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Private Road Networks with Uncertain Demand

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Abstract

We study the efficiency of private supply of roads under demand uncertainty and evaluate various regulatory policies. Due to demand uncertainty, capacity is decided before demand is known but tolls can be adjusted after demand is known. Policy implications can differ considerably from those under deterministic demand. For instance, for serial links, the toll in the second-best zero-profit case is no longer equal to the marginal external congestion cost. In the first-best scenario, the capacity under uncertain demand is higher than that under deterministic demand of the same expected value, though self-financing still holds in expected terms. Regulation by competitive auction cannot replicate the second-best zero-profit result and thus leads to a lower welfare, whereas without uncertainty various forms of competitive auctions can attain this second-best optimum. For more complex networks, when private firms add capacity in turn, contrary to the case without demand uncertainty, some form of auction performs better than others with demand uncertainty.

Keywords: Traffic Congestion, Road Pricing, Uncertain Demand, Road Network, Private Supply, Auction

JEL codes: D63, H23, R41, R42

1 Introduction

There has been wide and growing interest in private supply of roads, in addition to public supply, as a solution to increasing traffic congestion. Various public-private partnerships and Build-Operate-Transfer projects have been found around the world, for example in Chile, Colombia, Mexico, China and United states(Fisher and Babbar (1996)). The often cited reasons are that governments in many countries have insufficient public funds to finance new road projects, and private firms are believed to manage projects more efficiently. A disadvantage, however, is that private suppliers have market power and tend to maximize their profits, resulting in a loss of social welfare compared to optimal pricing. The tradeoffs have been studied by numerous scholars: starting with a debate between Pigou (1920) and Knight (1924) considering a simple network, to De Vany, Arthur and Saving (1980) and de Palma (1992) on competing private toll roads, to de Palma and Lindsey (2000) on dynamic congestion and private roads, and Wang et al. (2011) on nonlinear pricing on private roads. Researchers have proposed several regulatory policies to overcome the disadvantages of private supply of roads. But they usually overlook a prevailing phenomenon in road construction: capacity is usually set when future demand is still uncertain and can only be estimated, even though tolls can be subsequently adjusted according to the realized demand.

The aim of the present paper is to study, under demand uncertainty, the impact of private supply of roads in a mixed network, and how to regulate this supply. Policy makers and practitioners alike can benefit from a better understanding of the impact of demand uncertainty on equilibrium tolls, capacities, profits and consumer surplus. We make the conventional assumptions that private firms maximize profit, while the public sector maximizes social welfare.[1]. We study both simple and complex networks. Simple networks have clear theoretical results, which can form a basis for rules of thumb in policy making, while complex network are more realistic, but we depend on simulation to solve equilibria. For simple networks, we distinguish between parallel links and serial links, and derive analytically the expected social welfare under various scenarios, which are used as benchmarks to assess the efficiency of private supply of road capacity. We also examine, in terms of impacts on social welfare, various competitive auctions to regulate the market. For more complex networks, we are interested in the development of private supply of road networks through competition. We run simulations to compare the equilibrium outcomes of free entry versus entry by regulation.

Our research shows that demand uncertainty adds new complexity to the economic assessment of private supply of roads. When there exists a free complementary road and the road provider needs at least zero profit, contrary to the case without demand uncertainty, the toll is no longer equal

[1] The reality is evidently more complex: for example, private firms may be restricted by several regulatory requirements, while the public sector and travelers may care about other aspects like pollution, fairness or local economic development. Nevertheless, our research can aid in establishing the most efficient way of regulating the private firms, while providing an objective evaluation benchmark of public projects based on the maximum social welfare attainable.

to the marginal external congestion cost. An especially relevant finding for policy makers is that, unlike in the case with deterministic demand, regulation by simple competitive auctions can no longer make the private suppliers adopt the socially desirable tolls and capacities. In other words, demand uncertainty makes regulating the private supply of roads more difficult and new forms of auctions need to be invented. The numerical simulation suggests that, with the help of competition, entry by regulation still works better than free entry. In addition, the "generalized price auction" generates higher expected social welfare than the "patronage auction" for the parameter ranges considered in the simulations.

Our study is mainly related to two strands of literature. The first one examines the effects of private supply of roads in a mixed network without demand uncertainty. Following Mohring and Harwitz (1962), Yang and Meng (2002) show that, if both the toll and the capacity are set optimally for every link of the network and neutral scale economies prevail, the Pigouvian toll is optimal and the private road is self-financing, meaning that the collected tolls cover the capacity costs. Obviously, not all roads are priced optimally in reality, and free public roads are common. Verhoef (2007) demonstrates when there is an untolled substitute road in the network, a road supplier who is forced to price and invest second-best optimally makes a loss, so a subsidy from the government is needed to achieve the desired second-best social welfare level. If such a subsidy is ruled out due to political or economic reasons, Verhoef (2008) derives the highest social welfare under the condition that the private firm makes at least a zero profit. This is a natural benchmark to compare the efficiency of various alternative ways of regulating private supply of roads through competitive auctions or free entry, since competitive auctions also typically drive profits of the winning bid to zero. The optimal toll in the second-best zero-profit case is again Pigouvian for both serial and parallel links. In addition, Verhoef (2008) found that among many possible regulatory schemes, two competitive auctions, namely the "patronage auction" and the "generalized price auction", are preferred, in the sense that they make the private firm choose the socially optimal tolls and capacity under the zero-profit condition. We study these two auctions in the case with uncertain demand, and find that they do not achieve the socially optimal goal any more.

The second strand of literature considers a single road with demand uncertainty. De Vany and Saving (1977), Kraus (1982), D'Ouville and McDonald (1990) and Arnott et al. (1996) all study different forms of demand uncertainty for a single link, and find that in general a larger investment in capacity is justified, because the optimal capacity with demand uncertainty is larger than the optimal capacity for a deterministic demand with the same expected value. Lindsey and de Palma (2014) prove that the cost recovery theorem holds also with uncertain demand. We confirm that the above intuition for a single road also apply to a public network. To date, however, models with demand uncertainty have not been applied to examine private supply of road in a mixed network.

The paper is organized as follows. Section 2 discusses the analytical model for simple networks. Section 3 shows the simulation results for more complex networks and section 4 concludes.

2 Analytical Model

In this section, we study small networks with either serial links or parallel links. This means that we consider the purest types of link interactions (i.e., complementarity versus substitutability), and thus can identify the mechanisms that will occur in real life networks in the cleanest possible way. We will show the basic model setup first, then discuss in detail four regimes for each network structure: the first-best case, the second-best case, the second-best zero-profit case and competitive auctions.

Under demand uncertainty, the assumed timing of the game is as follows: capacity is decided before the demand state is known, and tolls are decided after the demand uncertainty is resolved. This setting can represent not only long-term demand uncertainty due to unpredictable economic booms and busts, but also predictable seasonal demand fluctuations such as alternating peak and non-peak hours. The core assumption is that capacities are usually hard to change and independent of the demand state, but prices can be adjusted rather quickly and dependent on the demand state. We aim to model demand uncertainty in a general way. To that end, let p^i denote the probability of state i, N^i denote the total traffic flow in state i, and $D^i(N^i)$ denote the (decreasing) inverse demand in state i, so we can represent a variety of uncertainty distributions, and different inverse demand functions.

Our other assumptions are akin to those in standard traffic models without uncertainty. For simplicity, there is a single market with one origin and one destination, and the users are homogeneous, and risk neutral. The congestion cost is increasing and homogeneous of degree zero in flow and capacity. We assume the marginal capacity cost to be a constant, to represent neutral scale economies in road construction. The toll, capacity, congestion cost and marginal capacity cost of demand state i on link j are denoted by $\tau^i, K_j, c(N_j^i, K_j)$ and γ respectively.

