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On the 1-nucleolus for classes of cooperative games*

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Abstract

This paper analyzes the 1-nucleolus and, in particular, its relation to the nucleolus and compromise value. It is seen that the 1-nucleolus of a cooperative game can be characterized using a combination of standard bankruptcy rules for associated bankruptcy problems. In particular, for any zero-normalized balanced game, the 1-nucleolus coincides with the Aumann-Maschler rule (Aumann and Maschler, 1985) in this sense. From this result, not only necessary conditions on a compromise stable game are derived such that the 1-nucleolus and the nucleolus coincide, but also necessary and sufficient conditions such that the 1-nucleolus and the compromise value of exact games coincide.

Keywords: 1-nucleolus; Compromise stable games; Exact games; Aumann-Maschler rule; Nucleolus; Compromise value

1 Introduction

Cooperative transferable utility games (TU-games) have proven effective to analyze problems where the joint profits obtained by a joint collaboration have to be shared among the individuals involved (the grand coalition). In order to decide on a "fair" or "just" distribution of the joint profits (a solution), benchmarks are used: the joint profits that any subgroup of individuals (a coalition) could obtain by cooperation without any help from the other members of the grand coalition that are outside this subgroup. This means that the description of a cooperative game in general requires the computation of 2^n values, with n being the number of members of the grand coalition. Therefore, the computation of a cooperative game is NP-hard and a game with 6 players already requires 64 coalitional values. Due to this computational drawback, it

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is important to devise fair solutions that do not need much computational work. For instance, within the field of Operations Research games (cf. Borm et al., 2001), context specific solutions are constructed that only make use of an algorithm to find the value of the grand coalition and that lead to stable core elements of the cooperative game at the same time. In this way, we can solve both the optimization and allocation problem of the grand coalition simultaneously without the need to obtain all coalitional values. As examples, we mention the Owen set in linear production games (Owen , 1975), the Bird rule in minimum cost spanning trees (Bird, 1976), and the equal gain splitting rule in sequencing games (Curiel, Pederzoli and Tijs, 1989).

In a general framework, there are several solutions to TU-games available. Among the most studied are the core (Gillies, 1953), the nucleolus (Schmeidler, 1969), the Shapley value (Shapley, 1953) and the compromise value (Tijs , 1981). For TU-games with a nonempty imputation set, the nucleolus is a solution based on the idea that a fair distribution of the total worth should (lexicographically) minimize the sorted vector of the excesses (or complaints) associated with all possible coalitions. Given an imputation x and a coalition S, the excess measures the dissatisfaction of S at x. There are different algorithms to compute the nucleolus, see the Kopelowitz algorithm (Kopelowitz, 1967) or the Maschler-Peleg-Shapley algorithm (Maschler et al., 1979). The complexity of these algorithms, however, is exponential in the number of players, and therefore useful only for relatively small games. Still, there are classes of games, such as assignment games (Shapley and Shubik , 1972), where the complexity of these algorithms only grows polynomically in the number of players. That fact has allowed to develop special algorithms to obtain the nucleolus when the game has a special underlying structure. Nevertheless, in most applications where many players are involved, the task of computing the nucleolus can be very difficult, and it can be difficult even to attribute a value to each coalition. The computational complexity can be reduced by considering k-nucleoli that are based on the excesses of coalitions of size at most k and at least $n-k$. Clearly, the case where k is greater than or equal to the integer part of $\frac{n}{2}$ corresponds to the nucleolus, since all coalitions are considered.

This paper focuses on the 1-nucleolus. We characterize the 1-nucleolus using a combination of standard bankruptcy rules of an associated bankruptcy problem. Since only the values of the grand coalition and coalitions of size 1 and $n-1$ are needed, the 1-nucleolus can be computed in polynomial time. Moreover, we show that the 1-nucleolus of a balanced game, that is, a game with a nonempty core, corresponds to the Aumann-Maschler rule (Aumann and Maschler, 1985) of the associated bankruptcy problem. Besides, we analyze under which conditions the nucleolus and 1-nucleolus of compromise stable games coincide. This is done by taking into account the fact that the nucleolus of a compromise stable game can be also computed as the Aumann-Maschler rule of another associated bankruptcy problem (cf. Quant, Borm, Reijnierse, and van Velzen, 2005). Finally, we characterize the class of exact games (cf. Schmeidler, 1972) for which the 1-nucleolus and the compromise value (cf. Tijs , 1981) coincide. Here, we exploit the fact that the compromise value of an exact game only depends on the same coalitional values as the 1-nucleolus.

The outline of the paper is as follows. Section 2 recalls basic concepts and results that will be used throughout the paper. Section 3 formally introduces the k-nucleoli. In Section 4, we characterize the 1-nucleolus by means of a combination of bankruptcy rules. Section 5 analyzes the 1-nucleolus in relation with the nucleolus for compromise stable games, while Section 6 studies the 1-nucleolus in relation with the compromise value for exact games. The Appendix contains the proof of the characterization of the 1-nucleolus.

2 Preliminaries

In this section, we survey some well-known concepts and results that will be used in the subsequent sections.

For $x, y \in \mathbb{R}^n$, we say that x is *lexicographically smaller* than y, $x <_L y$, if there is $m \in \{1, \ldots, n\}$ such that $x_l = y_l$ for every $1 \leq l < m$ and $x_m < y_m$. Moreover, $x \leq_L y$ if either $x = y$, or $x <_L y$.

A *transferable utility game* (TU-game) is a pair (N, v) where N is a finite set of players and $v : 2^N \to \mathbb{R}$ satisfies $v(\emptyset) = 0$. In general, $v(S)$ represents the value of coalition S, that is, the joint payoff that can be obtained by this coalition when its members decide to cooperate. Let G^N be the set of all TU-games with player set N. Given $S \subseteq N$, let $|S|$ be the number of players in S.

The main focus within a cooperative setting is on how to share the total joint payoff obtained when all players decide to cooperate. Given a TU-game $v \in G^N$, the *imputation set* of v, $I(v)$, is the set of efficient allocations that are individually rational. Formally,

$$I(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), x_i \geq v(\{i\}) \text{ for all } i \in N \right\}.$$

Note that the imputation set is nonempty if, and only if,

$$\sum_{i \in N} v(\{i\}) \leq v(N).$$

We denote by I^N the set of all TU-games with player set N and nonempty imputation set.

The *core* of $v \in G^N$, $\mathcal{C}ore(v)$, was first introduced in Gillies (1953) and is defined as the set of efficient allocations that are stable, in the sense that no coalition has an incentive to deviate. Formally,

$$\mathcal{C}ore(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N \right\}.$$

Bondareva (1963) and Shapley (1967) established that a game $v \in G^N$ has a nonempty core if, and only if, it is balanced. Before introducing balanced games, we need to fix some notation.

