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Abstract

The paper considers the problem as to whether financial returns have a common volatility process in the framework of stochastic volatility models that were suggested by Harvey et al. (1994). We propose a stochastic volatility version of the ARCH test proposed by Engle and Susmel (1993), who investigated whether international equity markets have a common volatility process. The paper also checks the hypothesis of frictionless cross-market hedging, which implies perfectly correlated volatility changes, as suggested by Fleming et al. (1998). The paper uses the technique of Chesher (1984) in differentiating an integral that contains a degenerate density function in deriving the Lagrange Multiplier test statistic.

Keywords: Volatility comovement, Cross-market hedging, Spillovers, Contagion.
JEL Classification: C12, C58, G01, G11

1 Introduction

This paper considers the problem as to whether financial returns have a common volatility process in the framework of stochastic volatility models that were suggested by Harvey et al. (1994). We propose a stochastic volatility version of the ARCH test proposed by Engle and Susmel (1993), who investigated whether international equity markets have a common volatility process using a multivariate ARCH model. They found groups of countries that showed similar time-varying volatility.

Fleming et al. (1998) used the multivariate stochastic volatility model of Harvey et al. (1994), and estimated volatility linkages across stock, bond, and money markets, and found strong correlation between the markets. Fleming et al. (1998) also suggested that cross-market hedging in frictionless markets causes perfectly correlated volatility changes, extending the model of Tauchen and Pitts (1983). This linkage is stronger than the presence of a common factor in volatility changes in that it implies that the idiosyncratic part of stochastic volatility changes will disappear and have a common volatility process. They also conducted a Wald test, and rejected the null hypothesis of perfectly correlated volatility to conclude that cross-market hedging is imperfect.

Contrary to what has been presented, the use of the Wald and likelihood ratio tests is inappropriate for the null hypothesis of perfectly correlated volatility, as the asymptotic distribution of the Wald test statistics is different from the conventional chi-squared distribution, as shown, for example, in Chernoff (1954) . As the null hypothesis is on the boundary of the parameter space, the correlation estimator cannot be greater than one in absolute value, so that the distribution is asymmetric, and hence non-normal, when the true correlation coefficient is unity.

The paper proposes a new Lagrange multiplier test for the hypothesis that the volatility changes of a bivariate series are perfectly correlated. We use the framework of a multivariate stochastic volatility model proposed by Harvey et al. (1994), where the log-volatility follows vector autoregressive (VAR) process of order one with diagonal autoregressive coefficient matrix.

The Lagrange multiplier test principle is the only alternative for this problem in deriving the test statistics because it uses only the estimator of the unconstrained parameters, which are asymptotically normally distributed, and does not estimate the parameter on the boundary of the parameter space. Then the test statistic will follow the conventional chi-squared asymptotic distribution under the null hypothesis.

To the best of our knowledge, the Lagrange multiplier test statistic for the perfectly correlated volatility changes has not been proposed in the literature. It follows that the hypothesis of frictionless cross-market hedging has also not been tested, so that a new test for perfectly correlated volatility would be useful from a practical perspective.

It is not without reason why an LM test has not been proposed to date as the conventional method to obtain a score function that is used in constructing the LM test statistic is unworkable for the multivariate stochastic volatility model.

The derivative of the transition density is intractable in this integral under the null hypothesis, as the transition disturbance has zero variance, and the transition equation density degenerates. We express the score function analytically with respect to the degenerate parameter using the ingenious method devised by Chesher (1984), which is the main technical breakthrough in tackling this problem.

The new test is a stochastic volatility version of the ARCH test proposed by Engle and Susmel (1993) to investigate whether international equity markets have a common volatility process. The test can be regarded as a test for the number of stochastic volatility factors, in line with the definition of Harvey et al. (1994) and Cipollini and Kapetanios (2008), when the number of factors is one under the null hypothesis. Cipollini and Kapetanios (2008) used a linearized model for the log of squared returns, and used the principal component methodology of Stock and Watson (2002) in deciding the number of factors. Their method has the advantage in that it is applicable when the number of variables is large, even though it is not a statistical test. The new test developed in this paper is a unique statistical test for the null hypothesis of the number of stochastic volatility factors.

Although theoretically straightforward, a generalization to multi-factor models is left to for further research, as numerical calculation of the test statistic is extremely time consuming, even in the simple case given here. The bottleneck lies in the calculation of score functions by the conventional smoothing algorithm. Maximum likelihood estimation by means of the quadrature method proposed by Watanabe (1999) is efficient when the state variable is univariate. Moreover, Monte Carlo simulation methods would work well for the estimation of multivariate state space models, with some difficulty, as the filtering algorithm, which is required for estimation, is far faster than the smoothing algorithm. It should be possible to generalize the results in the paper to the multi-factor model by improving the smoothing algorithm, as well as by hardware advancement.

The remainder of the paper is as follows. Section 2 presents the model, Section 3 develops the LM test statistic, Monte Carlo experiments are presented in Section 4, the empirical analysis is given in Section 5, and some concluding remarks are in Section 6, followed by the Appendices.

2 Model

Under the alternative hypothesis, the observation vector $y_t = (y_{1t}, y_{2t})'$ can be expressed as:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \exp(\frac{h_1}{2}) & 0 \\ 0 & \exp(\frac{h_2}{2}) \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix} \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix}, t = 1, \cdots, T, \tag{1}$$

where the log-volatility, $(h_{1t}, h_{2t})'$, follows a stationary bivariate autoregressive process of order one, defined by:

$$\begin{pmatrix} h_{1t} \\ h_{2t} \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 0 & \psi \end{pmatrix} \begin{pmatrix} h_{1t-1} \\ h_{2t-1} \end{pmatrix} + \begin{pmatrix} b_1 & 0 \\ b_2 & \sqrt{c} \end{pmatrix} \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}, t = 1, \cdots, T, \tag{2}$$

and

$$\begin{pmatrix} h_{11} \\ h_{21} \end{pmatrix} = \begin{pmatrix} b_1/\sqrt{(1-\rho^2)} & 0 \\ b_2/\sqrt{(1-\psi^2)} & \sqrt{c/(1-\psi^2)} \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix},$$

$$(e_{1t}, e_{2t}, u_{1t}, u_{2t})' \sim NID\left(\mathbf{0}, \mathbf{I}_4\right).$$

The disturbance term of the measurement equation (1) is assumed to be contemporaneously uncorrelated, and the transition equation (2) is assumed to be contemporaneously correlated. This model was originally suggested by Harvey et al. (1994), and was examined in detail in **?** and Asai et al. (2006).

It is easy to see that, under the null hypothesis defined by:

$$\mathbf{H}_0 : \sqrt{c} \to 0, \psi = \rho, b_2 = b_1, \tag{3}$$

we have:

$$h_{1t} \equiv h_{2t}, \text{ for any } t.$$

This is the stochastic volatility factor model discussed in Harvey et al. (1994) and Cipollini and Kapetanios (2008) in the simple case when the number of factors is one, that is , $h_{1t} = h_{2t}$.

3 Implications for finance

According to Fleming et al. (1998), the strong linkage of volatility between markets has two sources: (i) the common information flow that affects expectations in more than one market simultaneously; (ii) the information spillovers caused by cross-market hedging. The new test, which will be given in the following, can be interpreted as a test for the joint null hypothesis of common information and frictionless information spillover between markets. If the null hypothesis is rejected, this means that the link between the two markets is not strong enough with respect to risk.

It might be possible to interpret that the common information between the markets is expressed by the the constraints $c \to 0$ and $\psi = \rho$ in (3), which means that the volatility equations of more than one market have identical innovations or shocks and that frictionless information spillovers are given by the constraint $b_1 = b_2$. However, it is not possible to decompose the test statistic into two components, which correspond to common information and information spillovers, as the Fisher information employed in constructing the χ^2-distributed test statistic is not diagonal. The identification of the two sources in order to construct two test statistics is left to further research.

4 LM Test statistic

We propose an LM test statistic for the null hypothesis for the observation series y_{1t} and y_{2t}. For the sake of computational simplicity and parameter parsimony, we set $a_1 = a_2 = 1$ by standardizing the returns variance. Define the unrestricted parameter vector as $\theta_1 = (c, \psi, b_2, \rho, b_1)$, and the restricted parameter vector as $\theta_0 = (0, \psi, b_2, \psi, b_2)$.

First, we obtain the maximum likelihood estimator of the constrained parameter, θ_0, of the state space system, (1) and (2), at time t. Denote $\mathbf{y}_1 = (y_{11}, y_{12}, \ldots, y_{1t})^{'}$, $\mathbf{y}_2 = (y_{21}, y_{22}, \ldots, y_{2t})^{'}$, $\mathbf{h}_1 = (h_{11}, h_{12}, \ldots, h_{1t})^{'}$, and $\mathbf{h}_2 = (h_{21}, h_{22}, \ldots, h_{2t})^{'}$. The likelihood function is expressed as:

$$f(\mathbf{y}) = \int f(\mathbf{h}, \mathbf{y}) d\mathbf{h} = \int f(\mathbf{y}_1, \mathbf{y}_2 | \mathbf{h}_1, \mathbf{h}_2) f(\mathbf{h}_2 | \mathbf{h}_1) f(\mathbf{h}_1) d\mathbf{h}_1 d\mathbf{h}_2,$$

where the specified densities $f(\mathbf{y}_1|\mathbf{h}_1), f(\mathbf{y}_2|\mathbf{y}_1, \mathbf{h}_1, \mathbf{h}_2), f(\mathbf{h}_2|\mathbf{h}_1)$, and $f(\mathbf{h}_1)$ are given in the Appendix.

Second, we derive the score function under the alternative hypothesis, and evaluate it under the null hypothesis. Denote:

$$\mathbf{y}_{1:t} = (y_{11}, y_{12}, \ldots, y_{1t},\ y_{21}, y_{22}, \ldots, y_{2t})^{'} ,\ \mathbf{y}_t = (\mathbf{y}_1, \mathbf{y}_2),\ f_t = f(\mathbf{y}_t),$$

and the score function as:

$$\frac{\partial \log f_t}{\partial \theta_1} = \left(\frac{\partial \log f_t}{\partial c}, \frac{\partial \log f_t}{\partial \psi}, \frac{\partial \log f_t}{\partial b_2}, \frac{\partial \log f_t}{\partial \rho}, \frac{\partial \log f_t}{\partial b_1} \right).$$

Note that $\log f(\mathbf{y}_t|\mathbf{y}_{1:t-1}) = \log f(\mathbf{y}_{1:t}) - \log f(\mathbf{y}_{1:t-1})$. Define the conditional score function as:

$$Q_t = \frac{\partial \log f(\mathbf{y}_t|\mathbf{y}_{1:t-1})}{\partial \theta_1'} = \frac{\partial \log f(\mathbf{y}_{1:t})}{\partial \theta_1'} - \frac{\partial \log f(\mathbf{y}_{1:t-1})}{\partial \theta_1'}.$$

The Fisher information matrix can be expressed as:

$$\mathcal{I}(\theta) = \frac{1}{T} \sum_{t=1}^{T} Q_t Q_t',$$

and the full information score function is given as:

$$\mathcal{U}(\theta) = \frac{1}{T}\sum_{t=1}^{T} Q_t = \frac{1}{T}\frac{\partial \log f(\mathbf{y}_{1:T})}{\partial \theta_1'},$$

which is evatulated at $\theta_1 = \theta$.Then we have the following proposition:

Proposition: *Define the LM test statistic as:*

$$LM = T\mathcal{U}'(\hat{\theta}_0)\mathcal{I}(\hat{\theta}_0)^{-1}\mathcal{U}(\hat{\theta}_0) \xrightarrow{L} \chi^2(3),$$

where $\hat{\theta}_0$ is the maximum likelihood estimator of θ_0 under the null hypothesis. The asymptotic χ^2-distribution has three degrees of freedom corresponding to the three restrictions under the null hypothesis.