We discuss four regimes, which are natural benchmarks for evaluating the efficiency of alternative ways of organizing the private supply of roads. In the first-best case, the social planner maximizes the expected social welfare, by setting the capacity and tolls of both links. This is the benchmark and the efficiency gain of any realistic policy can be valued against the gain of the first-best case. In the second-best case, the social planner faces a constraint that there is already a free link in the network, and optimizes over the capacity and toll of the other link. This case is more realistic than the first-best case, because tolling originally free road is more likely to meet resistance from the public than tolling new roads. It will typically result in lower social welfare. In the second-best zero-profit case, the social planner faces an additional constraint: the provider earns at least zero profit on the tolled link. This gives an upper limit for achievable social welfare under private operations, given that private firms will not operate under a loss, and competition will drive profits to zero. Finally, we study two competitive auctions as useful regulatory tools, which are proved to implement the second-best zero-profit result in the deterministic case (Verhoef (2007)). In the competitive auctions, the social planner tries to regulate the private firms by a patronage auction (the concession is awarded to the firm that will realize the highest level of use of the new road), or a generalized price auction (the concession is granted to the firm that will offer the lowest generalized price, as the sum of toll and time costs, of the new road). We will discuss in the remainder of the section the implications of the four regimes for both serial and parallel networks.

2.1 Serial Links

We first study, in this section, a network of two serial links, where a traveler must use both links to get from the origin to the destination. The serial links are thus perfect complements to each other, so that the total traffic flow equals the traffic flow on each link, i.e. $N^i = N_0^i = N_1^i$.

We will show that uncertain demand generates quite different policy implications from deterministic demand. For first-best and second-best cases, we show the analytical expressions of the equilibrium tolls directly. For second-best zero-profit and competing-auction cases, such expressions are omitted due to their complex forms, and we show instead the Lagrangian multipliers for ease of interpretation.

Here is a preview of results: in the first-best case, the capacity is larger than in the case without demand uncertainty. In the second-best zero-profit case, the optimal toll is not equal to the marginal external cost of congestion of both links. And in competitive auctions, neither the patronage auction nor the generalized price auction can implement the second-best zero-profit result. The results for the second-best case without a zero-profit constraint are more similar to those for the deterministic demand, where an expected profit is gained on the tolled link. We will discuss each case in more detail in the remainder of this section.

2.1.1 First-Best for Serial Links

The natural benchmark to evaluate the efficiency of private supply of roads is the first-best case. In the first-best case, the social planner maximizes the expected social welfare, which is the sum of the expected total consumer benefit minus the expected congestion cost and capacity cost. The choice variables of the social planner are capacities, state-dependent traffic flows, and state-dependent tolls. In addition, the user equilibrium constraint needs to be satisfied, where the generalized price of an active route, i.e. the sum of user congestion cost and toll, in any state is equal to the inverse demand in that state. So the social planner's problem can be expressed by the following Lagrangian[2]:

$$\max_{K_j, N^i, \tau^i, \lambda^i} \sum_i p^i \cdot \left(\int_0^{N^i} D^i(n)dn - N^i \cdot (c(N^i, K_0) + c(N^i, K_1)) \right) - \gamma \cdot (K_0 + K_1) + \sum_i \lambda^i \cdot (c(N^i, K_0) + c(N^i, K_1) + \tau^i - D^i(N^i)) \tag{1}$$

For this and the following Lagrangians, we will skip the first-order conditions, which are taken w.r.t. the quantities N^i, investments K_j, tolls τ^i, and the relevant Lagrangian multipliers. The first order conditions determine these same variables in the associated equilibrium. All variables, apart from the capacities and the multiplier for the zero-profit constraint (if relevant), are state-dependent. The solution is:

$$\tau^i = N^i \cdot (c_{N^i}(N^i, K_0) + c_{N^i}(N^i, K_1)) \tag{2}$$

The optimal toll in each state i is equal to the marginal external conges-

[2] Note that for the first-best case, the Lagrangian function may be more laborious than strictly needed, as all multipliers will be zero through optimization of the relevant toll levels (Small and Verhoef 2007). For comparison with second-best schemes later on, we however use a Lagrangian specification also for the first-best problem.

tion cost over the full trip in that state. It follows immediately that the two links are self-financing in expectation (Mohring and Harwitz (1962)). At the equilibrium, $\lambda^i = 0$, because the toll and traffic flows are already optimal and relaxing the user equilibrium constraint cannot improve the expected social welfare any further. For a linear inverse demand function and a BPR congestion function, we show in Appendix I that the optimal equilibrium capacity is larger than in the case without uncertainty. In other words: both the expected toll revenue and the total capacity cost are higher with uncertainty than without, but they are so in equal amounts, so that self-financing still prevails. The intuition why both are higher is that due to the convexity of the user cost function, the expected value of the marginal external cost over all states exceeds the marginal external cost for a deterministic traffic flow that is equal to the expected traffic flows under uncertainty. This raises the expected value of the toll, but also the optimal capacity of the road.

2.1.2 Second-Best for Serial Links

To compare the private supply of a new road with the overall first-best case can be less informative when some untolled roads already exist, and remain existent in the network. A better benchmark would then be the case where the social planner can only design and optimize the capacity and tolls of a new road. The resulting expected social welfare is generally lower than in the overall first-best case, because the capacity of the free road is not set optimally. When the pre-existing initial road is indexed 0 and the new serial road 1, the Lagrangian becomes:

$$\max_{K_1,N^i,\tau^i,\lambda^i} \sum_i p^i \cdot \left(\int_0^{N^i} D^i(n)dn - N^i \cdot (c(N^i, K_0) + c(N^i, K_1))\right) - \gamma \cdot (K_0 + K_1)$$
$$+ \sum_i \lambda^i \cdot (c(N^i, K_0) + c(N^i, K_1) + \tau^i - D^i(N^i)) \tag{3}$$

The solution is:

$$\tau^i = N^i \cdot (c_{N^i}(N^i, K_0) + c_{N^i}(N^i, K_1)) \tag{4}$$

The resulting toll in each demand state equals the marginal external congestion cost of the full trip, and the expected toll revenues naturally more than compensate the capacity cost of the toll link. In fact, the toll revenue would be sufficient to cover the cost of supplying both links at the optimal capacity, because with an unpriced perfectly complementary link, the second-best toll becomes equal to what would have been the first-best tolls for the two links together.

2.1.3 Second-Best Zero-Profit for Serial Links

The second-best zero-profit case is probably an even better benchmark for competing private firms, because with free entry, and for most types of competitive auctions, profit will be driven down to zero. It is also a good benchmark for regulation, because it identifies the most efficient outcome under the constraint that the toll revenue must cover capacity cost. In addition to the user equilibrium constraint, we also need to add the zero-profit constraint, where the expected profit summed over all states is equal to the capacity costs. The problem thus translates into the following Lagrangian:

$$\max_{K_1,N^i,\tau^i,\lambda^i,\lambda^{zp}} \sum_i p^i\cdot\Big(\int_0^{N^i} D^i(n)dn - N^i\cdot(c(N^i,K_0)+c(N^i,K_1))\Big) - \gamma\cdot(K_0+K_1)$$
$$+\sum_i \lambda^i\cdot(c(N^i,K_0)+c(N^i,K_1)+\tau^i - D^i(N^i)) + \lambda^{zp}\cdot\Big(\sum_i p^i\cdot N^i\cdot\tau^i - \gamma\cdot K_1\Big) \tag{5}$$

Note first λ^i is state-dependent, but λ^{zp} is not. After solving the first order conditions, we do not have a closed form solution for the tolls. However the Lagrangian multiplier for the zero-profit constraint implies that, contrary to the deterministic case, the optimal toll is not Pigouvian:

$$\lambda^{zp} = \frac{\tau^i - N^i \cdot (c_{N^i}(N^i, K_0) + c_{N^i}(N^i, K_1))}{-\tau^i - N^i \cdot (D^i_{N^i}(.) - c_{N^i}(N^i, K_0) - c_{N^i}(N^i, K_1))} \tag{6}$$

By the usual interpretation of a Lagrangian multiplier, λ^{zp} reflects how much the expected social welfare changes if we allow for a small expected surplus or deficit. The numerator is the derivative of the social welfare in state i with respect to the traffic volume in state i. It thus equals the height of the Harberger triangle that measures the deadweight loss due to inefficient tolling. The denominator is the derivative of the financial deficit in state i with respect to the traffic volume in state i. As the traffic volume in state i increases, the social planner gets more toll payment from the new marginal traveler but collects less toll from every original traveler.

