)$, and θ_0 represents the true parameter vector, where $\theta \in \Theta \subset \mathbb{R}^r$ with $r \in \mathbb{N}$. We estimate the true parameter θ_0 by using QMLE. This method is particularly relevant for GARCH models because it provides, under mild regularity conditions, consistent and asymptotically normal estimators for strictly stationary GARCH processes. In doing so, we assume the normality of the innovations $\{\eta_t\}$, while the actual distribution of the innovations may be non-Gaussian. In that case we have that ${}^{\epsilon_t}/_{\sigma_t(\theta)} = \eta_t(\theta) \sim N(0, 1)$ if $\theta = \theta_0$, where

$$\sigma_t(\theta) = \sigma(\epsilon_{t-1}, \epsilon_{t-2}, ..., \epsilon_1, \epsilon_0, \epsilon_{-1}, ...; \theta). \tag{3}$$

Since we generally do not have observations $(\epsilon_0, \epsilon_{-1}, ..., \epsilon_{-t}, ...)$, we replace these observations by arbitrary values $\tilde{\epsilon}_t$ for $t \leq 0$, such that we obtain:

$$\tilde{\sigma}_t(\theta) = \sigma(\epsilon_{t-1}, ..., \epsilon_1, \tilde{\epsilon}_0, \tilde{\epsilon}_{-1}, ...; \theta), \tag{4}$$

which we can use to approximate the process shown in (3). The quasi-maximum likelihood estimator $\hat{\theta}_n$ of θ_0 is defined by

$$\hat{\theta}_n = \arg\max_{\theta \in \Theta} \tilde{L}_n(\theta) \tag{5}$$

where n is the sample size and $\tilde{L}_n$ denotes the criterion function shown below:

$$\tilde{L}_n(\theta) = \frac{1}{n}\sum_{t=1}^{n} \tilde{\ell}_t(\theta), \qquad \text{where} \qquad \tilde{\ell}_t(\theta) = -\frac{1}{2}\left(\frac{\epsilon_t}{\tilde{\sigma}_t(\theta)}\right)^2 - \log \tilde{\sigma}_t(\theta).$$

This paper builds on the assumptions of Francq and Zakoïan (2016), hereafter FZ16 for convenience, which can be found in Appendix A. Theorem 1 recalls the consistency and asymptotic normality of the QMLE for univariate GARCH-type volatility models under dependent innovations, as proved by FZ16 (Theorem 1 with $k = 1$). An analogous result was proved by Escanciano (2009), but merely for semi-strong GARCH(p, q) models, while the results of FZ16 hold for a broader range of GARCH-type volatility models. In the remainder of this paper, we use x' to denote the transpose of a column vector x.

Theorem 1 (Consistency and Asymptotic Normality (FZ16))**.** *Under Assumptions 1, 2, 3, 4(i) and 5(i) the estimator in Equation (5) is strongly consistent, i.e.* $\hat{\theta}_n \stackrel{a.s.}{\to} \theta_0$*. If, in addition, Assumptions 4(ii), 5(ii), 6, 7 and 8 hold, we have*

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \stackrel{d}{\to} N(0, \Sigma), \tag{6}$$

where $\Sigma = J^{-1} I J^{-1}$ *and*

$$J = \mathbb{E}\left[D_t D_t'\right], \; I = \mathbb{E}\left[(\kappa_t - 1) D_t D_t'\right],$$

with $D_t(\theta) = \frac{1}{\sigma_t^2(\theta)} \frac{\partial \sigma_t^2(\theta)}{\partial \theta}$ *and* $\kappa_t := \mathbb{E}\left[\eta_t^4 | \mathcal{F}_{t-1}\right]$.

In the following, we set $D_t(\theta_0) = D_t$ to lighten notation. If the innovations are iid, the asymptotic variance in (6) reduces to $\Sigma_{iid} = (\kappa - 1) J^{-1}$ with $\kappa = \mathbb{E}\left[\eta_t^4\right]$. However, if the innovations are not iid, then the asymptotic covariance matrix will typically be different from Σ_{iid} and takes a so-called sandwich-form. Σ is commonly estimated by

$$\hat{\Sigma}_n = \hat{J}_n^{-1} \hat{I}_n \hat{J}_n^{-1}, \text{ where } \hat{J}_n = \frac{1}{n} \sum_{t=1}^{n} \hat{D}_t \hat{D}_t' \text{ and } \hat{I}_n = \frac{1}{n} \sum_{t=1}^{n} \left(\hat{\eta}_t^4 - 1\right) \hat{D}_t \hat{D}_t'. \tag{7}$$

For convenience, we use $\hat{D}_t = \tilde{D}_t(\hat{\theta}_n)$ with $\tilde{D}_t(\theta) = \frac{1}{\tilde{\sigma}_t^2(\theta)} \frac{\partial \tilde{\sigma}_t^2(\theta)}{\partial \theta}$, and $\hat{\eta}_t = \epsilon_t / \sqrt{\tilde{\sigma}_t^2(\hat{\theta}_n)}$ denotes the model residual with empirical distribution function (edf) $\hat{\mathbb{F}}_n(x) = \frac{1}{n} \sum_{t=1}^{n} \mathbb{1}_{\{\hat{\eta}_t \leq x\}}$. Combining Theorem 1 and the multivariate delta method, a $(1 - \gamma)\%$ conditional confidence interval can be constructed for the next period's volatility:

$$CI_{\sigma_{n+1}^2}(\gamma) = \sigma_{n+1}^2(\hat{\theta}_n) \pm \frac{\Phi^{-1}(1 - \gamma/2)}{\sqrt{n}} \sqrt{\frac{\partial \sigma_{n+1}^2(\hat{\theta}_n)}{\partial \theta'} \Sigma \frac{\partial \sigma_{n+1}^2(\hat{\theta}_n)}{\partial \theta}}. \tag{8}$$

The conditional confidence interval exhibited in Equation (8) can be estimated by

$$CI_{\tilde{\sigma}_{n+1}^2}(\gamma) = \tilde{\sigma}_{n+1}^2(\hat{\theta}_n) \pm \frac{\Phi^{-1}(1 - \gamma/2)}{\sqrt{n}} \sqrt{\frac{\partial \tilde{\sigma}_{n+1}^2(\hat{\theta}_n)}{\partial \theta'} \hat{\Sigma}_n \frac{\partial \tilde{\sigma}_{n+1}^2(\hat{\theta}_n)}{\partial \theta}}. \tag{9}$$

It is worth highlighting that in a GARCH context the next period's volatility $\tilde{\sigma}_{n+1}^2(\hat{\theta}_n)$ and its derivatives are known conditionally on $\mathcal{F}_n$, as they are functions of estimated parameters, the lagged return, and the (latent) lagged volatility. For a theoretical justification of conditional confidence intervals in time series models see Beutner et al. (2021).

3 Bootstrapping GARCH processes under dependent innovations

In this section, we discuss the inconsistency of the fixed-design bootstrap approach in the setting of dependent innovations. Next, we present the recursive-design block bootstrap method customized for GARCH processes under dependent innovations. Before we proceed with a more in-depth examination of the two bootstrap designs, let us first provide a brief introduction of the bootstrap methodology within the context of GARCH modelling. When QMLE is performed, one works under the assumption that the maximizer of the log-likelihood function can be found by setting the derivative of the log-likelihood function (inflated by $\sqrt{n}$) equal to zero. Then, a Taylor expansion yields

$$0 = \sqrt{n}\frac{\partial \tilde{L}_n(\hat{\theta}_n)}{\partial \theta} = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{\partial \tilde{\ell}_n}{\partial \theta}(\theta_0) + \left(\frac{1}{n}\sum_{t=1}^{n}\frac{\partial^2}{\partial\theta\partial\theta'}\tilde{\ell}_t(\breve{\theta}_n)\right)\sqrt{n}\left(\hat{\theta}_n - \theta_0\right),$$

where $\breve{\theta}_n$ is between $\hat{\theta}_n$ and θ_0. We can rewrite this, under the assumption that the Hessian of the log-likelihood function is invertible, to

$$\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) = \left(\frac{1}{n}\sum_{t=1}^{n}\frac{\partial^2}{\partial\theta\partial\theta'}\tilde{\ell}_t(\breve{\theta}_n)\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{\partial \tilde{\ell}_n}{\partial \theta}(\theta_0)\right).$$

FZ16 prove that the first part of the right-hand side converges to J^{-1} (see steps 3 and 4 of the proof of Theorem 3.1 in FZ16 for the asymptotic normality). Let us now have a closer look at the second part of the right-hand side:

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\frac{\partial}{\partial\theta}\tilde{\ell}_n(\theta_0) = \frac{1}{\sqrt{n}}\sum_{t=1}^{n}\tilde{D}_t\left(\frac{\epsilon_t^2}{\tilde{\sigma}_t^2(\theta_0)} - 1\right).$$

FZ16 also show that, with $I = \mathbb{E}\left[(\kappa_t - 1)D_t D_t'\right]$ and $\tilde{D}_t = \tilde{D}_t(\theta_0)$,

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\tilde{D}_t\left(\frac{\epsilon_t^2}{\tilde{\sigma}_t^2(\theta_0)} - 1\right) \xrightarrow{\mathcal{L}} N(0, I). \tag{10}$$

For a bootstrap procedure to ensure consistency, it is necessary that the conditional covariance matrix of the bootstrapped parameter estimates converges in probability to the asymptotic covariance matrix of the original parameter estimates. Specifically, in the context at hand, the conditional covariance matrix of the bootstrapped parameter estimates should converge to the asymptotic covariance matrix presented in Theorem 1. Therefore, we require that the bootstrap estimator $\hat{\theta}_n^\star$ fulfills the following:

$$\sqrt{n}(\hat{\theta}_n^\star - \hat{\theta}_n) \xrightarrow{d^\star} N(0, \Sigma),$$

in probability. Here * indicates that we work in the bootstrap world (by a slight abuse of notation we follow here the convention to denote a generic bootstrap procedure by a *, while in Section 3.2 * stands for the recursive design bootstrap). For this to be true, we need that the bootstrap analogue of (10) also holds in the bootstrap world (note that $\epsilon_t^2/\tilde{\sigma}_t^2(\theta_0) \approx \eta_t^2$):

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \hat{D}_t^\star \left(\eta_t^{\star 2} - 1 \right) \xrightarrow{\mathcal{L}} N(0, I), \tag{11}$$

where $\hat{D}_t^\star = \tilde{D}_t^\star(\hat{\theta}_n)$ and $\tilde{D}_t^\star(\theta) = \frac{1}{\tilde{\sigma}_t^{\star 2}(\theta)} \frac{\partial \tilde{\sigma}_t^{\star 2}(\theta)}{\partial \theta}$ with $\tilde{\sigma}_t^\star(\theta)$ and $\epsilon_t^\star$ denoting the bootstrap volatility process and the bootstrap observation at time t, respectively. In the remainder of this paper, we use $^\times$ to indicate that we work in the inconsistent fixed-design bootstrap world, whereas * denotes that we are considering the recursive-design bootstrap procedure.

3.1 Inconsistency of the fixed-design bootstrap

The fixed-design residual process for volatility models of the ARCH type, as proposed by Cavaliere et al. (2018), generates bootstrap samples that all have the same conditional volatility process as the estimated model. Accordingly, for every bootstrap replication, the volatility process at time t is known conditionally on $\mathcal{F}_n$ and thus a constant. This property significantly simplifies the asymptotic analysis. Furthermore, the fixed-design bootstrap method is asymptotically valid under less stringent conditions when compared to the recursive-design bootstrap methodology. For a broader range of GARCH-type volatility models and conditional Value-at-Risk estimation within the context of fixed-design bootstrap procedures, reference can be made to Beutner et al. (2024).

Clearly, one cannot expect that the fixed-design bootstrap combined with iid draws from the residuals is consistent if the true innovations are actually dependent. Intuitively, replacing the iid draws from the residuals, for instance, by a block bootstrap method or the stationary bootstrap, could provide a way out. However, while such a procedure may preserve the underlying dependency structure in the innovations, we will see below that it seems that the fixed-design bootstrap is, in general, unable to capture the dependency between the innovations and the volatility process induced by dependent innovations.

We will now elaborate on the inconsistency of some fixed-design bootstrap schemes. Note that, since the fixed design bootstrap keeps the volatility process fixed across all bootstrap replications,

$\tilde{D}_t^{\times}(\hat{\theta}_n)$ reduces to $\tilde{D}_t(\hat{\theta}_n) = \hat{D}_t$. We set

$$\sum_{t=1}^{n} Z_{n,t}^{\times} = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \hat{D}_t \left(\eta_t^{\times 2} - 1 \right),$$

and assume for the rest of this subsection that, given the data, the $\eta_t^{\times 2}$'s are scaled such that their second moments equal 1. Given that we are working in the bootstrap world, we need to calculate the variance conditional on the data, i.e. $\mathcal{F}_n$. In order for (11) to hold we need this variance to converge in probability to the matrix on the right-hand side of (11). For any fixed-design stationary bootstrap scheme, i.e. given the data $\eta_1^{\times}, \ldots, \eta_t^{\times}$ are stationary, we have the following result. Hereafter, we use $\mathbb{E}_{\times}$ and $\mathbb{V}\mathrm{ar}_{\times}$ to denote the expectation and variance conditional on $\mathcal{F}_n$.

Lemma 1. *The bootstrap variance of $\sum_{t=1}^{n} Z_{n,t}^{\times}$ denoted by $\mathbb{V}ar_{\times}\left[\sum_{t=1}^{n} Z_{n,t}^{\times}\right]$ is given by:*

$$\begin{aligned}\frac{1}{n}\sum_{t=1}^{n}\sum_{u=1}^{n} \hat{D}_t \hat{D}_u' \, \mathbb{E}_{\times}\left[(\eta_t^{\times 2} - 1)(\eta_u^{\times 2} - 1)\right] =& \mathbb{E}_{\times}\left[(\eta_1^{\times 2} - 1)^2\right] \frac{1}{n} \sum_{t=1}^{n} \hat{D}_t \hat{D}_t' \\ &+ \frac{1}{n} \sum_{t=1}^{n} \sum_{u=1, u \neq t}^{n} \hat{D}_t \hat{D}_u' \gamma(|t-u|),\end{aligned}$$

where $\gamma(s) = \mathbb{E}_{\times}\left[(\eta_1^{\times 2} - 1)(\eta_{s+1}^{\times 2} - 1)\right]$, $s = 1, \ldots, n-1$. Moreover, if the bootstrap mimics the conditional second moment of the observations, i.e., if $\mathbb{E}_{\times}\left[(\eta_t^{\times 2} - 1) | \mathcal{F}_{t-1}^{\times}\right] = 0$, where $\mathcal{F}_{t-1}^{\times}$ denotes the sigma-algebra generated by $\{\eta_1^{\times}, \ldots, \eta_{t-1}^{\times}\}$, then we have

$$\frac{1}{n}\sum_{t=1}^{n}\sum_{u=1}^{n} \hat{D}_t \hat{D}_u' \, \mathbb{E}_{\times}\left[(\eta_t^{\times 2} - 1)(\eta_u^{\times 2} - 1)\right] = \mathbb{E}_{\times}\left[(\eta_1^{\times 2} - 1)^2\right] \frac{1}{n} \sum_{t=1}^{n} \hat{D}_t \hat{D}_t'.$$

The proof of this lemma is given in Appendix B. Note that the convergence of $1/n \sum_{t=1}^{n} \hat{D}_t \hat{D}_t' \to J$ almost surely follows under Assumptions 4 and 7 (ii); see, for instance, Beutner et al. (2023)[Lemma 2]. It can be directly seen from the second displayed equation in Lemma 1 that any bootstrap procedure for which $\mathbb{E}_{\times}\left[(\eta_1^{\times 2} - 1)^2\right]$ converges in probability to the respective moment of η_1 will fail because the asymptotic covariance matrix will be

$$(\kappa - 1) J^{-1} \tag{12}$$

instead of the required $\Sigma = J^{-1} I J^{-1}$. Clearly, this claim remains true if the bootstrap procedure is such that it mimics the conditional second moment of η_t only asymptotically in the sense that uniformly in s we have $\gamma(s) \leq C o_{\mathbb{P}}(1)$ for some constant C independent of s. More generally, the fact that by construction the fixed-design bootstrap always allows to pull out the derivative of the volatility process from the bootstrap expectation of $Z_{n,t}^{\times}$ makes it very plausible that it must fail in general. As mentioned above, in Section 4 we combine the fixed-design bootstrap with the moving block bootstrap; see Appendix C Algorithm 3 for the fixed-design moving block bootstrap. The simulation results confirm this claim.

3.2 The recursive-design residual block bootstrap

We shall now advance to the recursive-design residual block bootstrap for GARCH processes under dependent innovations. Pascual et al. (2006) discuss the recursive-design residual bootstrap for GARCH$(1,1)$ processes and assess its finite sample properties employing a simulation study. We refer to Hidalgo and Zaffaroni (2007) and Jeong (2017) for theoretical results on the recursive-design residual bootstrap under iid innovations for ARCH(∞) and GARCH(p,q) processes, respectively. In contrast to the fixed-design procedure, the bootstrap volatility processes (and their derivatives) are now generated recursively. Consequently, conditionally on $\mathcal{F}_n$, the bootstrap volatility processes are random and, under dependent errors, not independent of the bootstrap innovations. The procedural steps outlined in Algorithm 1 show how this feature of the recursive-design procedure ensures that the dependence between the innovations and the volatility processes (and their derivatives) is not ignored. However, this feature also introduces a substantial level of complexity to the asymptotic analysis.

Algorithm 1. *(Recursive-design residual block bootstrap)*
First, construct $n-l$ *blocks* b_i *of length* l $\forall i \in \{1,...,n-l\}$ *such that* $b_i = \{\hat{\eta}_i, \hat{\eta}_{i+1}, ..., \hat{\eta}_{i+k}, ..., \hat{\eta}_{i+l}\}$.