As $\frac{\partial \log f(y_{1:t})}{\partial \theta_1'} = \frac{1}{f(y_{1:t})}\frac{\partial f(y_{1:t})}{\partial \theta_1'}$, we focus on how to obtain $\frac{\partial f(y_{1:t})}{\partial \theta_1'}$. A problem is that the score function derived in the usual way diverges as the parameter c approaches 0. In order to solve this singularity, we use the method proposed in Chesher (1984). Denote $y_{1:t}$ as $\mathbf{y}$. The score functions with respect to each parameter under the null hypothesis are given below:

$$\begin{aligned}
\frac{\partial \log f(\mathbf{y})}{\partial c}|_{\mathbf{H}_0} &= trE_{\mathbf{h}_1|\mathbf{y}}(\mathbf{J}_t), \\
\frac{\partial \log f(\mathbf{y})}{\partial \psi}|_{\mathbf{H}_0} &= \frac{1}{2}\mathbf{1}_{1\times t}\mathbf{V}_\rho^{1/2}\mathbf{Z}_\rho E_{\mathbf{h}_1|\mathbf{y}}[\mathbf{h}_1] - \frac{1}{2}tr\left[\mathbf{Y}_2\mathbf{V}_\rho^{1/2}\mathbf{Z}_\rho E_{\mathbf{h}_1|\mathbf{y}}[\mathbf{h}_1\exp(-\mathbf{h}_1')]\right], \\
\frac{\partial \log f(\mathbf{y})}{\partial b_2}|_{\mathbf{H}_0} &= -\frac{1}{2b_1}\mathbf{1}_{1\times t}E_{\mathbf{h}_1|\mathbf{y}}[\mathbf{h}_1] + \frac{1}{2b_1}tr\left[\mathbf{y}_2^{2'}E_{\mathbf{h}_1|\mathbf{y}}[\exp(-\mathbf{h}_1)\circ\mathbf{h}_1]\right], \\
\frac{\partial \log f(\mathbf{y})}{\partial \rho}|_{\mathbf{H}_0} &= -\frac{\partial \log f(\mathbf{y})}{\partial \psi}|_{\mathbf{H}_0} + \left[-\frac{\rho}{1-\rho^2} - \frac{1}{2}b_1^{-2}tr\left(\frac{\partial \mathbf{V}_\rho^{-1}}{\partial \rho}E_{\mathbf{h}_1|\mathbf{y}}\left(\mathbf{h}_1\mathbf{h}_1'\right)\right)\right], \\
\frac{\partial \log f(\mathbf{y})}{\partial b_1}|_{\mathbf{H}_0} &= -\frac{\partial \log f(\mathbf{y})}{\partial b_2}|_{\mathbf{H}_0} - \frac{t}{b_1} + \frac{1}{b_1^3}tr\left(\mathbf{V}_\rho^{-1}E_{\mathbf{h}_1|\mathbf{y}}\left(\mathbf{h}_1\mathbf{h}_1'\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{J}_t &= \frac{1}{8}\{-2\times\mathbf{1}_{1\times t}\mathbf{V}_\rho\mathbf{Y}_2\exp(-\mathbf{h}_1) + \mathbf{1}_{t\times t}\mathbf{V}_\rho \\
&+ \mathbf{V}_\rho\mathbf{Y}_2\exp(-\mathbf{h}_1)\exp(-\mathbf{h}_1')\mathbf{Y}_2 - 2\mathbf{V}_\rho\mathbf{Y}_2\mathbf{H}_1^{-1}\}.
\end{aligned}$$

Further details are given in the Appendix.

5 Monte Carlo Experiments

In order to confirm that the proposed new test statistic is asymptotically distributed as $\chi^2(3)$ under the null, and whether it has power to reject a false null hypothesis, we conduct two Monte Carlo experiments, as given below.

5.1 Asymptotic distribution

This experiment is to generate samples drawn under the null hypothesis, H_0, calculates the new test statistic, and obtains the empirical distribution of the test statistic.
In this test, the significance level corresponds to the probability of the rejection region for the upper-tailed distribution of $\chi^2(3)$, so the experiment calculates the rejection rate that is larger than the theoretical critical value. Using the calculated statistics, we obtain the empirical distribution of the statistic and use kernel estimation or a simple histogram to show that it follows the asymptotic $\chi^2(3)$ distribution.
First, we generate samples drawn from different null hypotheses, particularly for different values of the parameter, ψ, which is the autocorrelation coefficient of the state variable. The rejection rates correspond to different critical values, and are shown in Table 1. The parameter vector, θ, follows the same definition as in the previous section, namely $\theta_0 = (0, \psi, b_2, \psi, b_2)$. As the time length T increases, the rejection rate converges to the theoretical significance level.

Table 1: Rejection rates under the null hypothesis

Sampling from H_0					Rejection rates					
					T=100		T=200		T=500	
c	ψ	b_2	ψ	b_2	5%	1%	5%	1%	5%	1%
0	0.7	1	0.7	1	10.2%	4.3%	8.3%	2.7%	7.1%	1.3%
0	0.9	1	0.9	1	21.0%	9.5%	13.1%	4.8%	6.3%	1.7%
0	0.95	0.45	0.95	0.45	22.5%	10.2%	14.7%	5.6%	7.4%	1.6%

From the table, we can see that the rejection rate converges sufficiently well when the time length is 500, which suggests that we should use data with at least 500 observations in practice. The histogram of the samples statistics obtained when $T = 500$ is given in the Appendix.

5.2 Statistical power

In comparison with the previous experiments, we generate data drawn from the alternative hypothesis, H_1, and calculate the rejection rates to see whether the statistic has power to reject a false null hypothesis. The parameter vector under H_1 shifts from the parameter vector under H_0. The Monte Carlo results are shown in Table 2, where the vector of parameters under the null hypothesis is given as $\theta_0 = (0, 0.7, 0.32, 0.7, 0.32)$.
Given the accurate finite sample rejection rates, it was felt reasonable to perform the Monte Carlo simulations only 100 times.

Table 2: Rejection rates under the alternative hypothesis

Sampling from H_1					Rejection rates	
					T=500	
c	ψ	b_2	ρ	b_1	5%	1%
0.32	0.7	0.32	0.7	0.32	28%	13%
0.45	0.7	0.32	0.7	0.32	72%	43%
0	0.7	0.25	0.7	0.32	24%	7%
0	0.7	0.19	0.7	0.32	51%	28%
0	0.5	0.32	0.7	0.32	14%	7%
0	0.9	0.32	0.7	0.32	86%	72%

6 Empirical Analysis

6.1 Data adjustment

Before using the LM test statistic, it is worth recalling that the error terms (e_{1t}, e_{2t}) in the measurement equation are mutually independent. However, as the usual situation is that they are contemporaneously correlated, we need to adjust the data to eliminate the correlation between y_{1t} and y_{2t}. Instead of using the original data, y_{1t} and y_{2t}, we use a linear combination of y_{1t} and y_{2t}. A similar approach is used in Engle and Kozicki (1993). We illustrate the reason with the linear transformations, as follows.

First, under the null hypothesis, the measurement equation can be written simply as:

$$\begin{pmatrix} y''_{1t} \\ y''_{2t} \end{pmatrix} = \exp\left(\frac{h_1}{2}\right)\begin{pmatrix} a_1 & 0 \\ a_2 & a_2 \end{pmatrix}\begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix}.$$

It is easy to see that any linear operator applied to (y''_{1t}, y''_{2t}) does not change the state part, $\exp(\frac{h_1}{2})$, of the equation. If the null hypothesis is true, with linear transformation, we only alter the structure of the measurement noise, and the state part $\exp(\frac{h_1}{2})$ remains the same after the data adjustment.

Second, if the original data are drawn under the alternative hypothesis:

$$\begin{pmatrix} y''_{1t} \\ y''_{2t} \end{pmatrix} = \begin{pmatrix} \exp(\frac{h_1}{2}) & 0 \\ 0 & \exp(\frac{h_2}{2}) \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix}\begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix}, \quad (4)$$

then any linear transformation between (y''_{1t}, y''_{2t}) will retain the two "features" $\exp(\frac{h_i}{2})$, $i = 1, 2$, where the word "feature" is used in Engle and Kozicki (1993).

Finally, any linear combination between $(y_{1t}^{''}, y_{2t}^{''})$ has its own significant meaning in empirical finance. Notice that the original data, $(y_{1t}^{''}, y_{2t}^{''})$, denote the difference in the log-price, namely the financial returns of the assets:

$$\begin{aligned} y_{1t}^{''} &= \log(p_{1t}) - \log(p_{1t-1}), \\ y_{2t}^{''} &= \log(p_{2t}) - \log(p_{2t-1}). \end{aligned}$$

For example, consider two assets, S&P 500 and Nikkei 225, for which the linear combination of the return is the returns on the portfolio which shares the weights between S&P 500 and Nikkei 225. In this situation, we can also use the model to analyse two new assets which always contain two "features", which makes our test statistic useful for empirical analysis.

We need two steps to adjust the data, which are given as:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = a_0^{-1} \begin{pmatrix} y_{1t}^{'} \\ y_{2t}^{'} \end{pmatrix} = a_0^{-1} \Lambda^{-\frac{1}{2}} \begin{pmatrix} y_{1t}^{''} \\ y_{2t}^{''} \end{pmatrix}, \tag{5}$$

where estimation of Λ and a_0 are given by:

$$\hat{\Lambda} = \frac{1}{T} \begin{pmatrix} \sum y_{1t}^{''2} & \sum y_{1t}^{''} y_{2t}^{''} \\ \sum y_{1t}^{''} y_{2t}^{''} & \sum y_{2t}^{''2} \end{pmatrix}, \tag{6}$$

$$\hat{a_0} = \begin{pmatrix} \exp\left\{\left[\frac{1}{T}\left(\sum \log y_{1t}^{'2}\right) + 1.27\right]/2\right\} & 0 \\ 0 & \exp\left\{\left[\frac{1}{T}\left(\sum \log y_{2t}^{'2}\right) + 1.27\right]/2\right\} \end{pmatrix}. \tag{7}$$

If:

$$a_0^{-1} \Lambda^{-\frac{1}{2}} \xrightarrow{p} \begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix}^{-1}, \tag{8}$$

we can obtain a new data set, (y_{1t}, y_{2t}), that has the same distribution compared with the model that is used in the new test statistic, with the same asymptotic null distribution. Further adjustments are shown in the Appendix.

6.2 Empirical result

Using the proposed new statistical test in the previous section, we examine the relationship between different stock markets, and also investigate the exchange rate movements over different time periods.

6.2.1 Analysis of stock markets

First, we investigate whether there exists a common factor of volatility between different stock market indices. The data we use for analysis come from Yahoo finance, and we use the Adjusted-Close price p_t. The stock market indices list is given below.

Table 3: Stock market indices

Index number	Stock Market	Symbol	Country/Region
1	Dow Jones Industrial Average	**DOW**	United State
2	FTSE Index	**FTSE**	Unite Kingdom
3	DAX Index	**DAX**	Germany
4	Shanghai Composite Index	**SSCI**	China
5	Nikkei 225 Stock Average Index	**NIKKEI**	Japan
6	Hang Seng Index	**HSI**	Hong Kong
7	Straits Times Index	**STI**	Singapore
8	All Ordinaries Index	**AORD**	Australia

We obtained daily data from 1 January 2011 to 30 December 2014 and separated them into two sets to check the performance in different years. The test needs a combination of two indices, so there are 28 pairs. We excluded data whenever there were closed-market days in one market, such as holidays.

The data adjustment follows the two steps given in the previous section. The parameter estimates are shown in the Appendix as Tables 9,10,11,12, and the test outcomes for different pairs are shown in Tables 4 and 5.

Table 4: LM statistics between markets from 2011 to 2012

y_1 \ y_2	**DOW**	**FTSE**	**DAX**	**SSCI**	**NIKKEI**	**HSI**	**STI**	**AORD**
DOW		9.1*	9.19*	21.48**	24.22**	5	10.44*	4.07
FTSE	5.14		1.39	20.33**	25.33**	3.66	3.74	1.07
DAX	5.96	8.7*		17.84**	30.19**	2.52	6.22	6.82
SSCI	21.07**	7.63	15.72**		18.07**	4.7	7.92*	8.46*
NIKKEI	22.15**	8.88*	29.5**	15.23**		2.34	10.23*	5.22
HSI	4.9	2.73	5.88	4.14	40.52**		2.87	4.69
STI	7.69	2.99	7.03	3.17	34.31**	18.12**		1.98
AORD	2.46	5.56	12.88**	5.19	69.35**	15.88**	2.68	

Note: * significant at 5% level, ** significant at 1% level.

