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# The Fiction of Full BEKK*

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## Abstract

The purpose of the paper is to show that univariate GARCH is not a special case of multivariate GARCH, specifically the Full BEKK model, except under parametric restrictions on the off-diagonal elements of the random coefficient autoregressive coefficient matrix, provides the regularity conditions that arise from the underlying random coefficient autoregressive process, and for which the (quasi-) maximum likelihood estimates have valid asymptotic properties under the appropriate parametric restrictions. The paper provides a discussion of the stochastic processes, regularity conditions, and asymptotic properties of univariate and multivariate GARCH models. It is shown that the Full BEKK model, which in practice is estimated almost exclusively, has no underlying stochastic process, regularity conditions, or asymptotic properties.

**Keywords:** Random coefficient stochastic process, Off-diagonal parametric restrictions, Diagonal and Full BEKK, Regularity conditions, Asymptotic properties, Conditional volatility, Univariate and multivariate models.

**JEL:** C22, C32, C52, C58

## 1. Introduction

The most widely estimated univariate and multivariate models of time-varying volatility for financial data, as well as any high frequency data that are measured in days, hours and minutes, is the conditional volatility model. The stochastic processes, regularity conditions and asymptotic properties of the most popular univariate conditional volatility models, such as GARCH (see Engle (1982) and Bollerslev (1986)) and GJR (see Glosten et al. (1993)) are well established in the literature, though McAleer and Hafner (2014) have raised caveats regarding the existence of the stochastic process underlying exponential GARCH (EGARCH) (see Nelson (1990, 1991)).

However, the same cannot be said about multivariate conditional volatility models, specifically Full BEKK (see Baba et al. (1985) and Engle and Kroner (1995)), for which the underlying stochastic processes, regularity conditions and asymptotic properties have either not been established, or are simply assumed rather than derived. These conditions are essential for valid statistical analysis of empirical estimates.

The purpose of the paper is to show that the stochastic process underlying univariate GARCH is not a special case of that underlying multivariate GARCH, except under parametric restrictions on the off-diagonal elements of the random coefficient autoregressive coefficient matrix. The paper provides the regularity conditions that arise from the underlying random coefficient autoregressive process, and for which the (quasi-) maximum likelihood estimates (QMLE) have valid asymptotic properties under the appropriate parametric restrictions.

The Full BEKK model is estimated almost exclusively in practice, despite the fact that it has no underlying stochastic process, regularity conditions, or asymptotic properties, as shown in the proposition and three corollaries.

The plan of the paper is as follows. Section 2 provides a discussion of the stochastic processes, regularity conditions, and asymptotic properties of univariate and multivariate GARCH models. Section 3 shows that the Full BEKK model has no underlying stochastic process, regularity conditions, or asymptotic properties. Section 3 gives some concluding comments.

## 2. Univariate and Multivariate GARCH Models

### 2.1 Univariate Conditional Volatility Models

Consider the conditional mean of financial returns for commodity *i*, in a financial portfolio of *m* assets, as follows:

$$y_{it} = E(y_{it}|I_{t-1}) + \varepsilon_{it}\ , \qquad i = 1, 2, \dots, m, \qquad (1)$$

where the returns, $y_{it} = \Delta logP_{it}$, represent the log-difference in financial commodity prices, $P_t$, $I_{t-1}$ is the information set for all financial assets at time *t-1*, $E(y_{it}|I_{t-1})$ is the conditional expectation of returns, and $\varepsilon_{it}$ is a conditionally heteroskedastic error term.

In order to derive conditional volatility specifications, it is necessary to specify the stochastic processes underlying the returns shocks, $\varepsilon_{it}$. The most popular univariate conditional volatility model, GARCH model, is discussed below.

For a portfolio of *m* financial assets, consider the random coefficient autoregressive process underlying the returns shocks, $\varepsilon_{it}$, as follows:

$$\varepsilon_{it} = \phi_{it}\varepsilon_{it-1} + \eta_{it}\ , \qquad i = 1, 2, \dots, m,\ m > 1, \qquad (2)$$

where

$\phi_{it} \sim iid(0, \alpha_i)$, $\alpha_i \geq 0$,

$\eta_{it} \sim iid(0, \omega_i)$, $\omega_i \geq 0$,

$\eta_{it} = \varepsilon_{it}/\sqrt{h_{it}}$ is the standardized residual,

$h_{it}$ is the conditional volatility of financial asset *i*.