1. *Draw* $\lceil n/l \rceil$ *numbers* $U_1, ..., U_{\lceil n/l \rceil} \sim Uniform(1, n-l)$ *(with replacement) and create the bootstrap innovations* $\{\eta_1^\star, ..., \eta^\star_{\lceil n/l \rceil \times l}\} = \{b_{U_1}, b_{U_2}, ..., b_{U_{\lceil n/l \rceil}}\}$. *In the case that* $\lceil n/l \rceil \times l \neq n$, *truncate the series such that it has length* n. *Generate bootstrap observations* $\epsilon_t^\star = \tilde{\sigma}_t^\star \eta_t^\star$ *recursively, with* $\tilde{\sigma}_t^\star = \tilde{\sigma}_t^\star(\hat{\theta}_n)$ *and* $\tilde{\sigma}_t^\star(\theta) = \sigma_t(\epsilon^\star_{t-1}, \epsilon^\star_{t-2}, ..., \epsilon^\star_1, \tilde{\epsilon}_0, \tilde{\epsilon}_{-1}, \tilde{\epsilon}_{-2}, ...; \theta)$.

2. *Calculate the bootstrap estimator for the volatility process by QMLE*

$$\hat{\theta}_n^\star = \arg\max_{\theta \in \Theta} L_n^\star(\theta) \tag{13}$$

   with

$$L_n^\star(\theta) = \frac{1}{n}\sum_{t=1}^{n} \ell_t^\star(\theta) \qquad \textit{and} \qquad \ell_t^\star(\theta) = -\frac{1}{2}\left(\frac{\epsilon_t^\star}{\tilde{\sigma}_t^\star(\theta)}\right)^2 - \log \tilde{\sigma}_t^\star(\theta). \tag{14}$$

3. *Compute the next period's volatility using the original returns series and the bootstrap estimator* $\hat{\theta}_n^\star$

$$\hat{\sigma}_{n+1}^\star = \tilde{\sigma}_{n+1}(\hat{\theta}_n^\star), \tag{15}$$

where $\tilde{\sigma}_{n+1}(\hat{\theta}_n^\star) = \sigma_{n+1}(\epsilon_n, \epsilon_{n-1}, ..., \epsilon_1, \tilde{\epsilon}_0, \tilde{\epsilon}_{-1}, ...; \hat{\theta}_n^\star)$.

The difference between Algorithms 3 and 1 is that in the latter case, the bootstrap volatility processes are generated iteratively using the estimated volatility dynamics and the bootstrap observations. Therefore, the dependence between $\eta_t^{\star 4}$ and $\hat{D}_t^\star \hat{D}_t^{\star\prime}$ is not ignored. In Section 4 we substantiate this claim by combining the recursive-design bootstrap procedure with the moving block bootstrap.

3.3 Bootstrapping confidence intervals

Algorithm 2 presents how to compute the confidence intervals surrounding the parameter estimates. The intervals can be interpreted as follows: *given the past up to and including time* n there is a $100 \times (1-\gamma)\%$ probability that the $100 \times (1-\gamma)\%$ confidence interval contains the true parameter θ_0 (or σ_{n+1}^2).

Algorithm 2. *(Bootstrap confidence intervals for parameter estimates and the next period's volatility)*

1. *Generate* B *bootstrap replicates for* $\hat{\theta}_{i,n}^\star$ *with* $i \in \{1, ..., r\}$ *and* $\hat{\sigma}_{n+1}^{\star 2}$ *for* $b = 1, ..., B$ *by repeating Algorithms 1 (recursive-design) or 3 (fixed-design).*

2. *Calculate the reversed tails interval for the parameter estimates:*

$$\left[\hat{\theta}_{i,n}^\star + \frac{1}{\sqrt{n}} \hat{G}_{n,B,\theta_i}^{\star -1}(\gamma/2), \hat{\theta}_{i,n}^\star + \frac{1}{\sqrt{n}} \hat{G}_{n,B,\theta_i}^{\star -1}(1-\gamma/2)\right],$$

where $\hat{G}_{n,B,\theta_i}^{\star -1}(\cdot)$ *denotes the quantile function, or the generalized inverse, of* $\hat{G}_{n,B,\theta_i}^\star(x) = \frac{1}{B}\sum_{b=1}^B \mathbb{1}_{\{\sqrt{n}\left(\hat{\theta}_{i,n}^{\star(b)} - \hat{\theta}_{i,n}\right) \leq x\}}$.

3. *Calculate the reversed tails intervals for the next period's volatility:*

$$\left[\hat{\sigma}_{n+1}^{\star 2} + \frac{1}{\sqrt{n}} \hat{G}_{n,B,\sigma_{n+1}^2}^{\star -1}(\gamma/2), \hat{\sigma}_{n+1}^{\star 2} + \frac{1}{\sqrt{n}} \hat{G}_{n,B,\sigma_{n+1}^2}^{\star -1}(1-\gamma/2)\right],$$

where $\hat{G}_{n,B,\sigma_{n+1}^2}^{\star -1}(\cdot)$ *denotes the quantile function, or the generalized inverse, of* $\hat{G}_{n,B,\sigma_{n+1}^2}^\star(x) = \frac{1}{B}\sum_{b=1}^B \mathbb{1}_{\{\sqrt{n}\left(\hat{\sigma}_{n+1}^{\star 2(b)} - \hat{\sigma}_{n+1}^2\right) \leq x\}}$.

The reversed-tails bootstrap confidence interval is essentially the bootstrap analogue of the (uncentered) statistic $\hat{\sigma}^2_{n+1}$ and is often reported in reduced form, in which it ‘reduces’ to the $\gamma/2$ and $1-\gamma/2$ quantiles of $\frac{1}{B}\sum_{b=1}^{B}\mathbb{1}_{\{\hat{\sigma}^{\star 2(b)}_{n+1}\leq x\}}$. It is worth noting that we intentionally focus on the reversed-tails bootstrap confidence interval in this paper. In our simulation study, we included the equal-tailed percentile (EP) and the symmetric (SY) interval, but the reversed-tailed confidence intervals outperformed the EP and SY in all settings in terms of coverage and confidence interval length, which is in line with the results of Beutner et al. (2024). For a theoretical motivation, we refer to Falk and Kaufmann (1991).

4 Simulation study

In this section, we examine the finite sample performance of our proposed bootstrap method compared to the fixed-design procedure. We will conduct this analysis in scenarios featuring dependence within the innovations. In doing so, we make use of the following specifications for the volatility process σ_t:

$$\begin{aligned}\text{GARCH(1,1)}: \sigma_t^2 &= \beta_0 + \beta_1\epsilon_{t-1}^2 + \beta_2\sigma_{t-1}^2\\ \text{TGARCH(1,1,1)}: \sigma_t &= \beta_0 + \beta_1^+\epsilon_{t-1}^+ + \beta_1^-\epsilon_{t-1}^- + \beta_2\sigma_{t-1},\end{aligned}$$

with $\epsilon_t^- = \max(0, -\epsilon_t)$ and $\epsilon_t^+ = \max(0, \epsilon_t)$. For the innovations process, we adopt the GARCHSK model proposed by León et al. (2005) and subsequently refined by León and Ñíguez (2021), and the Autoregressive Conditional Kurtosis (ARCK) model with a conditional student-t distribution as proposed by Brooks et al. (2005). The first model enables the incorporation of time-varying conditional skewness and kurtosis through the utilization of a time-varying Transformed Gram-Charlier (tv-TGC) distribution. For a comprehensive analysis of the tv-TGC distribution, we direct readers to Appendix D. León and Ñíguez (2021) assume that the observables and the distributional parameters evolve according to the following dynamics:

$$\begin{aligned}\epsilon_t &= \mu_t + \sigma_t\eta_t,\\ \eta_t|\mathcal{F}_{t-1} &\sim TGC(0,1,k_t),\\ k_t &= \delta_0 + \delta_1\eta_{t-1}^4 + \delta_2 k_{t-1}.\end{aligned}\tag{16}$$

For the sake of simplicity and convenience, we will disregard the time-varying mean component (i.e. $\mu_t = 0\ \forall t \in \{1, ..., n\}$). The DGP of the volatility process is modelled using a GARCH(1, 1)

(TGARCH(1, 1, 1)) structure with high persistence, characterized by a true parameter vector denoted as $(\beta_0, \beta_1, \beta_2) = (0.1, 0.15, 0.8)$ $((\beta_0, \beta_1^+, \beta_1^-, \beta_2) = (0.1, 0.05, 0.1, 0.8))$. The conditional mean, variance, and skewness parameters of the tv-TGC distribution are held constant at 0, 1, and 0, respectively. In this context, the term k_t indirectly drives the conditional kurtosis of the tv-TGC distribution. The true parameters for the updating equation of k_t are set to $(\delta_0, \delta_1, \delta_2) = (2, 0.15, 0.4)$.

The ARCK framework is formulated as follows:

$$
\begin{aligned}
&\epsilon_t = \mu_0 + \sqrt{\frac{(\nu_t - 2)}{\nu_t}}\sigma_t \eta_t, \text{ with} \\
&\eta_t | \mathcal{F}_{t-1} \sim t(0, 1, \nu_t), \\
&k_t = \delta_0 + \delta_1 \eta_{t-1}^4 + \delta_2 k_{t-1}, \\
&\nu_t = \frac{2\,(2k_t - 3)}{k_t - 3}.
\end{aligned}
$$

Once more, we omit the mean component, that is, $\mu_0 = 0$, and proceed to model the volatility process using a GARCH(1, 1) (TGARCH(1, 1, 1)) structure with specific parameter values, namely $(\beta_0, \beta_1, \beta_2) = (0.1, 0.4, 0.55)$ $((\beta_0, \beta_1^+, \beta_1^-, \beta_2) = (0.1, 0.1, 0.3, 0.5))$. This choice is made in order to assess the performance of the model under conditions of low persistence. In this context, k_t serves as the driver for the degrees-of-freedom parameter ν_t, which in turn is employed to accommodate heavier tails in the distribution of the innovations. Specifically, as $\nu_t > 4$ decreases, the distribution exhibits heavier tails. Conversely, higher values of ν_t lead to lighter tails, ultimately converging to a normal distribution as ν_t tends towards infinity. The true parameter values for the updating equation of k_t in the ARCK setting are set to $(\delta_0, \delta_1, \delta_2) = (1.5, 0.4, 0.5)$.

The outcomes are generated through $S = 2{,}000$ Monte Carlo iterations and $B = 2{,}000$ bootstrap replications, and an initial burn-in phase of 500. If a (bootstrap) parameter estimate falls outside of the stationary region, both the process and the corresponding estimations are disregarded and replaced with a new realization. The length of the simulated time series, denoted as n, spans from 500 to 10,000. Correspondingly, the block length, denoted as l, is allowed to vary within the range of 1 to 30 for $n = 500$, 1 to 50 for $n \in \{1{,}000; 2{,}000\}$, 1 to 70 for $n = 5{,}000$, and 1 to 100 for $n = 10{,}000$. In the forthcoming tables, the abbreviation "ASY" denotes the asymptotic variance based on Theorem 1 within the recursive-design procedure. On the other hand, "ASY" denotes the value obtained through (12) within the framework of the fixed-design methodology. The bootstrapped residuals are subject to resampling using the moving block bootstrap procedure, as outlined in Algorithms 1 and 3 for the recursive- and fixed-design bootstrap, respectively. The various block lengths are illustrated along the horizontal axis for reference. We refer to Appendix E for a smaller-scale simulation study in which the fixed- and recursive-design bootstrap procedures are combined with the stationary bootstrap of Politis and Romano (1994). Here blocks of random length are generated using the geometric distribution with parameter $p = 1/l$. The simulation outcomes closely align with the results obtained by the moving block bootstrap.

Tables 1 and 2 (3 and 4) present the simulation results for parameter estimates of the GARCH(1,1) process with tv-TGC and ARCK errors, respectively, with nominal coverage of 0.9 for $\gamma = 0.1$ (0.95 for $\gamma = 0.05$). We will commence with the tv-TGC scenario, i.e. Tables 1 and 3. For both nominal coverage levels $1-\gamma$ and across all sample sizes, it can be seen that the empirical coverage for both the recursive-design and fixed-design procedures exhibits similarity when $l = 1$. More interestingly, it becomes clear that when the block size is increased, the length of the confidence intervals (indicated in parentheses) reduces under the recursive-design bootstrap procedure. Correspondingly, the empirical coverage rates approach their nominal value. The results indicate that, although having block lengths of reasonable size is essential, the specific block length itself does not significantly impact the outcome. This observation highlights the robustness of our methodology.

As anticipated, this trend is not observed in the case of the fixed-design bootstrap methodology. In this approach, employing resampled blocks rather than individual draws from the empirical distribution of residuals does not yield narrower (or wider) confidence intervals. Furthermore, it is worth noting that the accuracy of both bootstrap procedures improves with larger sample sizes. While the empirical coverage occasionally falls slightly below its nominal value for smaller sample sizes, this discrepancy diminishes for larger sample sizes. The results also demonstrate that for sample sizes $n \in \{5{,}000; 10{,}000\}$, both bootstrap designs appear to closely align with their asymptotic counterparts.

Tables 2 and 4 present the simulation results for the model with ARCK innovations. When considering innovations generated by an ARCK model, the results show similarities with the tv-TGC case, albeit with minor discrepancies. In the tv-TGC scenario, we noted that the fixed-design procedure tends to overestimate the uncertainty around parameter estimates, especially for β_1 and β_2. Conversely, within the current context, and for both nominal coverage levels $1-\gamma$, the outcomes suggest that the fixed-design procedure either underestimates (for β_1) or overestimates (for β_0) the uncertainty. Once again, the results of the recursive-design framework show that for an increase in block length l, the empirical coverage rates tend to approach their nominal value, and confidence intervals widen for β_1 (shrink for β_0). Note that in comparison to the results for the tv-TGC process, the bootstrap procedure (along with its asymptotic counterparts) demands larger sample sizes to attain empirical coverage rates that closely converge with their nominal values. This phenomenon can be attributed to our utilization of a student-t distribution with a time-varying degrees-of-freedom parameter $\nu_t > 4$, which may tend to approach 4, thereby inducing heavy tails.

Concerning the TGARCH(1,1,1) with tv-TGC and ARCK innovations presented in Tables 5 and 6, respectively, the overall outcomes align with the GARCH(1,1) configuration. This implies that the recursive-design bootstrap effectively approximates the nominal coverage for larger sample sizes and block lengths, whereas the fixed-design bootstrap procedure once again fails. A closer examination reveals that in the tv-TGC setting the under-coverage (over-coverage) for the asymptotic confidence intervals is more pronounced for β_0 and β_2 (β_1^+ and β_1^-) in comparison to the GARCH(1,1) scenario. Also, over-coverage is evident in smaller sample sizes for all parameters in both bootstrap procedures, a trend that diminishes with an increase in sample size. In the ARCK context, there is under-coverage observed for the asymptotic confidence bounds in small sample sizes, which diminishes as the sample size n increases. As for the empirical coverage rates determined by the bootstrap procedures, it is noticeable that for $n = 500$, there is over-coverage. Specifically, this over-coverage is observed for β_0 and β_2 in the recursive-design bootstrap procedure and for β_1^+ in the fixed-design bootstrap procedure.

		Recursive-design										Fixed-design									
n/l	β	ASY	1	5	10	15	20	30	50	70	100	ASY	1	5	10	15	20	30	50	70	100
500	β_0	0.886	0.894	0.880	0.875	0.868	0.864	0.855	-	-	-	0.906	0.906	0.909	0.907	0.905	0.911	0.913	-	-	-
		(0.235)	(0.312)	(0.318)	(0.326)	(0.324)	(0.322)	(0.319)	-	-	-	(0.240)	(0.261)	(0.261)	(0.262)	(0.263)	(0.264)	(0.264)	-	-	-
	β_1	0.871	0.943	0.918	0.914	0.904	0.898	0.890	-	-	-	0.918	0.940	0.940	0.943	0.941	0.943	0.942	-	-	-
		(0.171)	(0.174)	(0.164)	(0.163)	(0.161)	(0.160)	(0.158)	-	-	-	(0.158)	(0.174)	(0.174)	(0.175)	(0.175)	(0.175)	(0.177)	-	-	-
	β_2	0.864	0.918	0.901	0.896	0.885	0.875	0.864	-	-	-	0.903	0.905	0.909	0.910	0.908	0.909	0.908	-	-	-
		(0.245)	(0.306)	(0.305)	(0.310)	(0.307)	(0.304)	(0.300)	-	-	-	(0.233)	(0.273)	(0.274)	(0.275)	(0.275)	(0.276)	(0.278)	-	-	-
1,000	β_0	0.893	0.906	0.899	0.895	0.896	0.895	0.889	0.872	-	-	0.912	0.912	0.912	0.911	0.909	0.909	0.910	0.910	-	-
		(0.137)	(0.161)	(0.162)	(0.167)	(0.168)	(0.168)	(0.167)	(0.166)	-	-	(0.135)	(0.145)	(0.145)	(0.145)	(0.146)	(0.146)	(0.146)	(0.147)	-	-
	β_1	0.876	0.932	0.913	0.907	0.903	0.897	0.892	0.879	-	-	0.919	0.935	0.935	0.940	0.935	0.938	0.938	0.941	-	-
		(0.117)	(0.118)	(0.111)	(0.109)	(0.109)	(0.108)	(0.107)	(0.105)	-	-	(0.106)	(0.119)	(0.119)	(0.119)	(0.119)	(0.119)	(0.120)	(0.120)	-	-
	β_2	0.880	0.923	0.914	0.913	0.910	0.900	0.897	0.873	-	-	0.917	0.918	0.918	0.921	0.916	0.920	0.920	0.919	-	-
		(0.150)	(0.169)	(0.165)	(0.168)	(0.168)	(0.168)	(0.167)	(0.165)	-	-	(0.139)	(0.160)	(0.160)	(0.160)	(0.161)	(0.161)	(0.162)	(0.163)	-	-
2,000	β_0	0.896	0.901	0.891	0.884	0.883	0.882	0.877	0.874	0.870	-	0.911	0.911	0.908	0.908	0.910	0.911	0.908	0.914	0.910	-
		(0.090)	(0.097)	(0.096)	(0.096)	(0.096)	(0.096)	(0.095)	(0.095)	(0.094)	-	(0.088)	(0.092)	(0.092)	(0.092)	(0.092)	(0.092)	(0.092)	(0.093)	(0.093)	-
	β_1	0.894	0.931	0.915	0.905	0.905	0.901	0.899	0.893	0.881	-	0.930	0.936	0.936	0.940	0.938	0.935	0.941	0.938	0.940	-
		(0.082)	(0.082)	(0.077)	(0.075)	(0.075)	(0.074)	(0.074)	(0.073)	(0.073)	-	(0.074)	(0.083)	(0.083)	(0.083)	(0.083)	(0.083)	(0.083)	(0.084)	(0.084)	-
	β_2	0.894	0.926	0.916	0.910	0.906	0.902	0.899	0.893	0.882	-	0.928	0.922	0.921	0.923	0.923	0.922	0.921	0.924	0.925	-
		(0.102)	(0.107)	(0.102)	(0.101)	(0.101)	(0.100)	(0.100)	(0.099)	(0.098)	-	(0.094)	(0.104)	(0.105)	(0.105)	(0.105)	(0.105)	(0.105)	(0.105)	(0.106)	-
5,000	β_0	0.903	0.906	0.901	0.898	0.893	0.891	0.889	0.887	0.891	-	0.906	0.902	0.903	0.905	0.905	0.904	0.901	0.907	0.904	-
		(0.054)	(0.056)	(0.055)	(0.055)	(0.055)	(0.055)	(0.055)	(0.055)	(0.054)	-	(0.053)	(0.054)	(0.055)	(0.055)	(0.055)	(0.055)	(0.055)	(0.055)	(0.055)	-
	β_1	0.900	0.938	0.917	0.911	0.906	0.907	0.907	0.902	0.898	-	0.940	0.937	0.936	0.940	0.941	0.938	0.940	0.939	0.940	-
		(0.052)	(0.051)	(0.048)	(0.047)	(0.047)	(0.046)	(0.046)	(0.046)	(0.046)	-	(0.046)	(0.052)	(0.052)	(0.052)	(0.052)	(0.052)	(0.052)	(0.052)	(0.052)	-
	β_2	0.906	0.925	0.913	0.904	0.900	0.898	0.899	0.895	0.896	-	0.926	0.925	0.925	0.924	0.925	0.924	0.926	0.925	0.930	-
		(0.062)	(0.063)	(0.061)	(0.060)	(0.060)	(0.059)	(0.059)	(0.059)	(0.058)	-	(0.058)	(0.063)	(0.063)	(0.063)	(0.063)	(0.063)	(0.063)	(0.063)	(0.063)	-
10,000	β_0	0.903	0.910	0.899	0.896	0.894	0.894	0.892	0.894	0.897	0.893	0.912	0.909	0.910	0.907	0.908	0.906	0.908	0.910	0.907	0.910
		(0.038)	(0.038)	(0.038)	(0.038)	(0.038)	(0.038)	(0.038)	(0.038)	(0.038)	(0.037)	(0.037	(0.038)	(0.038)	(0.038)	(0.038)	(0.038)	(0.038)	(0.038)	(0.038)	(0.038)
	β_1	0.895	0.929	0.910	0.906	0.901	0.898	0.895	0.894	0.896	0.896	0.938	0.933	0.934	0.938	0.934	0.935	0.933	0.932	0.934	0.934
		(0.036)	(0.036)	(0.034)	(0.033)	(0.033)	(0.033)	(0.033)	(0.032)	(0.032)	(0.032)	(0.032	(0.036)	(0.037)	(0.037)	(0.037)	(0.036)	(0.037)	(0.037)	(0.037)	(0.037)
	β_2	0.893	0.917	0.901	0.892	0.890	0.889	0.894	0.889	0.891	0.892	0.921	0.917	0.919	0.919	0.918	0.919	0.917	0.920	0.919	0.921
		(0.044)	(0.044)	(0.042)	(0.041)	(0.041)	(0.041)	(0.041)	(0.041)	(0.041)	(0.041)	(0.040	(0.044)	(0.044)	(0.044)	(0.044)	(0.044)	(0.044)	(0.044)	(0.044)	(0.044)