As can be seen from the tables, even for different time periods, the group that contains the FTSE, STI and AORD stock markets share the same volatility factor. Stock markets in the

Table 5: LM statistics between markets from 2013 to 2014

y_1 \ y_2	**DOW**	**FTSE**	**DAX**	**SSCI**	**NIKKEI**	**HSI**	**STI**	**AORD**
DOW		0.39	8.05*	11.83**	15.43**	9.24*	4.97	5.21
FTSE	15.79**		9.9*	2.28	4.76	4.17	1.92	2.68
DAX	4.34	7.52		10.64*	21.47**	3.31	7.83*	10.22*
SSCI	11.53**	2.14	10.72*		3.39	12.84**	3.79	6.11
NIKKEI	15.05**	5.68	25.2**	2.58		9.92*	9.78*	9.88*
HSI	7.88*	1.54	7.13	7.47	12.39**		19.45**	2.58
STI	11.42**	1.91	23.23**	7.08	7.13	8.32*		1.63
AORD	6.11	0.8	8.03*	2.31	5.71	3.99	4.13	

Note: * significant at 5% level, ** significant at 1% level.

Asian region (such as China and Japan) appear to have a unique factor compared with other regions during 2011 and 2012. However, in 2013 and 2014, there appear to be more groups that share the same factor, namely:

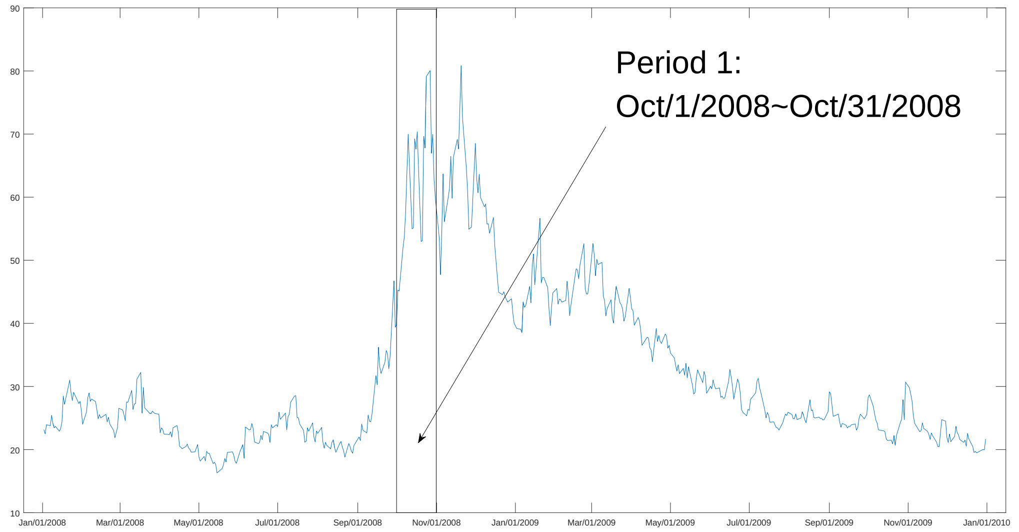
Group 1 : FTSE, SSCI, NIKKEI
Group 2 : FTSE, HSI, AORD
Group 2 : FTSE, STI, AORD

6.2.2 Analysis of exchange rates

We also use foreign exchange rates to evaluate the performance during extreme situations, especially when volatility is higher than usual. Comparing the performance for two time periods, namely the global financial crisis and normal times, we focus on the rates that are aggregated from the table instead of a single result between only two currency pairs.

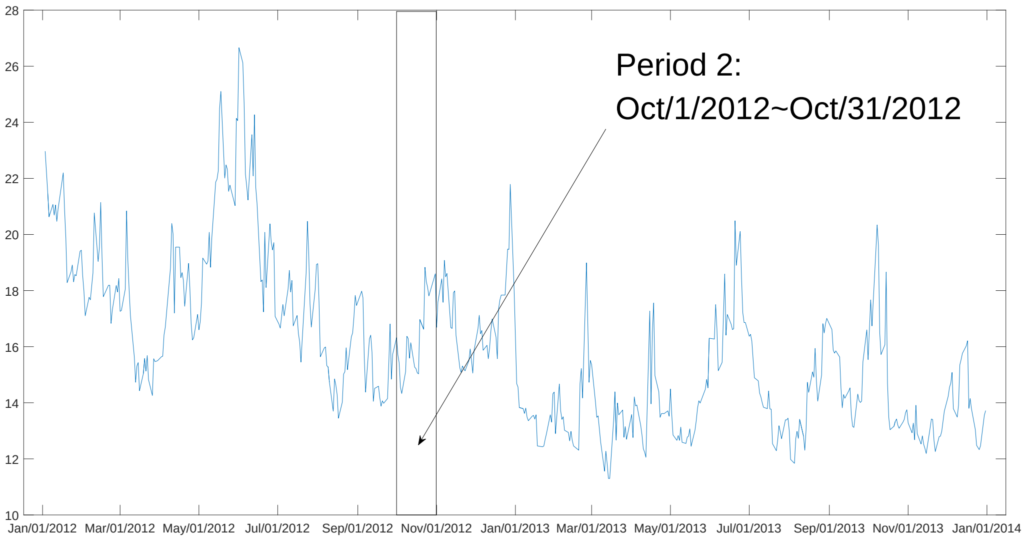
First, we define two time periods representing the financial crisis and normal times. We use the Chicago Board Options Exchange (CBOE) Volatility Index (VIX) as an indicator to detect when volatility is high. It is easy to see that volatility changed from the historical chart (see Figures 1 and 2). We choose Period 1: Oct/1/2008 $\sim$ Oct/31/2008 as the financial crisis, and Period 2: Oct/1/2012 $\sim$ Oct/31/2012 as normal times.

Figure 1: VIX during global financial crisis (2008)



Note: VIX data are downloaded from Yahoo Finance.

Figure 2: VIX during normal times (2012)



Note: VIX data are downloaded from Yahoo Finance.

Second, we use 6 major currency pairs that are traded widely. The pairs used are listed in Table 6, and all contain USD, so that all currencies are priced in USD.

We obtained hourly data for a month, which means roughly 500 data series. The estimated parameters are listed in the Appendix as Tables 13,14,15,16, and the statistical results are

Table 6: Currency Pairs list

Currency Pairs	EUR/USD	USD/JPY	GBP/USD	AUD/USD	USD/CHF	USD/CAD

shown in Tables 7 and 8.

Table 7: LM statistics for exchange rates during Period 1

y_1 \ y_2	**EURUSD**	**USDJPY**	**GBPUSD**	**AUDUSD**	**USDCHF**	**USDCAD**
EURUSD		23.73**	12.06**	34.09**	10.43*	17.89**
USDJPY	35.45**		17.28**	15.62**	28.23**	54.5**
GBPUSD	14.49**	14.34**		53.78**	18.74**	22.03**
AUDUSD	37.32**	33.16**	30.01**		25.47**	28.3**
USDCHF	24.47**	33.96**	43.9**	39.49**		23.63**
USDCAD	18.31**	37.44**	15.14**	19.02**	14.64**	

Note: * significant at 5% level, ** significant at 1% level. The exchange rate is downloaded from FXDD's historical data.

Table 8: LM statistics for exchange rates during Period 2

y_1 \ y_2	**EURUSD**	**USDJPY**	**GBPUSD**	**AUDUSD**	**USDCHF**	**USDCAD**
EURUSD		11.27*	5.17	7.56	6.61	25.69**
USDJPY	15.88**		28.67**	12.97**	16.73**	14.86**
GBPUSD	2.15	23.74**		20.35**	5.39	8.38*
AUDUSD	28.07**	18.32**	26.51**		14.97**	22.86**
USDCHF	3.22	9.53*	4.64	6.9		17.66**
USDCAD	5.5	6.47	4.83	5.82	4.41	

Note: * significant at 5% level, ** significant at 1% level. The exchange rate is downloaded from FXDD's historical data.

As can be seen from Tables 7 and 8, during the financial crisis volatility is larger than usual, so it is difficult to find a single asset to hedge volatility. The accepted rate is given as 0 in

Table 7. Conversely, it is easy to find currency pairs that potentially share the same volatility factor during normal times. The accepted rate is given as 43.3% in Table 8.

7 Conclusion

In this paper, we considered whether financial returns has a common volatility process in the framework of stochastic volatility models, and proposed a Lagrange Multiplier test statistic for the null hypothesis that the volatility changes of a bivariate series are perfectly correlated. It is useful in investigating the correlation between different markets, even for frictionless cross-market hedging.

In the empirical analysis of stock markets, we found some groups that potentially share common time-varying volatility, especially markets for the United Kingdom, Singapore and Australia. We also investigated the correlations between different major currencies when big events, such as financial crises, occurred. The empirical analysis suggested that, during high volatility periods, it is more difficult to find a common factor between currencies, compared with low volatility periods, so that it is harder to hedge with different currencies.

However, the approach adopted in the paper it is the simplest case of a multiple stochastic volatility model. The extension to a multi-factor model, even stochastic volatility with a fat-tailed distribution of the test, is left for further research.

Appendices

A Likelihood Function

In order to express the transition equation (2) in matrix form, we express the log volatilities and disturbance terms used in (1) and (2) in vector form, as follows:

$$\mathbf{h}_1 = (h_{11}, \ldots, h_{1t})', \ \mathbf{h}_2 = (h_{21}, \ldots, h_{2t})', \tag{9}$$

$$\mathbf{u}_1 = (u_{11}, \ldots, u_{1t})', \ \mathbf{u}_2 = (u_{21}, \ldots, u_{2t})', \tag{10}$$

$$\mathbf{e}_1 = (e_{11}, \ldots, e_{1t})', \ \mathbf{e}_2 = (e_{21}, \ldots, e_{2t})'. \tag{11}$$

Then the transition equation (2) is expressed as:

$$\mathbf{h}_1 = \mathbf{V}_\rho^{1/2}(b_1\mathbf{u}_1), \ \mathbf{h}_2 = \mathbf{V}_\psi^{1/2}(b_2\mathbf{u}_1 + \sqrt{c}\mathbf{u}_2) = \mathbf{V}_\psi^{1/2}(\mathbf{V}_\rho^{-1/2}\mathbf{h}_1 b_2/b_1 + \sqrt{c}\mathbf{u}_2), \tag{12}$$

where $\mathbf{V}_\rho$ and $\mathbf{V}_\psi$ are the covariance matrices of the autoregressive processes of order one, $\mathbf{h}_1$ and $\mathbf{h}_2$, respectively, and $\mathbf{V}_\rho^{1/2}$ and $\mathbf{V}_\psi^{1/2}$ are defined by their Cholesky decomposition:

$$\mathbf{V}_\rho = (\mathbf{V}_\rho^{1/2})(\mathbf{V}_\rho^{1/2})', \ \mathbf{V}_\psi = (\mathbf{V}_\psi^{1/2})(\mathbf{V}_\psi^{1/2})',$$

where

$$\mathbf{V}_\psi^{1/2} = \begin{pmatrix} 1/\sqrt{1-\psi^2} & 0 & \ldots & 0 & 0 \\ \psi/\sqrt{1-\psi^2} & 1 & \ldots & 0 & 0 \\ \psi^2/\sqrt{1-\psi^2} & \psi & \ldots & 0 & 0 \\ \vdots & & & & \\ \psi^{n-1}/\sqrt{1-\psi^2} & \psi^{n-2} & \ldots & \psi & 1 \end{pmatrix}, \ \mathbf{V}_\rho^{1/2} = \begin{pmatrix} 1/\sqrt{1-\rho^2} & 0 & \ldots & 0 & 0 \\ \rho/\sqrt{1-\rho^2} & 1 & \ldots & 0 & 0 \\ \rho^2/\sqrt{1-\rho^2} & \rho & \ldots & 0 & 0 \\ \vdots & & & & \\ \rho^{n-1}/\sqrt{1-\rho^2} & \rho^{n-2} & \ldots & \rho & 1 \end{pmatrix}, \tag{13}$$