At the optimum, the toll is set such that λ^{zp} will get the same value, independent of for which state i it it evaluated. The intuition is that at the optimum, a shift of a dollar revenue between two states should bring as much benefit in one state as damage in the other. As a result, contrary to the deterministic case, the toll cannot be equal to the marginal external congestion cost. For example, for linear inverse demand function such as $D^i(N^i) = d_0^i - d_1 \cdot N^i$ and two demand states such as $d_0^h > d_0^l$, the Pigouvian toll directly implies that $\frac{-c_{N^h}(N^h,K_0)}{c_{N^h}(N^h,K_0)+d_1} < \frac{-c_{N^l}(N^l,K_0)}{c_{N^l}(N^l,K_0)+d_1}$, and thus no solution for λ^{zp}. In other words, the social welfare is not maximized with the Pigouvian toll, because it can be increased further by decreasing N^h and increasing N^l.

2.1.4 Auctions for Serial Links

When a social planner is not sure about the optimal toll and capacity due to a lack of information on the congestion cost function or the capacity cost, a competitive auction may improve the efficiency of private road supply. In a competitive auction, all sellers have the same marginal capacity cost γ and full information of the congestion cost function, and they will bid until the profit is exhausted. It was shown before that in the deterministic case, two auctions can implement the second best zero profit outcome (Verhoef, 2007). These are the patronage auction, where firms bid in terms of committing to achieve the highest traffic flow on the toll road; and the generalized price auction, where they bid to realize the lowest generalized price. So we will now study how theses two auctions perform under uncertain demand.

The patronage auction maximizes the expected traffic flow, and the Lagrangian showing us the competitive zero-profit outcome is:

$$\max_{K_1, N^i, \tau^i, \lambda^i, \lambda^{zp}} \sum_i p^i \cdot N^i + \sum_i \lambda^i \cdot (c(N^i, K_0) + c(N^i, K_1) + \tau^i - D^i(N^i)) \\ + \lambda^{zp} \cdot (\sum_i p^i \cdot N^i \cdot \tau^i - \gamma \cdot K_1) \tag{7}$$

The toll does not have a closed-form solution, but we can gain insights into the nature of the toll from the Lagrangian multiplier for the zero-profit constraint.

$$\lambda^{zp} = \frac{1}{-\tau^i - N^i \cdot (D^i_{N^i}(.) - c_{N^i}(N^i, K_0) - c_{N^i}(N^i, K_1))} \tag{8}$$

λ^{zp} reflects how much the expected patronage, the objective in (7), changes if we allow for a small expected deficit and is the same across states. The numerator equals the derivative of the traffic volume in state i (directly entering the objective as the patronage) with respect to itself, and is therefore 1 now. The denominator is the derivative of the expected deficit in state i with respect to the traffic volume in that state, which is the same as in (6).

By similar reasoning as in the second-best zero-profit case, the toll cannot be the marginal external congestion cost of the tolled road in general. In addition, the results are also different from the second-best zero-profit case[3].

The generalized price auction minimizes the expected generalized price, so the Lagrangian is:

$$\min_{K_1, N^i, \tau^i, \lambda^i, \lambda^{zp}} \sum_i p^i \cdot D^i(N^i) + \sum_i \lambda^i (c(N^i, K_0) + c(N^i, K_1) + \tau^i - D^i(N^i)) + \lambda^{zp} (\sum_i p^i \cdot N^i \cdot \tau^i - \gamma \cdot K_1) \tag{9}$$

We now find the following Lagrangian multiplier for the zero-profit constraint:

$$\lambda^{zp} = \frac{D^i_{N^i}(.)}{-\tau^i - N^i \cdot (D^i_{N^i}(.) - c_{N^i}(N^i, K_0) - c_{N^i}(N^i, K_1))} \tag{10}$$

The numerator now equals the derivative of the expected generalized price in state i with respect to the traffic volume in state i, and the denominator is the same as in (6) and (8). For a linear inverse demand function, where

[3]When $D^i(N^i) = d^i_0 - d_1 \cdot N^i$ and $d^h_0 > d^l_0$, Pigouvian toll implies that $\frac{1}{N^h \cdot (d_1 + c_{N^h}(N^h, K_0))} < \frac{1}{N^l \cdot (d_1 + c_{N^l}(N^l, K_0))}$.

$D^i_{N^i}(.)$ is a constant, the solutions for the two auctions are the same. This is because, for linear demand function and serial links, the maximum expected total traffic flow corresponds to the minimum expected generalized price. For non-linear demands, the outcomes of the auctions will be different.

In sum, the patronage auction and the generalized price auction in general cannot replicate the result for the second-best zero-profit case with serial links if there is demand uncertainty. There are two ways to explain it. Firstly, the auctions and the second best zero profit case have different *expected* optimization objectives, which is the sum of the optimization objectives in each state weighted by the probability of that state. Note that the expected social welfare equals the expected consumer surplus under the zero profit constraint. For linear inverse demand functions, in each state, the consumer surplus is quadratic in the traffic flow while the patronage and the generalized price are linear in the traffic flow, so the highest *expected* consumer surplus corresponds to neither the highest *expected* patronage nor the lowest *expected* generalized price. Secondly, although the FOCs for τ^i, K_1, λ^i and λ^{zp} are the same for the two auctions and the second-best zero-profit case, they cannot determine a unique solution. The solution depends also on the FOCs for N^i, which in general differ across the cases. On the contrary, when there is no demand uncertainty, in terms of the optimization objective, maximization of the social welfare under the zero profit constraint is equivalent to maximization of the patronage and minimization of the generalized price. In terms of the resulting FOCs, the FOCs for τ, K_1, λ and λ^{zp} determine a unique combination of the traffic flows, capacity and toll (Wu et al. (2011) and Verhoef (2007)).

2.2 Parallel Links

In this section, we consider two parallel links, where both roads connect the same origin and destination and a traveler can use either of them. They are pure substitutes to each other, so the total traffic flow is the sum of the traffic flow of both links, i.e. $N^i = N_0^i + N_1^i$.

Here is a summary of the key results for parallel links. We show the equilibrium tolls for first-best and second-best cases, and only the Lagrangian multipliers for the remaining cases. For second-best zero-profit and competing-auction cases, the equilibrium toll is omitted due to its complex form, and we instead use the Lagrangian multipliers for model interpretation. As opposed to the deterministic case, neither the patronage auction nor the generalized price auction can implement the second-best zero-profit result when demand uncertainty prevails. In the first-best case, the capacity is larger than in the case without demand uncertainty. Other results are more in line with the deterministic case. In the second-best case, the social planner expects a loss. In the second-best zero-profit case, the toll is equal to the marginal external congestion cost of the toll link, which in general is not true for serial links under demand uncertainty.