Table 1: Presents the empirical coverage for the **parameter estimates** of the volatility process. The DGP is a **GARCH**$(1,1)$ with **tv-TGC** innovations. The bootstrapped residuals are subject to resampling using the **moving block** bootstrap procedure. The **nominal coverage** is **90**%. $S = 2{,}000$ Monte Carlo iterations and $B = 2{,}000$ bootstrap replications are performed. The volatility and kurtosis parameters are $(\beta_0, \beta_1, \beta_2) = (0.1, 0.15, 0.8)$ and $(\delta_0, \delta_1, \delta_2) = (2, 0.15, 0.4)$, respectively.

		Recursive-design										Fixed-design									
n/l	β	ASY	1	5	10	15	20	30	50	70	100	ASY	1	5	10	15	20	30	50	70	100
500	β_0	0.884	0.899	0.899	0.897	0.892	0.887	0.877	-	-	-	0.924	0.909	0.904	0.908	0.905	0.904	0.908	-	-	-
		(0.129)	(0.138)	(0.138)	(0.138)	(0.137)	(0.137)	(0.135)	-	-	-	(0.125)	(0.130)	(0.128)	(0.128)	(0.128)	(0.128)	(0.128)	-	-	-
	β_1	0.840	0.851	0.836	0.830	0.829	0.827	0.823	-	-	-	0.850	0.864	0.861	0.863	0.864	0.863	0.866	-	-	-
		(0.284)	(0.261)	(0.271)	(0.270)	(0.269)	(0.267)	(0.265)	-	-	-	(0.309)	(0.257)	(0.256)	(0.256)	(0.256)	(0.257)	(0.258)	-	-	-
	β_2	0.860	0.916	0.914	0.906	0.905	0.899	0.896	-	-	-	0.883	0.908	0.902	0.907	0.907	0.907	0.909	-	-	-
		(0.277)	(0.297)	(0.302)	(0.303)	(0.301)	(0.299)	(0.295)	-	-	-	(0.277)	(0.285)	(0.283)	(0.283)	(0.284)	(0.284)	(0.285)	-	-	-
1,000	β_0	0.892	0.890	0.880	0.876	0.879	0.881	0.878	0.867	-	-	0.924	0.905	0.902	0.902	0.901	0.900	0.901	0.903	-	-
		(0.088)	(0.088)	(0.088)	(0.088)	(0.088)	(0.088)	(0.088)	(0.087)	-	-	(0.084)	(0.086)	(0.086)	(0.085)	(0.085)	(0.085)	(0.085)	(0.086)	-	-
	β_1	0.860	0.864	0.859	0.861	0.855	0.855	0.854	0.847	-	-	0.862	0.871	0.868	0.869	0.873	0.871	0.868	0.875	-	-
		(0.210)	(0.193)	(0.204)	(0.204)	(0.204)	(0.204)	(0.203)	(0.200)	-	-	(0.231)	(0.192)	(0.191)	(0.191)	(0.191)	(0.191)	(0.192)	(0.192)	-	-
	β_2	0.870	0.899	0.892	0.887	0.881	0.883	0.883	0.874	-	-	0.890	0.891	0.893	0.893	0.889	0.891	0.892	0.895	-	-
		(0.191)	(0.194)	(0.197)	(0.199)	(0.199)	(0.199)	(0.198)	(0.195)	-	-	(0.192)	(0.191)	(0.190)	(0.190)	(0.190)	(0.190)	(0.191)	(0.192)	-	-
2,000	β_0	0.884	0.900	0.882	0.873	0.874	0.875	0.877	0.875	0.869	-	0.921	0.913	0.908	0.907	0.905	0.905	0.906	0.908	0.906	-
		(0.061)	(0.059)	(0.058)	(0.058)	(0.058)	(0.058)	(0.058)	(0.058)	(0.057)	-	(0.057)	(0.060)	(0.059)	(0.059)	(0.059)	(0.059)	(0.059)	(0.059)	(0.059)	-
	β_1	0.870	0.856	0.862	0.863	0.866	0.868	0.868	0.863	0.861	-	0.855	0.865	0.862	0.862	0.861	0.863	0.863	0.862	0.862	-
		(0.152)	(0.141)	(0.151)	(0.152)	(0.152)	(0.152)	(0.152)	(0.151)	(0.151)	-	(0.169)	(0.142)	(0.141)	(0.141)	(0.141)	(0.141)	(0.141)	(0.141)	(0.142)	-
	β_2	0.869	0.873	0.874	0.873	0.878	0.888	0.887	0.888	0.885	-	0.874	0.877	0.874	0.870	0.875	0.872	0.872	0.872	0.876	-
		(0.134)	(0.132)	(0.134)	(0.135)	(0.135)	(0.135)	(0.135)	(0.134)	(0.133)	-	(0.134)	(0.133)	(0.132)	(0.132)	(0.132)	(0.132)	(0.132)	(0.132)	(0.133)	-
5,000	β_0	0.893	0.906	0.886	0.871	0.873	0.877	0.878	0.886	0.887	-	0.926	0.916	0.912	0.914	0.915	0.917	0.913	0.916	0.915	-
		(0.039)	(0.037)	(0.036)	(0.036)	(0.036)	(0.036)	(0.036)	(0.036)	(0.036)	-	(0.036)	(0.038)	(0.038)	(0.038)	(0.038)	(0.038)	(0.038)	(0.038)	(0.038)	-
	β_1	0.884	0.850	0.874	0.882	0.885	0.884	0.888	0.888	0.884	-	0.851	0.852	0.856	0.853	0.852	0.852	0.852	0.853	0.856	-
		(0.099)	(0.094)	(0.103)	(0.104)	(0.105)	(0.105)	(0.105)	(0.105)	(0.105)	-	(0.113)	(0.095)	(0.095)	(0.095)	(0.095)	(0.095)	(0.095)	(0.095)	(0.095)	-
	β_2	0.884	0.877	0.875	0.872	0.875	0.878	0.880	0.888	0.883	-	0.887	0.880	0.880	0.879	0.881	0.880	0.882	0.881	0.881	-
		(0.085)	(0.083)	(0.085)	(0.085)	(0.085)	(0.085)	(0.085)	(0.085)	(0.085)	-	(0.086)	(0.084)	(0.084)	(0.084)	(0.084)	(0.084)	(0.084)	(0.084)	(0.084)	-
10,000	β_0	0.906	0.918	0.890	0.876	0.878	0.882	0.889	0.888	0.892	0.891	0.928	0.924	0.925	0.924	0.923	0.927	0.924	0.924	0.924	0.926
		(0.027)	(0.026)	(0.025)	(0.025)	(0.025)	(0.025)	(0.025)	(0.025)	(0.025)	(0.025)	(0.025)	(0.027)	(0.027)	(0.027)	(0.027)	(0.027)	(0.027)	(0.027)	(0.027)	(0.027)
	β_1	0.890	0.842	0.875	0.885	0.883	0.889	0.889	0.890	0.891	0.890	0.848	0.845	0.845	0.849	0.843	0.846	0.849	0.848	0.847	0.849
		(0.071)	(0.069)	(0.076)	(0.077)	(0.078)	(0.078)	(0.078)	(0.078)	(0.078)	(0.078)	(0.081)	(0.070)	(0.069)	(0.069)	(0.069)	(0.069)	(0.069)	(0.069)	(0.069)	(0.070)
	β_2	0.904	0.890	0.879	0.877	0.886	0.890	0.900	0.902	0.901	0.900	0.898	0.898	0.895	0.897	0.896	0.896	0.895	0.899	0.899	0.896
		(0.060)	(0.059)	(0.060)	(0.061)	(0.061)	(0.061)	(0.061)	(0.061)	(0.061)	(0.061)	(0.062)	(0.060)	(0.060)	(0.060)	(0.060)	(0.060)	(0.060)	(0.060)	(0.060)	(0.060)

Table 2: Presents the empirical coverage for the **parameter estimates** of the volatility process. The DGP is a **GARCH**$(1,1)$ with **ARCK** innovations. The bootstrapped residuals are subject to resampling using the **moving block** bootstrap procedure. The **nominal coverage** is **90**%. $S = 2{,}000$ Monte Carlo iterations and $B = 2{,}000$ bootstrap replications are performed. The volatility and kurtosis parameters are $(\beta_0, \beta_1, \beta_2) = (0.1, 0.4, 0.55)$ and $(\delta_0, \delta_1, \delta_2) = (1.5, 0.4, 0.5)$, respectively.

		Recursive-design									Fixed-design								
n/l	β	ASY	1	5	10	15	20	30	50	70	ASY	1	5	10	15	20	30	50	70
500	β_0	0.936	0.951	0.947	0.944	0.941	0.940	0.935	-	-	0.952	0.960	0.962	0.962	0.961	0.963	0.964	-	-
		(0.282)	(0.405)	(0.413)	(0.419)	(0.415)	(0.411)	(0.405)	-	-	(0.281)	(0.329)	(0.329)	(0.330)	(0.331)	(0.332)	(0.333)	-	-
	β_1	0.926	0.979	0.964	0.957	0.952	0.948	0.941	-	-	0.950	0.979	0.981	0.979	0.979	0.980	0.980	-	-
		(0.202)	(0.208)	(0.196)	(0.194)	(0.192)	(0.190)	(0.187)	-	-	(0.184)	(0.209)	(0.209)	(0.210)	(0.210)	(0.211)	(0.213)	-	-
	β_2	0.926	0.967	0.961	0.957	0.954	0.950	0.939	-	-	0.950	0.963	0.964	0.967	0.961	0.965	0.962	-	-
		(0.293)	(0.394)	(0.392)	(0.396)	(0.391)	(0.387)	(0.380)	-	-	(0.273)	(0.341)	(0.341)	(0.342)	(0.343)	(0.345)	(0.347)	-	-
1,000	β_0	0.944	0.961	0.958	0.953	0.951	0.948	0.944	0.935	-	0.954	0.962	0.965	0.964	0.966	0.965	0.968	0.965	-
		(0.163)	(0.200)	(0.202)	(0.209)	(0.211)	(0.212)	(0.212)	(0.211)	-	(0.162)	(0.178)	(0.178)	(0.178)	(0.178)	(0.178)	(0.179)	(0.180)	-
	β_1	0.940	0.974	0.964	0.957	0.957	0.958	0.954	0.952	-	0.962	0.976	0.975	0.975	0.976	0.977	0.976	0.975	-
		(0.140)	(0.142)	(0.133)	(0.131)	(0.130)	(0.130)	(0.128)	(0.126)	-	(0.126)	(0.143)	(0.143)	(0.143)	(0.143)	(0.143)	(0.144)	(0.145)	-
	β_2	0.946	0.974	0.967	0.964	0.961	0.957	0.953	0.943	-	0.962	0.969	0.968	0.967	0.966	0.968	0.970	0.972	-
		(0.180)	(0.210)	(0.205)	(0.210)	(0.211)	(0.211)	(0.210)	(0.207)	-	(0.168)	(0.196)	(0.196)	(0.196)	(0.196)	(0.197)	(0.198)	(0.199)	-
2,000	β_0	0.946	0.952	0.945	0.944	0.943	0.947	0.941	0.938	0.931	0.956	0.954	0.953	0.953	0.956	0.954	0.953	0.957	0.959
		(0.107)	(0.117)	(0.117)	(0.118)	(0.117)	(0.117)	(0.117)	(0.116)	(0.116)	(0.106)	(0.111)	(0.111)	(0.111)	(0.111)	(0.111)	(0.111)	(0.112)	(0.112)
	β_1	0.942	0.968	0.959	0.954	0.953	0.950	0.948	0.942	0.938	0.965	0.970	0.971	0.974	0.971	0.973	0.972	0.972	0.975
		(0.098)	(0.098)	(0.092)	(0.090)	(0.090)	(0.089)	(0.088)	(0.087)	(0.087)	(0.088)	(0.099)	(0.099)	(0.100)	(0.100)	(0.100)	(0.100)	(0.100)	(0.101)
	β_2	0.941	0.964	0.955	0.949	0.945	0.947	0.945	0.938	0.934	0.964	0.963	0.962	0.964	0.963	0.961	0.965	0.963	0.964
		(0.121)	(0.129)	(0.124)	(0.123)	(0.123)	(0.123)	(0.122)	(0.121)	(0.120)	(0.112)	(0.125)	(0.125)	(0.125)	(0.125)	(0.126)	(0.126)	(0.126)	(0.127)
5,000	β_0	0.944	0.946	0.944	0.944	0.937	0.935	0.937	0.937	0.928	0.946	0.943	0.942	0.939	0.943	0.943	0.941	0.946	0.944
		(0.065)	(0.067)	(0.066)	(0.066)	(0.066)	(0.066)	(0.066)	(0.066)	(0.066)	(0.064)	(0.066)	(0.066)	(0.066)	(0.066)	(0.066)	(0.066)	(0.066)	(0.066)
	β_1	0.944	0.968	0.954	0.952	0.947	0.950	0.948	0.946	0.945	0.966	0.972	0.969	0.971	0.970	0.966	0.972	0.967	0.971
		(0.062)	(0.061)	(0.057)	(0.056)	(0.056)	(0.055)	(0.055)	(0.055)	(0.055)	(0.055)	(0.062)	(0.062)	(0.062)	(0.062)	(0.062)	(0.062)	(0.062)	(0.062)
	β_2	0.943	0.962	0.950	0.951	0.945	0.944	0.941	0.942	0.940	0.964	0.963	0.960	0.960	0.960	0.960	0.961	0.961	0.961
		(0.074)	(0.076)	(0.073)	(0.072)	(0.071)	(0.071)	(0.071)	(0.071)	(0.070)	(0.069)	(0.075)	(0.075)	(0.075)	(0.075)	(0.075)	(0.075)	(0.076)	(0.076)

Table 3: Presents the empirical coverage for the **parameter estimates** of the volatility process. The DGP is a **GARCH**$(1,1)$ with **tv-TGC** innovations. The bootstrapped residuals are subject to resampling using the **moving block** bootstrap procedure. The **nominal coverage** is **95**%. $S = 2{,}000$ Monte Carlo iterations and $B = 2{,}000$ bootstrap replications are performed. The volatility and kurtosis parameters are $(\beta_0, \beta_1, \beta_2) = (0.1, 0.15, 0.8)$ and $(\delta_0, \delta_1, \delta_2) = (2, 0.15, 0.4)$, respectively.