It is easy to see that their inverses are decomposed as $\mathbf{V}_\psi{}^{-1} = (\mathbf{V}_\psi^{-\frac{1}{2}})'\mathbf{V}_\psi^{-\frac{1}{2}}, \ \mathbf{V}_\rho{}^{-1} = (\mathbf{V}_\rho^{-\frac{1}{2}})'\mathbf{V}_\rho^{-\frac{1}{2}}$, where:

$$\mathbf{V}_\psi^{-\frac{1}{2}} = \begin{pmatrix} \sqrt{1-\psi^2} & 0 & \ldots & 0 & 0 \\ -\psi & 1 & \ldots & 0 & 0 \\ 0 & -\psi & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \ldots & -\psi & 1 \end{pmatrix}, \ \mathbf{V}_\rho^{-\frac{1}{2}} = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & \ldots & 0 & 0 \\ -\rho & 1 & \ldots & 0 & 0 \\ 0 & -\rho & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \ldots & -\rho & 1 \end{pmatrix}. \tag{14}$$

Then the density functions of the transition and measurement equations of the model can be expressed as:

$$
\begin{aligned}
f(\mathbf{h}_1) &= \frac{1}{(2\pi)^{\frac{t}{2}} \left|\mathbf{V}_\rho^{1/2}\right| b_1^t} \exp\left(-\frac{1}{2} b_1^{-2} \mathbf{h}_1' \mathbf{V}_\rho^{-1} \mathbf{h}_1\right), && (15)\\
f(\mathbf{h}_2|\mathbf{h}_1) &= \frac{1}{(2\pi)^{\frac{t}{2}} \left|\mathbf{V}_\psi^{1/2}\right| (\sqrt{c})^t} \exp\left(-\frac{1}{2} \mathbf{u}_2' \mathbf{u}_2\right), && (16)\\
f(\mathbf{y}_1|\mathbf{h}_1) &= \frac{1}{(2\pi)^{\frac{t}{2}}} \frac{1}{\left|\mathbf{H}_1^{1/2}\right|} \exp\left(-\frac{1}{2} \mathbf{y}_1' \mathbf{H}_1^{-1} \mathbf{y}_1\right), && (17)\\
f(\mathbf{y}_2|\mathbf{h}_2) &= \frac{1}{(2\pi)^{\frac{t}{2}}} \frac{1}{\left|\mathbf{H}_2^{1/2}\right|} \exp\left(-\frac{1}{2} \mathbf{y}_2' \mathbf{H}_2^{-1} \mathbf{y}_2\right), && (18)
\end{aligned}
$$

where

$$
\mathbf{u}_2 = \left(\mathbf{V}_\psi^{-\frac{1}{2}} \mathbf{h}_2 - \mathbf{V}_\rho^{-\frac{1}{2}} \mathbf{h}_1 \frac{b_2}{b_1}\right) / \sqrt{c}, \tag{19}
$$

$$
\mathbf{H}_1 = diag(\exp(h_{11}), \ldots, \exp(h_{1t})),\ \mathbf{H}_2 = diag(\exp(h_{21}), \ldots, \exp(h_{2t})). \tag{20}
$$

Then we can rewrite the likelihood function, given by:

$$
f(\mathbf{y}_1, \mathbf{y}_2) = \int f(\mathbf{y}_2|\mathbf{h}_2) f(\mathbf{y}_1|\mathbf{h}_1) f(\mathbf{h}_2|\mathbf{h}_1) f(\mathbf{h}_1) d\mathbf{h}_2 d\mathbf{h}_1, \tag{21}
$$

as

$$
f(\mathbf{y}_1,\, \mathbf{y}_2) = \int f(\mathbf{y}_2|\mathbf{u}_2, \mathbf{h}_1) f(\mathbf{y}_1|\mathbf{h}_1) f(\mathbf{u}_2|\mathbf{h}_1) f(\mathbf{h}_1) d\mathbf{u}_2 d\mathbf{h}_1, \tag{22}
$$

where

$$
f(\mathbf{u}_2|\mathbf{h}_1) = \frac{1}{(2\pi)^{\frac{t}{2}}} \exp\left(-\frac{1}{2} \mathbf{u}_2' \mathbf{u}_2\right), \tag{23}
$$

in terms of $\mathbf{u}_2$, instead of $\mathbf{h}_2$, by the variable transformation given in (19).

B Score function with respect to c

We obtain the score function with respct to c from (22) as:

$$
\frac{\partial f(\mathbf{y})}{\partial c} = \int \frac{\partial f(\mathbf{y}_2|\mathbf{u}_2, \mathbf{h}_1)}{\partial c} f(\mathbf{y}_1|\mathbf{h}_1) f(\mathbf{u}_2|\mathbf{h}_1) f(\mathbf{h}_1) d\mathbf{u}_2 d\mathbf{h}_1, \tag{24}
$$

as the variance parameter c appears only in:

$$
f(\mathbf{y}_2|\mathbf{h}_1, \mathbf{u}_2) = \frac{1}{(2\pi)^{\frac{t}{2}}} \frac{1}{\left|\mathbf{H}_2^{1/2}\right|} \exp\left(-\frac{1}{2} \mathbf{y}_2' \mathbf{H}_2^{-1} \mathbf{y}_2\right)
$$

through $\mathbf{h}_2$ in $\mathbf{H}_2 = diag(\exp(\mathbf{h}_2))$ because, from (12), we have:

$$
\mathbf{h}_2 = \mathbf{V}_\psi^{-\frac{1}{2}} \left(\sqrt{c}\mathbf{u}_2 + \mathbf{V}_\rho^{-\frac{1}{2}} \frac{b_2}{b_1} \mathbf{h}_1\right). \tag{25}
$$

Then we obtain the derivative of $f(\mathbf{y}_2|\mathbf{h}_1, \mathbf{u}_2,)$ with respect to c, as follows. First, noting (25), we define:

$$f(\mathbf{y}_2|\mathbf{h}_1, \mathbf{u}_2,) = D_t F_t, \tag{26}$$

where

$$\begin{aligned} D_t &= \frac{1}{|\mathbf{H}_2|^{1/2}} = \exp\left(-\frac{1}{2}\mathbf{1}_{1\times t}\mathbf{h}_2\right) \\ &= \exp\left(-\frac{1}{2}\mathbf{1}_{1\times t}\mathbf{V}_\psi^{1/2}\left(\sqrt{c}\mathbf{u}_2 + \mathbf{V}_\psi^{-\frac{1}{2}}\frac{b_2}{b_1}\mathbf{h}_1\right)\right), \qquad (27) \\ F_t &= \exp\left(-\frac{1}{2}\mathbf{y}_2^{'}\mathbf{H}_2^{-1}\mathbf{y}_2\right) \\ &= \exp\left\{-\frac{1}{2}(\exp(-\mathbf{h}_2))'\mathbf{y}_2^2\right\}, \qquad (28) \end{aligned}$$

and, for notational convenience, we define:

$$\exp(-\mathbf{h}_2) = (\exp(-h_{21}), \ldots, \exp(-h_{2t}))',\ \mathbf{y}_2^2 = (y_{21}^2, y_{22}^2, \ldots, y_{2t}^2)',$$

and $\mathbf{h}_2$ denotes a function of $\mathbf{u}_2$ as the abbreviation of equation (25).

Then, from (24), we have:

$$\begin{aligned} B_t &= \lim_{c\to 0}\frac{\partial f(\mathbf{y})}{\partial c} \qquad (29) \\ &= \lim_{c\to 0}\int (other\ terms)(F_t\frac{\partial D_t}{\partial c} + D_t\frac{\partial F_t}{\partial c})d\mathbf{u}_2 d\mathbf{h}_1 \qquad (30) \\ &= \lim_{c\to 0}\frac{\sqrt{c}\int (other\ terms)(F_t\ M_{1t} + D_t\ M_{2t})d\mathbf{u}_2 d\mathbf{h}_1}{c}, \qquad (31) \end{aligned}$$

where we define:

$$M_{1t} = \frac{\partial D_t}{\partial c}\frac{1}{\sqrt{c}},\ M_{2t} = \frac{\partial F_t}{\partial c}\frac{1}{\sqrt{c}}. \tag{32}$$

We need $\sqrt{c}$ in the denominator of (32) as:

$$\begin{aligned} \frac{\partial D_t}{\partial c} &= -\frac{1}{2}D_t\ \frac{1}{2\sqrt{c}}\mathbf{1}_{1\times t}\mathbf{V}_\psi^{1/2}\mathbf{u}_2 \qquad (33) \\ &= -\frac{1}{4\sqrt{c}}D_t\ \mathbf{1}_{1\times t}\mathbf{V}_\psi^{1/2}\mathbf{u}_2, \qquad (34) \\ \frac{\partial F_t}{\partial c} &= -\frac{1}{2}F_t\ \left(\frac{\partial}{\partial c}\exp(-\mathbf{h}_2^{'})\mathbf{y}_2^2\right) \qquad (35) \\ &= -\frac{1}{2}F_t\ \left(-\frac{1}{2\sqrt{c}}\mathbf{u}_2^{'}\mathbf{V}_\psi^{1/2^{'}}\mathbf{H}_2^{-1}\mathbf{y}_2^2\right) \qquad (36) \\ &= \frac{1}{4\sqrt{c}}F_t\ G_t, \qquad (37) \\ G_t &= \mathbf{u}_2^{'}\mathbf{V}_\psi^{1/2^{'}}\mathbf{H}_2^{-1}\mathbf{y}_2^2, \qquad (38) \end{aligned}$$

as

$$\frac{\partial \mathbf{h}_2}{\partial c} = \frac{1}{2\sqrt{c}}\mathbf{V}_\psi^{1/2}\mathbf{u}_2, \frac{\partial \exp(-\mathbf{h}_2)}{\partial c} = -\frac{1}{2\sqrt{c}}\mathbf{H}_2^{-1}\mathbf{V}_\psi^{1/2}\mathbf{u}_2.$$

Note that the denominators of the derivatives (34) and (37) contain c, which converges to zero, and hence is intractable by conventional methods. It is convenient to use the method proposed by Chesher (1984). First, applying L'Hopital's rule to (31) with respect to c , we obtain:

$$B_t = \frac{1}{2}B_t + \lim_{c\to 0}\sqrt{c}\frac{\partial}{\partial c}\int (other\, terms)(F_t\ M_{1t} + D_t\ M_{2t})d\mathbf{u}_2 d\mathbf{h}_1. \tag{39}$$

Comparing both sides of equation (39), we have:

$$B_t = 2\lim_{c\to 0}\sqrt{c}\int (other\, terms)\left(\frac{\partial F_t}{\partial c}\ M_{1t} + \frac{\partial D_t}{\partial c}\ M_{2t} + F_t\frac{\partial M_{1t}}{\partial c} + D_t\frac{\partial M_{2t}}{\partial c}\right)d\mathbf{u}_2 d\mathbf{h}_1 \tag{40}$$

$$= 2\lim_{c\to 0}\sqrt{c}\int (other\, terms)\left(2M_{1t}\ M_{2t} + F_t\frac{\partial M_{1t}}{\partial c} + D_t\frac{\partial M_{2t}}{\partial c}\right)d\mathbf{u}_2 d\mathbf{h}_1. \tag{41}$$