2.2.1 First-Best for Parallel Links

We again first discuss the first-best case, where a social planner can design the capacities and tolls of the two parallel links to obtain the highest expected social welfare. The Lagrangian is the following:

$$\max_{K_j, N_j^i, \tau_j^i, \lambda_j^i} \sum_i p^i \cdot \left(\int_0^{N_0^i + N_1^i} D^i(n) dn - N_0^i \cdot c(N_0^i, K_0) - N_1^i \cdot c(N_1^i, K_1) \right) - \gamma \cdot (K_0 + K_1)$$
$$+ \sum_j \sum_i \lambda_j^i \cdot (c(N_j^i, K_j) + \tau_j^i - D^i(N_0^i + N_1^i)) \tag{11}$$

The result is:

$$\tau_j^i = N_j^i \cdot c_{N_j^i}(N_j^i, K_j) \tag{12}$$

The Pigouvian toll is levied on each link in each state, so the externality of congestion is internalized and the roads are self-financing in expectation. For linear inverse demand function and BPR congestion function, we show in Appendix I that also now the optimal capacity is larger than in the case without uncertainty.

2.2.2 Second-Best for Parallel Links

In the second-best case, now an untolled parallel link already exists. When the social planner optimizes the capacity and tolls of the new road, the Lagrangian is:

$$\max_{K_1, N_j^i, \tau_1^i, \lambda_j^i} \sum_i p^i \cdot \left(\int_0^{N_0^i + N_1^i} D^i(n) dn - N_0^i \cdot c(N_0^i, K_0) - N_1^i \cdot c(N_1^i, K_1) \right) - \gamma \cdot (K_0 + K_1)$$
$$+ \sum_i \lambda_0^i \cdot (c(N_0^i, K_0) - D^i(N_0^i + N_1^i)) + \sum_i \lambda_1^i \cdot (c(N_1^i, K_1) + \tau_1^i - D^i(N_0^i + N_1^i)) \tag{13}$$

The solution is:

$$\tau_1^i = N_1^i \cdot c_{N_1^i}(N_1^i, K_1) + N_0^i \cdot c_{N_0^i}(N_0^i, K_0) \frac{D^i_{N_1^i}(N_0^i + N_1^i)}{c_{N_0^i}(N_0^i, K_0) - D^i_{N_0^i}(N_0^i + N_1^i)} \tag{14}$$

Similar to the case without uncertainty, the toll in each demand state equals the sum of the marginal external congestion cost and a negative network spillover effect. The social planner should expect a loss on the toll road, because of a downward adjustment of the toll from the Pigouvian toll(Verhoef (2007)).

2.2.3 Second-Best Zero-Profit for Parallel Links

As just shown, in the second-best case with parallel links, the operator of the toll road makes a loss. The logic followup question is that: what is the highest expected social welfare if this road has to be self-financing? This is a relevant question, because it is a natural benchmark for private supply (with or without regulation) when there is an untolled alternative parallel road: we cannot expect the private road to make a loss, so the best we can hope for is the setting where welfare is maximized under the constraint that the firm makes (at least) a zero profit. Mathematically, we solve the following

Lagrangian problem:

$$\max_{K_1,N_j^i,\tau_1^i,\lambda_j^i,\lambda^{zp}} \sum_i p^i\cdot(\int_0^{N_0^i+N_1^i} D^i(n)dn - N_0^i\cdot c(N_0^i,K_0) - N_1^i\cdot c(N_1^i,K_1)) - \gamma\cdot(K_0+K_1)$$
$$+\sum_i \lambda_0^i\cdot(c(N_0^i,K_0) - D^i(N_0^i+N_1^i)) + \sum_i \lambda_1^i\cdot(c(N_1^i,K_1)+\tau_1^i - D^i(N_0^i+N_1^i))$$
$$+\lambda^{zp}\cdot(\sum_i p^i\cdot\tau_1^i\cdot N_1^i - \gamma\cdot K_1) \tag{15}$$

After simplification, we can show that the equilibrium toll is Pigouvian through solving the Lagrangian multiplier:

$$\lambda^{zp} = \frac{\tau_1^i - N_1^i\cdot c_{N_1^i}(.) - N_0^i\cdot c_{N_0^i}(.)\frac{D^i_{N_1^i}(.)}{c_{N_0^i}(.)-D^i_{N_0^i}(.)}}{-\tau_1^i - N_1^i\cdot(D^i_{N_1^i}(.) + D^i_{N_0^i}(.)\frac{D^i_{N_1^i}(.)}{c_{N_0^i}(.)-D^i_{N_0^i}(.)} - c_{N_1^i}(.))} \tag{16}$$

The numerator of λ^{zp} is the derivative of the social welfare in state i with respect to the traffic flow in state i on the toll road.[4] It equals the sum of the height of the Harberger triangle of both links, where that of untold link is weighted to reflect the substitution between equilibrium use of the two links. The denominator is the derivative of the deficit in state i with respect to the traffic flow in state i on the toll link. As the traffic volume on the toll link

[4]If N_1^i increases by a small amount Δ, N_0^i will decrease by $\frac{D^i_{N_1^i}(.)}{c_{N_0^i}(.)-D^i_{N_0^i}(.)}\Delta$, because on the untolled road, user cost must equal inverse demand.

increases, the road provider gets more toll payment from the new traveler on the tolled road but collects less toll from every original traveler due to the diversion of traffic to the untolled link.

In contrast to what we found for the serial links, the Pigouvian toll, i.e. $\tau^i = N_1^i \cdot c_{N_1^i}(N_1^i, K_1)$, guarantees that $\lambda^{zp} = \frac{N_0^i}{N_1^i}$ is the same across states. Note that this result is for a general from of inverse demand function and demand uncertainty. If the social planner allows an expected deficit on the tolled road, the positive λ^{zp} shows that the expected social welfare will increase. This is because, the Pigouvian toll on the toll road does not account for the congestion spill-over on the untolled road like in the second-best case. Lowering the toll below the Pigouvian level raises welfare. Such increase in welfare is larger when there are more travelers on the untolled road and less travelers on the toll road. In the extreme case of zero traffic flow on the untolled road, the effect is zero because we already achieve the highest social welfare by optimum tolling.

2.2.4 Auctions for Parallel Links

Without demand uncertainty, the patronage auction and the generalized price auction prove to replicate the second-best zero-profit outcome also for parallel links (Verhoef 2007). We show in this subsection that this is no longer true with demand uncertainty. For policy makers thinking of how to regulate the private firms, more sophisticated mechanisms are needed.

Patronage Auction for Parallel Links To find the outcome for the patronage auction, we set up the Lagrangian that maximizes the expected flow on the tolled link subject to a zero-profit constraint:

$$\max_{K_1, N_j^i, \tau_1^i, \lambda_j^i, \lambda^{zp}} \sum_i p^i \cdot N_1^i + \sum_i \lambda_0^i \cdot (c(N_0^i, K_0) - D^i(N_0^i + N_1^i)) \\ + \sum_i \lambda_1^i \cdot (c(N_1^i, K_1) + \tau_1^i - D^i(N_0^i + N_1^i)) + \lambda^{zp} \cdot (\sum_i p^i \cdot \tau_1^i \cdot N_1^i - \gamma \cdot K_1) \tag{17}$$

The FOCs for $\tau_1^i, K_1, \lambda_j^i$ and λ^{zp}, which specify the investment rule, the Wardropian user equilibrium conditions and the zero profit constraint, are the same as those of the second-best zero-profit case. However, unlike in the case without uncertainty Wu et al. (2011), they cannot determine a unique solution, because demand uncertainty brings in more choice variables now. The Lagrangian multiplier for the zero-profit constraint shows that the toll, which has no closed-form solution, is not the same as in the second-best zero-profit case:

$$\lambda^{zp} = \frac{1}{-\tau_1^i - N_1^i \cdot (D^i_{N_1^i}(.) + D^i_{N_0^i}(.)\frac{D^i_{N_1^i}(.)}{c_{N_0^i}(.) - D^i_{N_0^i}(.)} - c_{N_1^i}(.))} \tag{18}$$

The numerator of λ^{zp} is the derivative of the traffic volume on the tolled road in state i with respect to itself, thus its value is again 1. The denominator is the same as in the second-best zero-profit case in (16). Contrary to the case without demand uncertainty, the resulting toll cannot be equal to the marginal external congestion cost on the toll road[5].