		Recursive-design									Fixed-design								
n/l	β	ASY	1	5	10	15	20	30	50	70	ASY	1	5	10	15	20	30	50	70
500	β_0	0.925	0.938	0.933	0.938	0.938	0.934	0.927	-	-	0.952	0.945	0.939	0.942	0.939	0.942	0.943	-	-
		(0.142)	(0.170)	(0.169)	(0.170)	(0.170)	(0.170)	(0.168)	-	-	(0.154)	(0.158)	(0.156)	(0.155)	(0.155)	(0.155)	(0.156)	-	-
	β_1	0.870	0.894	0.877	0.871	0.869	0.867	0.867	-	-	0.889	0.910	0.909	0.909	0.908	0.910	0.912	-	-
		(0.353)	(0.312)	(0.323)	(0.323)	(0.321)	(0.319)	(0.316)	-	-	(0.340)	(0.308)	(0.305)	(0.305)	(0.306)	(0.306)	(0.308)	-	-
	β_2	0.894	0.950	0.956	0.952	0.954	0.952	0.938	-	-	0.916	0.941	0.940	0.938	0.936	0.940	0.941	-	-
		(0.315)	(0.362)	(0.368)	(0.370)	(0.369)	(0.366)	(0.361)	-	-	(0.329)	(0.344)	(0.341)	(0.341)	(0.341)	(0.342)	(0.344)	-	-
1,000	β_0	0.934	0.931	0.929	0.924	0.930	0.929	0.923	0.921	-	0.959	0.944	0.942	0.941	0.942	0.939	0.942	0.942	-
		(0.104)	(0.106)	(0.104)	(0.105)	(0.105)	(0.105)	(0.105)	(0.104)	-	(0.098)	(0.103)	(0.102)	(0.101)	(0.101)	(0.101)	(0.102)	(0.102)	-
	β_1	0.905	0.910	0.899	0.898	0.897	0.898	0.892	0.887	-	0.906	0.917	0.918	0.916	0.916	0.916	0.918	0.918	-
		(0.249)	(0.229)	(0.240)	(0.241)	(0.241)	(0.241)	(0.240)	(0.238)	-	(0.272)	(0.228)	(0.227)	(0.226)	(0.226)	(0.227)	(0.227)	(0.228)	-
	β_2	0.910	0.932	0.934	0.929	0.932	0.927	0.928	0.920	-	0.926	0.931	0.928	0.928	0.928	0.930	0.929	0.932	-
		(0.226)	(0.232)	(0.236)	(0.237)	(0.238)	(0.237)	(0.236)	(0.234)	-	(0.224)	(0.229)	(0.227)	(0.227)	(0.227)	(0.227)	(0.228)	(0.229)	-
2,000	β_0	0.952	0.958	0.945	0.939	0.938	0.941	0.943	0.939	0.930	0.966	0.962	0.960	0.957	0.958	0.960	0.957	0.958	0.958
		(0.068)	(0.071)	(0.070)	(0.070)	(0.070)	(0.070)	(0.070)	(0.069)	(0.069)	(0.073)	(0.072)	(0.071)	(0.071)	(0.071)	(0.071)	(0.071)	(0.071)	(0.071)
	β_1	0.926	0.917	0.925	0.923	0.924	0.922	0.922	0.919	0.915	0.916	0.922	0.923	0.925	0.923	0.924	0.922	0.923	0.925
		(0.205)	(0.170)	(0.182)	(0.183)	(0.184)	(0.183)	(0.183)	(0.183)	(0.182)	(0.184)	(0.170)	(0.169)	(0.168)	(0.169)	(0.169)	(0.169)	(0.169)	(0.169)
	β_2	0.936	0.941	0.935	0.934	0.933	0.934	0.939	0.940	0.938	0.943	0.939	0.939	0.936	0.938	0.940	0.938	0.940	0.940
		(0.160)	(0.159)	(0.162)	(0.163)	(0.163)	(0.163)	(0.163)	(0.162)	(0.161)	(0.160)	(0.159)	(0.158)	(0.158)	(0.158)	(0.158)	(0.158)	(0.159)	(0.159)
5,000	β_0	0.947	0.961	0.944	0.936	0.934	0.939	0.942	0.947	0.946	0.968	0.964	0.965	0.963	0.963	0.961	0.961	0.964	0.965
		(0.046)	(0.044)	(0.043)	(0.043)	(0.043)	(0.043)	(0.043)	(0.043)	(0.042)	(0.043)	(0.045)	(0.045)	(0.045)	(0.045)	(0.045)	(0.045)	(0.045)	(0.045)
	β_1	0.934	0.910	0.927	0.935	0.935	0.937	0.938	0.934	0.932	0.916	0.913	0.917	0.918	0.915	0.916	0.916	0.913	0.919
		(0.119)	(0.112)	(0.123)	(0.124)	(0.124)	(0.124)	(0.124)	(0.124)	(0.124)	(0.135)	(0.114)	(0.113)	(0.113)	(0.113)	(0.113)	(0.113)	(0.113)	(0.113)
	β_2	0.943	0.941	0.933	0.926	0.929	0.933	0.940	0.942	0.943	0.944	0.945	0.947	0.943	0.944	0.945	0.946	0.944	0.947
		(0.101)	(0.099)	(0.101)	(0.102)	(0.102)	(0.102)	(0.102)	(0.102)	(0.101)	(0.103)	(0.100)	(0.100)	(0.100)	(0.100)	(0.100)	(0.100)	(0.100)	(0.100)

Table 4: Presents the empirical coverage for the **parameter estimates** of the volatility process. The DGP is a **GARCH**$(1,1)$ with **ARCK** innovations. The bootstrapped residuals are subject to resampling using the **moving block** bootstrap procedure. The **nominal coverage** is **95**%. $S = 2{,}000$ Monte Carlo iterations and $B = 2{,}000$ bootstrap replications are performed. The volatility and kurtosis parameters are $(\beta_0, \beta_1, \beta_2) = (0.1, 0.4, 0.44)$ and $(\delta_0, \delta_1, \delta_2) = (1.5, 0.4, 0.5)$, respectively.

		Recursive-design									Fixed-design								
n/l	β	ASY	1	5	10	15	20	30	50	70	ASY	1	5	10	15	20	30	50	70
500	β_0	0.823	0.940	0.925	0.921	0.914	0.915	0.893	-	-	0.870	0.919	0.917	0.917	0.919	0.915	0.918	-	-
		(0.520)	(0.495)	(0.481)	(0.466)	(0.459)	(0.453)	(0.444)	-	-	(0.389)	(0.356)	(0.357)	(0.357)	(0.357)	(0.359)	(0.359)	-	-
	β_1^+	0.952	0.970	0.951	0.936	0.929	0.931	0.921	-	-	0.961	0.935	0.934	0.936	0.939	0.940	0.938	-	-
		(0.208)	(0.172)	(0.164)	(0.164)	(0.164)	(0.164)	(0.163)	-	-	(0.188)	(0.171)	(0.171)	(0.171)	(0.171)	(0.171)	(0.173)	-	-
	β_1^-	0.920	0.963	0.937	0.930	0.929	0.919	0.917	-	-	0.938	0.936	0.937	0.937	0.936	0.935	0.938	-	-
		(0.234)	(0.213)	(0.203)	(0.201)	(0.200)	(0.199)	(0.196)	-	-	(0.208)	(0.206)	(0.206)	(0.206)	(0.207)	(0.207)	(0.209)	-	-
	β_2	0.816	0.924	0.904	0.904	0.899	0.894	0.874	-	-	0.869	0.875	0.876	0.875	0.880	0.877	0.879	-	-
		(0.817)	(0.790)	(0.763)	(0.739)	(0.727)	(0.716)	(0.701)	-	-	(0.618)	(0.582)	(0.583)	(0.584)	(0.585)	(0.588)	(0.589)	-	-
1,000	β_0	0.809	0.918	0.919	0.918	0.919	0.913	0.913	0.902	-	0.846	0.913	0.909	0.913	0.909	0.912	0.914	0.911	-
		(0.441)	(0.368)	(0.380)	(0.380)	(0.378)	(0.376)	(0.371)	(0.368)	-	(0.283)	(0.279)	(0.279)	(0.280)	(0.280)	(0.280)	(0.281)	(0.282)	-
	β_1^+	0.930	0.965	0.941	0.937	0.934	0.928	0.923	0.921	-	0.943	0.957	0.955	0.957	0.957	0.954	0.953	0.955	-
		(0.132)	(0.120)	(0.118)	(0.119)	(0.119)	(0.119)	(0.119)	(0.119)	-	(0.127)	(0.118)	(0.118)	(0.118)	(0.118)	(0.118)	(0.119)	(0.119)	-
	β_1^-	0.895	0.945	0.923	0.918	0.916	0.913	0.916	0.907	-	0.923	0.913	0.915	0.915	0.914	0.918	0.918	0.919	-
		(0.154)	(0.147)	(0.141)	(0.141)	(0.140)	(0.140)	(0.139)	(0.137)	-	(0.143)	(0.145)	(0.145)	(0.145)	(0.145)	(0.145)	(0.145)	(0.146)	-
	β_2	0.800	0.913	0.912	0.910	0.912	0.899	0.898	0.888	-	0.857	0.905	0.905	0.910	0.900	0.905	0.903	0.906	-
		(0.690)	(0.586)	(0.604)	(0.603)	(0.598)	(0.596)	(0.587)	(0.582)	-	(0.453)	(0.454)	(0.455)	(0.455)	(0.456)	(0.456)	(0.457)	(0.460)	-
2,000	β_0	0.847	0.912	0.925	0.914	0.914	0.917	0.909	0.902	0.894	0.863	0.912	0.912	0.907	0.908	0.912	0.911	0.909	0.908
		(0.193)	(0.209)	(0.226)	(0.239)	(0.243)	(0.246)	(0.247)	(0.247)	(0.247)	(0.174)	(0.191)	(0.191)	(0.191)	(0.191)	(0.190)	(0.191)	(0.192)	(0.192)
	β_1^+	0.907	0.936	0.925	0.923	0.927	0.927	0.929	0.925	0.923	0.922	0.942	0.947	0.945	0.948	0.947	0.944	0.950	0.948
		(0.082)	(0.083)	(0.081)	(0.082)	(0.082)	(0.083)	(0.083)	(0.083)	(0.083)	(0.085)	(0.082)	(0.082)	(0.082)	(0.082)	(0.082)	(0.083)	(0.083)	(0.083)
	β_1^-	0.920	0.939	0.923	0.921	0.922	0.923	0.924	0.931	0.924	0.932	0.934	0.940	0.938	0.937	0.934	0.936	0.937	0.939
		(0.097)	(0.101)	(0.097)	(0.097)	(0.097)	(0.098)	(0.098)	(0.097)	(0.097)	(0.101)	(0.102)	(0.102)	(0.102)	(0.102)	(0.102)	(0.102)	(0.102)	(0.103)
	β_2	0.851	0.917	0.915	0.912	0.916	0.920	0.915	0.909	0.898	0.871	0.909	0.910	0.908	0.904	0.908	0.909	0.908	0.911
		(0.312)	(0.340)	(0.364)	(0.383)	(0.389)	(0.392)	(0.394)	(0.394)	(0.395)	(0.286)	(0.312)	(0.312)	(0.313)	(0.313)	(0.312)	(0.313)	(0.314)	(0.315)
5,000	β_0	0.890	0.890	0.889	0.892	0.893	0.895	0.903	0.893	0.895	0.895	0.898	0.899	0.904	0.896	0.895	0.898	0.900	0.902
		(0.105)	(0.104)	(0.108)	(0.113)	(0.116)	(0.117)	(0.118)	(0.120)	(0.120)	(0.100)	(0.105)	(0.106)	(0.106)	(0.106)	(0.106)	(0.106)	(0.106)	(0.106)
	β_1^+	0.898	0.916	0.899	0.894	0.895	0.902	0.901	0.894	0.890	0.917	0.918	0.914	0.917	0.917	0.914	0.919	0.916	0.917
		(0.049)	(0.052)	(0.050)	(0.049)	(0.049)	(0.049)	(0.049)	(0.049)	(0.050)	(0.052)	(0.053)	(0.053)	(0.053)	(0.053)	(0.053)	(0.053)	(0.053)	(0.053)
	β_1^-	0.889	0.926	0.896	0.896	0.892	0.892	0.897	0.892	0.897	0.920	0.926	0.927	0.925	0.924	0.926	0.922	0.927	0.925
		(0.059)	(0.063)	(0.060)	(0.059)	(0.059)	(0.059)	(0.059)	(0.059)	(0.059)	(0.063)	(0.063)	(0.063)	(0.063)	(0.063)	(0.063)	(0.063)	(0.064)	(0.064)
	β_2	0.888	0.900	0.891	0.884	0.890	0.888	0.896	0.890	0.885	0.898	0.904	0.904	0.905	0.899	0.902	0.903	0.906	0.905
		(0.172)	(0.173)	(0.177)	(0.185)	(0.188)	(0.190)	(0.192)	(0.194)	(0.194)	(0.166)	(0.175)	(0.175)	(0.175)	(0.175)	(0.175)	(0.175)	(0.175)	(0.176)

Table 5: Presents the empirical coverage for the **parameter estimates** of the volatility process. The bootstrapped residuals are subject to resampling using the **moving block** bootstrap procedure. The DGP is a **TGARCH**$(1,1,1)$ with **tv-TGC** innovations. The **nominal coverage** is **90**%. $S = 2{,}000$ Monte Carlo iterations and $B = 2{,}000$ bootstrap replications are performed. The volatility and kurtosis parameters are $(\beta_0, \beta_1^+, \beta_1^-, \beta_2) = (0.1, 0.05, 0.1, 0.8)$ and $(\delta_0, \delta_1, \delta_2) = (2, 0.15, 0.4)$, respectively.

		Recursive-design									Fixed-design								
n/l	β	ASY	1	5	10	15	20	30	50	70	ASY	1	5	10	15	20	30	50	70
500	β_0	0.802	0.915	0.923	0.928	0.923	0.915	0.909	-	-	0.829	0.907	0.905	0.900	0.905	0.896	0.910	-	-
		(0.142)	(0.133)	(0.134)	(0.135)	(0.134)	(0.133)	(0.131)	-	-	(0.132)	(0.122)	(0.121)	(0.121)	(0.122)	(0.122)	(0.122)	-	-
	β_1^+	0.890	0.888	0.897	0.893	0.882	0.872	0.874	-	-	0.883	0.936	0.934	0.935	0.937	0.935	0.939	-	-
		(0.240)	(0.186)	(0.191)	(0.193)	(0.193)	(0.193)	(0.191)	-	-	(0.220)	(0.186)	(0.185)	(0.185)	(0.185)	(0.186)	(0.187)	-	-
	β_1^-	0.846	0.834	0.848	0.849	0.843	0.846	0.829	-	-	0.827	0.848	0.847	0.846	0.847	0.846	0.844	-	-
		(0.291)	(0.261)	(0.267)	(0.267)	(0.265)	(0.264)	(0.261)	-	-	(0.272)	(0.259)	(0.258)	(0.258)	(0.258)	(0.259)	(0.260)	-	-
	β_2	0.780	0.905	0.919	0.926	0.914	0.909	0.895	-	-	0.812	0.899	0.896	0.890	0.885	0.890	0.899	-	-
		(0.565)	(0.533)	(0.542)	(0.543)	(0.537)	(0.534)	(0.526)	-	-	(0.522)	(0.489)	(0.488)	(0.488)	(0.489)	(0.490)	(0.491)	-	-
1,000	β_0	0.823	0.875	0.875	0.881	0.883	0.886	0.878	0.869	-	0.847	0.873	0.874	0.873	0.877	0.873	0.872	0.873	-
		(0.105)	(0.098)	(0.100)	(0.101)	(0.101)	(0.101)	(0.101)	(0.100)	-	(0.098)	(0.096)	(0.095)	(0.095)	(0.096)	(0.096)	(0.096)	(0.096)	
	β_1^+	0.859	0.832	0.853	0.861	0.862	0.859	0.856	0.854	-	0.828	0.845	0.849	0.850	0.846	0.849	0.852	0.853	-
		(0.169)	(0.142)	(0.148)	(0.151)	(0.151)	(0.151)	(0.151)	(0.149)	-	(0.157)	(0.141)	(0.141)	(0.141)	(0.141)	(0.141)	(0.142)	(0.142)	
	β_1^-	0.844	0.835	0.848	0.851	0.856	0.854	0.850	0.844	-	0.835	0.844	0.838	0.840	0.835	0.840	0.843	0.839	-
		(0.216)	(0.195)	(0.203)	(0.204)	(0.205)	(0.204)	(0.203)	(0.201)	-	(0.201)	(0.194)	(0.194)	(0.194)	(0.194)	(0.194)	(0.195)	(0.195)	
	β_2	0.818	0.863	0.873	0.874	0.875	0.876	0.879	0.859	-	0.825	0.857	0.859	0.861	0.865	0.862	0.867	0.866	-
		(0.419)	(0.391)	(0.400)	(0.407)	(0.407)	(0.406)	(0.405)	(0.400)	-	(0.389)	(0.381)	(0.379)	(0.379)	(0.381)	(0.381)	(0.381)	(0.383)	
2,000	β_0	0.861	0.870	0.872	0.872	0.874	0.876	0.881	0.878	0.863	0.861	0.874	0.869	0.872	0.876	0.869	0.871	0.875	0.869
		(0.071)	(0.068)	(0.070)	(0.072)	(0.072)	(0.072)	(0.073)	(0.072)	(0.072)	(0.069)	(0.068)	(0.068)	(0.068)	(0.068)	(0.068)	(0.068)	(0.068)	(0.068)
	β_1^+	0.872	0.834	0.852	0.857	0.857	0.862	0.853	0.848	0.855	0.845	0.848	0.846	0.848	0.846	0.843	0.844	0.844	0.848
		(0.120)	(0.107)	(0.114)	(0.115)	(0.116)	(0.116)	(0.116)	(0.116)	(0.115)	(0.113)	(0.107)	(0.107)	(0.107)	(0.107)	(0.107)	(0.107)	(0.107)	(0.107)
	β_1^-	0.870	0.834	0.854	0.861	0.866	0.866	0.865	0.860	0.858	0.836	0.833	0.834	0.833	0.833	0.836	0.832	0.837	0.836
		(0.160)	(0.142)	(0.152)	(0.153)	(0.153)	(0.153)	(0.153)	(0.152)	(0.152)	(0.145)	(0.142)	(0.142)	(0.142)	(0.142)	(0.142)	(0.142)	(0.142)	(0.142)
	β_2	0.851	0.863	0.859	0.864	0.873	0.879	0.878	0.879	0.864	0.851	0.859	0.864	0.860	0.863	0.865	0.871	0.870	0.863
		(0.289)	(0.274)	(0.283)	(0.289)	(0.292)	(0.293)	(0.294)	(0.293)	(0.291)	(0.274)	(0.272)	(0.272)	(0.272)	(0.272)	(0.272)	(0.272)	(0.273)	(0.273)
5,000	β_0	0.886	0.873	0.870	0.885	0.890	0.890	0.889	0.884	0.887	0.875	0.875	0.871	0.875	0.876	0.874	0.872	0.871	0.878
		(0.044)	(0.042)	(0.043)	(0.044)	(0.044)	(0.044)	(0.044)	(0.044)	(0.044)	(0.043)	(0.043)	(0.043)	(0.043)	(0.042)	(0.042)	(0.042)	(0.043)	(0.043)
	β_1^+	0.876	0.840	0.874	0.876	0.877	0.880	0.877	0.878	0.875	0.851	0.852	0.846	0.849	0.849	0.845	0.848	0.845	0.845
		(0.078)	(0.070)	(0.076)	(0.076)	(0.077)	(0.077)	(0.077)	(0.077)	(0.077)	(0.071)	(0.070)	(0.070)	(0.070)	(0.070)	(0.070)	(0.070)	(0.070)	(0.070)
	β_1^-	0.893	0.850	0.878	0.887	0.884	0.886	0.888	0.884	0.884	0.851	0.849	0.853	0.849	0.850	0.848	0.847	0.850	0.843
		(0.103)	(0.092)	(0.099)	(0.100)	(0.101)	(0.101)	(0.101)	(0.101)	(0.101)	(0.093)	(0.092)	(0.092)	(0.092)	(0.092)	(0.092)	(0.092)	(0.092)	(0.092)
	β_2	0.883	0.869	0.869	0.886	0.895	0.896	0.896	0.891	0.888	0.871	0.873	0.864	0.867	0.865	0.868	0.869	0.867	0.868
		(0.182)	(0.171)	(0.176)	(0.180)	(0.181)	(0.182)	(0.182)	(0.182)	(0.182)	(0.171)	(0.171)	(0.171)	(0.171)	(0.170)	(0.171)	(0.171)	(0.171)	(0.171)

Table 6: Presents the empirical coverage for the **parameter estimates** of the volatility process. The DGP is a **TGARCH**$(1,1,1)$ with **ARCK** innovations. The bootstrapped residuals are subject to resampling using the **moving block** bootstrap procedure. The **nominal coverage** is **90**%. $S = 2{,}000$ Monte Carlo iterations and $B = 2{,}000$ bootstrap replications are performed. The volatility and kurtosis parameters are $(\beta_0, \beta_1^+, \beta_1^-, \beta_2) = (0.1, 0.1, 0.3, 0.5)$ and $(\delta_0, \delta_1, \delta_2) = (1.5, 0.4, 0.5)$, respectively.