Defining $\mathbf{Y}_2 = diag(\mathbf{y}_2^2)$, the terms in the integrand can be expressed as follows:

$$\begin{aligned}
M_{1t}\ M_{2t} &= \sqrt{c}\frac{\partial D_t}{\partial c}\sqrt{c}\frac{\partial F_t}{\partial c}\\
&= \left(-\frac{1}{4}D_t\ \mathbf{1}_{1\times t}\mathbf{V}_\psi^{1/2}\mathbf{u}_2\right)\left(\frac{1}{4}F_t\ \mathbf{u}_2^{'}\mathbf{V}_\psi^{1/2'}\mathbf{Y}_2\exp(-\mathbf{h}_2)\right)\\
&= -\frac{1}{16}D_t F_t\left(\mathbf{1}_{1\times t}\mathbf{V}_\psi^{1/2}\mathbf{u}_2\mathbf{u}_2^{'}\mathbf{V}_\psi^{1/2'}\mathbf{Y}_2\exp(-\mathbf{h}_2)\right),\\
\frac{\partial M_{1t}}{\partial c} &= -\frac{1}{4}\frac{\partial D_t}{\partial c}\mathbf{1}_{1\times t}\mathbf{V}_\psi^{1/2}\mathbf{u}_2\\
&= -\frac{1}{4}\left(-\frac{\sqrt{c}}{4c}\right)D_t\ \mathbf{1}_{1\times t}\mathbf{V}_\psi^{1/2}\mathbf{u}_2\mathbf{u}_2^{'}\mathbf{V}_\psi^{1/2'}\mathbf{1}_{t\times 1}\\
&= \frac{1}{16\sqrt{c}}D_t\ tr\left(\mathbf{1}_{1\times t}\mathbf{V}_\psi^{1/2}\mathbf{u}_2\mathbf{u}_2^{'}\mathbf{V}_\psi^{1/2'}\mathbf{1}_{t\times 1}\right)\\
&= \frac{1}{16\sqrt{c}}D_t\ tr(\mathbf{1}_{t\times t}\mathbf{V}_\psi^{1/2}\mathbf{u}_2\mathbf{u}_2^{'}\mathbf{V}_\psi^{1/2'}),\\
\frac{\partial M_{2t}}{\partial c} &= \frac{1}{4}\frac{\partial F_t}{\partial c}G_t + \frac{1}{4}F_t\frac{\partial G_t}{\partial c}\\
&= \frac{1}{16\sqrt{c}}F_t G_t^2 + \frac{1}{4}F_t\frac{\partial G_t}{\partial c},\\
\frac{\partial G_t}{\partial c} &= \mathbf{u}_2^{'}\mathbf{V}_\psi^{1/2'}\mathbf{Y}_2\frac{\partial \exp(-\mathbf{h}_2)}{\partial c}\\
&= -\frac{1}{2\sqrt{c}}\mathbf{u}_2^{'}\mathbf{V}_\psi^{1/2'}\mathbf{Y}_2\mathbf{H}_2^{-1}\mathbf{V}_\psi^{1/2}\mathbf{u}_2\\
&= -\frac{1}{2\sqrt{c}}tr\left(\mathbf{u}_2^{'}\mathbf{V}_\psi^{1/2'}\mathbf{Y}_2\mathbf{H}_2^{-1}\mathbf{V}_\psi^{1/2}\mathbf{u}_2\right)\\
&= -\frac{1}{2\sqrt{c}}tr\left(\mathbf{V}_\psi^{1/2'}\mathbf{Y}_2\mathbf{H}_2^{-1}\mathbf{V}_\psi^{1/2}\mathbf{u}_2\mathbf{u}_2^{'}\right),
\end{aligned}$$

$$
\begin{aligned}
G_t^2 &= \mathbf{u}_2^{'}\mathbf{V}_\psi^{1/2'}\mathbf{Y}_2\exp(-\mathbf{h}_2)\exp(-\mathbf{h}_2^{'})\mathbf{Y}_2\mathbf{V}_\psi^{1/2}\mathbf{u}_2\\
&= tr\left(\mathbf{u}_2^{'}\mathbf{V}_\psi^{1/2'}\mathbf{Y}_2\exp(-\mathbf{h}_2)\exp(-\mathbf{h}_2^{'})\mathbf{Y}_2\mathbf{V}_\psi^{1/2}\mathbf{u}_2\right)\\
&= tr\left(\mathbf{V}_\psi^{1/2'}\mathbf{Y}_2\exp(-\mathbf{h}_2)\exp(-\mathbf{h}_2^{'})\mathbf{Y}_2\mathbf{V}_\psi^{1/2}\mathbf{u}_2\mathbf{u}_2^{'}\right).
\end{aligned}
$$

Then we have:

$$
\begin{aligned}
B_t &= \frac{1}{8}\lim_{c\to 0}\int f(\mathbf{y}_1|\mathbf{h}_1)\frac{1}{(2\pi)^{\frac{t}{2}}}D_tF_t\\
&\left\{-2tr\left(\mathbf{1}_{1\times t}\mathbf{V}_\psi^{1/2}\mathbf{u}_2\mathbf{u}_2^{'}\mathbf{V}_\psi^{1/2'}\mathbf{Y}_2\exp(-\mathbf{h}_2)\right)\right.\\
&+ tr\left(\mathbf{1}_{t\times t}\mathbf{V}_\psi^{1/2}\mathbf{u}_2\mathbf{u}_2^{'}\mathbf{V}_\psi^{1/2'}\right)\\
&+ tr\left(\mathbf{V}_\psi^{1/2'}\mathbf{Y}_2\exp(-\mathbf{h}_2)\exp(-\mathbf{h}_2^{'})\mathbf{Y}_2\mathbf{V}_\psi^{1/2}\mathbf{u}_2\mathbf{u}_2^{'}\right)\\
&- \left. 2tr\left(\mathbf{V}_\psi^{1/2'}\mathbf{Y}_2\mathbf{H}_2^{-1}\mathbf{V}_\psi^{1/2}\mathbf{u}_2\mathbf{u}_2^{'}\right)\right\} f(\mathbf{u}_2|\mathbf{h}_1)f(\mathbf{h}_1)d\mathbf{u}_2 d\mathbf{h}_1. \qquad (42)
\end{aligned}
$$

We can perform the integration with respect to $\mathbf{u}_2$ in (42) analytically. As $\mathbf{u}_2|\mathbf{h}_1$ follows the t-dimensitonal standard normal distribution:

$$
\int \mathbf{u}_2\mathbf{u}_2^{'}f(\mathbf{u}_2|\mathbf{h}_1)d\mathbf{u}_2 = \mathbf{I}_t, \qquad (43)
$$

under the null hypothesis $\mathbf{h}_2 = \mathbf{h}_1$ and $\psi = \rho$, equation (42) is expressed as:

$$
\begin{aligned}
B_t &= \frac{1}{8}\int f(\mathbf{y}_1|\mathbf{h}_1)\frac{1}{(2\pi)^{\frac{t}{2}}}D_tF_t\\
&\left\{-2tr\left(\mathbf{1}_{1\times t}\mathbf{V}_\rho^{1/2}\mathbf{V}_\rho^{1/2'}\mathbf{Y}_2\exp(-\mathbf{h}_1)\right)\right.\\
&+ tr\left(\mathbf{1}_{t\times t}\mathbf{V}_\rho^{1/2}\mathbf{V}_\rho^{1/2'}\right)\\
&+ tr\left(\mathbf{V}_\rho^{1/2'}\mathbf{Y}_2\exp(-\mathbf{h}_1)\exp(-\mathbf{h}_1^{'})\mathbf{Y}_2\mathbf{V}_\rho^{1/2}\right)\\
&- \left. 2tr\left(\mathbf{V}_\rho^{1/2'}\mathbf{Y}_2\mathbf{H}_1^{-1}\mathbf{V}_\rho^{1/2}\right)\right\} f(\mathbf{h}_1)d\mathbf{h}_1. \qquad (44)
\end{aligned}
$$

Noting that $\mathbf{V}_\rho = \mathbf{V}_\rho^{1/2}\mathbf{V}_\rho^{1/2'}$, and applying the cyclic property of the trace operator to simplify equation (44), we have:

$$
B_t = \left.\frac{\partial f(\mathbf{y})}{\partial c}\right|_{\mathbf{H}_0} = \int tr\mathbf{J}_t f(\mathbf{y},\mathbf{h}_1)d\mathbf{h}_1, \qquad (45)
$$

where

$$
\begin{aligned}
\mathbf{J}_t &= \frac{1}{8}\{-2\times\mathbf{1}_{1\times t}\mathbf{V}_\rho\mathbf{Y}_2\exp(-\mathbf{h}_1) + \mathbf{1}_{t\times t}\mathbf{V}_\rho \qquad (46)\\
&+ \mathbf{V}_\rho\mathbf{Y}_2\exp(-\mathbf{h}_1)\exp(-\mathbf{h}_1^{'})\mathbf{Y}_2 - 2\mathbf{V}_\rho\mathbf{Y}_2\mathbf{H}_1^{-1}\}. \qquad (47)
\end{aligned}
$$

It follows that:

$$\left.\frac{\partial \log f(\mathbf{y})}{\partial c}\right|_{\mathbf{H}_0} = \lim_{c\to 0}\frac{1}{f(\mathbf{y})}\frac{\partial f(\mathbf{y})}{\partial c} \tag{48}$$

$$= \int tr\mathbf{J}_t\frac{1}{f(\mathbf{y})}f(\mathbf{h}_1,\mathbf{y})d\mathbf{h}_1 \tag{49}$$

$$= tr\int \mathbf{J}_t f(\mathbf{h}_1|\mathbf{y})d\mathbf{h}_1 \tag{50}$$

$$= trE_{\mathbf{h}_1|\mathbf{y}}(\mathbf{J}_t), \tag{51}$$

as $f(\mathbf{h}_1|\mathbf{y}) = f(\mathbf{h}_1,\mathbf{y})/f(\mathbf{y})$. From (46), we have only to evaluate $E_{\mathbf{h}_1|\mathbf{y}}\left[\exp(-\mathbf{h}_1)\right]$ and

$$E_{\mathbf{h}_1|\mathbf{y}}\left[\exp(-\mathbf{h}_1)\exp(-\mathbf{h}_1)'\right]$$

to obtain the score function with respect to ψ. These expected values have no analytic expressions, so that they will need to be evaluated numerically.

C Score function with respect to ψ

In the log-likelihood function, ψ appears only in $f(\mathbf{y}_1|\mathbf{h}_1,\mathbf{u}_2) = D_tF_t$, as shown in (22) and (26). The partial derivative of the likelihood with respect to ψ can be expressed as :

$$\frac{\partial f(\mathbf{y})}{\partial \psi} = \int \left(\frac{\partial D_t}{\partial \psi}D_t^{-1} + \frac{\partial F_t}{\partial \psi}F_t^{-1}\right) f(\mathbf{y},\mathbf{u}_2,\mathbf{h}_1)d\mathbf{u}_2 d\mathbf{h}_1 \tag{52}$$

$$= \int \left(\frac{\partial D_t}{\partial \psi}D_t^{-1} + \frac{\partial F_t}{\partial \psi}F_t^{-1}\right) f(\mathbf{y},\mathbf{h}_1)d\mathbf{h}_1, \tag{53}$$

since, as will be seen later, $\mathbf{u}_2$ can be integrated out in $\left(\frac{\partial D_t}{\partial \psi}D_t^{-1} + \frac{\partial F_t}{\partial \psi}F_t^{-1}\right)$. Then we have:

$$\left.\frac{\partial \log f(\mathbf{y})}{\partial \psi}\right|_{\mathbf{H}_0} = \int \left(\frac{\partial D_t}{\partial \psi}D_t^{-1} + \frac{\partial F_t}{\partial \psi}F_t^{-1}\right) f(\mathbf{h}_1|\mathbf{y})d\mathbf{h}_1 = E_{\mathbf{h}_1|\mathbf{y}}\left(\frac{\partial D_t}{\partial \psi}D_t^{-1} + \frac{\partial F_t}{\partial \psi}F_t^{-1}\right), \tag{54}$$

as

$$f(\mathbf{h}_1|\mathbf{y}) = f(\mathbf{h}_1,\mathbf{y})/f(\mathbf{y}).$$

First, using the formula:

$$\frac{\partial \mathbf{V}_\psi^{1/2}}{\partial \psi} = -\mathbf{V}_\psi^{1/2}\mathbf{Z}_\psi\mathbf{V}_\psi^{1/2}, \tag{55}$$

where

$$\mathbf{Z}_\psi = \frac{\partial \mathbf{V}_\psi^{-1/2}}{\partial \psi},$$

note that

$$\frac{\partial D_t}{\partial \psi} = -\frac{1}{2}D_t\left[\mathbf{1}_{1\times t}\frac{\partial \mathbf{V}_{\psi}^{1/2}}{\partial \psi}\left(\sqrt{c}\mathbf{u}_2 + \mathbf{V}_{\rho}^{-\frac{1}{2}}\frac{b_2}{b_1}\mathbf{h}_1\right)\right] \quad (56)$$

$$= \frac{1}{2}D_t\left[\mathbf{1}_{1\times t}\mathbf{V}_{\psi}^{1/2}\mathbf{Z}_{\psi}\mathbf{V}_{\psi}^{1/2}\left(\sqrt{c}\mathbf{u}_2 + \mathbf{V}_{\rho}^{-\frac{1}{2}}\frac{b_2}{b_1}\mathbf{h}_1\right)\right], \quad (57)$$

