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Polluted River Problems and Games with a Permission Structure

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Abstract

Polluted rivers are harmful to human, animals and plants living along it. To reduce the harm, cleaning costs are generated. However, when the river passes through several different countries or regions, a relevant question is how should the costs be shared among the agents. Ni and Wang (2007) first consider this problem as *cost sharing problems on a river network*, shortly called *polluted river problems*. They consider rivers with one spring which was generalized by Dong, Ni, and Wang (2012) to rivers with multiple springs. They introduce and axiomatize three cost sharing methods: the Local Responsibility Sharing (LRS) method, the Upstream Equal Sharing (UES) method and the Downstream Equal Sharing (DES) method.

In this paper, we show that the UES and DES methods can also be obtained as the *conjunctive permission value* of an associated *game with a permission structure*, where the permission structure corresponds to the river structure and the game is determined by the cleaning costs. Then, we show that several axiomatizations of the conjunctive permission value also give axiomatizations of the UES and DES methods, of which one is comparable with the one from Dong, Ni, and Wang (2012). Besides, by applying another solution, the disjunctive permission value, to polluted river games with a permission structure we obtain a new cost allocation method for polluted river problems. We axiomatize this solution and compare it with the UES method.

Keywords: Polluted river, cost sharing, axiomatization, permission values.

JEL code: C71; D61; D62

1 Introduction

The allocation of (clean) river water has gained attention in the recent literature. In particular, there is a growing literature on applying game theory to such allocation problems, see e.g. Ambec and Sprumont (2002), Parrachino, Dinar, and Patrone (2006), van den Brink, van der Laan, and Vasil'ev (2007), Ambec and Ehlers (2008), Khmelnitskaya (2010), Wang (2011), Ansink and Weikard (2012), van den Brink, van der Laan, and Moes (2012) and van den Brink et al. (2014). Typically, the goal is to obtain an efficient allocation of water over the agents along the river, where water can stream from upstream to downstream agents against a possible monetary compensation from downstream to upstream agents to support this allocation.

Besides the allocation of available river water, Ni and Wang (2007) introduced a model of a situation where a river is polluted, and in order to consume the water cleaning costs must be made to clean the water. When the river passes through several different countries or regions, a natural question is how should the costs be shared among the agents. An extreme solution is that each country just pays for the cleaning cost at its own region. However, if upstream countries are also partly responsible for the pollution at a certain river segment, then it seems reasonable that upstream countries share in the pollution cost of their downstream countries. On the other hand, since downstream countries benefit from upstream countries cleaning the river, it might be reasonable that downstream countries contribute in the cleaning cost of upstream countries.

These issues are considered initially by Ni and Wang (2007) for single spring rivers, and generalized by Dong, Ni, and Wang (2012) for rivers with multiple springs. They introduced the so-called *cost sharing problem on a river network*, shortly called *polluted river problem*, where besides a river structure, for every river segment a fixed cleaning cost is given.[1] They introduce and axiomatize three cost sharing methods reflecting the three different forms of responsibility mentioned above: the Local Responsibility Sharing (LRS) method, the Upstream Equal Sharing (UES) method and the Downstream Equal Sharing (DES) method. They also show that these methods can be obtained as the Shapley value of associated games.

In this paper, we first show that the UES and DES methods coincide with the *conjunctive permission value* (Gilles and Owen (1994), van den Brink and Gilles (1996)) of an associated *game with a permission structure*. Games with a permission structure model situations where players in a cooperative transferable utility game belong to some hierarchical structure where players need permission from some of their superiors before they can cooperate with other players. The polluted river problems correspond to games with a permission structure where the game is the inessential game where the worth of each coalition is the sum of the cleaning costs for all agents in the coalition (which is the Local Responsibility game used by Dong, Ni, and Wang (2012) to

[1] Alcalde-Unzu, Gómez-Rúa, and Molis (2015) recently extended this model by having transfer rates about how pollution flows through the river, so one can take more precise care about who is responsible for the pollution in a river segment.

obtain the LRS method), and the digraph (permission structure) is the sink tree corresponding to the river structure with the arcs oriented from upstream to downstream agents.