Let $\emptyset \neq S \subseteq N$ and let $e^S \in \mathbb{R}^N$ be the characteristic vector of S, defined as $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$. A *family* $\mathcal{B}$ of nonempty subcoalitions of S is called *balanced on* S if there are positive weights $\delta = \{\delta_T\}_{T \in \mathcal{B}}$, $\delta_T > 0$ for all $T \in \mathcal{B}$, such that $\sum_{T \in \mathcal{B}} \delta_T e^T = e^S$ or, equivalently, $\sum_{T\in\mathcal{B}:T\ni i}\delta_T = 1$ for all $i\in S$ and $\sum_{T\in\mathcal{B}:T\ni i}\delta_T = 0$ for all $i \in N\setminus S$. We denote by $\mathcal{F}(S)$ the set of balanced families of S. Given a balanced family $\mathcal{B}$, we denote by $\Delta(\mathcal{B})$ the set of positive weights satisfying the balancedness condition. A *game* $v\in G^N$ is called *balanced* if for all balanced families $\mathcal{B}\in\mathcal{F}(N)$ and all $\{\delta_S\}_{S\in\mathcal{B}}\in\Delta(\mathcal{B})$, $\sum\limits_{S\in\mathcal{B}}\delta_S v(S)\leq v(N)$.

A well-established one-point solution concept is the nucleolus, introduced in Schmeidler (1969). Let $v\in I^N$ and let $x\in I(v)$. We denote the *excess* of coalition $S\in 2^N$ with respect to x by

$$e(S,x) = v(S) - \sum_{j\in S} x_j.$$

Moreover, we denote by $\theta(x)\in\mathbb{R}^{2^{|N|}}$ the vector whose coordinates are the excesses $e(S,x)$ arranged in non-increasing order, that is, $\theta_l(x)\geq\theta_m(x)$ for every $1\leq l\leq m\leq 2^{|N|}$. The *nucleolus* of $v\in G^N$, $nuc(v)$, is defined as

$$nuc(v) = \{x\in I(v)|\theta(x)\leq_L\theta(y)\text{ for all }y\in I(v)\}.$$

Schmeidler (1969) showed that the nucleolus of a game with a nonempty imputation set exists and is unique. The nucleolus is invariant with respect to positive affine transformations, i.e. for $v\in I^N$, $\alpha>0$, and $a\in\mathbb{R}^N$, it follows $nuc(\alpha v + a) = \alpha nuc(v) + a$ with $(\alpha v + a)(S) = \alpha v(S) + \sum_{j\in S} a_j$ for every $S\in 2^N$.

Kohlberg (1971) characterizes the nucleolus by means of collections of coalitions. Let $v\in I^N$ and let $x\in I(v)$. Let $B_0(x,v) = \{\{i\}\subseteq N | x_i = v(\{i\})\}$ and define recursively

$$B_l(x,v) = \left\{S\in 2^N\setminus(\cup_{m=1}^{l-1}B_m(x,v))|e(S,x)\geq e(R,x)\text{ for every }R\in 2^N\setminus(\cup_{m=1}^{l-1}B_m(x,v))\right\}$$

for $l\in\{1,\dots,p\}$, with p such that $B_p(x,v)\neq\emptyset$ and $\{B_1(x,v),\dots,B_p(x,v)\}$ forms a partition of the set of coalitions of N. For $l\in\{1,\dots,p\}$, let $\mathcal{B}^l(x,v) = \cup_{m=1}^{l}B_m(x,v)$.

Theorem 2.1 (cf. Kohlberg (1971))**.** *Let $v\in I^N$. Then, x is the nucleolus of v if, and only if, for every $l\in\{1,\dots,p\}$, there exists $B_0^l(x,v)\subseteq B_0(x,v)$ such that $B_0^l(x,v)\cup\mathcal{B}^l(x,v)$ is balanced.*

Tijs and Lipperts (1982) introduced the core cover. Let $v\in I^N$ and $i\in N$. The *utopia* value of player i, $M_i(v)$, is defined as

$$M_i(v) = v(N) - v(N\setminus\{i\}).$$

The *minimal right* of player i, $m_i(v)$, is defined as

$$m_i(v) = \max_{S\subseteq N\setminus\{i\}}\left\{v(S\cup\{i\}) - \sum_{j\in S}M_j(v)\right\}.$$

The *utopia vector* is given by $M(v) = (M_i(v))_{i\in N}$ and the *minimal right* vector is given by $m(v) =$

$(m_i(v))_{i\in N}$. The *core cover* of $v \in G^N$, $\mathcal{CC}(v)$, is defined as

$$\mathcal{CC}(v) = \{x \in I(v) \mid m(v) \leq x \leq M(v)\}.$$

It can be verified that $\mathcal{C}ore(v) \subseteq \mathcal{CC}(v) \subseteq I(v)$. A TU game $v \in I^N$ is *compromise admissible* if $\mathcal{CC}(v) \neq \emptyset$. A compromise admissible game is *compromise stable* if $\mathcal{C}ore(v) = \mathcal{CC}(v)$. Quant et al. (2005) characterized the family of compromise stable games.

Theorem 2.2 (Quant et al. (2005)). *Let $v \in I^N$ be compromise admissible. Then, v is compromise stable if, and only if, $v(S) \leq \max\{\sum_{j\in S} m_j(v), v(N) - \sum_{j\in N\setminus S} M_j(v)\}$ for every $S \subseteq N$.*

Tijs (1981) introduced the compromise value as a one-point solution for compromise admissible games. Let $v \in G^N$ be a compromise admissible game. The *compromise value* of v, $\tau(v)$, is defined as

$$\tau(v) = m(v) + \alpha(M(v) - m(v))$$

with $\alpha > 0$ such that $\sum_{i\in N} \tau_i(v) = v(N)$.

An important subclass of balanced games is the class of exact games, which were introduced in Schmeidler (1972). A game $v \in G^N$ is *exact* if for every $S \subseteq N$ there exists $x \in \mathcal{C}ore(v)$ such that $\sum_{i\in S} x_i = v(S)$. A well-known subclass of exact games are convex games, as introduced in Shapley (1971). A game $v \in G^N$ is *convex* if $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for every $i \in N$ and $S \subset T \subseteq N \setminus \{i\}$.

A *bankruptcy problem* is described by (N, E, c), with N a finite set of players, $E > 0$, and $c \in \mathbb{R}^N$ such that $c_i \geq 0$ for all $i \in N$ and $\sum_{i\in N} c_i \geq E$. O'Neill (1982) defines the *bankruptcy game* associated to a bankruptcy problem (N, E, c), as

$$v_{E,c}(S) = \max\left\{0, E - \sum_{i\in N\setminus S} c_i\right\} \text{ for every } S \in 2^N.$$

In fact, Quant et al. (2005) show that any convex and compromise stable game is S-equivalent[1] to a bankruptcy game. Aumann and Maschler (1985) show that the nucleolus of a bankruptcy game corresponds to the Aumann-Maschler rule of the corresponding bankruptcy problem.