$$\frac{\partial F_t}{\partial \psi} = -\frac{1}{2}F_t\,\frac{\partial}{\partial \psi}\left[\mathbf{y}_2^{2'}\exp(-\mathbf{h}_2)\right] \quad (58)$$

$$= \frac{1}{2}F_t\,\left[\mathbf{y}_2^{2'}\mathbf{H}_2^{-1}\frac{\partial \mathbf{V}_{\psi}^{1/2}}{\partial \psi}\left(\sqrt{c}\mathbf{u}_2 + \mathbf{V}_{\rho}^{-\frac{1}{2}}\frac{b_2}{b_1}\mathbf{h}_1\right)\right] \quad (59)$$

$$= -\frac{1}{2}F_t\,\left[\mathbf{y}_2^{2'}\mathbf{H}_2^{-1}\mathbf{V}_{\psi}^{1/2}\mathbf{Z}_{\psi}\mathbf{V}_{\psi}^{1/2}\left(\sqrt{c}\mathbf{u}_2 + \mathbf{V}_{\rho}^{-\frac{1}{2}}\frac{b_2}{b_1}\mathbf{h}_1\right)\right], \quad (60)$$

as we have:

$$\mathbf{h}_2 = \mathbf{V}_{\psi}^{1/2}\left(\sqrt{c}\mathbf{u}_2 + \mathbf{V}_{\rho}^{-\frac{1}{2}}\frac{b_2}{b_1}\mathbf{h}_1\right), \quad (61)$$

so that:

$$\frac{\partial}{\partial \psi}\mathbf{h}_2 = \frac{\partial \mathbf{V}_{\psi}^{1/2}}{\partial \psi}\left(\sqrt{c}\mathbf{u}_2 + \mathbf{V}_{\rho}^{-\frac{1}{2}}\frac{b_2}{b_1}\mathbf{h}_1\right), \quad (62)$$

$$D_t = \exp\left(-\frac{1}{2}\mathbf{1}_{1\times t}\mathbf{V}_{\psi}^{1/2}\left(\sqrt{c}\mathbf{u}_2 + \mathbf{V}_{\rho}^{-\frac{1}{2}}\frac{b_2}{b_1}\mathbf{h}_1\right)\right),$$

$$F_t = \exp\left\{-\frac{1}{2}(\exp(-\mathbf{h}_2))'\mathbf{y}_2^2\right\}.$$

We have used $-\mathbf{V}_{\psi}^{1/2}\mathbf{Z}_{\psi}\mathbf{V}_{\psi}^{1/2}$ rather than $\frac{\partial \mathbf{V}_{\psi}^{1/2}}{\partial \psi}$ because it is easy to generate computationally. Evaluating each term under the null hypothesis $c = 0$ and $b_1 = b_2$, we have:

$$\frac{\partial D_t}{\partial \psi}|_{\mathbf{H}_0} = \frac{1}{2}D_t\mathbf{1}_{1\times t}\mathbf{V}_{\psi}^{1/2}\mathbf{Z}_{\psi}\mathbf{h}_1, \quad (63)$$

$$\frac{\partial F_t}{\partial \psi}|_{\mathbf{H}_0} = -\frac{1}{2}F_t\;\mathbf{y}_2^{2'}\mathbf{H}_1^{-1}\mathbf{V}_{\psi}^{1/2}\mathbf{Z}_{\psi}\mathbf{h}_1 \quad (64)$$

$$= -\frac{1}{2}F_t\;tr\left[\exp(-\mathbf{h}_1^{'})\mathbf{Y}_2\mathbf{V}_{\psi}^{1/2}\mathbf{Z}_{\psi}\mathbf{h}_1\right] \quad (65)$$

$$= -\frac{1}{2}F_t\;tr\left[\mathbf{Y}_2\mathbf{V}_{\psi}^{1/2}\mathbf{Z}_{\psi}\mathbf{h}_1\exp(-\mathbf{h}_1^{'})\right],$$

using the identity:

$$\mathbf{y}_2^{2'}\mathbf{H}_1^{-1} = \exp(-\mathbf{h}_1^{'})\mathbf{Y}_2.$$

From (54), we have:

$$\left.\frac{\partial \log f(\mathbf{y})}{\partial \psi}\right|_{\mathbf{H}_0} = \frac{1}{2}\times\mathbf{1}_{1\times t}\mathbf{V}_{\rho}^{1/2}\mathbf{Z}_{\rho}E_{\mathbf{h}_1|\mathbf{y}}[\mathbf{h}_1] - \frac{1}{2}tr\left[\mathbf{Y}_2\mathbf{V}_{\rho}^{1/2}\mathbf{Z}_{\rho}E_{\mathbf{h}_1|\mathbf{y}}[\mathbf{h}_1\exp(-\mathbf{h}_1^{'})]\right]. \quad (66)$$

Note that the matrix $\mathbf{Y}_2\mathbf{V}_{\rho}^{1/2}\mathbf{Z}_{\rho}$ is lower triangular, and we have only to calculate the upper triangular part of the matrix $E_{\mathbf{h}_1|\mathbf{y}}[\mathbf{h}_1\exp(-\mathbf{h}_1^{'})]$ in evaluating the score function (66).

D Score function with respect to b_2

First, note that, in the log-likelihood function, b_2 appears only in $f(\mathbf{y}_2|\mathbf{h}_1,\mathbf{u}_2) = D_tF_t$, through:

$$\mathbf{h}_2 = \mathbf{V}_\psi^{1/2}\left(\sqrt{c}\mathbf{u}_2 + \mathbf{V}_\rho^{-\frac{1}{2}}\frac{b_2}{b_1}\mathbf{h}_1\right), \tag{67}$$

as shown in (22) and (26). Then we have the formula:

$$\frac{\partial \log f(\mathbf{y})}{\partial b_2} = E_{\mathbf{u}_2,\mathbf{h}_1|\mathbf{y}}\left(\frac{\partial D_t}{\partial b_2}D_t^{-1} + \frac{\partial F_t}{\partial b_2}F_t^{-1}\right), \tag{68}$$

using:

$$\frac{\partial f(\mathbf{y})}{\partial b_2} = \int (\mathbf{y}_1|\mathbf{h}_1)\frac{\partial f(\mathbf{y}_2|\mathbf{h}_1,\mathbf{u}_2)}{\partial b_2}f(\mathbf{h}_1)f(\mathbf{u}_2|\mathbf{h}_1)d\mathbf{u}_2\mathbf{h}_1 = \int\left(\frac{\partial D_t}{\partial b_2}D_t^{-1} + \frac{\partial F_t}{\partial b_2}F_t^{-1}\right)f(\mathbf{y},\mathbf{h}_1)d\mathbf{h}_1,$$

as we have:

$$\frac{\partial f(\mathbf{y}_2|\mathbf{h}_1,\mathbf{u}_2)}{\partial b_2} = \frac{\partial D_t}{\partial b_2}F_t + D_t\frac{\partial F_t}{\partial b_2} = \left(\frac{\partial D_t}{\partial b_2}D_t^{-1} + \frac{\partial F_t}{\partial b_2}F_t^{-1}\right)f(\mathbf{y}_2|\mathbf{h}_1,\mathbf{u}_2). \tag{69}$$

The partial derivatives of D_t and F_t are:

$$\begin{aligned}
\frac{\partial D_t}{\partial b_2} &= -\frac{1}{2}D_t\mathbf{1}_{1\times t}\mathbf{V}_\psi^{1/2}\left(\mathbf{V}_\rho^{-\frac{1}{2}}\frac{1}{b_1}\mathbf{h}_1\right), &(70)\\
\frac{\partial F_t}{\partial b_2} &= -\frac{1}{2}F_t\ \frac{\partial}{\partial b_2}\left[\mathbf{y}_2^{2\prime}\exp(-\mathbf{h}_2)\right] &(71)\\
&= \frac{1}{2}F_t\ tr\left[\mathbf{y}_2^{2\prime}\mathbf{H}_2\mathbf{V}_\psi^{1/2}\left(\mathbf{V}_\rho^{-\frac{1}{2}}\frac{1}{b_1}\mathbf{h}_1\right)\right]. &(72)
\end{aligned}$$

Note that, under the null hypothesis, $\rho = \psi$, $\mathbf{h}_1 = \mathbf{h}_2$, and $\mathbf{V}_\rho = \mathbf{V}_\psi$, so that we have:

$$\begin{aligned}
\left.\frac{\partial D_t}{\partial b_2}\right|_{\mathbf{H}_0} &= -\frac{1}{2}D_t\mathbf{1}_{1\times t}\frac{1}{b_1}\mathbf{h}_1, &(73)\\
\left.\frac{\partial F_t}{\partial b_2}\right|_{\mathbf{H}_0} &= -\frac{1}{2}F_t\ \frac{\partial}{\partial b_2}\mathbf{y}_2^{2\prime}\exp(-\mathbf{h}_2)\Big|_{\mathbf{H}_0}, &(74)\\
&= \frac{1}{2}F_t\ tr\left[\mathbf{y}_2^{2\prime}\mathbf{H}_2\frac{1}{b_1}\mathbf{h}_1\right]\Big|_{\mathbf{H}_0}, &(75)
\end{aligned}$$

as

$$\frac{\partial}{\partial b_2}\exp(-\mathbf{h}_2) = -\mathbf{H}_2^{-1}b_2/b_1.$$

Then we have:

$$\frac{\partial \log f(\mathbf{y})}{\partial b_2}|_{\mathbf{H}_0} = -\frac{1}{2b_1}\mathbf{1}_{1\times t}E_{\mathbf{h}_1|\mathbf{y}}\left[\mathbf{h}_1\right] + \frac{1}{2b_1}tr\left[\mathbf{y}_2^{2\prime}E_{\mathbf{h}_1|\mathbf{y}}\left[\exp(-\mathbf{h}_1)\circ\mathbf{h}_1\right]\right], \tag{76}$$

where $\circ$ denotes the Hadamard (or element-by-element) product.

E Score function with respect to ρ

In the log-likelihood function, ρ appears only in $f(y_1|\mathbf{h}_1, \mathbf{u}_2) = D_t F_t$ and $f(\mathbf{h}_1)$, as shown in (15) and (26). Then we have the derivative using the formula:

$$\frac{\partial \log f(\mathbf{y})}{\partial \rho}\Big|_{\mathbf{H}_0} = E_{\mathbf{h}_1|\mathbf{y}} \left(\frac{\partial D_t}{\partial \rho} D_t^{-1} + \frac{\partial F_t}{\partial \rho} F_t^{-1} + \frac{\partial f(\mathbf{h}_1)}{\partial \rho} f(\mathbf{h}_1)^{-1} \right), \tag{77}$$

analogously to that of (54). As:

$$\begin{aligned} D_t &= \exp\left(-\frac{1}{2} \mathbf{1}_{1\times t} \mathbf{V}_\psi^{1/2} \left(\sqrt{c}\mathbf{u}_2 + \mathbf{V}_\rho^{-\frac{1}{2}} \frac{b_2}{b_1} \mathbf{h}_1 \right) \right), & (78)\\ F_t &= \exp\left(-\frac{1}{2} \exp(-\mathbf{h}_2^{'}) \mathbf{y}_2^2 \right), & (79)\\ f(\mathbf{h}_1) &= \frac{1}{(2\pi)^{\frac{t}{2}} \left| \mathbf{V}_\rho^{1/2} \right| b_1^t} \exp\left(-\frac{1}{2} b_1^{-2} \mathbf{h}_1^{'} \mathbf{V}_\rho^{-1} \mathbf{h}_1 \right), & (80) \end{aligned}$$

defining

$$\mathbf{Z}_\rho = \frac{\partial \mathbf{V}_\rho^{-1/2}}{\partial \rho}, \tag{81}$$

their derivatives are expressed as:

$$\begin{aligned} \frac{\partial D_t}{\partial \rho} &= -\frac{1}{2} D_t \left[\mathbf{1}_{1\times t} \mathbf{V}_\psi^{1/2} \mathbf{Z}_\rho \frac{b_2}{b_1} \mathbf{h}_1 \right], & (82)\\ \frac{\partial F_t}{\partial \rho} &= \frac{1}{2} F_t \left[\mathbf{y}_2^{2'} \mathbf{H}_1^{-1} \mathbf{V}_\psi^{1/2} \mathbf{Z}_\rho \frac{b_2}{b_1} \mathbf{h}_1 \right], & (83)\\ \frac{\partial f(\mathbf{h}_1)}{\partial \rho} &= f(\mathbf{h}_1) \left[-\frac{\rho}{1-\rho^2} - \frac{1}{2} b_1^{-2} tr \left(\mathbf{h}_1^{'} \frac{\partial \mathbf{V}_\rho^{-1}}{\partial \rho} \mathbf{h}_1 \right) \right] \\ &= f(\mathbf{h}_1) \left[-\frac{\rho}{1-\rho^2} - \frac{1}{2} b_1^{-2} tr \left(\frac{\partial \mathbf{V}_\rho^{-1}}{\partial \rho} \mathbf{h}_1 \mathbf{h}_1^{'} \right) \right]. & (84) \end{aligned}$$

We have used $(\partial/\partial\rho) \left| \mathbf{V}_\rho^{1/2} \right| = 1/\sqrt{1-\rho^2}$ in deriving the first term of equation (84).