Generalized Price Auction for Parallel Links For the generalized price auction, a firm minimizes the expected generalized price, such that the toll road beaks even. The Lagrangian for the problem with parallel links is:

[5]For a linear inverse demand function $D^i(N^i) = d_0^i - d_1 N^i$ and $d_0^h > d_0^l$, Pigouvian toll implies $\frac{1}{N_1^h \cdot d_1 \frac{c_{N_0^h}(.)}{c_{N_0^h}(.)+d_1}} < \frac{1}{N_1^l \cdot d_1 \frac{c_{N_0^l}(.)}{c_{N_0^l}(.)+d_1}}$,

$$\min_{K_1, N_j^i, \tau_1^i, \lambda_j^i, \lambda^{zp}} \sum_i p^i \cdot D^i(N_0^i + N_1^i) + \sum_i \lambda_0^i \cdot (c(N_0^i, K_0) - D^i(N_0^i + N_1^i))$$
$$+ \sum_i \lambda_1^i \cdot (c(N_1^i, K_1) + \tau_1^i - D^i(N_0^i + N_1^i)) + \lambda^{zp} \cdot (\sum_i p^i \cdot \tau_1^i \cdot N_1^i - \gamma \cdot K_1) \tag{19}$$

After some substitutions, λ^{zp} can be expressed as follows:

$$\lambda^{zp} = \frac{D^i_{N_1^i}(.) + D^i_{N_0^i}(.) \frac{D^i_{N_1^i}(.)}{c_{N_0^i}(.) - D^i_{N_0^i}(.)}}{-\tau_1^i - N_1^i \cdot (D^i_{N_1^i}(.) + D^i_{N_0^i}(.) \frac{D^i_{N_1^i}(.)}{c_{N_0^i}(.) - D^i_{N_0^i}(.)} - c_{N_1^i}(.))} \tag{20}$$

The numerator is the derivative of the generalized price in state i with respect to the traffic volume on the tolled road in state i, taking into consideration the induced traffic volume on the untolled link. The denominator is the same as in the earlier two cases (16) and (18). The tolls again cannot be Pigouvian[6].

In sum, neither the patronage auction nor the generalized price auction can in general replicate the result of the second-best zero-profit case. The

[6]For a linear inverse demand function $D^i(N^i) = d_0^i - d_1 \cdot N^i$ and $d_0^h > d_0^l$, Pigouvian toll implies $\frac{-1}{N_1^h} > \frac{-1}{N_1^l}$.

traffic volumes that maximize the expected social welfare under the appropriate constraints neither maximize the expected patronage on the toll road nor minimize the expected generalized price.

3 Numerical Analysis

The analytical results for the two simple networks show that demand uncertainty is likely to affect the efficiency of private supply of roads, where equilibrium with competitive auctions differs from zero-profit second-best outcomes. For a more general network, which is also more realistic, clear-cut analytical results are hard to obtain, and we rely on numerical simulations to gain further insights. Moreover, with the help of simulation, we may not only examine statically how private roads perform in a given mixed network, but can also learn dynamically how to regulate private provision of roads through efficient network formation.

Similar to Verhoef (2008), we assume there are two serial segments a and b in a network connecting one origin and one destination. The initial links on the two segments, denoted as $a0$ and $b0$ respectively, are untolled, which represent a pre-existing free public road network. Private firms can add capacities on each segment one at a time, and then charge tolls. For example, if the first firm adds a link in section a, we denote the new link as *a1*. *a1* is now parallel to the existing link *a0*, and serial to the existing link *b0*. In this way, we can model both parallel and serial competition and the development of the network.

The timing of the game is as follows: since construction takes time, at the beginning of each round, there is uncertainty about the future demand. Firms compete to add a capacity to one section of the network without knowing the realized demand. After the demand uncertainty is resolved, the firms can no longer change their capacities, but they can decide on tolls to charge on their own links. Then there is again demand uncertainty and a new round begins. The sequential game continues until there is no profit for a new entry. Alternatively, the model can describe the situation where after the opening of a new link, there remains demand variability over the days with state i occurring on a fraction p_i of the relevant (working) days.

For simplicity, we assume that within each round, firms are forward looking and rational, so the capacity decision takes into account the equilibrium toll setting in the next stage. But between rounds, firms are assumed myopic, in the sense that they take every round as being the last, until they are surprised by new comers who change the network structure. We make these assumptions to avoid unsolvable dynamic games, and we think they may also represent the slow and lump-sum development in infrastructure in real life. Because it usually takes considerable time for new roads to materialize, a firm can focus on competing with the existing firms for now and not worry too much about possible new competitors in the future. But we admit it is a simplification.[7] Another assumption is that when the uncertainty is assumed to be resolved after completing a investment, there is again uncertain demand in a new round. The replication of the same type of uncertainty when every round begins is of course unrealistic, but helpful in our setting that it avoids making ad hoc assumptions on how uncertainty itself evolves over time. In addition, this setting can also be applied to cases of peak-load pricing when the demand fluctuates regularly within a period in which capacity is fixed.

We will discuss the basic benchmarks first, then compare and contrast two regimes, namely the unregulated free-entry regime and the regulated entry-by-auction regime. In the free-entry regime, the firm with the highest expected profit adds a capacity on the most desired section. After the capacity is built, the demand is known and all firms in the network play a Bertrand

[7] In general, if firms can anticipate new entries, they will set larger capacities to preempt the entry.

price setting game, i.e. every firm sets its own toll simultaneously while taking the tolls of the other road operators as given. In the entry-by-auction regime, the winner of an auction can add a capacity. Due to the perfect competition in auctions, any firm that adds a capacity earns zero profit in expectation. The auction can be on either the expected patronage of the new road, or the expected generalized price. To be comparable to the deterministic case in Verhoef (2008), when demand is known, all existing firms charge tolls as promised in the auction and stick to these over successive rounds, so there is no direct toll competition in the entry-by-auction regime.[8]

The parameters of the numerical simulation is as follows. To be comparable with Verhoef (2008), the inverse demand function is linear and $D^i(N^i) = \delta_0^i - \delta_1 \cdot N^i$. The demand uncertainty is in the intercept, which is δ_0^h with probability p^h and δ_0^l with probability $1-p^h$. We set $\delta_1 = 0.01167, \delta_0^h = 74.11, \delta_0^l = 49.41, p^h = 0.5$, which means compared with Verhoef 2008, the reservation price can go up or down by 20% with equal probability. [9] The congestion cost function is of the familiar BPR form, i.e. $c(N_j^i, K_j) = \alpha \cdot t_f \cdot (1 + \beta \cdot (\frac{N_j^i}{K_j})^x)$. The parameter for the value of time, α, is set at 7.5. t_f is the free-flow travel

[8] If Bertrand competition is allowed, tolls have to decrease faster to meet the promised patronage, and the entry game also stops sooner.