3 k-nucleoli

In this section, we introduce the k-nucleolus of a game by considering only coalitions of size at most k and at least $|N| - k$. In order to formally define the k-nucleolus, we need to fix some notation. Let N be a finite set and $k \leq |N|$, we denote

$$\mathcal{C}^k(N) = \{S \in 2^N \mid |S| \leq k \text{ or } |S| \geq |N| - k\}.$$

[1]Two games $v, w \in G^N$ are S-equivalent if there exists $\alpha > 0$ and $a \in \mathbb{R}^N$ such that $v(S) = \alpha w(S) + \sum_{i\in S} a_i$ for every $S \subseteq N$.

If no confusion arises, we write $\mathcal{C}^k$ instead of $\mathcal{C}^k(N)$. We denote $n_k = 2\sum_{l=0}^{k}\binom{|N|}{l}$ if $k < \frac{|N|}{2}$ and $n_k = 2^{|N|}$ if $k \geq \frac{|N|}{2}$. Given $v \in I^N$ and $x \in I(v)$, we write $\theta^k(x) \in \mathbb{R}^{n_k}$ the vector whose coordinates are the excesses $e(S,x)$, with $S \in \mathcal{C}^k$, arranged in non-increasing order, that is, $\theta_l^k(x) \geq \theta_m^k(x)$ for every $1 \leq l \leq m \leq n_k$.

Definition 3.1. *Let $v \in I^N$. The* k-nucleolus *is defined by*

$$nuc^k(v) = \{x \in I(v) | \theta^k(x) \leq_L \theta^k(y) \text{ for all } y \in I(v)\}.$$

Note that for[2] $k \geq \lfloor \frac{|N|}{2} \rfloor$, $nuc^k(v) = nuc(v)$ since $\mathcal{C}^k(N) = 2^N$. So clearly, the 1-nucleolus and nucleolus of 3-players games coincide. Moreover, just like the nucleolus, the k-nucleolus is invariant with respect to positive affine transformations.

Theorem 3.2 (cf. Schmeidler (1969))**.** *Let $v \in I^N$. Then, $nuc^k(v)$ exists and is unique for every $k \in \{1, \ldots, |N|\}$.*

The characterization in Kohlberg (1971) can be translated to the k-nucleolus as already pointed out in Maschler et al. (1992). Similarly as in Section 2, we need to obtain a partition of the coalitions in N of size at most k or at least $|N| - k$. Let $v \in I^N$ and let $x \in I(v)$. Let $B_0^k(x,v) = \{\{i\} \subseteq N | x_i = v(\{i\})\}$ and define recursively

$$B_l^k(x,v) = \left\{S \in \mathcal{C}^k(N) \setminus (\cup_{m=1}^{l-1} B_m^k(x,v)) | e(S,x) \geq e(R,x) \text{ for every } R \in \mathcal{C}^k(N) \setminus (\cup_{m=1}^{l-1} B_m(x,v))\right\}$$

for $l \in \{1, \ldots, p\}$, with p such that $B_p^k(x,v) \neq \emptyset$ and $\langle B_1^k(x,v), \ldots, B_p^k(x,v)\rangle$ forms a partition of the set of coalitions of $\mathcal{C}^k(N)$. For $l \in \{1, \ldots, p\}$, let $\mathcal{B}^{k,l}(x,v) = \cup_{m=1}^{l} B_m^k(x,v)$.

Theorem 3.3 (cf. Kohlberg (1971) and Maschler et al. (1992))**.** *Let $v \in I^N$ and let $k \in \{1, \ldots, |N|\}$. Then, x is the k-nucleolus of v if, and only if, for every $l \in \{1, \ldots, p\}$, there exists $B_0^{k,l}(x,v) \subseteq B_0^k(x,v)$ such that $B_0^{k,l}(x,v) \cup \mathcal{B}^{k,l}(x,v)$ is balanced.*

4 1-nucleolus and bankruptcy

The 1-nucleolus only takes into account the information provided by the value of the singletons (individual coalitions), the value of the $|N| - 1$ player coalitions, and the value of the grand coalition. Thus, the information needed stems from $2|N| + 1$ coalitions.

This section shows that the 1-nucleolus is related to the Aumann-Maschler rule of bankruptcy problems (see Aumann and Maschler, 1985), the constrained equal losses rule for bankruptcy problems, and the equal share rule. Moreover, the 1-nucleolus of a balanced game can be described as the Aumann-Maschler rule of an associated bankruptcy problem. As a consequence, it turns out that the 1-nucleolus and the nucleolus of bankruptcy games coincide (see O'Neill , 1982; Aumann and Maschler, 1985; Quant et al., 2005).

[2]For each $r \in \mathbb{R}$, $\lfloor r \rfloor$ denotes the largest integer smaller than or equal to r.

We recall some well-known bankruptcy rules in the literature. The *equal share rule* of a bankruptcy problem (N, E, c), $\mathrm{ES}(N, E, c)$, assigns

$$\mathrm{ES}_j(N, E, c) = \frac{E}{|N|} \text{ for every } j \in N.$$

The *constrained equal awards rule* of a bankruptcy problem (N, E, c), $\mathrm{CEA}(N, E, c)$, assigns

$$\mathrm{CEA}_j(N, E, c) = \min\{\lambda, c_j\} \text{ to every } j \in N,$$

with $\lambda \in \mathbb{R}_+$ chosen such that $\sum_{j \in N} \mathrm{CEA}_j(N, E, c) = E$. The *Aumann-Maschler rule* of a bankruptcy problem (N, E, c), $\mathrm{AM}(N, E, c)$, is given by

$$\mathrm{AM}(N, E, c) = \begin{cases} \mathrm{CEA}(N, E, \frac{1}{2}c) & \text{if } E \leq \frac{1}{2}\sum_{j \in N} c_j, \\ c - \mathrm{CEA}(N, \sum_{j \in N} c_j - E, \frac{1}{2}c) & \text{otherwise.} \end{cases}$$

Aumann and Maschler (1985) showed that the Aumann-Maschler rule of a bankruptcy problem corresponds to the nucleolus of the associated bankruptcy game. To conclude, the *constrained equal losses rule* of a bankruptcy problem (N, E, c), $\mathrm{CEL}(N, E, c)$, is defined as

$$\mathrm{CEL}_j(N, E, c) = \max\{0, c_j - \lambda\} \text{ for every } j \in N,$$

where λ is chosen such that $\sum_{j \in N} \mathrm{CEL}_j(N, E, c) = E$.

For $v \in I^N$, we define the *zero-normalization* of v, $v_0 \in I^N$, as

$$v_0(S) = v(S) - \sum_{j \in S} v(\{j\}) \text{ for every } S \in 2^N.$$

Note that $nuc^1(v_0) = nuc^1(v) - (v(\{j\}))_{j \in N}$. Therefore, when describing the 1-nucleolus, we can assume that $v = v_0$, that is, that v is zero-normalized.