Tsay (1987) derived the following conditional volatility of financial asset *i* as an ARCH process (see Engle, 1982):

$$E(\varepsilon_{it}^2|I_{t-1}) \equiv h_{it} = \omega_i + \alpha_i \varepsilon_{it-1}^2, \quad i = 1, 2, \dots, m, \; m > 1, \qquad (3)$$

where $h_t$ represents conditional volatility, and $I_{t-1}$ is the information set available at time *t-1*. A lagged dependent variable, $h_{t-1}$, is typically added to equation (3) to improve the sample fit:

$$h_{it} \equiv E(\varepsilon_{it}^2|I_{t-1}) = \omega_i + \alpha_i \varepsilon_{it-1}^2 + \beta_i h_{t-1}, \quad \beta_i \in (-1, 1). \qquad (4)$$

From the specification of equation (2), it is clear that both $\omega_i$ and $\alpha_i$ should be positive as they are the unconditional variances of two different stochastic processes. In equation (4), which is a GARCH(1,1) model for commodity *i* (see Bollerslev, 1986), the stability condition requires that $\beta_i \in (-1, 1)$.

The stochastic process can be extended to asymmetric conditional volatility models (see, for example, McAleer (2014)), and to give higher-order lags and a larger number of alternative commodities, namely up to *m*-1. However, the symmetric process considered here is sufficient to focus the key ideas associated with the purpose of the paper.

As the stochastic process in equation (2) follows a random coefficient autoregressive process, under normality (non-normality) of the random errors, the maximum likelihood estimators (quasi- maximum likelihood estimators, QMLE) of the parameters will be consistent and asymptotically normal. It is worth emphasizing that the regularity conditions include invertibility, which is obvious from equation (2), as:

$$\varepsilon_{it} - \phi_{it}\varepsilon_{it-1} = \eta_{it},$$

so that the standardized shocks can be expressed in terms of the empirical data through equations (1) and (2).

Ling and McAleer (2003) and McAleer et al. (2008) provide general proofs of the asymptotic properties of univariate and multivariate conditional volatility models based on satisfying the regularity conditions in Jeantheau (1998) for consistency, and in Theorem 4.1.3 in Amemiya (1985) for asymptotic normality.

## 2.2 Multivariate Conditional Volatility Models

The multivariate extension of the univariate ARCH and GARCH models is given in Baba et al. (1985) and Engle and Kroner (1995). It is useful to define the multivariate extension of the relationship between the returns shocks and the standardized residuals, that is, $\eta_{it} = \varepsilon_{it}/\sqrt{h_{it}}$ . The multivariate extension of equation (1), namely $y_t = E(y_t|I_{t-1}) + \varepsilon_t$, can remain unchanged by assuming that the three components are now $m \times 1$ vectors, where $m$ is the number of financial assets.

Consider the vector random coefficient autoregressive process of order one, which is the multivariate extension of the univariate process given in equation (2):

$$\varepsilon_t = \Phi_t \varepsilon_{t-1} + \eta_t, \qquad (5)$$

where

$\varepsilon_t$ and $\eta_t$ are $m \times 1$ vectors,

$\Phi_t$ is an $m \times m$ matrix of random coefficients,

$\Phi_t \sim iid(0, A)$, $A$ is positive definite,

$\eta_t \sim iid(0, C)$, $C$ is an $m \times m$ matrix.

Vectorization of a full matrix $A$ to *vec* $A$ can have dimension as high as $m^2 \times m^2$, whereas vectorization of a symmetric matrix $A$ to *vech* $A$ can have a smaller dimension of $m(m+1)/2 \times m(m+1)/2$.

In the case where $A$ is a diagonal matrix, with $a_{ii} > 0$ for all $i = 1,\ldots,m$ and $|b_{jj}| < 1$ for all $j = 1,\ldots,m$, so that $A$ has dimension $m \times m$, McAleer et al. (2008) showed that the multivariate extension of GARCH(1,1) from equation (5) is given as the Diagonal BEKK model, namely:

$$Q_t = CC' + A\varepsilon_{t-1}\varepsilon'_{t-1}A' + BQ_{t-1}B', \tag{6}$$

where $A$ and $B$ are both diagonal matrices. The diagonality of the positive definite matrix $A$ is essential for matrix multiplication as $\varepsilon_{t-1}\varepsilon'_{t-1}$ is an $m \times m$ matrix; otherwise equation (6) could not be derived from the vector random coefficient autoregressive process in equation (5).