[9] According to Flyvbjerg et al. (2006), actual traffic deviates from the forecasted ones by more than 20% for half of roads projects.

time and is set at 0.25, implying a total trip length of 60 kilometers for a highway with 120 kilometer/hour speed. β and x take their conventional values of 0.15 and 4 respectively. The marginal capacity cost is set at 3.5 for both segments, and represents the hourly capital cost per unit of capacity. We assume the initial capacities are $K_{a0} = K_{b0} = 1500$.

3.1 Benchmark

As a benchmark, Table 1 summarizes the characteristics of the base equilibrium, the first-best case, the second-best case and the second-best zero-profit case. The results we are interested in are: the social welfare(S), relative efficiency(ω) [10], profit(π), capacity(K), toll(τ), congestion cost(c), generalized price(P) and traffic volume(N). The superscript $h(l)$ denotes the high(low) demand state. The subscript $a(b)$ is for section *a(b)*, while 0(1) is for the initial (newly-added) link. E is for expectation. The results are fully consistent with the analytical model.

The base equilibrium with the two untolled links is quite congested, and its expected social welfare is only half the first-best value. Since no toll is

[10] The relative efficiency is defined as the gain in the expected social welfare in the regime, divided by the gain achieved when moving from the base equilibrium to the first-best outcome.

Table 1: Benchmark Equilibria

Variables	Base	First-best	Second-best	Second-best zero-profit
S^h	76513	167573	166325	146383
S^l	43864	59908	58061	60099
$E[S]$	60189	113741	112193	103241
ω	0	1	0.971	0.804
π^h_{a0}, π^h_{b0}	-5250	8685	-5250	-5250
π^h_{a1}, π^h_{b1}	-	-	-7461	3340
π^l_{a0}, π^l_{b0}	-5250	-8685	-5250	-5250
π^l_{a1}, π^l_{b1}	-	-	-8625	-3340
$E[\pi_{a0}], E[\pi_{b0}]$	-5250	0	-5250	-5250
$E[\pi_{a1}], E[\pi_{b1}]$	-	-	-8043	0
K_{a0}, K_{b0}	1500	3644	1500	1500
K_{a1}, K_{b1}	-	-	2479	1401
τ^h_{a0}, τ^h_{b0}	0	4.226	0	0
τ^h_{a1}, τ^h_{b1}	-	-	0.350	4.226
τ^l_{a0}, τ^l_{b0}	0	1.119	0	0
τ^l_{a1}, τ^l_{b1}	-	-	0.022	1.119
c^h_{a0}, c^h_{b0}	14.230	2.931	3.313	7.157
c^h_{a1}, c^h_{b1}	-	-	2.963	2.931
c^l_{a0}, c^l_{b0}	6.697	2.155	2.131	3.273
c^l_{a1}, c^l_{b1}	-	-	2.110	2.155
$D^h = P^h$	28.459	14.315	6.626	14.315
$D^l = P^l$	13.395	6.546	4.263	6.546
N^h_{a0}, N^h_{b0}	3862	5074	2256	3123
N^h_{a1}, N^h_{b1}	-	-	3477	1951
N^h	3862	5074	5733	5074
N^l_{a0}, N^l_{b0}	3052	3639	1466	2240
N^l_{a1}, N^l_{b1}	-	-	2369	1399
N^l	3052	3639	3835	3639

charged, the government expects a loss on the two initial links. In both states, congestion cost is much higher than that of the first-best case, so there is a large room for improvement from the base equilibrium.

In the first-best case, the capacity more than doubles from the base case and the congestion costs in both states decrease. The expected profit is zero, because the profit in the high state and the loss in the low state cancel out. Compared with the deterministic case discussed in Verhoef (2008), the first-best capacity is larger under uncertainty.

The second-best case can achieve 97.1% of the increase in the expected social welfare from the base equilibrium to the first-best case. However, this generates a considerable loss for the two newly-added parallel link in either realized demand state, because the capacity expansion is too large for the toll revenues to cover the cost.

As predicted by the theoretical result for the parallel links, the second-best zero-profit case has the same toll and generalized price as in the first-best case in both demand states. It can achieve 80.4% of the increase in social welfare.

3.2 Entry Game

Table 2: Free Entry

Round	Investing Firm	K_a	K_b	$E(S)$	ω	$E(\pi)$	$D^h = p^h$	$D^l = p^l$
0		1500	1500	60189	0	-	-	-
1	I	1862	1500	66436	0.117	1145	26.678	12.364
2	I	1862	1960	77972	0.332	1837	23.606	10.761
3	II	2176	1960	82920	0.432	721	22.043	9.953
4	II	2176	2283	88639	0.531	759	20.231	9.049
5	III	2426	2283	91887	0.592	380	19.073	8.511
6	III	2426	2518	94892	0.648	321	17.963	8.017
7	IV	2607	2518	96824	0.684	172	17.194	7.689
8	IV	2607	2677	98448	0.714	128	16.523	7.412

Free Entry In the free entry regime, the firm with the highest expected profit sets a capacity on the segment of his choice. Then demand uncertainty is resolved and firms set tolls simultaneously. We allow both old and new firms to add capacity, so the double marginalization problem is mitigated, in the sense that one may expect firms to be active on both serial segments, and competition between such firms on parallel segments to drive down tolls.

Table 2 shows the network developments for free entry. We assume without loss of generality that capacity is first added on section a. In equilibrium, the firm that has added capacity on section a will, in the next round, add capacity on section b. Then a new firm adds capacity on section a, then section b, and so on. This pattern emerges because (1) the same firm can better coordinate the tolls on both sections, so it is the same firm rather than a new one which will invest on a serial segment in the even round; (2) when the capacity on one section is expanded, it is more profitable to add capacity on the complementary section. In addition, the capacity addition in section b is always larger than that in section a in the previous round, because of increased demand due to capacity expansion on the other link. A new firm always adds capacity in section a in the next round because if an old firm does so, it will end up competing with its own capacities in section a. After eights rounds of capacity building, the total capacity on both section are above that of the first-best case in the numerical example, reflecting that competition will cause over-investment in capacity in the end. The expected profit of the entrant decreases as more firms join the network, and so does the expected profit of the incumbent firms. The generalized price in each state falls over time due to increased competition, but it remains higher than the second-best zero-profit case's level owing to the market power of the firms. The expected social welfare is 98433 after eight rounds, and it can achieve 71.4% improvement in social welfare from the base equilibrium to the first-best case. The qualitative patterns match those described in Verhoef(2008) for deterministic demand.

Table 3: Entry by Patronage Auction

Round	Investing Firm	K_a	K_b	$E(S)$	ω	$D^h = p^h$	$D^l = p^l$
0		1500	1500	60189	0	-	-
1	I	2275	1500	69744	0.178	25.315	11.311
2	I	2275	2764	96109	0.671	17.419	6.802
3	II	2886	2764	101567	0.773	15.501	6.006
4	II	2886	3154	102353	0.787	14.946	5.753
5	III	2930	3154	102503	0.790	14.870	5.725
6	III	2930	3159	102503	0.790	14.865	5.723
7	IV	2930	3159	102503	0.790	14.864	5.723
8	IV	2930	3159	102503	0.790	14.864	5.723

Entry by Patronage Auction In the entry by patronage auction regime, the firm which offers the highest expected traffic flow on the new link is allowed to add the link. Every time a new firm enters, it makes zero expected profit due to the competitive nature of the auction. Afterwards, it keeps the toll scheme (conditional on the realized demand) unchanged. It may not collect enough tolls to cover the capacity cost if later too many firms enter with low tolls.