The following result fully describes the 1-nucleolus by means of a combination of standard bankruptcy solutions to associated bankruptcy problems.

Theorem 4.1. *Let $v \in I^N$ with $v = v_0$. Let $E = v(N)$ and let $c \in \mathbb{R}^N$ be defined by $c_j = v(N) - v(N \setminus \{j\})$ for every $j \in N$.*

(i) If $c_j \geq 0$ for every $j \in N$, then,

$$nuc^1(v) = \begin{cases} AM(N, E, c) & \text{if } E \leq \sum_{j \in N} c_j, \\ c + ES(N, E - \sum_{j \in N} c_j, c) & \text{if } E > \sum_{j \in N} c_j. \end{cases}$$

(ii) If $c_j < 0$ for some $j \in N$, let $c^+ \in \mathbb{R}^N$ be defined by $c_j^+ = \max\{0, c_j\}$ for every $j \in N$ and let

$c^{\min} \in \mathbb{R}^N$ *be defined as* $c_j^{\min} = c_j - \min\{c_l | l \in N\}$ *for every* $j \in N$. *Then,*

$$nuc^1(v) = \begin{cases} AM(N, E, c^+) & \text{if } E \leq \sum_{j \in N} c_j^+, \\ CEL(N, E, c^{\min}) & \text{if } \sum_{j \in N} c_j^+ < E \leq \sum_{j \in N} c_j^{\min}, \\ c + ES(N, E - \sum_{j \in N} c_j, c) & \text{if } E > \sum_{j \in N} c_j^{\min}. \end{cases}$$

Proof. See Appendix. □

The next result provides an explicit connection of the 1-nucleolus for balanced games to the Aumann-Maschler rule.

Theorem 4.2. *Let* $v \in I^N$ *be a balanced game with* $v = v_0$. *Then,*

$$nuc^1(v) = AM(N, E, c)$$

with $E = v(N)$ *and* $c \in \mathbb{R}^N$ *defined as* $c_j = v(N) - v(N \setminus \{j\})$ *for every* $j \in N$.

Proof. By Theorem 4.1 (i), it suffices to show that $c_j \geq 0$ for every $j \in N$ and that $E \leq \sum_{j \in N} c_j$.

Let $j \in N$. Since v is balanced, we have that $v(N) \geq v(\{j\}) + v(N \setminus \{j\}) = v(N \setminus \{j\})$. Therefore, $c_j = v(N) - v(N \setminus \{j\}) \geq 0$.

Moreover, since v is balanced, we have that $\sum_{j \in N} \frac{1}{|N|-1} v(N \setminus \{j\}) \leq v(N)$. Therefore,

$$\begin{aligned} E = v(N) &= v(N) + (|N| - 1) \sum_{j \in N} \frac{1}{|N| - 1} v(N \setminus \{j\}) - \sum_{j \in N} v(N \setminus \{j\}) \\ &\leq v(N) + (|N| - 1) v(N) - \sum_{j \in N} v(N \setminus \{j\}) = \sum_{j \in N} (v(N) - v(N \setminus \{j\})) = \sum_{j \in N} c_j. \end{aligned}$$

□

As a consequence, we have

Theorem 4.3. *Let* (N, E, c) *be a bankruptcy problem and let* $(N, v_{E,c})$ *be the corresponding bankruptcy game. Then,* $nuc(v_{E,c}) = nuc^1(v_{E,c})$.

Proof. Let $w \in G^N$ be the zero-normalization of $v_{E,c}$, that is $w(S) = v_{E,c}(S) - \sum_{j \in S} v_{E,c}(\{j\})$ for all $S \in 2^N$. Then,

$$\begin{aligned} nuc^1(v_{E,c}) &= (v_{E,c}(\{j\}))_{j \in N} + nuc^1(w) \\ &= (v_{E,c}(\{j\}))_{j \in N} + \mathrm{AM}(N, w(N), (w(N) - w(N \setminus \{j\}))_{j \in N}) \\ &= (v_{E,c}(\{j\}))_{j \in N} + \mathrm{AM}\left(N, v_{E,c}(N) - \sum_{j \in N} v_{E,c}(\{j\}), M(v_{E,c}) - (v_{E,c}(\{j\}))_{j \in N}\right) \\ &= m(v_{E,c}) + \mathrm{AM}(N, v_{E,c}(N) - \sum_{j \in N} m_j(v_{E,c}), M(v_{E,c}) - m(v_{E,c})) \\ &= \mathrm{AM}(N, v_{E,c}(N), M(v_{E,c})) \end{aligned}$$

$$= nuc(v_{E,c})$$

where the third equality follows from $M_i(w) = w(N) - w(N \setminus \{i\}) = M_i(v_{E,c}) - v_{E,c}(\{i\})$ for every $i \in N$, the fourth equality is a direct consequence of $v_{E,c}(\{j\}) = m_j(v_{E,c})$ for every $j \in N$, and the fifth equality follows from the fact that the Aumann-Maschler rule satisfies the property of minimal rights first (see Thomson, 2003). $\square$

As a consequence of Theorem 4.3, the nucleolus and 1-nucleolus of convex and compromise stable games coincide since every convex and compromise stable game is S-equivalent to a bankruptcy game (cf. Quant et al., 2005).

5 1-nucleolus and compromise stability

Quant et al. (2005) characterize the nucleolus of compromise stable games.

Theorem 5.1 (Quant et al. (2005))**.** *Let $v \in I^N$ be a compromise stable game. Then,*

$$nuc(v) = m(v) + AM(N, v(N) - \sum_{j \in N} m_j(v), M(v) - m(v)).$$

The following example shows that the 1-nucleolus might not belong to the core cover of a compromise stable game. Furthermore, it illustrates that the 1-nucleolus and the nucleolus of such a game need not coincide.

Example 5.2. *Consider $v \in I^N$ with $N = \{1, 2, 3, 4\}$,*

$$\begin{gathered} v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0,\ v(\{4\}) = 0, \\ v(\{1,2\}) = 1,\ v(\{1,3\}) = 0,\ v(\{1,4\}) = 4,\ v(\{2,3\}) = 0,\ v(\{2,4\}) = 0,\ v(\{3,4\}) = 3, \\ v(\{1,2,3\}) = 1,\ v(\{1,2,4\}) = 5,\ v(\{1,3,4\}) = 5,\ v(\{2,3,4\}) = 0,\ v(N) = 5. \end{gathered}$$

Here, $m(v) = (1, 0, 0, 3)$ and $M(v) = (5, 0, 0, 4)$. One readily verifies (using Theorem 2.2) that v is compromise stable. Using Theorem 5.1, we have

$$nuc(v) = (1, 0, 0, 3) + AM(N, 1, (4, 0, 0, 1)) = (1.5, 0, 0, 3.5) \in \mathcal{CC}(v).$$

However, using Theorem 4.1, we have

$$nuc^1(v) = AM(N, 5, (5, 0, 0, 4)) = (3, 0, 0, 2) \notin \mathcal{CC}(v).$$

Notice that in the example above, both the nucleolus and the 1-nucleolus are obtained through the Aumann-Maschler rule, but they provide different allocations. This difference arises from the fact that we first allocate the minimal rights in the nucleolus and then we apply the Aumann-Maschler rule, while in the case of the 1-nucleolus we first allocate the vector $(v(\{i\})_{i \in N}$ and then we apply the Aumann-Maschler rule. Thus, some of the coordinates of the

1-nucleolus may be smaller than the corresponding coordinates of the minimal rights vector. Precisely that difference makes the 1-nucleolus of a compromise stable game easier to compute than the nucleolus, since one does not need the minimal rights vector. Next, we provide some conditions for the nucleolus and 1-nucleolus of a compromise stable game to coincide.