Noting that $\exp(-\mathbf{h}_1{'})\mathbf{Y}_2 = (\mathbf{y}_2^2)'\mathbf{H}_1^{-1}$ under the null hypothesis, the above can be expressed as:

$$\begin{aligned} \frac{\partial D_t}{\partial \rho}|_{\mathbf{H}_0} &= -\frac{1}{2} D_t \left[\mathbf{1}_{1\times t} \mathbf{V}_\rho^{1/2} \mathbf{Z}_\rho \frac{b_2}{b_1} \mathbf{h}_1 \right] & (85)\\ &= -\frac{\partial D_t}{\partial \psi}|_{\mathbf{H}_0}, & (86)\\ \frac{\partial F_t}{\partial \rho}|_{\mathbf{H}_0} &= -\frac{1}{2} F_t \; tr \left[\mathbf{Y}_2 \mathbf{V}_\rho^{1/2} \mathbf{Z}_\rho \frac{b_2}{b_1} \mathbf{h}_1 \exp(-\mathbf{h}_1^{'}) \right] & (87)\\ &= -\frac{\partial F_t}{\partial \psi}|_{\mathbf{H}_0}. & (88) \end{aligned}$$

The score function with respect to ρ can be expressed as:

$$\frac{\partial \log f(\mathbf{y})}{\partial \rho}|_{\mathbf{H}_0} = -\frac{\partial \log f(\mathbf{y})}{\partial \psi}|_{\mathbf{H}_0} - \frac{\rho}{1-\rho^2} - \frac{1}{2} b_1^{-2} tr\left(\frac{\partial \mathbf{V}_\rho^{-1}}{\partial \rho} E_{\mathbf{h}_1|\mathbf{y}}\left(\mathbf{h}_1 \mathbf{h}_1^{'}\right)\right). \tag{89}$$

F Score function with respect to standard deviation b_1

In the likelihood function, b_1 appears only in:

$$D_t = \exp\left(-\frac{1}{2}\mathbf{1}_{1\times t}\mathbf{V}_\psi^{1/2}\left(\sqrt{c}\mathbf{u}_2 + \mathbf{V}_\rho^{-\frac{1}{2}}\frac{b_2}{b_1}\mathbf{h}_1\right)\right), \tag{90}$$

$$F_t = \exp\left(-\frac{1}{2}\exp(-\mathbf{h}_2^{'})\mathbf{y}_2^2\right), \tag{91}$$

$$f(\mathbf{h}_1) = \frac{1}{(2\pi)^{\frac{t}{2}}\left|\mathbf{V}_\rho^{1/2}\right| b_1^t}\exp\left(-\frac{1}{2}b_1^{-2}\mathbf{h}_1^{'}\mathbf{V}_\rho^{-1}\mathbf{h}_1\right). \tag{92}$$

Then we can derive the score function with respect to b_1 using the formula analogous to that of ρ given in (77), with ρ replaced by b_1. We can easily show from (27) and(28) that, under the null hypothesis $b_2 = b_1$, the derivatives of D_t and F_t with respect to b_1 are equal to the negative of the derivatives with respect to b_2 , namely:

$$\frac{\partial D_t}{\partial b_1}|_{\mathbf{H}_0} = -\frac{\partial D_t}{\partial b_2}|_{\mathbf{H}_0}, \tag{93}$$

$$\frac{\partial F_t}{\partial b_1}|_{\mathbf{H}_0} = -\frac{\partial F_t}{\partial b_2}|_{\mathbf{H}_0}, \tag{94}$$

so that no additional calculations are necessary. From (15), the derivative of $f(\mathbf{h}_1)$ can be expressed as:

$$\frac{\partial f(\mathbf{h}_1)}{\partial b_1} = f(\mathbf{h}_1)\left[-\frac{t}{b_1} + \frac{1}{b_1^3}tr\left(\mathbf{h}_1^{'}\mathbf{V}_\rho^{-1}\mathbf{h}_1\right)\right] \tag{95}$$

$$= f(\mathbf{h}_1)\left[-\frac{t}{b_1} + \frac{1}{b_1^3}tr\left(\mathbf{V}_\rho^{-1}\mathbf{h}_1\mathbf{h}_1^{'}\right)\right]. \tag{96}$$

Using the formula:

$$\frac{\partial \log f(\mathbf{y})}{\partial b_1}\Big|_{\mathbf{H}_0} = E_{\mathbf{h}_1|\mathbf{y}}\left(\frac{\partial D_t}{\partial b_1}D_t^{-1} + \frac{\partial F_t}{\partial b_1}F_t^{-1} + \frac{\partial f(\mathbf{h}_1)}{\partial b_1}f(\mathbf{h}_1)^{-1}\right), \tag{97}$$

where the derivation is analogous to that of (77), and comparing it with the formula (68), we have:

$$\frac{\partial \log f(\mathbf{y})}{\partial b_1}|_{\mathbf{H}_0} = -\frac{\partial \log f(\mathbf{y})}{\partial b_2}|_{\mathbf{H}_0} - \frac{t}{b_1} + \frac{1}{b_1^3}tr\left(\mathbf{V}_\rho^{-1}E_{\mathbf{h}_1|\mathbf{y}}\left(\mathbf{h}_1\mathbf{h}_1^{'}\right)\right). \tag{98}$$

G Monte Carlo Results

Histogram of LM statistic distribution:

Figure 3: Histogram of LM statistic for ψ=0.7

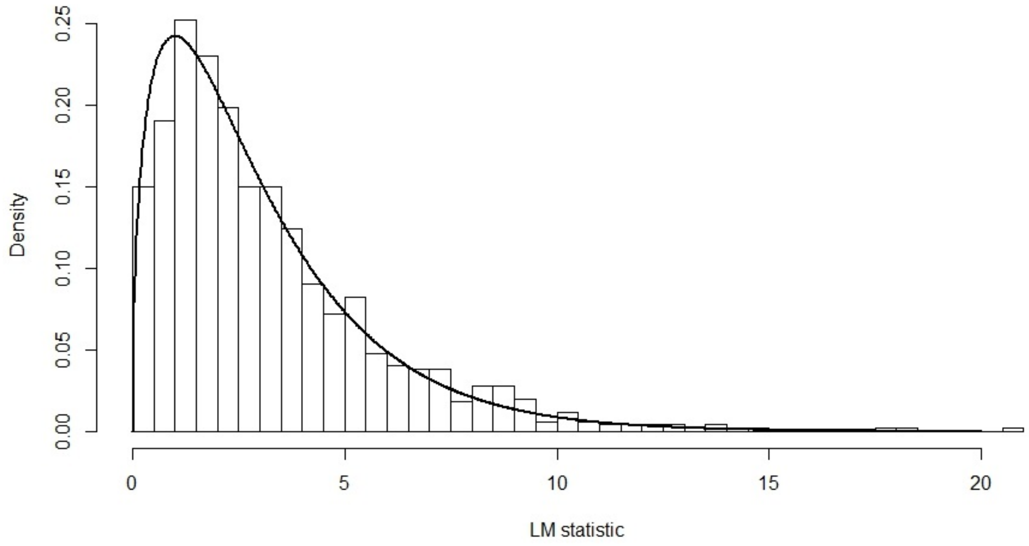


Figure 4: Histogram of LM statistic for ψ=0.9

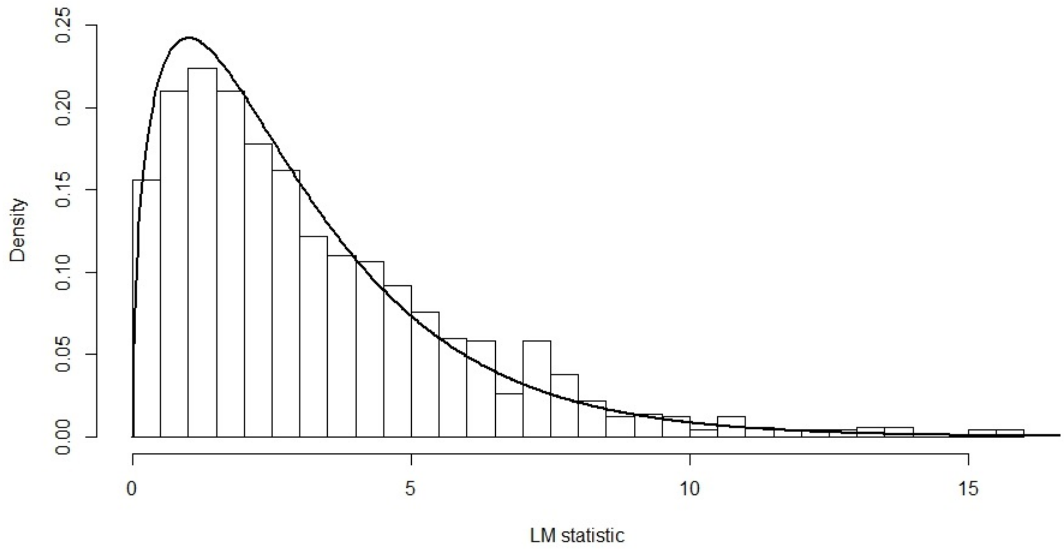
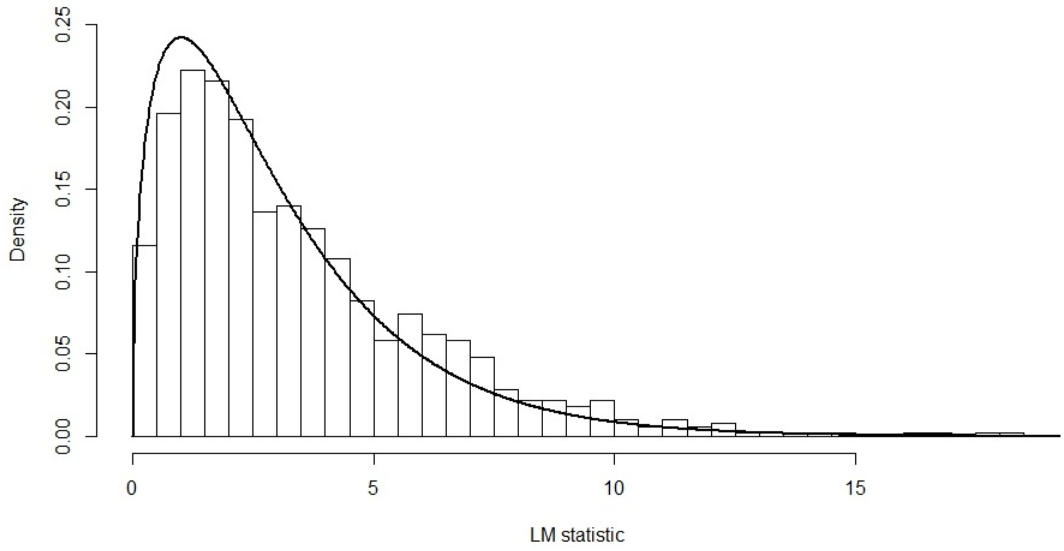


Figure 5: Histogram of LM statistic for ψ=0.95



H Data adjustment

The model to which we can apply the statistic test is:

$$\begin{pmatrix} y_{1t}^{''} \\ y_{2t}^{''} \end{pmatrix} = \begin{pmatrix} \exp(\frac{h_1}{2}) & 0 \\ 0 & \exp(\frac{h_2}{2}) \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix} \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix}. \tag{99}$$