Tables 7 and 8 (9 and 10) present the findings from the Monte Carlo simulation study for the next period's volatility in the context of the GARCH(1,1) process with tv-TGC and ARCK innovations, respectively, with nominal coverage $\gamma = 0.1$ ($\gamma = 0.05$). For the tv-TGC setting, increasing the block length leads to a contraction of confidence intervals within the recursive-design block bootstrap approach, whereas the fixed-design counterpart displays no adaptability. Conversely, in the case of ARCK innovations, using larger block lengths leads to larger confidence intervals, exclusively for the recursive-design procedure. When employing the recursive-design procedure in the tv-TGC setting, the selection of large block lengths l results in overly narrow confidence intervals. As a consequence, the empirical coverage falls short of its nominal value. When dealing with samples of modest size ($n = 500$), the empirical coverage achieved by the recursive-design bootstrap markedly falls short in both settings, but when the sample size is enlarged, the empirical coverage rates approach their nominal value. Once more, in the context at hand, the coverage rates for the tv-TGC process are closer to their nominal values in smaller sample sizes compared to those obtained in the ARCK setting, which can be attributed to the pronounced heavy-tailed distribution inherent to the latter setting.

Concerning the TGARCH(1,1,1) illustrated in Tables 11 and 12 for the tv-TGC and ARCK configurations, respectively, the findings closely parallel those of the GARCH(1,1) framework. Specifically, in the context of tv-TGC (ARCK), the confidence intervals decrease (increase) with the block length exclusively within the recursive-design framework, while the fixed-design procedure once again falls short in capturing the interdependence between innovations and volatility processes.

	n/l	ASY	1	5	10	15	20	30	50	70	100
	500	0.869	0.876	0.869	0.865	0.854	0.854	0.844	-	-	-
		(0.965)	(0.978)	(0.956)	(0.954)	(0.952)	(0.950)	(0.948)	-	-	-
	1,000	0.893	0.900	0.891	0.888	0.886	0.881	0.876	0.861	-	-
		(0.692)	(0.719)	(0.701)	(0.699)	(0.698)	(0.697)	(0.695)	(0.690)	-	-
Recursive-	2,000	0.875	0.899	0.890	0.890	0.889	0.885	0.886	0.883	0.880	-
design		(0.456)	(0.474)	(0.463)	(0.461)	(0.460)	(0.459)	(0.460)	(0.459)	(0.457)	-
	5,000	0.896	0.911	0.902	0.903	0.899	0.898	0.896	0.895	0.895	-
		(0.324)	(0.338)	(0.328)	(0.327)	(0.326)	(0.326)	(0.325)	(0.324)	(0.323)	-
	10,000	0.901	0.906	0.905	0.903	0.899	0.898	0.898	0.898	0.895	0.895
		(0.207)	(0.214)	(0.209)	(0.208)	(0.208)	(0.207)	(0.207)	(0.206)	(0.205)	(0.205)
	500	0.890	0.891	0.892	0.886	0.889	0.887	0.887	-	-	-
		(1.003)	(0.918)	(0.918)	(0.921)	(0.920)	(0.921)	(0.921)	-	-	-
	1,000	0.904	0.908	0.906	0.908	0.907	0.906	0.902	0.904	-	-
		(0.726)	(0.695)	(0.696)	(0.696)	(0.696)	(0.698)	(0.698)	(0.698)	-	-
Fixed-	2,000	0.892	0.895	0.897	0.896	0.897	0.899	0.900	0.897	0.899	-
design		(0.474)	(0.471)	(0.471)	(0.471)	(0.472)	(0.470)	(0.471)	(0.471)	(0.470)	-
	5,000	0.908	0.908	0.906	0.902	0.904	0.905	0.904	0.907	0.905	-
		(0.339)	(0.339)	(0.339)	(0.339)	(0.339)	(0.339)	(0.339)	(0.338)	(0.338)	-
	10,000	0.909	0.906	0.906	0.908	0.907	0.908	0.904	0.906	0.907	0.906
		(0.215)	(0.215)	(0.215)	(0.215)	(0.215)	(0.215)	(0.215)	(0.215)	(0.215)	(0.215)

Table 7: Presents the empirical coverage rates for the **next period's volatility** σ^2_{t+1} with **tv-TGC innovations**. The bootstrapped residuals are subject to resampling using the **moving block** bootstrap procedure. The **nominal coverage** is $(1-\gamma)\times 100 = \mathbf{90}\%$. For each bootstrap procedure, sample size n and block length l, S = 2,000 Monte Carlo iterations and B = 2,000 bootstrap replications are performed. The DGP is a **GARCH**$(1,1)$ with parameters $(\beta_0, \beta_1, \beta_2) = (0.1, 0.15, 0.8)$ for the volatility process, and $(\delta_0, \delta_1, \delta_2) = (2, 0.15, 0.4)$.

	n/l	ASY	1	5	10	15	20	30	50	70	100
	500	0.861	0.847	0.834	0.829	0.823	0.820	0.815	-	-	-
		(0.596)	(0.513)	(0.519)	(0.521)	(0.522)	(0.524)	(0.521)	-	-	-
	1,000	0.889	0.882	0.871	0.865	0.865	0.864	0.858	0.853	-	-
Recursive-		(0.498)	(0.425)	(0.440)	(0.442)	(0.444)	(0.445)	(0.448)	(0.445)	-	-
design	2,000	0.888	0.887	0.883	0.873	0.869	0.867	0.870	0.859	0.855	-
		(0.542)	(0.450)	(0.465)	(0.485)	(0.500)	(0.507)	(0.517)	(0.516)	(0.503)	-
	5,000	0.898	0.899	0.899	0.894	0.893	0.893	0.893	0.895	0.893	-
		(0.354)	(0.297)	(0.306)	(0.310)	(0.313)	(0.311)	(0.312)	(0.312)	(0.308)	-
	10,000	0.893	0.890	0.892	0.889	0.893	0.892	0.894	0.897	0.890	0.889
		(0.337)	(0.272)	(0.300)	(0.304)	(0.309)	(0.310)	(0.314)	(0.315)	(0.316)	(0.318)
	500	0.878	0.863	0.850	0.848	0.846	0.844	0.839	-	-	-
		(0.582)	(0.497)	(0.491)	(0.490)	(0.489)	(0.488)	(0.487)	-	-	-
	1,000	0.907	0.889	0.890	0.884	0.885	0.883	0.884	0.879	-	-
Fixed-		(0.487)	(0.421)	(0.417)	(0.415)	(0.415)	(0.415)	(0.415)	(0.414)	-	-
design	2,000	0.896	0.893	0.891	0.885	0.884	0.884	0.881	0.882	0.878	-
		(0.497)	(0.447)	(0.448)	(0.446)	(0.444)	(0.448)	(0.448)	(0.446)	(0.442)	-
	5,000	0.898	0.904	0.899	0.901	0.898	0.899	0.896	0.896	0.898	-
		(0.325)	(0.305)	(0.301)	(0.303)	(0.301)	(0.298)	(0.300)	(0.301)	(0.300)	-
	10,000	0.894	0.892	0.889	0.893	0.888	0.891	0.894	0.892	0.888	0.888
		(0.290)	(0.278)	(0.280)	(0.283)	(0.277)	(0.276)	(0.277)	(0.277)	(0.277)	(0.278)

Table 8: Presents the empirical coverage rates for the **next period's volatility** σ^2_{t+1} with **ARCK innovations**. The bootstrapped residuals are subject to resampling using the **moving block** bootstrap procedure. The **nominal coverage** is $(1-\gamma)\times 100 = \mathbf{90}\%$. For each bootstrap procedure, sample size n and block length l, $S = 2{,}000$ Monte Carlo iterations and $B = 2{,}000$ bootstrap replications are performed. The DGP is a $\mathbf{GARCH}(1,1)$ with parameters $(\beta_0, \beta_1, \beta_2) = (0.1, 0.4, 0.55)$ for the volatility process, and $(\delta_0, \delta_1, \delta_2) = (1.5, 0.4, 0.5)$.

	n/l	ASY	1	5	10	15	20	30	50	70
Recursive-design	500	0.919	0.927	0.921	0.912	0.911	0.910	0.900	-	-
		(1.164)	(1.201)	(1.186)	(1.191)	(1.190)	(1.186)	(1.179)	-	-
	1,000	0.934	0.946	0.940	0.936	0.941	0.938	0.934	0.922	-
		(0.828)	(0.870)	(0.851)	(0.853)	(0.853)	(0.851)	(0.849)	(0.841)	-
	2,000	0.931	0.950	0.943	0.942	0.938	0.935	0.930	0.922	0.923
		(0.632)	(0.662)	(0.644)	(0.641)	(0.639)	(0.638)	(0.638)	(0.636)	(0.631)
	5,000	0.944	0.950	0.943	0.940	0.943	0.939	0.938	0.940	0.939
		(0.362)	(0.377)	(0.367)	(0.366)	(0.365)	(0.366)	(0.366)	(0.365)	(0.364)
Fixed-design	500	0.931	0.939	0.939	0.936	0.935	0.934	0.938	-	-
		(1.210)	(1.119)	(1.118)	(1.117)	(1.120)	(1.120)	(1.119)	-	-
	1,000	0.945	0.951	0.952	0.949	0.947	0.951	0.947	0.948	-
		(0.874)	(0.839)	(0.839)	(0.838)	(0.839)	(0.839)	(0.840)	(0.839)	-
	2,000	0.944	0.949	0.948	0.947	0.947	0.947	0.947	0.946	0.946
		(0.663)	(0.656)	(0.656)	(0.656)	(0.655)	(0.655)	(0.656)	(0.654)	(0.653)
	5,000	0.952	0.953	0.950	0.949	0.950	0.953	0.951	0.948	0.948
		(0.378)	(0.378)	(0.378)	(0.379)	(0.378)	(0.378)	(0.378)	(0.378)	(0.378)

Table 9: Presents the empirical coverage rates for the **next period's volatility** σ^2_{t+1} with **tv-TGC innovations**. The bootstrapped residuals are subject to resampling using the **moving block** bootstrap procedure. The **nominal coverage** is $(1-\gamma)\times 100 = \mathbf{95}\%$. For each bootstrap procedure, sample size n and block length l, $S = 2{,}000$ Monte Carlo iterations and $B = 2{,}000$ bootstrap replications are performed. The DGP is a **GARCH**$(1,1)$ with parameters $(\beta_0, \beta_1, \beta_2) = (0.1, 0.15, 0.8)$ for the volatility process, and $(\delta_0, \delta_1, \delta_2) = (2, 0.15, 0.4)$.

	n/l	ASY	1	5	10	15	20	30	50	70
Recursive-design	500	0.926	0.921	0.913	0.908	0.907	0.901	0.892	-	-
		(1.095)	(1.024)	(1.028)	(1.031)	(1.027)	(1.025)	(1.017)	-	-
	1,000	0.933	0.929	0.922	0.920	0.918	0.915	0.910	0.906	-
		(1.073)	(0.850)	(0.876)	(0.915)	(0.941)	(0.950)	(0.998)	(0.995)	-
	2,000	0.938	0.931	0.928	0.926	0.923	0.926	0.925	0.926	0.922
		(0.662)	(0.597)	(0.615)	(0.618)	(0.621)	(0.619)	(0.620)	(0.623)	(0.620)
	5,000	0.951	0.947	0.950	0.950	0.948	0.949	0.950	0.948	0.948
		(0.331)	(0.289)	(0.301)	(0.303)	(0.305)	(0.305)	(0.306)	(0.305)	(0.305)
Fixed-design	500	0.933	0.942	0.936	0.933	0.932	0.933	0.930	-	-
		(1.078)	(0.975)	(0.960)	(0.956)	(0.957)	(0.957)	(0.957)	-	-
	1,000	0.944	0.936	0.932	0.932	0.927	0.932	0.928	0.930	-
		(1.016)	(0.834)	(0.815)	(0.810)	(0.819)	(0.821)	(0.823)	(0.824)	-
	2,000	0.936	0.939	0.936	0.933	0.932	0.932	0.930	0.930	0.930
		(0.634)	(0.606)	(0.603)	(0.599)	(0.599)	(0.596)	(0.596)	(0.595)	(0.592)
	5,000	0.951	0.951	0.951	0.952	0.950	0.951	0.951	0.950	0.951
		(0.311)	(0.293)	(0.291)	(0.291)	(0.290)	(0.290)	(0.290)	(0.291)	(0.290)

Table 10: Presents the empirical coverage rates for the **next period's volatility** σ^2_{t+1} with **ARCK innovations**. The bootstrapped residuals are subject to resampling using the **moving block** bootstrap procedure. The **nominal coverage** is $(1-\gamma)\times 100 = \mathbf{95}\%$. For each bootstrap procedure, sample size n and block length l, S = 2,000 Monte Carlo iterations and B = 2,000 bootstrap replications are performed. The DGP is a **GARCH**$(1,1)$ with parameters $(\beta_0,\beta_1,\beta_2) = (0.1, 0.4, 0.55)$ for the volatility process, and $(\delta_0,\delta_1,\delta_2) = (1.5, 0.4, 0.5)$.

	n/l	ASY	1	5	10	15	20	30	50	70
Recursive-design	500	0.875	0.912	0.908	0.902	0.896	0.890	0.878	-	-
		(0.246)	(0.248)	(0.246)	(0.246)	(0.246)	(0.245)	(0.243)	-	-
	1,000	0.879	0.910	0.891	0.890	0.892	0.893	0.885	0.875	-
		(0.175)	(0.179)	(0.177)	(0.178)	(0.179)	(0.179)	(0.177)	(0.175)	-
	2,000	0.880	0.913	0.904	0.912	0.918	0.914	0.914	0.912	0.909
		(0.125)	(0.133)	(0.133)	(0.134)	(0.135)	(0.135)	(0.135)	(0.135)	(0.134)
	5,000	0.910	0.908	0.905	0.902	0.902	0.912	0.906	0.904	0.896
		(0.081)	(0.083)	(0.082)	(0.082)	(0.082)	(0.083)	(0.083)	(0.083)	(0.082)
Fixed-design	500	0.896	0.921	0.918	0.920	0.914	0.921	0.915	-	-
		(0.259)	(0.244)	(0.244)	(0.244)	(0.244)	(0.243)	(0.243)	-	-
	1,000	0.894	0.912	0.920	0.914	0.917	0.913	0.909	0.911	-
		(0.182)	(0.178)	(0.177)	(0.178)	(0.177)	(0.177)	(0.177)	(0.177)	-
	2,000	0.890	0.911	0.913	0.910	0.914	0.913	0.917	0.914	0.910
		(0.130)	(0.135)	(0.135)	(0.135)	(0.135)	(0.135)	(0.135)	(0.135)	(0.135)
	5,000	0.923	0.910	0.913	0.905	0.911	0.913	0.909	0.913	0.906
		(0.084)	(0.085)	(0.085)	(0.085)	(0.085)	(0.085)	(0.085)	(0.085)	(0.085)

Table 11: Presents the empirical coverage rates for the **next period's volatility** σ^2_{t+1} with **tv-TGC innovations**. The bootstrapped residuals are subject to resampling using the **moving block** bootstrap procedure. The **nominal coverage** is $(1-\gamma)\times 100 = \mathbf{90}\%$. For each bootstrap procedure, sample size n and block length l, S = 2,000 Monte Carlo iterations and B = 2,000 bootstrap replications are performed. The DGP is a **TGARCH**$(1,1,1)$ with parameters $(\beta_0, \beta_1^+, \beta_1^- \beta_2) = (0.1, 0.05, 0.1, 0.8)$ for the volatility process, and $(\delta_0, \delta_1, \delta_2) = (2, 0.15, 0.4)$.

	n/l	ASY	1	5	10	15	20	30	50	70
	500	0.859	0.879	0.876	0.873	0.866	0.860	0.855	-	-
		(0.045)	(0.041)	(0.041)	(0.041)	(0.041)	(0.041)	(0.040)	-	-
	1,000	0.867	0.866	0.876	0.880	0.878	0.881	0.880	0.857	-
		(0.036)	(0.031)	(0.032)	(0.032)	(0.032)	(0.032)	(0.032)	(0.032)	-
Recursive-design	2,000	0.880	0.873	0.878	0.880	0.877	0.875	0.879	0.875	0.869
		(0.026)	(0.022)	(0.023)	(0.024)	(0.024)	(0.024)	(0.024)	(0.023)	(0.023)
	5,000	0.886	0.871	0.886	0.891	0.889	0.886	0.883	0.881	0.879
		(0.016)	(0.015)	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)	(0.016)
	500	0.868	0.887	0.889	0.887	0.881	0.888	0.875	-	-
		(0.045)	(0.041)	(0.040)	(0.040)	(0.040)	(0.040)	(0.040)	-	-
	1,000	0.872	0.881	0.877	0.872	0.875	0.881	0.873	0.876	-
		(0.035)	(0.032)	(0.032)	(0.031)	(0.032)	(0.031)	(0.031)	(0.031)	-
Fixed-design	2,000	0.873	0.879	0.881	0.880	0.879	0.880	0.879	0.879	0.879
		(0.025)	(0.023)	(0.023)	(0.022)	(0.022)	(0.022)	(0.022)	(0.022)	(0.022)
	5,000	0.880	0.876	0.876	0.879	0.877	0.878	0.873	0.875	0.876
		(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)	(0.015)

Table 12: Presents the empirical coverage rates for the **next period's volatility** σ^2_{t+1} with **ARCK innovations**. The bootstrapped residuals are subject to resampling using the **moving block** bootstrap procedure. The **nominal coverage** is $(1-\gamma)\times 100 = \mathbf{90}\%$. For each bootstrap procedure, sample size n and block length l, S = 2,000 Monte Carlo iterations and B = 2,000 bootstrap replications are performed. The DGP is a **TGARCH**$(1,1,1)$ with parameters $(\beta_0, \beta_1^+, \beta_1^- \beta_2) = (0.1, 0.1, 0.3, 0.5)$ for the volatility process, and $(\delta_0, \delta_1, \delta_2) = (1.5, 0.4, 0.5)$.