Theorem 5.3. *Let $v \in I^N$ be compromise stable. Let $E = v(N) - \sum_{j \in N} v(\{j\})$ and $c_j = M_j(v) - v(\{j\})$ for every $j \in N$.*

(i) If $m_j(v_{E,c}) = m_j(v) - v(\{j\})$ for every $j \in N$, then, $nuc^1(v) = nuc(v)$.

(ii) If $m_j(v) = \max\{v(\{j\}), v(N) - \sum_{k \in N \setminus \{j\}} M_k(v)\}$ for every $j \in N$, then, $nuc^1(v) = nuc(v)$.

(iii) If either $m(v) < M(v)$, or $m(v) = M(v)$, then, $nuc^1(v) = nuc(v)$.

Proof. We assume, without loss of generality, that $v = v_0$. Note that $E = v(N) - \sum_{j \in N} v(\{j\}) = v(N)$ and $c_j = M_j(v) - v(\{j\}) = M_j(v)$ for every $j \in N$. Since v is compromise stable, v is balanced and, therefore, $nuc^1(v) = \text{AM}(N, E, c)$ by Theorem 4.2.

(i) Let $m_j(v_{E,c}) = m_j(v) - v(\{j\}) = m_j(v)$ for every $j \in N$. Then,

$$\begin{aligned} nuc^1(v) &= \text{AM}(N, E, c) \\ &= m(v_{E,c}) + \text{AM}(N, E - \sum_{j \in N} m_j(v_{E,c}), c - m(v_{E,c})) \\ &= m(v) + \text{AM}(N, v(N) - \sum_{j \in N} m_j(v), M(v) - m(v)) \\ &= nuc(v) \end{aligned}$$

where the second equality follows from the fact that the Aumann-Maschler rule satisfies minimal rights first (see Thomson, 2003), the third one is a direct consequence of $m_j(v_{E,c}) = m_j(v) - v(\{j\}) = m_j(v)$ for every $j \in N$, and the last one follows from Theorem 5.1.

(ii) Let $m_j(v) = \max\left\{v(\{j\}), v(N) - \sum_{k \in N \setminus \{j\}} M_k(v)\right\}$ for every $j \in N$. We show that $m_j(v_{E,c}) = m_j(v) - v(\{j\}) = m_j(v)$ for every $j \in N$.

Since $(N, v_{E,c})$ is convex, we have $m_j(v_{E,c}) = v_{E,c}(\{j\})$ for every $j \in N$. Then, for $j \in N$,

$$\begin{aligned} m_j(v_{E,c}) &= v_{E,c}(\{j\}) \\ &= \max\left\{0, E - \sum_{k \in N \setminus \{j\}} c_k\right\} \\ &= \max\left\{0, v(N) - \sum_{k \in N} v(\{k\}) - \sum_{k \in N \setminus \{j\}} (M_k(v) - v(\{k\}))\right\} \\ &= \max\left\{0, v(N) - \sum_{k \in N \setminus \{j\}} M_k(v)\right\} \end{aligned}$$

$$= \max\left\{v(\{j\}), v(N) - \sum_{k\in N\setminus\{j\}} M_k(v)\right\}$$

$$= m_j(v) - v(\{j\})$$

Then, by (i), we have that $nuc^1(v) = nuc(v)$.

(iii) First, let $m(v) < M(v)$. We show that $m_j(v) = \max\{v(\{j\}), v(N) - \sum_{k\in N\setminus\{j\}} M_k(v)\}$ for every $j \in N$. By (ii), we then have that $nuc^1(v) = nuc(v)$. On the contrary, suppose that there exists $i \in N$ and $S \in 2^N \setminus \{\emptyset, N\}$ with $S \ni i$ such that $m_i(v) = v(S) - \sum_{j\in S\setminus\{i\}} M_j(v) > \max\{v(\{i\}), v(N) - \sum_{j\in N\setminus\{i\}} M_j(v)\}$. Then, we arrive at a contradiction since

$$\begin{aligned}
m_i(v) &= v(S) - \sum_{j\in S\setminus\{i\}} M_j(v) \\
&\leq \max\left\{\sum_{j\in S} m_j(v), v(N) - \sum_{j\in N\setminus S} M_j(v)\right\} - \sum_{j\in S\setminus\{i\}} M_j(v) \\
&= \max\left\{m_i(v) + \sum_{j\in S\setminus\{i\}} (m_j(v) - M_j(v)), v(N) - \sum_{j\in N\setminus\{i\}} M_j(v)\right\} \\
&< m_i(v)
\end{aligned}$$

where the first inequality follows from Theorem 2.2 and the second one is a direct consequence of $M(v) > m(v)$ and our supposition.

Second, let $m(v) = M(v)$. Since $v \in I^N$ is a compromise stable game and $m(v) = M(v)$, it follows that $\sum_{i\in N} m_i(v) = v(N) = \sum_{i\in N} M_i(v)$ and $nuc(v) = M(v) = \text{AM}(N, E, c) = nuc^1(v)$.

□

Remark 5.1. As a consequence of Theorem 5.3, we can identify several well-known classes of compromise admissible games for which the nucleolus and the 1-nucleolus coincide: big boss games, clan games (see Potters et al., 1989), 1-convex games (see Driessen , 1983) and 2-convex games (see Driessen , 1983).

6 1-nucleolus and exactness

In this section, we examine the relationship between the 1-nucleolus and the compromise value (Tijs , 1981) for exact games.

The following result characterizes the class of (zero-normalized) exact games where the 1-nucleolus and the compromise value coincide.

Theorem 6.1. *Let $v \in G^N$ be an exact game with $v = v_0$. Then, $nuc^1(v) = \tau(v)$ if, and only if, one of the following conditions hold.*

(i) $2v(N) = \sum_{j\in N} M_j(v)$.

(ii) $v(N\setminus\{i\}) = v(N\setminus\{j\})$, *for every* $i,j\in N$.

(iii) $v(N) = \sum_{j\in N} M_j(v)$.