McAleer et al. (2008) showed that the QMLE of the parameters of the Diagonal BEKK model were consistent and asymptotically normal, so that standard statistical inference on testing hypotheses is valid.

## 3. The Fiction of Full BEKK

Consider element $i$ of equation (5), which is given as:

$$\varepsilon_{it} = \sum_{j=1}^{m} \phi_{ijt}\varepsilon_{ijt-1} + \eta_{it}\,, \quad i = 1, 2, \ldots, m,\ m > 1 \tag{7}$$

which is not equivalent to equation (2) unless $\phi_{ijt} = 0\ \forall\ j \neq i$. The stochastic equation (7) is not a random coefficient autoregressive process because of the presence of an additional $m$-1 random coefficients. Importantly, equation (7) is **not** invertible as the random processes cannot be connected to the data, which requires $m$ equations, such as in equation (5). Consequently, the stochastic process underlying univariate ARCH is not a special case of that underlying multivariate ARCH unless $\phi_{ijt} = 0\ \forall\ j \neq i$.

The same condition holds for all $i = 1,\ldots,m$, which leads to the following:

**Proposition:** For the stochastic process underlying univariate ARCH to be a special case of the stochastic process underlying multivariate ARCH requires the restrictions:

$$\phi_{ijt} = 0 \ \forall j \neq i.$$

A similar condition holds for univariate GARCH and multivariate GARCH.

The Proposition leads to the following corollaries:

**Corollary 1:** The $m \times m$ matrix of random coefficients, $\Phi_t$, is a diagonal matrix.

**Corollary 2:** From Corollary 1, it follows that the $m \times m$ weight matrix of (co)variances, $A$, is a diagonal matrix.

**Corollary 3:** Corollaries 1 and 2 show that a Full BEKK model, namely where there are no restrictions on the off-diagonal elements in $\Phi_t$, and hence no restrictions in the off-diagonal elements in $A$, is not possible if univariate ARCH is to be a special case of its multivariate counterpart, BEKK.

**Corollary 4:** As there are no underlying regularity conditions for Full BEKK, including invertibility, the model cannot be estimated. Therefore, there is no likelihood function, and hence there are also no asymptotic properties of the QMLE of the unknown parameters in Full BEKK.

Corollary 4 is consistent with the proof in McAleer et al. (2008) that the QMLE of Full BEKK has no asymptotic properties, whereas the QMLE of Diagonal BEKK is consistent and asymptotically normal.

For all intents and purposes, Full BEKK does not exist, except by assumption.

## 4. Conclusion

The Full BEKK model in Baba et al. (1985) and Engle and Kroner (1995), who do not derive the model from an underlying stochastic process, was presented as equation (6), with *A* and *B* given as full matrices, with no restrictions on the off-diagonal elements. The Full BEKK model is estimated almost exclusively in practice, despite the fact that it has no underlying stochastic process, regularity conditions, or asymptotic properties, as shown in the proposition and three corollaries.

The full BEKK model can be replaced by the triangular or Hadamard (element-by-element multiplication) BEKK models, with similar problems of identification and (lack of) existence. The full, triangular and Hadamard BEKK models cannot be derived from any known underlying stochastic processes, which means there are no regularity conditions (except by assumption) for checking the internal consistency of the alternative models, and consequently no valid asymptotic properties of the QMLE of the associated parameters (except by assumption).

Moreover, as the number of parameters in a full BEKK model can be as much as $3m(m+1)/2$, the "curse of dimensionality" will be likely to arise, which means that convergence of the estimation algorithm can become problematic and less reliable when there is a large number of parameters to be estimated. As a matter of fact, estimation of the full BEKK can be problematic even when $m$ is as low as 5 financial assets. Such computational difficulties do not arise for the diagonal BEKK model. Convergence of the estimation algorithm is more likely when the number of commodities is less than 4, though this is nevertheless problematic in terms of interpretation.

The purpose of the paper was to show that univariate GARCH is not a special case of multivariate GARCH, specifically the Full BEKK model, except under parametric restrictions on a random coefficient autoregressive coefficient matrix. The paper provided the regularity conditions that arise from the underlying random coefficient autoregressive process, and for which the (quasi-) maximum likelihood estimates have valid asymptotic properties under the appropriate parametric restrictions, for the univariate and multivariate GARCH models. It was shown that the Full BEKK model has no underlying stochastic process, regularity conditions, or asymptotic properties.

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