5 Empirical application

Based on the simulation study outlined in Section 4, it is evident that when there exists dependence in the higher moments of the distribution of the innovations, the confidence intervals for the next period's volatility generated by the fixed-design bootstrap procedure can either surpass or fall short of the intervals derived from the recursive-design bootstrap procedure proposed in this paper. This discrepancy hinges on the specific DGP. This section aims to demonstrate the practical implications of our findings regarding differences in the confidence bounds of parameter estimates and, consequently, the next period's volatility. To achieve this, we analyze data extracted from the International Carbon Action Partnership, in particular, the price of CO2 in the EU Emission Trading System (ETS), spanning from 18 January 2018 to 18 January 2023. Also, we use daily closing prices of the USD/EUR exchange rate from 22 October 2018 until 19 October 2023. The log-returns are computed using the formula $\epsilon_t = 100 \times \log(p_t/p_{t-1})$, where p_t represents the price at time t. The resulting data is graphically presented in Figure 1. We implement the model specification test for GARCH$(1,1)$ processes by Leucht et al. (2015) to test whether a GARCH(1,1) specification fits the data well. The test statistic computed by performing 2,000 bootstrap replications is equal to -0.160 (-0.675) for the ETS (USD/EUR) data, with a corresponding p-value of 0.7625 (0.967), which implies that we cannot reject the GARCH(1,1) model specification in both settings. To calculate the next period's volatility, we apply a GARCH$(1,1)$ model to the preceding 933 (1,000) log-returns. We calculate the parameter estimates by QMLE, and the uncertainty surrounding these parameter estimates is quantified through the utilization of Algorithms 1 and 3.

In empirical settings, it is unknown whether dependence within the innovations is present. In the iid setting, the fixed-design bootstrap procedure may be computationally more appealing, but it is inconsistent in dependent settings. Therefore, we performed the testing procedure proposed by Huo and Cho (2021) to test for the sandwich-form of the covariance matrix. The p-values of the test are 0.016 and 0.009 for the ETS and USD/EUR, respectively, and are calculated using 2,000 bootstrap replications. Therefore, for both time series, the null hypotheses of the equality of the covariance matrices under dependent and independent innovations are rejected at the 5% level. Hence, based on the test, the sandwich-form covariance matrix appears to be a more suitable option.

Parameter estimates and corresponding confidence interval lengths of the bootstrap procedures are provided in Table 13. The confidence intervals obtained by the recursive-design bootstrap procedure

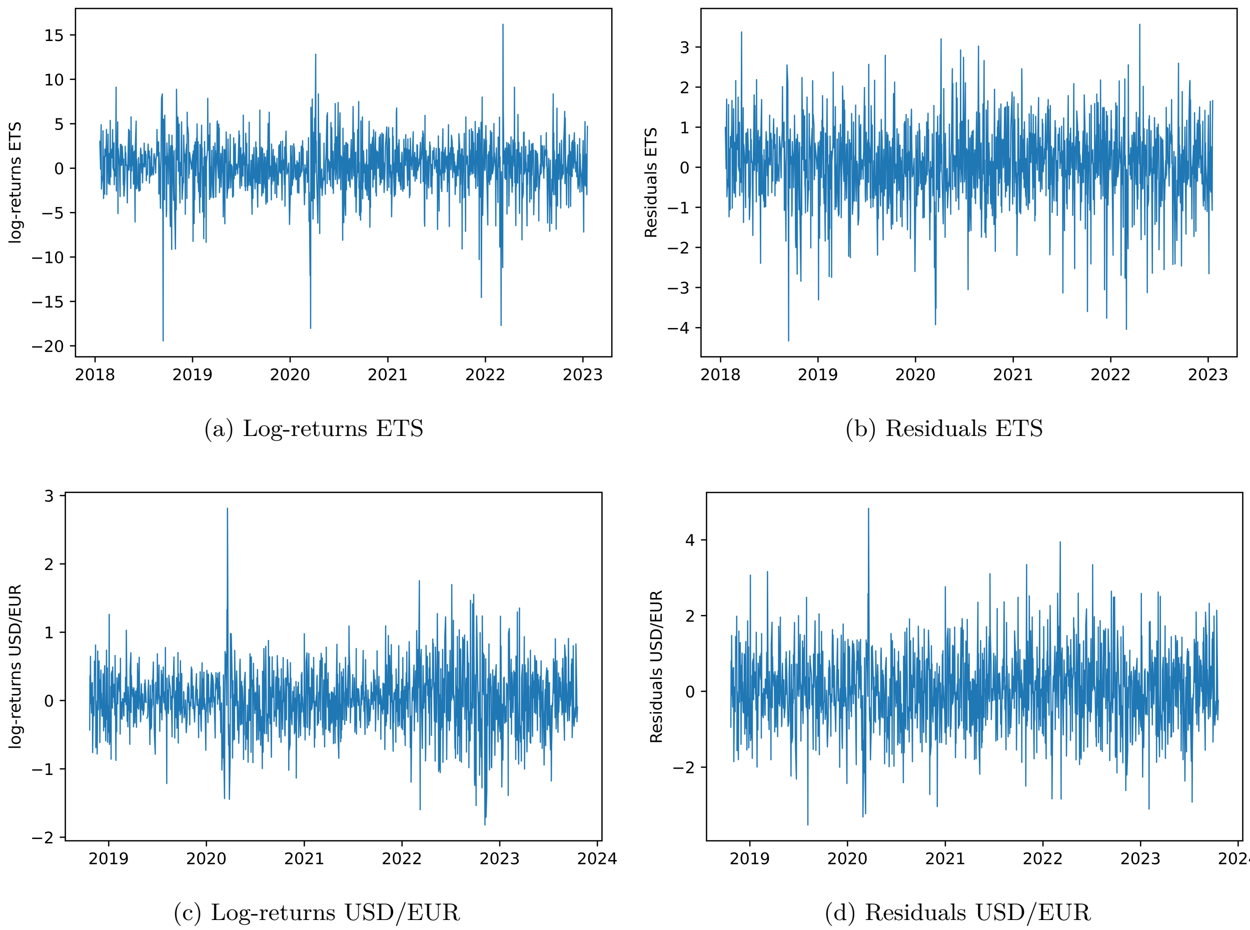


Figure 1: Plots of the log-returns and residuals for the ETS (a&b) and USD/EUR (c&d) in the periods from 28 January 2018 until 28 February 2023 (ETS) and 22 October 2018 until 19 October 2023 (USD/EUR). The residuals in plots (b) and (d) are obtained after fitting a GARCH(1,1) model on the corresponding log-returns.

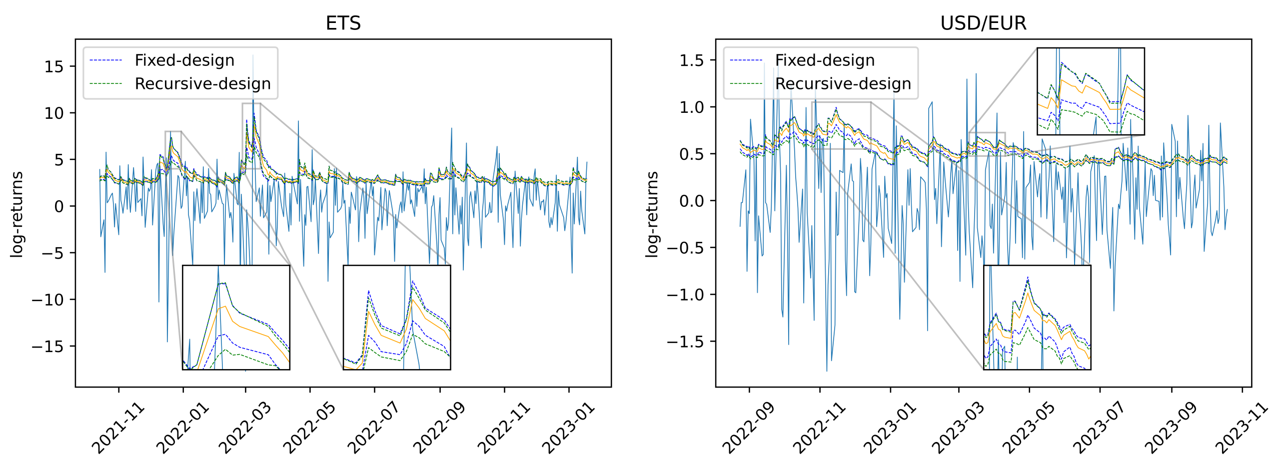
and the estimated sandwich-form asymptotic covariance matrix are notably larger (smaller) for β_1 and β_2 (β_0) compared to the intervals derived from the fixed-design bootstrap procedure and the estimated asymptotic covariance matrix assuming iid innovations in the ETS setting. When examining the USD/EUR data, the disparities in confidence interval lengths are narrower, but the confidence intervals for the recursive-design procedure are still larger when compared to the fixed-design procedure. These findings align with the results obtained from our simulation study, especially within the ARCK context.

	ETS			USD/EUR		
	β_0	β_1	β_2	β_0	β_1	β_2
point estimates	0.840	0.124	0.785	0.002	0.053	0.936
recursive-design	1.048	0.127	0.188	0.006	0.051	0.062
fixed-design	1.056	0.112	0.180	0.005	0.048	0.060
sandwich-form	0.851	0.144	0.180	0.005	0.067	0.078
iid	0.946	0.105	0.168	0.004	0.047	0.056

Table 13: Presents the GARCH$(1,1)$ parameter estimates $\hat{\beta}$ and 90% confidence interval lengths calculated by Algorithm 2 with $B = 2{,}000$. RD and FD denote the recursive- and fixed-design procedures, respectively. Sandwich-form represents the sandwich-form estimated asymptotic covariance matrix and iid refers to the estimated asymptotic covariance matrix assuming iid innovations.

The visual representations in Figure 2 are based on a rolling window analysis applied to the preceding 933 and 1,000 log-returns for the ETS and USD/EUR data, respectively. This analysis is utilized to compute the next period's volatility using a GARCH(1,1) model. The next period's volatility is depicted by the orange line, while the blue and green dotted lines portray the 90% confidence intervals for the fixed-design and recursive-design block bootstrap methodologies, respectively. Note that a block length of $l = 15$ is employed for both bootstrap procedures and both time series. While the choice of block length does not significantly impact our findings, it is worth highlighting that this selection can be justified by the results of our simulation study. Specifically, for a sample size of $n = 1{,}000$, the empirical coverage rates demonstrate an increase up to $l = 15$ followed by a decrease (see Tables 7 and 8). Therefore, it appears that $l = 15$ is well-suited for our sample size.

Over the full sample, the confidence intervals show a notable degree of similarity. However, when considering scenarios involving substantial squared returns, it becomes clear that the fixed-design bootstrap procedure tends to underestimate the level of uncertainty surrounding the next period's volatility. In contrast, the recursive-design moving block bootstrap method yields larger confidence intervals following substantial squared returns. This pattern corresponds with the outcomes from our simulation study with ARCK errors detailed in Section 4. This phenomenon can be linked to the narrower confidence bounds around the parameter estimates within the fixed-design bootstrap approach, which subsequently leads to smaller confidence intervals for the next period's volatility. This effect is particularly prominent for β_1, a factor that is multiplied by the previous return in the GARCH updating equation. Consequently, the fixed-design bootstrap procedure results in, presumably, too small confidence intervals, a pattern verified both by our simulation study and by the observations presented in Figure 2.



(a) ETS

(b) USD/EUR

Figure 2: Plots of the log-returns for the ETS (a) and USD/EUR (b) in the periods from 28 January 2018 until 28 February 2023 (ETS) and 22 October 2018 until 19 October 2023 (USD/EUR) with the next period's volatility (in orange) based on a rolling window analysis using 1,500 preceding observations. The 90% confidence intervals are computed according to Algorithm 2 with $B = 2{,}000$.

6 Concluding remarks

This paper introduces a recursive-design residual block bootstrap method designed for GARCH processes under dependent innovations. The procedure involves sampling random blocks from the empirical distribution of residuals to recursively generate new log-return paths. With each replication in the bootstrap, parameter estimates are computed using QMLE. These parameter estimates are then employed to calculate the volatility for the subsequent period. Our findings highlight a significant limitation in the fixed-design bootstrap procedure, as it fails to capture the relationship between the time-varying conditional fourth moment, the volatility process, and its derivatives. In contrast, through a simulation study, we demonstrate that the recursive-design residual block bootstrap procedure successfully captures this dependency. Consequently, it provides a more accurate quantification of the uncertainty surrounding parameter estimates and the next period's volatility.

Interestingly, our simulation study reveals that when there is dependence within the higher moments of the innovations, the length of the confidence intervals derived under the assumption of iid innovations may either surpass or fall short of the confidence intervals obtained by our bootstrap procedure. This implies that our approach provides a more finely-tuned way of estimating the uncertainty associated with parameter estimates and the next period's volatility.

The consistency between our empirical findings and the outcomes of the simulation study is noteworthy. For both financial time series, the empirical investigation demonstrates that, in most instances, the confidence intervals derived from both fixed- and recursive-design moving block bootstrap procedures exhibit a reasonable degree of similarity. However, an interesting disparity emerges in the case of large lagged squared returns. In such scenarios, the fixed-design procedure appears to underestimate the uncertainty, leading to narrower confidence intervals.

While our assertions find support in an extensive simulation study, it is essential to note that providing formal proof for the asymptotic validity of our bootstrap procedure goes beyond the current scope of this paper. This area of investigation is left open for further research.

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Appendices

A Assumptions

For completeness, the assumptions of FZ16 that we use are presented below. Note that they are rewritten such that they apply to the univariate setting and that Assumption 1 is not presented as such in FZ16 but is mentioned on p. 616-617.

Assumption 1. *Θ is a compact subset of $\mathbb{R}^r$.*

Assumption 2. *(ϵ_t) is a strictly stationary and ergodic process satisfying (1), with $\mathbb{E}\left[|\epsilon_t|^s < \infty\right]$ for some $s > 0$. Moreover, $\mathbb{E}\left[\log(\sigma_t^2)\right] < \infty$.*

Assumption 3. *We have $\sigma_t(\cdot) > \underline{\omega}$ for some $\underline{\omega}$ and for any real sequence $(e_i)_{i\geq 1}$, the function $\theta \mapsto \sigma(e_1, e_2, ...; \theta)$ is continuous. Also, we have $\sigma_t(\theta_0) = \sigma_t(\theta)$ almost surely if and only if $\theta_0 = \theta$.*

Assumption 4. *Let $C > 0$ and $0 < \rho < 1$ be generic constants, where C is allowed to depend on variables anterior to $t = 0$.*

(i) We have $\sup_{\theta\in\Theta} |\tilde{\sigma}_t(\theta) - \sigma_t(\theta)| \leq C\rho^t$ almost surely.

(ii) For any real sequence $(e_i)_{i\geq 1}$, the function $\theta \mapsto \sigma(e_1, e_2, ...; \theta)$ has continuous second order derivatives satisfying

$$\sup_{\theta\in\mathcal{V}(\theta_0)} \left\| \frac{\partial\tilde{\sigma}_t(\theta)}{\partial\theta} - \frac{\partial\sigma_t(\theta)}{\partial\theta} \right\| \leq C\rho^t, \text{ and } \sup_{\theta\in\mathcal{V}(\theta_0)} \left\| \frac{\partial^2\tilde{\sigma}_t(\theta)}{\partial\theta\partial\theta'} - \frac{\partial^2\sigma_t(\theta)}{\partial\theta\partial\theta'} \right\| \leq C\rho^t,$$

almost surely.

Assumption 5. *The innovations $\{\eta_t\}$ satisfy*

(i) $\{\eta_t\}$ is a sequence of strictly stationary and ergodic random variables satisfying $\mathbb{E}\left[\eta_t|\mathcal{F}_{t-1}\right] = 0$, $\mathbb{E}\left[\eta_t^2|\mathcal{F}_{t-1}\right] = 1$;

(ii) $\mathbb{E}\left[|\eta_t|^{4(1+\delta)}\right] < \infty$ for some $\delta > 0$.