After establishing that the UES method can be obtained as conjunctive permission value, we apply the axiomatization of the conjunctive permission value of van den Brink and Gilles (1996) to the class of polluted river problems which essentially are games with a permission structure where the game is inessential and the digraph is a sink tree. We show that this yields an axiomatization of the UES method and discuss the differences and similarities with that of Dong, Ni, and Wang (2012). Comparing these two axiomatic systems, we find that the advantage of introducing an axiomatization by games with a permission structure is threefold: (i) it splits one allocation principle into two others which is in line with the goal of axiomatization, (ii) by putting it in a more general context, we will see that new axiomatizations and even new cost allocation rules appear, and (iii) we can do without a strong independence axiom. Also, it turns out that the axioms have a good interpretation in terms of water allocation principles in International Water Law.

Kilgour and Dinar (1995) studied general principles to resolve water allocation disputes resulting from International Water Law, which leads a direction of the implications of the method. Two important principles are Absolute Territorial Sovereignty (shortly ATS, also known as the Harmon doctrine) and Territorial Integration of all Basin States (shortly TIBS). Absolute Territorial Sovereignty (ATS) states that every country has the absolute sovereignty over the inflow of the river on its own territory. Territorial Integration of all Basin states (TIBS) states that 'the water of an international watercourse belongs to all basin states combined, no matter where it enters the watercourse. It does not make any country the legal owner of water. Each basin state is entitled to a reasonable and equitable share in the optimal use of the available water' (, see Lipper (1967) and McCaffrey (2001)). TIBS can be interpreted in several ways. For the allocation of clean river water, Ambec and Sprumont (2002) take the Unlimited Territorial Integrity (UTI) interpretation saying that a state has the right to demand the natural flow of an international watercourse into its territory that is undiminished by its upstream states (stated in the rules of the Helsinki Convention on water rights of the International Law Association (1966)).

A problem with water allocation principles as described above is that often they can be interpreted in several ways, or are in conflict with each other. Here, cooperative game theory has an important contribution since one of the main objectives of cooperative game theory is to find axioms of solutions that are compatible, preferably yielding a unique solution. Applied to polluted river problems we wish that the axioms that characterize a cost allocation method are compatible and have a good interpretation in terms of water allocation principles from International Water Law. It turns out that the axioms underlying the UES method that are derived from the axioms characterizing the conjunctive permission value have a good interpretation of such water allocation principles, in particular of UTI.

Another advantage of studying the UES method as a conjunctive permission value for a specific class of games with a permission structure is that other axiomatizations of the conjunctive permission value can be applied. In this way, we find that a new axiomatization of the UES method by applying the axiomatization in van den Brink (1999) yielding a new axiom for polluted river problems. This new axiom is called *externality fairness* and reflects what happens if one agent stops to participate in the cleaning cost agreement among all agents. Specifically, suppose that the subriver consisting of i and all its upstream agents retreat from the agreement and only pay their own cost and do not contribute anymore in the cleaning cost of the others, in particular not for its downstream neighbour j and the other agents. Of course, then those other agents will not contribute to the cleaning cost of i and its upstream agents. Then the complement should pay its own cost. Externality fairness requires that in this case the change (increase) of the contribution of j in the cost of its component (in the new cooperation structure) should be equal to the change in the contribution of any of its other upstream neighbours. So, the refusal of an upstream neighbour of j to contribute to the cleaning cost in the river component with j affects the contributions of the other upstream neighbours of j by the same amount as j. This reflects UTI in the sense that an agent and an upstream neighbour are equally responsible in the extra contribution that has to be made when another upstream neighbour stops cooperation,. This also reflects the principles of *Equitable Utilization of River Water* implying that each state can use the river water unless this use negatively affects other states, and *The Mutual Use Principle* stating that a state may object to another state's use of river water, unless it receives a reasonable direct compensation.

Another advantage of the relation between polluted river problems and games with a permission structure is that other solutions for games with a permission structure can be applied. For example, by applying the *disjunctive permission value* of Gilles and Owen (1994), axiomatized in van den Brink (1997), we obtain a new cost allocation method, called the *Upstream Limited Sharing* (ULS) method. We apply the axiomatization of the disjunctive permission value to obtain an axiomatization of this new cost sharing method, yielding a new axiom, which is called *participation fairness*, and reflects what happens if one agent stops to participate in the cleaning cost agreement among all agents in a different way than externality fairness of the UES method. We also show that the ULS method can be obtained as the Shapley value of another newly defined game on the polluted river problems with multiple springs. This result can be used either for evaluating or as an alternative (direct) definition of the ULS method. We compare the ULS and UES methods with an example of Dong, Ni, and Wang (2012), and show that ULS emphasizes more local responsibility than upstream responsibility, which is in line with the water allocation principles and therefore easier to be implemented.