The key characteristics of the resulting equilibrium are shown in Table 3. After four rounds already, the expected social welfare is higher than that after eight rounds in the free entry regime. Capacity addition is rapid initially, and after five rounds the capacity addition is negligible, which means a stable network is more or less formed. The relative efficiency, $\omega = 0.790$, is much

Table 4: Entry by Generalized Price Auction

Round	Investing Firm	K_a	K_b	$E(S)$	ω	$D^h = p^h$	$D^l = p^l$
0		1500	1500	60189	0	-	-
1	I	2277	1500	69818	0.180	25.223	11.382
2	I	2277	2767	96906	0.687	16.802	7.174
3	II	2888	2767	102517	0.790	14.793	6.447
4	II	2888	2916	103137	0.802	14.502	6.346
5	III	2907	2916	103203	0.803	14.468	6.335
6	III	2907	2918	103210	0.803	14.464	6.334
7	IV	2908	2918	103211	0.803	14.464	6.333
8	IV	2908	2918	103211	0.803	14.464	6.333

higher than that of the free entry regime and close to that of the second-best zero-profit case (0.804). Judging from these criteria, the entry by patronage auction performs well.

Entry by Generalized-Price Auction In the entry by generalized-price auction regime, the firm which offers the lowest expected generalized price can add the link. Afterwards it keeps the toll scheme (conditional on the realized demand) unchanged. As shown in Table 4, the characteristics of the equilibrium are similar to those in the entry by patronage auction regime. The expected social welfare is even higher and $\omega = 80.3\%$. Both auctions achieve a high ω after only five rounds, so regulation by auction is a big improvement, and a faster road to efficiency, compared to the free entry regime.

3.3 Sensitivity Analysis

The numerical simulation seems to suggest that entry by either auction performs better than free entry. In addition, entry by the generalized price auction seems to generate higher efficiency than the patronage auction when demand is uncertain. In this section we test the sensitivity of such results with respect to the degree of uncertainty and the price elasticity of demand.

In the numerical simulation, the inverse demand function is represented by $D^i(N^i) = \delta_0^i - \delta_1 \cdot N^i$ and the degree of uncertainty is represented by a, where $\delta_0^h = (1 + a) \cdot \delta_0$ for the high demand state and $\delta_0^l = (1 - a) \cdot \delta_0$ for the low demand state. $a = 0$ means demand is completely certain, and as a increases, demand becomes more uncertain. $a = 0.2$ is used in the numerical simulation in the previous section. Figure 1 shows the relative efficiency of the three regimes after five rounds of entry corresponding with different degrees of demand uncertainty. For $0 \leq a \leq 0.35$, the two auctions clearly perform better than free entry, because capacity addition is quicker with auctions. The efficiencies of the two auctions are similar, which is consistent with the case without demand uncertainty. As the degree of uncertainty increases, the relative efficiency of all three regulatory regimes increases. The main reason is that when the degree of uncertainty increases, the optimal capacity

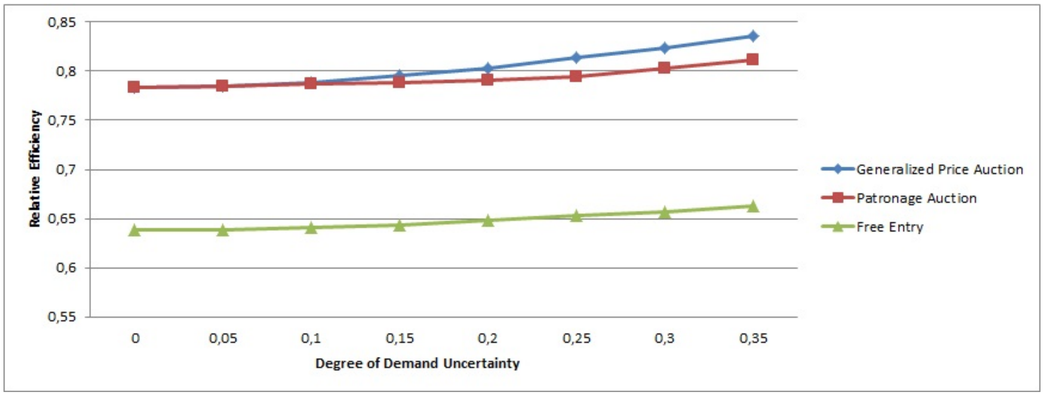

Figure 1: Sensitivity Analysis: Degree of Uncertainty

in the first-best case increases, so the social welfare of the base equilibrium decreases.

To study the robustness of the results with respect to demand elasticity, we vary the demand elasticity by changing the intercept and slope of the inverse demand function, keeping the base equilibrium unchanged. In other words, we change the value of b, where the new intercept is $\bar{\delta_0^i} = \delta_0^i + b \cdot \delta_1 \cdot N^i$ and the new slope is $\bar{\delta_1} = (1 + b) \cdot \delta_1$. For the simulation in the previous section, we set $b = 0$ and the resulting demand elasticity is 0.50. Figure 2 shows the relative efficiency of the three regimes after five rounds of entry corresponding with different demand elasticity. When b changes from -0.5 to 0.5, the demand elasticity changes from -1.01 to -0.34. For the parameter range in the simulation, it seems that the two auctions generate similar

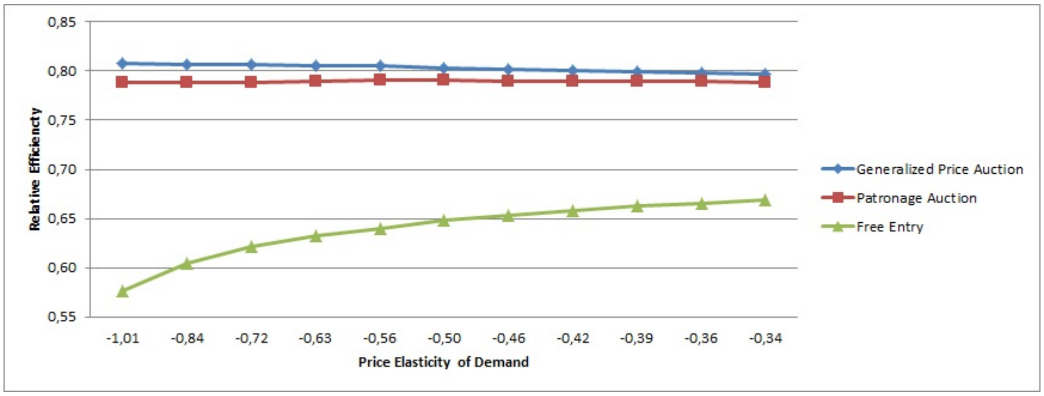

Figure 2: Sensitivity Analysis: Demand Elasticity

social welfare, with the generalized price auction performing only slightly better. They both perform much better than free entry, due to quick capacity addition. As the demand elasticity increases (i.e. becomes closer to zero reflecting less elastic demand), due to larger capacity adjustment under the first-best case and the auctions, the expected social welfare of the three cases increases less significantly than that of the base equilibrium and the free-entry case. As a result, the relative efficiency of the auctions decreases and that of the free-entry case increases.

4 Concluding Remarks

This paper investigates how demand uncertainty influences the efficiency of private supply of roads in a mixed network. We compare different benchmarks and evaluate the efficiency of regulatory policies for both simple static networks and more complex dynamic ones.