Assumption 6. *θ_0 belongs to the interior of Θ denoted as int(Θ).*

Assumption 7. *There is a neighbourhood $\mathcal{V}(\theta_0)$ of θ_0 such that*

$$(i) \quad \sup_{\theta \in \mathcal{V}(\theta_0)} \left\| \frac{1}{\sigma_t(\theta)} \frac{\partial \sigma_t(\theta)}{\partial \theta} \right\|^{4(1+1/\delta)}, \quad (ii) \quad \sup_{\theta \in \mathcal{V}(\theta_0)} \left\| \frac{1}{\sigma_t(\theta)} \frac{\partial^2 \sigma_t(\theta)}{\partial \theta \partial \theta'} \right\|^{2(1+1/\delta)}, \ \text{and } (iii) \quad \sup_{\theta \in \mathcal{V}(\theta_0)} \left| \frac{\sigma_t(\theta_0)}{\sigma_t(\theta)} \right|^4$$

have finite expectations.

Assumption 8. *There does not exist a nonzero $\lambda \in \mathbb{R}^r$ such that $\lambda' \frac{\partial \sigma_t(\theta_0)}{\partial \theta} = 0$ almost surely.*

B Proof

Proof of Lemma 1. We start with the first claim. Note first that by construction $\hat{D}_t$ is constant w.r.t. the bootstrap measure which implies together with the fact that $\mathbb{E}_\times\left[\eta_t^{\times 2}\right] = 1$ that

$$\mathbb{E}_\times\left[\hat{D}_t(\eta_t^{\times 2} - 1)\right] = \hat{D}_t\mathbb{E}_\times\left[\eta_t^{\times 2} - 1\right] = 0.$$

Hence we have

$$\mathbb{V}\mathrm{ar}_\times\left[\sum_{t=1}^{n} Z_{n,t}^\times\right] = \frac{1}{n}\sum_{t=1}^{n}\sum_{u=1}^{n}\hat{D}_t\hat{D}_u'\,\mathbb{E}_\times\left[(\eta_t^{\times 2} - 1)(\eta_u^{\times 2} - 1)\right], \tag{17}$$

using again that $\hat{D}_t$ and $\hat{D}_u$ are constant w.r.t. the bootstrap measure. The first displayed formula of Lemma 1 now follows by stationarity. Regarding the second displayed formula of this Lemma notice that the assumption $\mathbb{E}_\times\left[\eta_t^{\times 2} - 1|\mathcal{F}_{t-1}^\times\right] = 0$ implies that for $(u < t)$

$$\mathbb{E}_\times\left[\hat{D}_t(\eta_t^{\times 2} - 1)\hat{D}_u(\eta_u^{\times 2} - 1)\right] = \hat{D}_u\hat{D}_t'\,\mathbb{E}_\times\left[(\eta_u^{\times 2} - 1)\mathbb{E}_\times\left[(\eta_t^{\times 2} - 1)|\mathcal{F}_{t-1}^\times\right]\right] = 0,$$

where we used again that $\hat{D}_t$ and $\hat{D}_u$ are constant w.r.t. the bootstrap measure. Therefore, (17) reduces to ${}^1\!/{}_n\sum_{t=1}^{n}\hat{D}_t\hat{D}_t'\,\mathbb{E}_\times\left[(\eta_t^{\times 2} - 1)^2\right]$ which by stationarity becomes $\mathbb{E}_\times\left[(\eta_1^{\times 2} - 1)^2\right]{}^1\!/{}_n\sum_{t=1}^{n}\hat{D}_t\hat{D}_t'$.

$\square$

C Algorithm

Algorithm 3. *(Fixed-design moving block bootstrap) First, construct $n-l$ blocks b_i of length l $\forall i \in \{1, ..., n-l\}$ such that $b_i = \{\hat{\eta}_i, \hat{\eta}_{i+1}, ..., \hat{\eta}_{i+k}, ..., \hat{\eta}_{i+l}\}$.*

1. *Draw $\lceil n/l \rceil$ numbers $U_1, ..., U_{\lceil n/l \rceil} \sim Uniform(1, n-l)$ (with replacement) and create the bootstrap innovations $\{\eta_1^\times, ..., \eta_{\lceil n/l \rceil \times l}^\times\} = \{b_{U_1}, b_{U_2}, ..., b_{U_{\lceil n/l \rceil}}\}$. In the case that $\lceil n/l \rceil \times l \neq n$, truncate the series such that it has length n. Generate bootstrap observations $\epsilon_t^\times = \tilde{\sigma}_t(\hat{\theta}_n)\eta_t^\times$.*

2. *Calculate the bootstrap estimator for the volatility process by QMLE*

$$\hat{\theta}_n^\times = \arg\max_{\theta \in \Theta} L_n^\times(\theta) \tag{18}$$

with

$$L_n^\times(\theta) = \frac{1}{n}\sum_{t=1}^{n} \ell_t^\times(\theta) \qquad \text{and} \qquad \ell_t^\times(\theta) = -\frac{1}{2}\left(\frac{\epsilon_t^\times}{\tilde{\sigma}_t(\theta)}\right)^2 - \log \tilde{\sigma}_t(\theta). \tag{19}$$

3. *Compute the next period's volatility using the original returns series and the bootstrap estimator $\hat{\theta}_n^\times$*

$$\hat{\sigma}_{n+1}^\times = \tilde{\sigma}_{n+1}(\hat{\theta}_n^\times), \tag{20}$$

where $\tilde{\sigma}_{n+1}(\hat{\theta}_n^\times) = \sigma_{n+1}(\epsilon_n, \epsilon_{n-1}, ..., \epsilon_1, \tilde{\epsilon}_0, \tilde{\epsilon}_{-1}, ...; \hat{\theta}_n^\times)$.

D The time-varying Transformed Gram-Charlier process

Set $H_3 := H_3(\eta_t) = \eta_t^3 - 3\eta_t$ and $H_4 := H_4(\eta_t) = \eta_t^4 - 6\eta_t^2 + 3$. Then:

$$g(\eta_t|\mathcal{F}_{t-1}) = \phi(\eta_t)\left[1 + \frac{s_t}{3!}H_3 + \frac{k_t - 3}{4!}H_4\right] =: \phi(\eta_t)\Psi(\eta_t). \tag{21}$$

Here $\phi(\cdot)$ denotes the pdf of the standard normal distribution. As this distribution may be negative for particular values of η_t and the integral of $g(\cdot|\mathcal{F}_{t-1})$ may not be equal to 1, León et al. (2005) propose the following transformation:

$$\begin{aligned} f(\eta_t|\mathcal{F}_{t-1}) &= \frac{\phi(\eta_t)\Psi^2(\eta_t)}{\Gamma_t}, \text{ with} \\ \Gamma_t &= 1 + \frac{s_t^2}{3!} + \frac{(k_t-3)^2}{4!}. \end{aligned} \tag{22}$$

In order to obtain $\mathbb{E}\left[\eta_t|\mathcal{F}_{t-1}\right] = 0$ and $\mathbb{E}\left[\eta_t^2|\mathcal{F}_{t-1}\right] = 1$, the distribution is standardized as follows[1]:

$$q(\eta_t|\mathcal{F}_{t-1}) = \sigma_{t|t-1} f(\sigma_{t|t-1}\eta_t + \mu_{t|t-1}|\mathcal{F}_{t-1}).$$

Here, $\mu_{t|t-1}$ and $\sigma_{t|t-1}$ denote the conditional mean and standard deviation of the innovations, respectively. León and Ñíguez (2021) show that the conditional mean and moment of the transformed Gram-Charlies series expansion truncated at the fourth moment can be expressed as

$$\mathbb{E}[\eta_t|\mathcal{F}_{t-1}] = 4\frac{s_t(k_t-3)}{\sqrt{3!}\sqrt{4!}\Gamma_t} := \mu_{t|t-1} \qquad \text{and} \qquad \mathbb{E}[\eta_t^2|\mathcal{F}_{t-1}] = 1 + \frac{6\frac{s_t^2}{3!} + \frac{8(k_t-3)^2}{4!}}{\Gamma_t}.$$

The analytical expressions for the conditional first and second moment imply that the first moment equals 0 if either the conditional skewness is 0, the conditional kurtosis k_t is 3, or both. Furthermore, the conditional second moment equals 1 in case both $s_t = 0$ and $k_t = 3$.

To obtain a distribution with conditional second moment equal to 1 when s_t and k_t deviate from 0 and 3, respectively, we use

$$\sigma_{t|t-1} = \sqrt{\mathbb{V}\text{ar}(\eta_t|\mathcal{F}_{t-1})} = \sqrt{\mathbb{E}[\eta_t^2|\mathcal{F}_{t-1}] - \mu_{t|t-1}^2} = \sqrt{1 + \frac{6\frac{s_t^2}{3!} + \frac{8(k_t-3)^2}{4!}}{\Gamma_t} - \mu_{t|t-1}^2}.$$

[1] For f a continuous real-valued function defined on a closed interval $[a, b]$, take standardized random variable $\frac{x-\mu}{\sigma}$ or $x = z\sigma + \mu$. We then have $\mathbb{P}(X \leq \sigma z + \mu) = F(\sigma z + \mu) = \int_a^{\sigma z+\mu} f(x)dx$. Subsequently, by the first fundamental theorem of calculus: $F'(\sigma z + \mu) = \sigma f(\sigma z + \mu)$.

E Tables stationary bootstrap

		Recursive-design									Fixed-design								
n/l	β	ASY	1	5	10	15	20	30	50	70	ASY	1	5	10	15	20	30	50	70
500	β_0	0.895	0.932	0.910	0.906	0.895	0.889	0.876	-	-	0.920	0.921	0.920	0.922	0.924	0.922	0.927	-	-
		(0.222)	(0.320)	(0.336)	(0.341)	(0.340)	(0.339)	(0.331)	-	-	(0.234)	(0.260)	(0.262)	(0.263)	(0.264)	(0.266)	(0.268)	-	-
	β_1	0.880	0.939	0.917	0.918	0.914	0.901	0.907	-	-	0.908	0.936	0.941	0.938	0.944	0.946	0.949	-	-
		(0.149)	(0.175)	(0.166)	(0.165)	(0.163)	(0.161)	(0.157)	-	-	(0.168)	(0.171)	(0.172)	(0.174)	(0.175)	(0.175)	(0.177)	-	-
	β_2	0.855	0.953	0.929	0.919	0.909	0.911	0.904	-	-	0.900	0.914	0.916	0.918	0.925	0.921	0.926	-	-
		(0.220)	(0.320)	(0.327)	(0.328)	(0.326)	(0.322)	(0.312)	-	-	(0.242)	(0.271)	(0.273)	(0.274)	(0.276)	(0.278)	(0.280)	-	-
1,000	β_0	0.889	0.913	0.904	0.902	0.898	0.894	0.889	0.875	-	0.906	0.905	0.908	0.904	0.912	0.913	0.909	0.915	-
		(0.130)	(0.158)	(0.162)	(0.167)	(0.167)	(0.168)	(0.168)	(0.165)	-	(0.136)	(0.144)	(0.144)	(0.145)	(0.145)	(0.146)	(0.147)	(0.148)	-
	β_1	0.891	0.922	0.917	0.907	0.908	0.903	0.900	0.891	-	0.918	0.921	0.924	0.923	0.926	0.927	0.923	0.931	-
		(0.104)	(0.118)	(0.111)	(0.110)	(0.109)	(0.109)	(0.108)	(0.105)	-	(0.116)	(0.118)	(0.118)	(0.119)	(0.119)	(0.119)	(0.120)	(0.121)	-
	β_2	0.879	0.935	0.920	0.905	0.897	0.898	0.897	0.882	-	0.902	0.922	0.921	0.920	0.919	0.923	0.924	0.927	-
		(0.137)	(0.169)	(0.168)	(0.170)	(0.170)	(0.170)	(0.169)	(0.166)	-	(0.150)	(0.159)	(0.160)	(0.161)	(0.161)	(0.161)	(0.163)	(0.164)	-
2,000	β_0	0.909	0.922	0.904	0.899	0.906	0.906	0.898	0.890	0.879	0.921	0.924	0.925	0.920	0.919	0.922	0.928	0.929	0.930
		(0.087)	(0.096)	(0.096)	(0.096)	(0.096)	(0.095)	(0.095)	(0.094)	(0.093)	(0.090)	(0.092)	(0.093)	(0.092)	(0.093)	(0.093)	(0.093)	(0.094)	(0.094)
	β_1	0.885	0.927	0.916	0.906	0.902	0.896	0.890	0.882	0.879	0.922	0.931	0.936	0.931	0.931	0.932	0.934	0.933	0.934
		(0.073)	(0.082)	(0.076)	(0.075)	(0.074)	(0.074)	(0.073)	(0.072)	(0.071)	(0.082)	(0.083)	(0.083)	(0.083)	(0.083)	(0.083)	(0.084)	(0.084)	(0.084)
	β_2	0.892	0.926	0.915	0.915	0.907	0.903	0.893	0.882	0.876	0.925	0.915	0.923	0.918	0.923	0.920	0.921	0.921	0.920
		(0.094)	(0.107)	(0.103)	(0.102)	(0.101)	(0.100)	(0.100)	(0.098)	(0.097)	(0.102)	(0.104)	(0.104)	(0.104)	(0.105)	(0.105)	(0.105)	(0.106)	(0.107)
5,000	β_0	0.906	0.913	0.898	0.902	0.908	0.906	0.906	0.911	0.901	0.915	0.920	0.915	0.918	0.915	0.919	0.918	0.921	0.923
		(0.053)	(0.056)	(0.055)	(0.055)	(0.055)	(0.055)	(0.054)	(0.054)	(0.054)	(0.054)	(0.055)	(0.055)	(0.055)	(0.055)	(0.055)	(0.055)	(0.055)	(0.055)
	β_1	0.900	0.938	0.911	0.913	0.906	0.905	0.899	0.898	0.898	0.939	0.940	0.938	0.937	0.938	0.940	0.936	0.938	0.940
		(0.046)	(0.051)	(0.048)	(0.047)	(0.046)	(0.046)	(0.046)	(0.046)	(0.046)	(0.052)	(0.052)	(0.052)	(0.052)	(0.052)	(0.052)	(0.052)	(0.052)	(0.052)
	β_2	0.898	0.919	0.904	0.898	0.900	0.898	0.896	0.894	0.887	0.923	0.917	0.921	0.918	0.920	0.919	0.920	0.921	0.918
		(0.057)	(0.064)	(0.060)	(0.060)	(0.059)	(0.059)	(0.059)	(0.059)	(0.058)	(0.062)	(0.063)	(0.063)	(0.063)	(0.063)	(0.063)	(0.063)	(0.063)	(0.064)

Table 14: Presents the empirical coverage for the **parameter estimates** of the volatility process. The bootstrapped residuals are subject to resampling using the **stationary** bootstrap procedure. The DGP is a **GARCH**(1, 1) with **tv-TGC** innovations. The **nominal coverage** is **90**%. $S = 1{,}000$ Monte Carlo iterations and $B = 1{,}000$ bootstrap replications are performed for every average block length $l = {}^1/_p$. The volatility and kurtosis parameters are $(\beta_0, \beta_1, \beta_2) = (0.1, 0.15, 0.8)$ and $(\delta_0, \delta_1, \delta_2) = (2, 0.15, 0.4)$, respectively.

		Recursive-design									Fixed-design								
n/l	β	ASY	1	5	10	15	20	30	50	70	ASY	1	5	10	15	20	30	50	70
500	β_0	0.889	0.935	0.932	0.938	0.934	0.926	0.915	-	-	0.919	0.922	0.930	0.927	0.929	0.926	0.932	-	-
		(0.230)	(0.330)	(0.336)	(0.339)	(0.337)	(0.337)	(0.332)	-	-	(0.244)	(0.267)	(0.265)	(0.267)	(0.268)	(0.268)	(0.271)	-	-
	β_1	0.815	0.852	0.863	0.868	0.858	0.854	0.849	-	-	0.822	0.877	0.884	0.880	0.883	0.888	0.887	-	-
		(0.177)	(0.166)	(0.176)	(0.177)	(0.176)	(0.175)	(0.172)	-	-	(0.166)	(0.165)	(0.164)	(0.165)	(0.166)	(0.167)	(0.168)	-	-
	β_2	0.850	0.945	0.942	0.936	0.922	0.919	0.909	-	-	0.879	0.923	0.916	0.921	0.923	0.921	0.927	-	-
		(0.240)	(0.318)	(0.328)	(0.330)	(0.328)	(0.327)	(0.320)	-	-	(0.247)	(0.271)	(0.270)	(0.271)	(0.272)	(0.274)	(0.276)	-	-
1,000	β_0	0.889	0.931	0.924	0.916	0.915	0.905	0.895	0.888	-	0.913	0.921	0.920	0.924	0.921	0.923	0.923	0.925	-
		(0.140)	(0.165)	(0.165)	(0.166)	(0.167)	(0.166)	(0.165)	(0.162)	-	(0.148)	(0.155)	(0.154)	(0.154)	(0.154)	(0.155)	(0.155)	(0.157)	-
	β_1	0.851	0.847	0.854	0.863	0.856	0.856	0.855	0.842	-	0.837	0.872	0.872	0.876	0.870	0.872	0.878	0.873	-
		(0.127)	(0.115)	(0.124)	(0.125)	(0.124)	(0.124)	(0.123)	(0.121)	-	(0.118)	(0.118)	(0.117)	(0.117)	(0.117)	(0.118)	(0.118)	(0.119)	-
	β_2	0.871	0.921	0.911	0.902	0.895	0.899	0.888	0.879	-	0.891	0.911	0.905	0.910	0.904	0.908	0.906	0.912	-
		(0.158)	(0.172)	(0.176)	(0.178)	(0.177)	(0.177)	(0.175)	(0.171)	-	(0.160)	(0.167)	(0.167)	(0.167)	(0.167)	(0.168)	(0.168)	(0.170)	-
2,000	β_0	0.913	0.920	0.910	0.902	0.909	0.902	0.899	0.894	0.889	0.930	0.928	0.925	0.925	0.926	0.929	0.927	0.925	0.927
		(0.094)	(0.102)	(0.101)	(0.102)	(0.101)	(0.101)	(0.101)	(0.100)	(0.099)	(0.100)	(0.101)	(0.100)	(0.101)	(0.100)	(0.101)	(0.101)	(0.101)	(0.102)
	β_1	0.859	0.826	0.844	0.846	0.851	0.851	0.854	0.846	0.839	0.830	0.847	0.849	0.844	0.841	0.848	0.844	0.847	0.851
		(0.096)	(0.083)	(0.091)	(0.092)	(0.093)	(0.093)	(0.092)	(0.092)	(0.091)	(0.086)	(0.085)	(0.085)	(0.085)	(0.085)	(0.086)	(0.086)	(0.086)	(0.087)
	β_2	0.888	0.895	0.891	0.886	0.890	0.895	0.880	0.876	0.868	0.892	0.901	0.896	0.900	0.896	0.899	0.903	0.897	0.899
		(0.111)	(0.111)	(0.114)	(0.115)	(0.115)	(0.115)	(0.114)	(0.113)	(0.112)	(0.110)	(0.112)	(0.111)	(0.111)	(0.111)	(0.112)	(0.112)	(0.112)	(0.113)
5,000	β_0	0.885	0.909	0.889	0.891	0.882	0.883	0.885	0.882	0.887	0.907	0.913	0.900	0.907	0.901	0.912	0.904	0.907	0.910
		(0.056)	(0.059)	(0.059)	(0.059)	(0.058)	(0.058)	(0.058)	(0.058)	(0.058)	(0.060)	(0.060)	(0.060)	(0.060)	(0.060)	(0.060)	(0.060)	(0.060)	(0.060)
	β_1	0.873	0.813	0.830	0.844	0.853	0.856	0.860	0.861	0.862	0.829	0.830	0.834	0.836	0.832	0.835	0.835	0.832	0.831
		(0.062)	(0.052)	(0.058)	(0.059)	(0.059)	(0.060)	(0.060)	(0.060)	(0.059)	(0.055)	(0.054)	(0.054)	(0.054)	(0.054)	(0.054)	(0.054)	(0.054)	(0.054)
	β_2	0.891	0.883	0.887	0.891	0.886	0.887	0.894	0.889	0.887	0.889	0.895	0.894	0.899	0.895	0.897	0.897	0.896	0.898
		(0.069)	(0.066)	(0.069)	(0.069)	(0.069)	(0.069)	(0.069)	(0.069)	(0.068)	(0.068)	(0.068)	(0.068)	(0.068)	(0.068)	(0.068)	(0.068)	(0.068)	(0.068)

Table 15: Presents the empirical coverage for the **parameter estimates** of the volatility process. The bootstrapped residuals are subject to resampling using the **stationary** bootstrap procedure. The DGP is a **GARCH**$(1,1)$ with **ARCK** innovations. The **nominal coverage** is **90**%. $S = 1{,}000$ Monte Carlo iterations and $B = 1{,}000$ bootstrap replications are performed for every average block length $l = 1/p$. The volatility and kurtosis parameters are $(\beta_0, \beta_1, \beta_2) = (0.1, 0.4, 0.55)$ and $(\delta_0, \delta_1, \delta_2) = (1.5, 0.4, 0.5)$, respectively.