Finally, by reversing the orientation of the arcs in the permission structure, orienting them from downstream to upstream, applying the conjunctive permission value we obtain the DES method. Since for games with a permission structure where the permission structure is a rooted tree, the conjunctive and disjunctive permission values coincide, the DES method can also be obtained as the disjunctive permission value of the associated game with a permission structure. Also for this method we obtain an axiomatization from the literature on games with a permission structure and compare it with the one of Dong, Ni, and Wang (2012).

The paper is organized as follows. Section 2 contains preliminaries on games with a permission structure (being the tool that we will use) and polluted river problems (being the allocation problem to which we will apply this tool). In Section 3 we show that the UES method coincides with the conjunctive permission value of an associated game with a permission structure, and provide axiomatizations. In Section 4 we apply the disjunctive permission value yielding the new ULS method for polluted river problems, and provide an axiomatization. In Section 5, we show that by reversing the orientation of the arcs we obtain the DES method as conjunctive as well as disjunctive permission value. We end with concluding remarks.

2 Preliminaries

2.1 Cooperative TU-games, graphs and digraphs

2.1.1 TU-games

A situation in which a finite set of players $N \subset \mathbb{N}$ can generate certain payoffs by cooperation can be described by a *cooperative game with transferable utility* (or simply a TU-game), being a pair (N, v) where $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* on N satisfying $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S) \in \mathbb{R}$ is the *worth* of coalition S, i.e. the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate. If there is no confusion about the player set, we denote a TU-game (N, v) just by its characteristic function v. We denote the collection of all TU-games by $\mathcal{G}$ and the collection of all characteristic functions on player set N by $\mathcal{G}^N$.

A *payoff vector* for game $(N, v) \in \mathcal{G}$ is an $|N|$-dimensional vector $x \in \mathbb{R}^N$ assigning a payoff $x_i \in \mathbb{R}$ to any player $i \in N$. A (single-valued) *solution* for TU-games is a function $f : \mathcal{G} \to \mathbb{R}^N$ that assigns a payoff vector to every TU-game. One of the most famous solutions for TU-games is the *Shapley value* (Shapley (1953)) given by

$$Sh_i(N, v) = \sum_{S \subseteq N : i \in S} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} \left(v(S) - v(S \setminus \{i\})\right),$$

A game v is *additive* or *inessential* if $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$.

2.1.2 Digraphs

A directed graph or *digraph* is a pair (N, D) where $N \subset \mathbb{N}$ is a finite set of nodes (representing the players) and $D \subseteq N \times N$ is a binary relation on N. We assume the digraph to be irreflexive, i.e., $(i,i) \notin D$ for all $i \in N$. Again, if there is no confusion about the set of nodes N, we denote a digraph (N, D) just by its binary relation D. We denote the collection of all binary relations on N by $\mathcal{D}^N$. For $i \in N$, the nodes in $P_D(i) := \{j \in N \mid (j,i) \in D\}$ are called the *predecessors* of i in D, and the nodes in $P_D^{-1}(i) := \{j \in N \mid i \in P_D(j)\} = \{j \in N \mid (i,j) \in D\}$ are called the *successors* of i. For given $D \in \mathcal{D}^N$, a (directed) *path* from i to j in N is a sequence of distinct nodes $(h_1, \ldots, h_t)$ such that $h_1 = i$, $h_{k+1} \in P_D^{-1}(h_k)$ for $k = 1, \ldots, t-1$, and $h_t = j$. The *transitive closure* of $D \in \mathcal{D}^N$ is the digraph $tr(D)$ given by $(i,j) \in tr(D)$ if and only if there is a directed path (in D) from i to j. By $\widehat{P}_D(i) = P_{tr(D)}(i)$ we denote the set of predecessors of i in the transitive closure of D, and refer to these players as the *superiors* of i in D. We refer to the players in $\widehat{P}_D^{-1}(i) = \{j \in N \mid i \in \widehat{P}_D(j)\}$ as the *subordinates* of i in D. A digraph $D \in \mathcal{D}^N$ is transitive if $D = tr(D)$. For a set of players $S \subseteq N$ we denote by $P_D(S) = \bigcup_{i \in S} P_D(i)$, respectively, $P_D^{-1}(S) = \bigcup_{i \in S} P_D^{-1}(i)$, the sets of predecessors, respectively successors of players in coalition S. Also, for $S \subseteq N$, we denote $\widehat{P}_D(S) = \bigcup_{i \in S} \widehat{P}_D(i)$ and $\widehat{P}_D^{-1}(S) = \bigcup_{i \in S} \widehat{P}_D^{-1}(i)$.