Adding to the correlation between e_{1t} and e_{2t} we have the measurement equation given as:

$$\begin{pmatrix} y_{1t}^{''} \\ y_{2t}^{''} \end{pmatrix} = \begin{pmatrix} a_1 \exp(\frac{h_1}{2}) & 0 \\ 0 & a_2 \exp(\frac{h_2}{2}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix}, \tag{100}$$

where e_{1t} and e_{2t} remain independent, and λ represents the correlation coefficient. Owing to the identification between (a_1, a_2) and the variance of (e_{1t}, e_{2t}), we set the variance of (e_{1t}, e_{2t}) equal to 1. Recall that the null hypothesis is $h_{1t} = h_{2t}$ for any t, so that the measurement equation can be rewritten as:

$$\begin{pmatrix} y_{1t}^{''} \\ y_{2t}^{''} \end{pmatrix} = \exp\left(\frac{h_1}{2}\right) \begin{pmatrix} a_1 & 0 \\ \lambda a_2 & a_2 \end{pmatrix} \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix}, \tag{101}$$

and the product and moment of $(y_{1t}^{''}, y_{2t}^{''})$ would be given as:

$$\begin{aligned}
\begin{pmatrix} y_{1t}^{''2} & y_{1t}^{''} y_{2t}^{''} \\ y_{1t}^{''} y_{2t}^{''} & y_{2t}^{''2} \end{pmatrix} &= \exp(h_1) \begin{pmatrix} a_1 & 0 \\ \lambda a_2 & a_2 \end{pmatrix} \begin{pmatrix} e_{1t}^2 & e_{1t} e_{2t} \\ e_{1t} e_{2t} & e_{2t}^2 \end{pmatrix} \begin{pmatrix} a_1 & \lambda a_2 \\ 0 & a_2 \end{pmatrix}, && (102) \\
E \begin{pmatrix} y_{1t}^{''2} & y_{1t}^{''} y_{2t}^{''} \\ y_{1t}^{''} y_{2t}^{''} & y_{2t}^{''2} \end{pmatrix} &= E(\exp(h_1)) \begin{pmatrix} a_1 & 0 \\ \lambda a_2 & a_2 \end{pmatrix} \begin{pmatrix} a_1 & \lambda a_2 \\ 0 & a_2 \end{pmatrix} && (103) \\
&= E(\exp(h_1)) \begin{pmatrix} a_1^2 & \lambda a_1 a_2 \\ \lambda a_1 a_2 & a_2^2 \end{pmatrix} && (104) \\
&\equiv \Lambda, && (105)
\end{aligned}$$

where Λ is defined as:

$$\Lambda = \Lambda^{\frac{1}{2}} \left(\Lambda^{\frac{1}{2}}\right)', \ \Lambda^{\frac{1}{2}} = (E(\exp(h_1)))^{\frac{1}{2}} \begin{pmatrix} a_1 & 0 \\ \lambda a_2 & a_2 \end{pmatrix}. \tag{106}$$

It follows that:

$$\Lambda^{-\frac{1}{2}} \begin{pmatrix} y_{1t}^{''} \\ y_{2t}^{''} \end{pmatrix} = \exp\left(\frac{h_1}{2}\right) (E(\exp(h_1)))^{-\frac{1}{2}} \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix}. \tag{107}$$

At the first adjustment, we define:

$$\begin{pmatrix} y_{1t}^{'} \\ y_{2t}^{'} \end{pmatrix} = \Lambda^{-\frac{1}{2}} \begin{pmatrix} y_{1t}^{''} \\ y_{2t}^{''} \end{pmatrix} = a_0 \exp\left(\frac{h_1}{2}\right) \begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix}, \tag{108}$$

where

$$a_0 = (E(\exp(h_1)))^{-\frac{1}{2}}, \tag{109}$$

and the second adjustment is given as:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = a_0^{-1} \begin{pmatrix} y'_{1t} \\ y'_{2t} \end{pmatrix}. \tag{110}$$

Note that:

$$\begin{pmatrix} \log y'^2_{1t} \\ \log y'^2_{2t} \end{pmatrix} = 2\log a_0 + h_1 + \begin{pmatrix} \log e^2_{1t} \\ \log e^2_{2t} \end{pmatrix}, \tag{111}$$

$$E\begin{pmatrix} \log y'^2_{1t} \\ \log y'^2_{2t} \end{pmatrix} = 2\log a_0 - 1.27, \tag{112}$$

since $E(h_1) = 0$ and $E(\log(e^2)) = -1.27$ approximately, as shown by Harvey et al. (1994). Therefore, we reach the conclusion:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = a_0^{-1} \begin{pmatrix} y'_{1t} \\ y'_{2t} \end{pmatrix} = a_0^{-1}\Lambda^{-\frac{1}{2}} \begin{pmatrix} y''_{1t} \\ y''_{2t} \end{pmatrix}, \tag{113}$$

where the estimates of Λ and a_0 are given as:

$$\hat{\Lambda} = \frac{1}{T}\begin{pmatrix} \sum y''^2_{1t} & \sum y''_{1t}y''_{2t} \\ \sum y''_{1t}y''_{2t} & \sum y''^2_{2t} \end{pmatrix}, \tag{114}$$

$$\hat{a_0} = \begin{pmatrix} \exp\left\{\left[\frac{1}{T}\sum \log y'^2_{1t} + 1.27\right]/2\right\} & 0 \\ 0 & \exp\left\{\left[\frac{1}{T}\sum \log y'^2_{2t} + 1.27\right]/2\right\} \end{pmatrix}. \tag{115}$$

Then the adjusted data (y_{1t}, y_{2t}) can be applied to the proposed test.

I Empirical estimates

Table 9: Empirical estimates between stock markets correspond to $\hat{b}$ from 2011 to 2012

Stocks	**DOW**	**FTSE**	**DAX**	**SSCI**	**NIKKEI**	**HSI**	**STI**	**AORD**
DOW		0.1711	0.2103	0.147	0.3169	0.1998	0.1819	0.211
FTSE	0.165		0.1605	0.6314	0.2718	0.1879	0.1384	0.1495
DAX	0.199	0.1632		0.1089	0.3065	0.1676	0.1374	0.157
SSCI	0.1464	0.1016	0.1043		0.6901	0.7408	0.7272	0.7892
NIKKEI	0.318	0.2688	0.3078	0.685		0.3443	0.2865	0.3301
HSI	0.2	0.1904	0.1695	0.7045	0.343		0.1839	0.1576
STI	0.1858	0.1359	0.1374	0.7086	0.2871	0.1968		0.1137
AORD	0.2093	0.1469	0.1577	0.7487	0.3617	0.1768	0.1129	

Table 10: Empirical estimates between stock markets correspond to ψ from 2011 to 2012

Stocks	**DOW**	**FTSE**	**DAX**	**SSCI**	**NIKKEI**	**HSI**	**STI**	**AORD**
DOW		0.9717	0.9634	0.9717	0.9178	0.9699	0.9732	0.9674
FTSE	0.9714		0.9793	0.4727	0.9164	0.9732	0.9825	0.982
DAX	0.9657	0.9795		0.9811	0.9141	0.9821	0.9858	0.9833
SSCI	0.9711	0.9817	0.984		0.2652	0.0901	0.09	0.09
NIKKEI	0.9157	0.9147	0.915	0.2833		0.8388	0.8938	0.8648
HSI	0.9694	0.9725	0.9815	0.0902	0.8438		0.9662	0.9724
STI	0.9713	0.9829	0.9856	0.0901	0.8961	0.9609		0.9888
AORD	0.9674	0.9826	0.983	0.09	0.8449	0.9647	0.9885	

Table 11: Empirical estimates between stock markets correspond to b from 2013 to 2014

Stocks	DOW	FTSE	DAX	SSCI	NIKKEI	HSI	STI	AORD
DOW		0.3496	0.3341	0.3013	0.2941	0.2959	0.31	0.3048
FTSE	0.3665		0.289	0.2017	0.2479	0.2441	0.2416	0.2448
DAX	0.3328	0.2922		0.2458	0.1955	0.8115	0.3015	0.2804
SSCI	0.2966	0.1943	0.2476		0.1187	0.7535	0.2666	0.3395
NIKKEI	0.2957	0.2493	0.2046	0.1205		0.116	0.1465	0.1975
HSI	0.2861	0.2459	0.8285	0.1746	0.6946		0.7223	0.4359
STI	0.3108	0.2475	0.3074	0.6258	0.1478	0.6895		0.1905
AORD	0.3053	0.2433	0.2765	0.3251	0.2031	0.4082	0.1878	

Table 12: Empirical estimates between stock markets correspond to ψ from 2013 to 2014

Stocks	DOW	FTSE	DAX	SSCI	NIKKEI	HSI	STI	AORD
DOW		0.8696	0.8769	0.8857	0.9065	0.8611	0.8792	0.8857
FTSE	0.8633		0.8633	0.9502	0.9337	0.8932	0.918	0.9179
DAX	0.8776	0.8672		0.9294	0.9542	0.1669	0.8881	0.8981
SSCI	0.8908	0.9574	0.9269		0.9841	0.24	0.9123	0.8596
NIKKEI	0.9041	0.9342	0.9467	0.983		0.972	0.968	0.9451
HSI	0.8816	0.8928	0.1741	0.9523	0.2281		0.09	0.7186
STI	0.8751	0.9151	0.8847	0.6001	0.9662	0.09		0.9391
AORD	0.8856	0.9182	0.8984	0.8594	0.9419	0.7373	0.9395	

Table 13: Empirical estimates for exchange rates correspond to b in financial crisis

Curencies	**EURUSD**	**USDJPY**	**GBPUSD**	**AUDUSD**	**USDCHF**	**USDCAD**
EURUSD		0.322	0.6375	0.4357	0.5225	0.4753
USDJPY	0.3004		0.4268	0.4564	0.3658	0.5701
GBPUSD	0.5962	0.4318		0.4854	0.5239	0.6284
AUDUSD	0.4413	0.4552	0.4797		0.4342	0.535
USDCHF	0.5745	0.349	0.5816	0.4315		0.5294
USDCAD	0.4799	0.5578	0.6279	0.5191	0.5133	

Table 14: Empirical estimates for exchange rates correspond to ψ in financial crisis

Currencies	**EURUSD**	**USDJPY**	**GBPUSD**	**AUDUSD**	**USDCHF**	**USDCAD**
EURUSD		0.9014	0.7209	0.8183	0.7963	0.8463
USDJPY	0.9101		0.872	0.8523	0.8703	0.8261
GBPUSD	0.7437	0.8703		0.8449	0.8033	0.8038
AUDUSD	0.8089	0.8527	0.8475		0.8239	0.8616
USDCHF	0.768	0.8854	0.7953	0.8368		0.8243
USDCAD	0.8445	0.8281	0.8029	0.8661	0.8259	

Table 15: Empirical estimates for exchange rates correspond to b in normal times

Currencies	**EURUSD**	**USDJPY**	**GBPUSD**	**AUDUSD**	**USDCHF**	**USDCAD**
EURUSD		0.8462	0.8073	0.7858	0.918	0.8892
USDJPY	0.8599		0.8206	0.6888	0.85	0.8186
GBPUSD	0.8037	0.8333		0.7449	0.7302	0.7736
AUDUSD	0.8515	0.7424	0.8307		0.8041	0.8936
USDCHF	0.9163	0.8165	0.7198	0.6969		0.8686
USDCAD	0.8833	0.8113	0.818	0.8534	0.9306	

Table 16: Empirical estimates for exchange rates correspond to ψ in normal times

Currencies	**EURUSD**	**USDJPY**	**GBPUSD**	**AUDUSD**	**USDCHF**	**USDCAD**
EURUSD		0.404	0.644	0.4239	0.4179	0.6104
USDJPY	0.4325		0.5258	0.467	0.4317	0.5272
GBPUSD	0.6537	0.4644		0.5188	0.6825	0.677
AUDUSD	0.4202	0.4001	0.4774		0.4384	0.3643
USDCHF	0.4613	0.4794	0.7036	0.5476		0.6441
USDCAD	0.6384	0.5173	0.6559	0.3455	0.5727	

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