		Recursive-design									Fixed-design								
n/l	β	ASY	1	5	10	15	20	30	50	70	ASY	1	5	10	15	20	30	50	70
500	β_0	0.829	0.951	0.944	0.937	0.924	0.921	0.900	-	-	0.836	0.945	0.952	0.957	0.956	0.953	0.961	-	-
		(0.131)	(0.139)	(0.137)	(0.135)	(0.133)	(0.132)	(0.129)	-	-	(0.134)	(0.125)	(0.126)	(0.126)	(0.127)	(0.127)	(0.127)	-	-
	β_1^+	0.881	0.969	0.930	0.918	0.905	0.904	0.892	-	-	0.937	0.971	0.975	0.977	0.973	0.976	0.974	-	-
		(0.197)	(0.201)	(0.196)	(0.194)	(0.193)	(0.191)	(0.187)	-	-	(0.220)	(0.199)	(0.199)	(0.200)	(0.201)	(0.202)	(0.203)	-	-
	β_1^-	0.888	0.939	0.921	0.917	0.904	0.912	0.890	-	-	0.912	0.939	0.942	0.942	0.946	0.946	0.947	-	-
		(0.262)	(0.289)	(0.279)	(0.274)	(0.271)	(0.268)	(0.262)	-	-	(0.288)	(0.284)	(0.285)	(0.286)	(0.288)	(0.288)	(0.289)	-	-
	β_3	0.830	0.947	0.946	0.935	0.918	0.908	0.896	-	-	0.835	0.938	0.941	0.948	0.946	0.949	0.949	-	-
		(0.522)	(0.564)	(0.557)	(0.546)	(0.538)	(0.531)	(0.520)	-	-	(0.532)	(0.508)	(0.511)	(0.513)	(0.515)	(0.516)	(0.517)	-	-
1,000	β_0	0.839	0.884	0.894	0.897	0.888	0.886	0.881	0.872	-	0.844	0.882	0.878	0.887	0.881	0.883	0.888	0.887	-
		(0.094)	(0.099)	(0.100)	(0.101)	(0.101)	(0.101)	(0.100)	(0.098)	-	(0.094)	(0.096)	(0.096)	(0.097)	(0.097)	(0.097)	(0.097)	(0.098)	-
	β_1^+	0.874	0.922	0.910	0.903	0.898	0.894	0.889	0.873	-	0.911	0.943	0.942	0.941	0.944	0.944	0.946	0.948	-
		(0.140)	(0.151)	(0.144)	(0.143)	(0.142)	(0.141)	(0.140)	(0.138)	-	(0.156)	(0.149)	(0.149)	(0.149)	(0.150)	(0.150)	(0.150)	(0.151)	-
	β_1^-	0.876	0.921	0.903	0.896	0.889	0.883	0.876	0.869	-	0.910	0.918	0.920	0.919	0.924	0.917	0.918	0.916	-
		(0.189)	(0.208)	(0.198)	(0.196)	(0.194)	(0.194)	(0.191)	(0.186)	-	(0.206)	(0.206)	(0.206)	(0.206)	(0.207)	(0.208)	(0.208)	(0.208)	-
	β_2	0.839	0.883	0.893	0.890	0.885	0.888	0.871	0.872	-	0.843	0.880	0.879	0.880	0.879	0.885	0.881	0.888	-
		(0.375)	(0.403)	(0.405)	(0.407)	(0.407)	(0.406)	(0.403)	(0.396)	-	(0.376)	(0.387)	(0.388)	(0.389)	(0.390)	(0.390)	(0.392)	(0.394)	-
2,000	β_0	0.879	0.898	0.891	0.890	0.886	0.882	0.884	0.876	0.867	0.883	0.897	0.898	0.901	0.900	0.896	0.898	0.903	0.905
		(0.067)	(0.068)	(0.070)	(0.071)	(0.071)	(0.071)	(0.071)	(0.070)	(0.070)	(0.066)	(0.068)	(0.068)	(0.068)	(0.069)	(0.069)	(0.069)	(0.069)	(0.069)
	β_1^+	0.887	0.928	0.899	0.893	0.893	0.887	0.884	0.881	0.875	0.918	0.924	0.924	0.927	0.924	0.923	0.922	0.927	0.925
		(0.099)	(0.111)	(0.104)	(0.103)	(0.103)	(0.103)	(0.102)	(0.101)	(0.099)	(0.111)	(0.110)	(0.110)	(0.111)	(0.111)	(0.111)	(0.111)	(0.111)	(0.111)
	β_1^-	0.895	0.922	0.907	0.902	0.902	0.898	0.897	0.888	0.889	0.922	0.920	0.921	0.924	0.922	0.922	0.922	0.922	0.924
		(0.136)	(0.148)	(0.140)	(0.138)	(0.138)	(0.137)	(0.137)	(0.135)	(0.134)	(0.147)	(0.147)	(0.147)	(0.147)	(0.147)	(0.148)	(0.148)	(0.148)	(0.148)
	β_2	0.868	0.892	0.898	0.899	0.897	0.890	0.883	0.877	0.877	0.875	0.889	0.891	0.892	0.898	0.886	0.891	0.896	0.898
		(0.268)	(0.279)	(0.282)	(0.286)	(0.287)	(0.287)	(0.287)	(0.285)	(0.282)	(0.265)	(0.275)	(0.276)	(0.275)	(0.277)	(0.276)	(0.277)	(0.278)	(0.279)
5,000	β_0	0.894	0.897	0.900	0.904	0.907	0.903	0.905	0.902	0.896	0.892	0.899	0.896	0.894	0.902	0.896	0.901	0.894	0.899
		(0.042)	(0.042)	(0.042)	(0.043)	(0.043)	(0.043)	(0.043)	(0.043)	(0.043)	(0.041)	(0.042)	(0.042)	(0.042)	(0.042)	(0.042)	(0.042)	(0.042)	(0.042)
	β_1^+	0.894	0.930	0.907	0.902	0.896	0.898	0.893	0.897	0.893	0.930	0.931	0.931	0.928	0.928	0.930	0.931	0.927	0.933
		(0.062)	(0.070)	(0.065)	(0.064)	(0.064)	(0.063)	(0.063)	(0.063)	(0.062)	(0.070)	(0.070)	(0.070)	(0.070)	(0.070)	(0.071)	(0.070)	(0.071)	(0.071)
	β_1^-	0.894	0.915	0.896	0.896	0.894	0.895	0.889	0.883	0.885	0.914	0.915	0.917	0.914	0.916	0.916	0.917	0.911	0.920
		(0.086)	(0.094)	(0.088)	(0.087)	(0.087)	(0.087)	(0.087)	(0.086)	(0.086)	(0.093)	(0.093)	(0.093)	(0.093)	(0.093)	(0.093)	(0.093)	(0.094)	(0.094)
	β_2	0.886	0.896	0.896	0.897	0.899	0.894	0.892	0.891	0.885	0.888	0.891	0.893	0.892	0.891	0.892	0.895	0.896	0.895
		(0.169)	(0.171)	(0.171)	(0.173)	(0.173)	(0.174)	(0.174)	(0.174)	(0.173)	(0.167)	(0.169)	(0.169)	(0.170)	(0.169)	(0.169)	(0.170)	(0.170)	(0.170)

Table 16: Presents the empirical coverage for the **parameter estimates** of the volatility process. The bootstrapped residuals are subject to resampling using the **stationary** bootstrap procedure. The DGP is a **TGARCH**$(1,1,1)$ with **tv-TGC** innovations. The **nominal coverage** is **90**%. $S=1{,}000$ Monte Carlo iterations and $B=1{,}000$ bootstrap replications are performed for every average block length $l = {}^1\!/_p$. The volatility and kurtosis parameters are $(\beta_0, \beta_1^+, \beta_1^-, \beta_2) = (0.1, 0.05, 0.1, 0.8)$ and $(\delta_0, \delta_1, \delta_2) = (2, 0.15, 0.4)$, respectively.

		Recursive-design									Fixed-design								
n/l	β	ASY	1	5	10	15	20	30	50	70	ASY	1	5	10	15	20	30	50	70
500	β_0	0.807	0.912	0.933	0.917	0.910	0.899	0.886	-	-	0.836	0.913	0.911	0.909	0.916	0.911	0.921	-	-
		(0.126)	(0.133)	(0.135)	(0.134)	(0.133)	(0.131)	(0.129)	-	-	(0.131)	(0.121)	(0.121)	(0.121)	(0.122)	(0.122)	(0.123)	-	-
	β_1^+	0.828	0.909	0.896	0.887	0.887	0.883	0.872	-	-	0.853	0.943	0.939	0.938	0.944	0.940	0.945	-	-
		(0.206)	(0.185)	(0.189)	(0.191)	(0.190)	(0.189)	(0.186)	-	-	(0.209)	(0.184)	(0.184)	(0.185)	(0.186)	(0.186)	(0.187)	-	-
	β_1^-	0.821	0.847	0.869	0.868	0.860	0.859	0.847	-	-	0.828	0.833	0.837	0.847	0.849	0.845	0.848	-	-
		(0.272)	(0.264)	(0.271)	(0.269)	(0.266)	(0.264)	(0.260)	-	-	(0.268)	(0.260)	(0.260)	(0.261)	(0.262)	(0.262)	(0.263)	-	-
	β_2	0.780	0.903	0.921	0.903	0.893	0.880	0.870	-	-	0.817	0.890	0.890	0.892	0.896	0.895	0.898	-	-
		(0.502)	(0.532)	(0.542)	(0.539)	(0.534)	(0.527)	(0.519)	-	-	(0.518)	(0.486)	(0.487)	(0.488)	(0.490)	(0.491)	(0.493)	-	-
1,000	β_0	0.837	0.871	0.885	0.889	0.888	0.885	0.883	0.870	-	0.850	0.875	0.874	0.880	0.872	0.871	0.880	0.881	-
		(0.093)	(0.096)	(0.100)	(0.101)	(0.101)	(0.101)	(0.100)	(0.099)	-	(0.094)	(0.095)	(0.094)	(0.094)	(0.095)	(0.095)	(0.095)	(0.095)	-
	β_1^+	0.836	0.849	0.866	0.868	0.867	0.869	0.861	0.846	-	0.836	0.856	0.862	0.861	0.859	0.865	0.876	0.870	-
		(0.156)	(0.142)	(0.149)	(0.150)	(0.151)	(0.151)	(0.149)	(0.147)	-	(0.152)	(0.141)	(0.141)	(0.141)	(0.141)	(0.142)	(0.142)	(0.142)	-
	β_1^-	0.845	0.849	0.875	0.877	0.877	0.877	0.876	0.861	-	0.847	0.856	0.853	0.859	0.857	0.858	0.861	0.855	-
		(0.207)	(0.194)	(0.203)	(0.204)	(0.204)	(0.204)	(0.202)	(0.199)	-	(0.197)	(0.194)	(0.193)	(0.194)	(0.194)	(0.194)	(0.195)	(0.195)	-
	β_2	0.817	0.862	0.873	0.877	0.889	0.887	0.883	0.874	-	0.824	0.865	0.864	0.864	0.863	0.867	0.868	0.868	-
		(0.376)	(0.387)	(0.401)	(0.406)	(0.408)	(0.408)	(0.405)	(0.399)	-	(0.374)	(0.377)	(0.376)	(0.377)	(0.377)	(0.378)	(0.380)	(0.381)	-
2,000	β_0	0.856	0.868	0.889	0.900	0.895	0.888	0.885	0.889	0.876	0.859	0.871	0.878	0.868	0.879	0.875	0.877	0.876	0.876
		(0.067)	(0.068)	(0.070)	(0.071)	(0.072)	(0.072)	(0.072)	(0.071)	(0.070)	(0.067)	(0.068)	(0.068)	(0.068)	(0.068)	(0.068)	(0.068)	(0.068)	(0.068)
	β_1^+	0.844	0.838	0.859	0.865	0.863	0.863	0.860	0.854	0.854	0.839	0.843	0.845	0.845	0.844	0.849	0.849	0.844	0.849
		(0.117)	(0.107)	(0.114)	(0.115)	(0.115)	(0.115)	(0.115)	(0.114)	(0.113)	(0.110)	(0.107)	(0.107)	(0.107)	(0.107)	(0.107)	(0.107)	(0.108)	(0.108)
	β_1^-	0.869	0.850	0.882	0.886	0.886	0.879	0.883	0.888	0.870	0.856	0.850	0.850	0.856	0.848	0.854	0.850	0.855	0.854
		(0.154)	(0.142)	(0.151)	(0.152)	(0.152)	(0.152)	(0.152)	(0.151)	(0.149)	(0.143)	(0.141)	(0.142)	(0.142)	(0.142)	(0.142)	(0.142)	(0.142)	(0.143)
	β_2	0.852	0.857	0.884	0.891	0.889	0.887	0.887	0.887	0.880	0.841	0.858	0.854	0.854	0.859	0.857	0.866	0.858	0.870
		(0.274)	(0.272)	(0.284)	(0.289)	(0.291)	(0.291)	(0.291)	(0.289)	(0.286)	(0.266)	(0.270)	(0.271)	(0.271)	(0.271)	(0.271)	(0.272)	(0.273)	(0.273)
5,000	β_0	0.872	0.878	0.884	0.881	0.892	0.887	0.888	0.883	0.882	0.877	0.877	0.878	0.877	0.884	0.879	0.883	0.882	0.881
		(0.044)	(0.042)	(0.044)	(0.044)	(0.045)	(0.045)	(0.045)	(0.045)	(0.044)	(0.043)	(0.043)	(0.043)	(0.043)	(0.043)	(0.043)	(0.043)	(0.043)	(0.043)
	β_1^+	0.898	0.859	0.897	0.895	0.896	0.903	0.899	0.894	0.890	0.879	0.870	0.863	0.870	0.875	0.865	0.870	0.868	0.865
		(0.078)	(0.070)	(0.076)	(0.077)	(0.077)	(0.077)	(0.077)	(0.077)	(0.077)	(0.071)	(0.070)	(0.071)	(0.071)	(0.071)	(0.071)	(0.071)	(0.071)	(0.071)
	β_1^-	0.883	0.852	0.879	0.875	0.880	0.877	0.875	0.879	0.876	0.852	0.852	0.853	0.850	0.847	0.851	0.856	0.858	0.852
		(0.102)	(0.092)	(0.099)	(0.100)	(0.100)	(0.101)	(0.100)	(0.100)	(0.100)	(0.093)	(0.092)	(0.092)	(0.092)	(0.092)	(0.092)	(0.092)	(0.092)	(0.092)
	β_2	0.878	0.875	0.880	0.890	0.900	0.898	0.901	0.894	0.895	0.876	0.877	0.878	0.876	0.884	0.882	0.885	0.875	0.882
		(0.179)	(0.172)	(0.179)	(0.181)	(0.182)	(0.182)	(0.183)	(0.182)	(0.182)	(0.171)	(0.172)	(0.172)	(0.171)	(0.172)	(0.172)	(0.172)	(0.172)	(0.172)

Table 17: Presents the empirical coverage for the **parameter estimates** of the volatility process. The bootstrapped residuals are subject to resampling using the **stationary** bootstrap procedure. The DGP is a **TGARCH**$(1,1,1)$ with **ARCK** innovations. The **nominal coverage** is **90**%. $S = 1{,}000$ Monte Carlo iterations and $B = 1{,}000$ bootstrap replications are performed for every average block length $l = 1/p$. The volatility and kurtosis parameters are $(\beta_0, \beta_1^+, \beta_1^-, \beta_2) = (0.1, 0.1, 0.3, 0.5)$ and $(\delta_0, \delta_1, \delta_2) = (1.5, 0.4, 0.5)$, respectively.