A directed path $(i_1, \ldots, i_t)$, $t \geq 2$, in D is a *cycle* in D if $(i_t, i_1) \in D$. We call digraph D *acyclic* if it does not contain any cycle. We denote the class of all acyclic digraphs on N by $\mathcal{D}_A^N$. Note that acyclicity of digraph D implies that D has at least one node that does not have a predecessor, and at least one node that does not have a successor. We denote $T(D) = \{i \in N \mid P_D(i) = \emptyset\}$ the set of nodes that do not have a predecessor, and $B(D) = \{i \in N \mid P_D^{-1}(i) = \emptyset\}$ the set of nodes that does not have a successor.

A digraph $D \in \mathcal{D}^N$ is a rooted tree if and only if there is an $i_0 \in N$ such that (i) $T(D) = \{i_0\}$, (ii)$\widehat{P}_D^{-1}(i_0) = N \setminus \{i_0\}$, and (iii) $|P_D(i)| = 1$ for all $i \in N \setminus \{i_0\}$. In this case, i_0 is called the *root* of the tree. Note that this implies that D is acyclic.

A digraph $D \in \mathcal{D}^N$ is a sink tree if and only if there is an $i_s \in N$ such that (i) $B(D) = \{i_s\}$, (ii) $\widehat{P}_D(i_s) = N \setminus \{i_s\}$, and (iii) $|P_D^{-1}(i)| = 1$ for all $i \in N \setminus \{i_s\}$. Note that this also implies that D is acyclic. In this case, i_s is called the *sink* of the tree.

2.2 Games with a permission structure

A game with a permission structure describes a situation where some players in a TU-game need permission from other players before they are allowed to cooperate within a coalition. A permission structure can be described by a directed graph on N.

A triple (N, v, D) with $N \subset \mathbb{N}$ a finite set of players, $v \in \mathcal{G}^N$ a TU-game and $D \in \mathcal{D}^N$ a digraph on N is called a *game with a permission structure*. We denote by $\mathcal{GP}$ the collection of all games with a permission structure.

In the *conjunctive approach* as introduced in Gilles, Owen, and van den Brink (1992) and van den Brink and Gilles (1996) it is assumed that a player needs permission from all its predecessors in order to cooperate with other players. Therefore a coalition is feasible if and only if for any player in the coalition all its predecessors are also in the coalition. So, for permission structure D the set of *conjunctive feasible coalitions* is given by

$$\Phi_D^c = \{S \subseteq N \mid P_D(i) \subseteq S \text{ for all } i \in S\}.$$

Since Φ_D^c is union closed, i.e. the union of any two feasible coalitions is also feasible, every coalition has a unique largest feasible subset. The induced *conjunctive restricted* game of the game with permission structure (N, v, D) is the game $r_{v,D}^c \colon 2^N \to \mathbb{R}$, given by

$$r_{v,D}^c(S) = v\left(\bigcup_{\{T \in \Phi_D^c \mid T \subseteq S\}} T\right) \text{ for all } S \subseteq N, \tag{2.1}$$

i.e., the restricted game $r_{v,D}^c$ assigns to each coalition $S \subseteq N$ the worth of its largest conjunctive feasible subset. Then the *conjunctive permission value* φ^c is the solution that assigns to every game with a permission structure the Shapley value of the conjunctive restricted game, thus

$$\varphi^c(N, v, D) = Sh(N, r_{v,D}^c) \text{ for all } (N, v, D) \in \mathcal{GP}.$$