Proof. Since v is an exact game, $m_i(v) = v(\{i\}) = 0$ for every $i\in N$. Moreover, since every exact game is balanced, $M_i(v)\geq 0$ for every $i\in N$. Note that if $c = M(v) = 0$, then, $v(N) = 0$, $\tau(v) = 0 = nuc^1(v)$, and (i), (ii), and (iii) hold. We assume, without loss of generality, that $c\neq 0$. Then, for every $i\in N$,

$$\tau_i(v) = \frac{v(N)}{\sum_{j\in N} c_j} c_i.$$

First, for the only if part, assume that $nuc^1(v) = \tau(v)$. Since (N,v) is balanced, we have that $nuc^1(v) = \mathrm{AM}(N,E,c)$ with $E = v(N)$ and $c = M(v)$ by Theorem 4.2. Following the definition of the Aumann-Maschler rule, we have:

- if $E\leq \frac{1}{2}\sum_{j\in N} c_j$, then, if $|N|\frac{\min\{c_k|k\in N\}}{2} > E$,

  $$nuc^1_j(v) = \frac{E}{|N|} \text{ for all } j\in N$$

  and if $|N|\frac{\min\{c_k|k\in N\}}{2}\leq E$, there exists $i\in N$ such that

  $$nuc^1_j(v) = \begin{cases} \frac{c_j}{2} & \text{for all } j\in N \text{ with } c_j\leq c_i,\\ \frac{E-\sum_{k\in N|c_k\leq c_i}\frac{c_k}{2}}{|\{k\in N|c_k>c_i\}|} & \text{for all } j\in N \text{ with } c_j > c_i.\end{cases}$$

- if $E > \frac{1}{2}\sum_{j\in N} c_j$, then, if $|N|\frac{\min\{c_k|k\in N\}}{2} > \sum_{k\in N} c_k - E$,

  $$nuc^1_j(v) = c_j - \frac{\sum_{k\in N} c_k - E}{|N|} \text{ for all } j\in N$$

  and if $|N|\frac{\min\{c_k|k\in N\}}{2}\leq \sum_{k\in N} c_k - E$, there exists $i\in N$ such that

  $$nuc^1_j(v) = \begin{cases} \frac{c_j}{2} & \text{for all } j\in N \text{ with } c_j\leq c_i,\\ c_j - \frac{\sum_{k\in N} c_k - E - \sum_{k\in N|c_k\leq c_i}\frac{c_k}{2}}{|\{k\in N|c_k>c_i\}|} & \text{for all } j\in N \text{ with } c_j > c_i.\end{cases}$$

Then, we consider three cases: (i) there is $i\in N$ with $nuc^1_i(v) = \frac{c_i}{2}$, (ii) $nuc^1_i(v) = \frac{E}{|N|} = \frac{v(N)}{|N|}$ for every $i\in N$, and (iii) $nuc^1_i(v) = c_i - \frac{\sum_{j\in N} c_j - E}{|N|} = c_i - \frac{\sum_{j\in N} c_j - v(N)}{|N|}$ for every $i\in N$.

(i) There is $i\in N$ with $nuc^1_i(v) = \frac{c_i}{2}$. Since $nuc^1_i(v) = \tau_i(v)$, we have

$$\frac{c_i}{2} = \frac{v(N)}{\sum_{j\in N} c_j} c_i.$$

Thus, $2v(N) = \sum_{j\in N} c_j$ and Condition (i) holds.

(ii) For every $i \in N$, we have $nuc_i^1(v) = \frac{v(N)}{|N|}$. Then, $\frac{c_i}{2} \geq \frac{v(N)}{|N|}$ for every $i \in N$. Besides, since $nuc^1(v) = \tau(v)$, we have, for every $i, j \in N$,

$$\tau_i(v) = \frac{v(N)}{\sum_{k \in N} c_k} c_i = \frac{v(N)}{|N|} = \frac{v(N)}{\sum_{k \in N} c_k} c_j = \tau_j(v),$$

which implies $c_i = c_j$ for every $i, j \in N$. Thus, $v(N \setminus \{i\}) = v(N \setminus \{j\})$ for every $i, j \in N$ and Condition (ii) holds.

(iii) For every $i \in N$, we have $nuc_i^1(v) = c_i - \frac{\sum_{j \in N} c_j - v(N)}{|N|}$. Then, $2v(N) > \sum_{j \in N} c_j$ and $nuc_i^1(v) = c_i - \frac{\sum_{j \in N} c_j - v(N)}{|N|} \geq \frac{c_i}{2}$ for every $i \in N$. Then, the coincidence of the compromise value and the 1-nucleolus gives

$$\frac{v(N)}{\sum_{j \in N} c_j} c_i = c_i - \frac{\sum_{j \in N} c_j - v(N)}{|N|} \quad \text{for every } i \in N.$$

This implies that

$$\frac{\sum_{j \in N} c_j - v(N)}{|N|} = c_i \left(1 - \frac{v(N)}{\sum_{j \in N} c_j}\right) = \frac{c_i}{\sum_{j \in N} c_j} \left(\sum_{j \in N} c_j - v(N)\right)$$

for every $i \in N$. Thus, either $v(N) = \sum_{j \in N} c_j$, or $|N| c_i = \sum_{j \in N} c_j$ for every $i \in N$. That is to say, either $v(N) = \sum_{j \in N} c_j = \sum_{j \in N} M_j(v)$, or $M_i(v) = c_i = c_j = M_j(v)$ for every $i, j \in N$. Then, Condition *(ii)* or Condition *(iii)* holds.

Second, with respect to the if part, assume that one of the three conditions hold.

(i) If $2v(N) = \sum_{j \in N} M_j(v)$, then, $\tau(v) = \frac{v(N)}{\sum_{j \in N} M_j(v)} M(v) = \frac{v(N)}{2v(N)} M(v) = \frac{1}{2} M(v)$ and $nuc^1(v) = \text{CEA}(N, v(N), \frac{1}{2} M(v)) = \frac{1}{2} M(v)$. Consequently, $\tau(v) = nuc^1(v)$.

(ii) If $v(N \setminus \{i\}) = v(N \setminus \{j\})$, for every $i, j \in N$, then, $M_i(v) = M_j(v)$ for all $i, j \in N$ and $\tau_i(v) = \frac{v(N)}{|N|}$ for all $i \in N$. Moreover, since $\sum_{j \in N} M_j(v) = |N| M_i(v) \geq v(N)$ and the Aumann-Maschler rule satisfies equal treatment of equals (see Thomson, 2003), we have $nuc_i^1(v) = \frac{v(N)}{|N|}$ for all $i \in N$. Consequently, $\tau(v) = nuc^1(v)$.

(iii) If $v(N) = \sum_{j \in N} M_j(v)$, then, $\tau(v) = M(v) = nuc^1(v)$.