We find that demand uncertainty indeed brings new challenges for policy makers. For simple networks, taking into consideration of demand uncertainty, the optimal capacity for the first-best case is larger than its deterministic counterpart. In the second-best zero-profit case, the toll for serial links are no longer Pigouvian. The patronage and the generalized price auction can no longer achieve the second-best zero-profit result. For more complex networks with dynamic new link formation, if the firms with the highest expected profit can add capacity in turn, there is usually over-investment. When we control the process by the patronage auction or the generalized price auction, the expected social welfare increases much quicker and reaches a steady state rather closely after only five rounds. Unlike in the case of deterministic demand, the generalized price auction performs better than the patronage auction with demand uncertainty.

Our findings can be useful for policy makers and practitioners who have to make decisions about road pricing and capacity investment despite prevailing demand uncertainty. For Pigouvian toll to correct the externality in many markets such as the transport, telecommunication and energy market, caution needs to be exercised because demand uncertainty will render some policies less effective in mixed networks. In addition, demand uncertainty may justify "over-investment" observed in many projects, on the ground that the benefit of a larger capacity when economy is good on average outweighs the cost of it when economy is bad. Moreover, market competition itself is not enough to guarantee fast formation of efficient roads networks. Instead regulation combined with market competition forces, especially the generalized price auction, is proven in theory to be a reliable way of providing good public service by private firms. It can form the basis for public-private joint projects in more complex and realistic settings.

In sum, demand uncertainty complicates the evaluation and regulation of private supply of roads in mixed networks. For future research, we will consider more general networks, dynamic games of capacity addition, user heterogeneity and optimal auction design.

Appendix I

We prove first the case of one road, then show how to extend the proof to two serial and two parallel links.

Consider the simplest case with only one road. Demand state i happens with probability p^i. Let K, N^i, τ^i, γ denote the capacity, traffic flow, toll and unit capacity cost respectively. The congestion cost function is $c(N^i, K) = \alpha\cdot t_f\cdot (1+\beta\cdot (\frac{N^i}{K})^x)$ with $x > 0$, and the inverse demand function is $D^i(N^i) = \delta_0^i - \delta_1\cdot N^i$. The intercept of the inverse demand function is δ_0^i for state i.

The social planner solves the following constrained optimization problem:

$$\max_{K,\tau^i,N^i} \sum_i p^i\cdot (\int_0^{N^i} D^i(n)dn - c(N^i, K)N^i) - \gamma\cdot K$$
$$s.t. D^i(N^i) = c(N^i, K) + \tau^i \tag{21}$$

At the optimum:

$$\delta_0^i - \delta_1\cdot N^i - \alpha\cdot t_f\cdot (1 + \beta\cdot (\frac{N^i}{K})^x\cdot (1 + x)) = 0$$
$$\sum_i p^i\cdot \alpha\cdot t_f\cdot \beta\cdot x\cdot (\frac{N^i}{K})^{1+x} - \gamma = 0$$

The first equation is an implicit function for $N^i(K, \delta_0^i)$. Denote the resulting social surplus in state i as $W^i(K, \delta_0^i) = \int_0^{N^i} D^i(n)dn - c(N^i, K)\cdot N^i$. We can prove that the marginal benefit of capacity expansion, i.e. $\frac{\partial W^i(K,\delta_0^i)}{\partial K}$ is convex in δ_0^i, as shown below:

$$\frac{\partial \frac{\partial W^i(K,\delta_0^i)}{\partial K}}{\partial \delta_0} = \frac{\alpha \cdot t_f \cdot \beta \cdot x(1+x) \cdot N^{i1+x}}{\delta_1 \cdot N^i + \alpha \cdot \beta \cdot t_f \cdot (1+x) \cdot x \cdot (\frac{N^i}{K})^x} > 0$$
$$\frac{\partial^2 \frac{\partial W^i(K,\delta_0^i)}{\partial K}}{\partial \delta_0^2} = \frac{\alpha \cdot t_f \cdot \beta \cdot x \cdot (N^{i1+x} \cdot x \cdot (1+x)(\delta_1 \cdot N^i + (1+x) \cdot \alpha \cdot t_f \cdot \beta \cdot (\frac{N^i}{K})^x))}{(\delta_1 \cdot N^i + (1+x) \cdot x \cdot \alpha \cdot t_f \cdot \beta \cdot (\frac{N^i}{K})^x)^3} > 0 \tag{22}$$

The convexity implies that $\sum_i p_i \frac{\partial W^i(K,\delta_0^i)}{\partial K} > \frac{\partial W^i(K,\sum_i p_i \delta_0^i)}{\partial K}$. In other words, for any chosen capacity, the expected marginal benefit is higher with demand uncertainty. So the social planner should choose higher capacity under demand uncertainty.

For two parallel links, the first-best solution is the same as that of a single link with $K = K_0 + K_1$ and $N^i = N_0^i + N_1^i$, so the previous result still holds. For two serial link, the first-best solution is the same as that of a single link with α doubled, so the reasoning is similar.

References

Arnott, R., A. de Palma, and R. Lindsey (1996). Information and Usage of Free-Access Congestible Facilities with Stochastic Capacity and Demand.

International Economic Review 37(No. 1 (Feb. 1996)), 181–203.

de Palma, A. (1992). American Economic Association A Game-Theoretic Approach to the Analysis of Simple Congested Networks. *The American Economic Review 82*(2), 494–500.

de Palma, A. and R. Lindsey (2000). Private toll roads: Competition under various ownership regimes. *The Annals of Regional Science 34*(1), 13–35.

De Vany, a. S. and T. R. Saving (1977). Product Quality, Uncertainty, and Regulation: The Trucking Industry. *American Economic Review 67*(4), 583–594.

De Vany, Arthur and Saving, T. R. (1980). Competition and highway pricing for stochastic traffic. *Journal of Business*.

D'Ouville, E. L. and J. F. McDonald (1990). Effects of demand uncertainty on optimal capacity and congestion tolls for urban highways. *Journal of Urban Economics 28*(1), 63–70.

Fisher, G. and S. Babbar (1996). Private Financing of Toll Roads. *RMC Discussion Paper Series* (117), 1–47.

Flyvbjerg, B., M. K. Skamris Holm, and S. L. Buhl (2006). Inaccuracy in Traffic Forecasts. *Transport Reviews 26*(1), 1–24.

Knight, F. H. (1924). Some Fallacies in the Interpretation of Social Cost. *The Quarterly Journal of Economics 38*(4), 582–606.

Kraus, M. (1982). Highway pricing and capacity choice under uncertain demand. *Journal of Urban Economics 12*(1), 122–128.

Lindsey, R. and A. de Palma (2014). Cost recovery from congestion tolls with long-run uncertainty. *Economics of Transportation 3*(2), 119–132.

Mohring, H. and M. Harwitz (1962). *Highway benefits: An analytical framework.*

Pigou, A. C. (1920). *The Economics of Welfare, 1920.* London: Macmillan.

Verhoef, E. T. (2007). Second-best road pricing through highway franchising. *Journal of Urban Economics 62*(2), 337–361.

Verhoef, E. T. (2008). Private roads: Auctions and competition in networks. *Journal of Transport Economics and Policy 42*(3), 463–493.

Wang, J. Y. T., R. Lindsey, and H. Yang (2011). Nonlinear pricing on private roads with congestion and toll collection costs. *Transportation Research Part B: Methodological 45*(1), 9–40.

Wu, D., Y. Yin, and H. Yang (2011). The independence of volume-capacity ratio of private toll roads in general networks. *Transportation Research Part B: Methodological 45*(1), 96–101.

Yang, H. and Q. Meng (2002). A note on "highway pricing and capacity choice in a road network under a build-operate-transfer scheme". *Transportation Research Part A: Policy and Practice 36*(7), 659–663.

Small, K. A., Verhoef, E. T. (2007). The Economics of Urban Transportation. Routledge.