□

Remark 6.1. Since every convex game is also exact, Proposition 6.1 can be used to characterize the class of convex games for which the 1-nucleolus and the compromise value coincide, too.

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Appendix

Proof of Theorem 4.1.
We assume, without loss of generality, that $N = \{1, \ldots, n\}$ and $c_1 \leq c_2 \ldots \leq c_n$. Note that if $x \in I(v)$, then,

$$e(\{j\}, x) = v(\{j\}) - x_j = -x_j$$

and

$$e(N \setminus \{j\}, x) = v(N \setminus \{j\}) - \sum_{l \in N \setminus \{j\}} x_l = v(N \setminus \{j\}) - (v(N) - x_j) = x_j - c_j.$$

(i) We have $c_j \geq 0$ for every $j \in N$.

Case (i.a): $E \leq \sum_{j \in N} c_j$.

We show that $nuc^1(v) = \mathrm{AM}(N, E, c)$. Note that (N, E, c) is a bankruptcy problem. We distinguish between two situations: $E \leq \frac{1}{2} \sum_{j \in N} c_j$ and $E > \frac{1}{2} \sum_{j \in N} c_j$.

Case (i.a.1): $E \leq \frac{1}{2} \sum_{j \in N} c_j$.

By definition of the Aumann-Maschler rule, $\mathrm{AM}(N, E, c) = \mathrm{CEA}(N, E, \frac{1}{2}c)$ where $\mathrm{CEA}_j(N, E, \frac{1}{2}c) = \min\{\lambda, \frac{1}{2}c_j\}$ for every $j \in N$ and $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{j \in N} \mathrm{CEA}_j(N, E, \frac{1}{2}c_j) = E$.
Let $c_0 = 0$ and let $i \in N$ satisfy

$$\frac{c_{i-1}}{2} \leq \lambda < \frac{c_i}{2}.$$

Then, $\lambda = \frac{1}{|N| - i + 1}(E - \sum_{l=0}^{i-1} \frac{c_l}{2})$ and

$$\mathrm{AM}_j(N, E, c) = \begin{cases} \dfrac{c_j}{2} & \text{if } 1 \leq j \leq i-1, \\ \lambda & \text{if } i \leq j \leq n. \end{cases}$$

Let $x = \mathrm{AM}(N, E, c)$. Then,

$$e(\{j\}, x) = -x_j = \begin{cases} -\dfrac{c_j}{2} & \text{if } 1 \leq j \leq i-1, \\ -\lambda & \text{if } i \leq j \leq n \end{cases}$$

and

$$e(N \setminus \{j\}, x) = x_j - c_j = \begin{cases} -\dfrac{c_j}{2} & \text{if } 1 \leq j \leq i-1, \\ \lambda - c_j & \text{if } i \leq j \leq n. \end{cases}$$

Therefore,

$$\begin{aligned} e(\{1\}, x) = e(N \setminus \{1\}, x) \geq \ldots \geq e(\{i-1\}, x) = e(N \setminus \{i-1\}, x) = -\frac{c_{i-1}}{2} \geq -\lambda \\ = e(\{i\}, x) = \ldots = e(\{n\}, x) = -\lambda > -\lambda + 2\lambda - \frac{c_i}{2} = \lambda - \frac{c_i}{2} \end{aligned}$$

$$= e(N \setminus \{i\}, x) \geq \ldots \geq e(N \setminus \{n\}, x).$$

Then, by Theorem 3.3, we have that $x = nuc^1(v)$.

Case (i.a.2): $E > \frac{1}{2}\sum_{j\in N} c_j$.

By definition of the Aumann-Maschler rule, $\text{AM}(N, E, c) = c - \text{CEA}(N, \sum_{l\in N} c_l - E, \frac{1}{2}c)$ where $\text{CEA}_j(N, \sum_{l\in N} c_l - E, \frac{1}{2}c) = \min\{\lambda, \frac{1}{2}c_j\}$ for every $j \in N$ and $\lambda \in \mathbb{R}_+$ is chosen such that $\sum_{j\in N} \text{CEA}_j(N, \sum_{l\in N} c_l - E, \frac{1}{2}c) = \sum_{l\in N} c_l - E$.

Let $c_0 = 0$ and let $i \in N$ satisfy

$$\frac{c_{i-1}}{2} \leq \lambda < \frac{c_i}{2}.$$

Then, $\lambda = \frac{1}{|N|-i+1}\left(\sum_{l=1}^{|N|} c_l - E - \sum_{l=0}^{i-1} \frac{c_l}{2}\right)$ and

$$\text{AM}_j(N, E, c) = \begin{cases} \dfrac{c_j}{2} & \text{if } 1 \leq j \leq i-1, \\ c_j - \lambda & \text{if } i \leq j \leq n. \end{cases}$$

Let $x = \text{AM}(N, E, c)$. Then,

$$e(\{j\}, x) = -x_j = \begin{cases} -\dfrac{c_j}{2} & \text{if } 1 \leq j \leq i-1, \\ -c_j + \lambda & \text{if } i \leq j \leq n \end{cases}$$

and

$$e(N \setminus \{j\}, x) = x_j - c_j = \begin{cases} -\dfrac{c_j}{2} & \text{if } 1 \leq j \leq i-1, \\ -\lambda & \text{if } i \leq j \leq n. \end{cases}$$

Therefore,

$$\begin{aligned} e(\{1\}, x) = e(N \setminus \{1\}, x) \geq \ldots \geq e(\{i-1\}, x) = e(N \setminus \{i-1\}, x) &= -\frac{c_{i-1}}{2} \geq -\lambda \\ = e(N \setminus \{i\}, x) = \ldots = e(N \setminus \{n\}, x) &= -\lambda > -\lambda + 2\lambda - c_i = \lambda - c_i \\ = e(\{i\}, x) \geq \ldots \geq e(\{n\}, x). \end{aligned}$$

Then, by Theorem 3.3, we have that $x = nuc^1(v)$.

Case (i.b): $E > \sum_{j\in N} c_j$.

We show that $nuc^1(v) = c + \text{ES}(N, E - \sum_{j\in N} c_j, c)$. Let $x = c + \text{ES}(N, E - \sum_{j\in N} c_j, c)$. Then, $x_j = c_j + \frac{E - \sum_{l\in N} c_l}{|N|}$,

$$e(\{j\}, x) = -x_j = -c_j - \frac{E - \sum_{l\in N} c_l}{|N|}, \quad \text{and} \quad e(N \setminus \{j\}, x) = x_j - c_j = \frac{E - \sum_{l\in N} c_l}{|N|}$$

for every $j \in N$. Therefore,

$$e(N \setminus \{1\}, x) = \ldots = e(N \setminus \{n\}, x) > e(\{1\}, x) \geq \ldots \geq e(\{n\}, x)$$

where the strict inequality is a direct consequence of the fact that $\frac{E-\sum_{l\in N} c_l}{|N|} > 0 > -c_1 - \frac{E-\sum_{l\in N} c_l}{|N|}$. Then, by Theorem 3.3, we have that $x = nuc^1(v)$.

(ii) We have $c_j < 0$ for some $j \in N$.
Assume, without loss of generality, that $c_1 \leq \ldots \leq c_{\bar{k}} < 0 \leq c_{\bar{k}+1} \leq \ldots \leq c_n$ with $\bar{k} \in \{1, \ldots, n\}$.

Case (ii.a): $E \leq \sum_{j\in N} c_j^+$.
We show that $nuc^1(v) = \mathrm{AM}(N, E, c^+)$. Note that $c_1^+ = \ldots = c_{\bar{k}}^+ = 0$ and $c_j^+ = c_j$ for every $j \in \{\bar{k}+1, \ldots, n\}$. Moreover, (N, E, c^+) is a bankruptcy problem. By definition of the Aumann-Maschler rule, $\mathrm{AM}_j(N, E, c^+) = 0$ for every $j \in \{1, \ldots, \bar{k}\}$. Let $x = \mathrm{AM}(N, E, c^+)$. Then,

$$e(\{j\}, x) = -x_j = 0 \text{ and } e(N \setminus \{j\}, x) = x_j - c_j = -c_j > 0 \text{ for every } j \in \{1, \ldots, \bar{k}\}$$

and $B_0(x, v) \supseteq \{1, \ldots, \bar{k}\}$. Following the same lines as in Case (i.a) of this proof, we can show that $x = nuc^1(v)$.

Case (ii.b): $\sum_{j\in N} c_j^+ < E \leq \sum_{j\in N} c_j^{\min}$.
We show that $nuc^1(v) = \mathrm{CEL}(N, E, c^{\min})$, with $\mathrm{CEL}_j(N, E, c^{\min}) = \max\{0, c_j^{\min} - \lambda\}$ for every $j \in N$ and $\lambda \in \mathbb{R}_+$ chosen such that $\sum_{j\in N} \mathrm{CEL}_j(N, E, c^{\min}) = E$. Note that $c_j^{\min} = c_j - \min\{c_l | l \in N\} = c_j - c_1$ for every $j \in N$ and $0 = c_1^{\min} \leq c_2^{\min} \leq \ldots \leq c_n^{\min}$. Moreover, it follows that $(N, E, c^{\min})$ is a bankruptcy problem.

Let $i \in N$ satisfy

$$c_{i-1}^{\min} \leq \lambda < c_i^{\min}.$$

Then, $\lambda = \frac{1}{|N|-i+1}\left(\sum_{l=i}^{|N|} c_l^{\min} - E\right)$ and

$$\mathrm{CEL}_j(N, E, c^{\min}) = \begin{cases} 0 & \text{if } 1 \leq j \leq i-1, \\ c_j^{\min} - \lambda & \text{if } i \leq j \leq n. \end{cases}$$

Let $x = \mathrm{CEL}(N, E, c^{\min})$. Then,

$$e(\{j\}, x) = -x_j = \begin{cases} 0 & \text{if } 1 \leq j \leq i-1, \\ -c_j^{\min} + \lambda & \text{if } i \leq j \leq n \end{cases}$$

and

$$e(N \setminus \{j\}, x) = x_j - c_j = x_j - c_j^{\min} - c_1 = \begin{cases} -c_j & \text{if } 1 \leq j \leq i-1, \\ -c_1 - \lambda & \text{if } i \leq j \leq n. \end{cases}$$

Before we write the excesses in non-increasing order, we show that

$$i - 1 \leq \bar{k}. \tag{1}$$

First, note that

$$-c_1 > \lambda$$

since

$$\begin{aligned}\lambda &= \frac{1}{|N|-i+1}\left(\sum_{l=i}^{|N|} c_l^{\min} - E\right) = \frac{1}{|N|-i+1}\left(\sum_{l=i}^{|N|}(c_l - c_1) - E\right)\\ &= \frac{1}{|N|-i+1}\left(\sum_{l=i}^{|N|} c_l - E\right) - c_1 \leq \frac{1}{|N|-i+1}\left(\sum_{l\in N} c_l^+ - E\right) - c_1 < -c_1,\end{aligned}$$

where the weak inequality is a direct consequence of the definition of c^+ and the strict inequality follows by the assumption $\sum_{l\in N} c_l^+ < E$.

Next, we show that $i-1 \leq \bar{k}$ by contradiction. Suppose, on the contrary, that $i-1 > \bar{k}$. Then, $c_{i-1} > 0$ by definition of $\bar{k}$ and $c_{i-1}^{\min} = c_{i-1} - c_1 > -c_1 > \lambda$. This establishes a contradiction with the definition of i. Therefore, we have $i-1 \leq \bar{k}$. Then,

$$B_0(x,v) = \{1,\ldots,i-1\}$$

and

$$\begin{aligned}e(N\setminus\{1\},x) \geq \ldots \geq e(N\setminus\{i-1\},x) \geq e(N\setminus\{i\},x) = \ldots = e(N\setminus\{n\},x)\\ > e(\{1\},x) = \ldots = e(\{i-1\},x) > e(\{i\},x) \geq \ldots \geq e(\{n\},x)\end{aligned}$$

where $e(N\setminus\{i-1\},x) \geq e(N\setminus\{i\},x)$ because $-c_{i-1} = -c_1 - c_{i-1}^{\min} \geq -c_1 - \lambda$ by definition of i; $e(N\setminus\{n\},x) > e(\{1\},x)$ since $-c_1 > \lambda$ and, then, $-c_1 - \lambda > 0$; $e(\{i-1\},x) > e(\{i\},x)$ since $c_i^{\min} > \lambda$ by definition of i. Then, by Theorem 3.3, we have that $x = nuc^1(v)$.

<u>Case (ii.c)</u>: $E > \sum_{j\in N} c_j^{\min}$.

We show that $nuc^1(v) = c + \mathrm{ES}(N, E - \sum_{j\in N} c_j, c)$. Let $x = c + \mathrm{ES}(N, E - \sum_{j\in N} c_j, c)$. Then, $x_j = c_j + \frac{E-\sum_{l\in N} c_l}{|N|}$,

$$e(\{j\},x) = -x_j = -c_j - \frac{E-\sum_{l\in N} c_l}{|N|}, \quad \text{and} \quad e(N\setminus\{j\},x) = x_j - c_j = \frac{E-\sum_{l\in N} c_l}{|N|}$$

for every $j \in N$. Therefore,

$$e(N\setminus\{1\},x) = \ldots = e(N\setminus\{n\},x) > e(\{1\},x) \geq \ldots \geq e(\{n\},x)$$

where the strict inequality is a direct consequence of the fact that $\frac{E-\sum_{l\in N} c_l}{|N|} > 0 > -c_1 - \frac{E-\sum_{l\in N} c_l}{|N|}$. Then, by Theorem 3.3, we have that $x = nuc^1(v)$. □

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