Alternatively, in the *disjunctive approach* to acyclic permission structures as introduced in Gilles and Owen (1994) and van den Brink (1997) it is assumed that a player needs permission from at least one of its predecessors (if it has any) in order to cooperate with other players. Therefore a coalition is feasible if and only if for any player in the coalition at least one of its predecessors (if it has any) is also in the coalition. So, for permission structure D the set of *disjunctive feasible coalitions* is given by

$$\Phi_D^d = \{S \subseteq N \mid P_D(i) \cap S \neq \emptyset \text{ for all } i \in S, i \notin T(D)\}.$$

Again, by union closedness of Φ_D^d we can define the induced *disjunctive restricted* game of the game with permission structure (N, v, D) as the game $r_{v,D}^d \colon 2^N \to \mathbb{R}$, given by

$$r_{v,D}^d(S) = v\left(\bigcup_{\{T \in \Phi_D^d \mid T \subseteq S\}} T\right) \text{ for all } S \subseteq N, \tag{2.2}$$

i.e., the restricted game $r_{v,D}^d$ assigns to each coalition $S \subseteq N$ the worth of its largest disjunctive feasible subset. Then the *disjunctive permission value* φ^d is the solution that assigns to every game with a permission structure the Shapley value of the disjunctive restricted game, thus

$$\varphi^d(N, v, D) = Sh(N, r_{v,D}^d) \text{ for all } (N, v, D) \in \mathcal{GP}.$$

Player $i \in N$ is *inessential* in game with permission structure (N, v, D) if i and all its subordinates are *null* players in (N, v), i.e., if $v(S) = v(S \setminus \{j\})$ for all $S \subseteq N$ and $j \in \{i\} \cup \widehat{P}_D^{-1}(i)$. Player $i \in N$ is called *necessary* in game (N, v) if $v(S) = 0$ for all $S \subseteq N \setminus \{i\}$. A TU-game $(N, v) \in \mathcal{G}$ is *monotone* if $v(S) \leq v(T)$ for all $S \subseteq T \subseteq N$. The class of all monotone games is denoted by $\mathcal{G}_M$. Next we recall some axiomatizations of the permission values.[2]

Efficiency For every $(N, v, D) \in \mathcal{GP}$, it holds that $\sum_{i \in N} f_i(N, v, D) = v(N)$.

Additivity For every $(N, v, D), (N, w, D) \in \mathcal{GP}$, it holds that $f(N, v + w, D) = f(N, v, D) + f(N, w, D)$, where $(v + w) \in \mathcal{G}^N$ is given by $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$.

Inessential player property For every $(N, v, D) \in \mathcal{GP}$, if $i \in N$ is an inessential player in (N, v, D) then $f_i(N, v, D) = 0$.

Necessary player property For every $(N, v, D) \in \mathcal{GP}$ with $(N, v) \in \mathcal{G}_M$, if $i \in N$ is a necessary player in (N, v) then $f_i(N, v, D) \geq f_j(N, v, D)$ for all $j \in N$.

Structural monotonicity For every $(N, v, D) \in \mathcal{GP}$ with $(N, v) \in \mathcal{G}_M$, if $i \in N$ and $j \in P_D^{-1}(i)$ then $f_i(N, v, D) \geq f_j(N, v, D)$.

These five axioms characterize the conjunctive permission value.

Theorem 2.1 (van den Brink and Gilles (1996))**.** *A solution f on $\mathcal{GP}$ is equal to the conjunctive Shapley permission value φ^c if and only if it satisfies efficiency, additivity, the inessential player property, the necessary player property and structural monotonicity.*

On the class of games with an acyclic permission structure, the disjunctive permission value satisfies all axioms except structural monotonicity.[3] It satisfies a weaker monotonicity requiring the inequality only if player $j \in N$ dominates player $i \in N$ *completely* in the sense that all directed 'permission paths' from a top-player in $T(D)$ to player i contain player j. We denote the set of players that completely dominate player i by $\overline{P}_D(i)$, i.e.,

$$\overline{P}_D(i) = \left\{ j \in \widehat{P}_D(i) \;\middle|\; \begin{array}{c} j \in \{h_1, \ldots, h_{t-1}\} \text{ for every sequence of nodes } h_1, \ldots, h_t \\ \text{such that } h_1 \in T(D),\ h_k \in P_D(h_{k+1}) \text{ for} \\ k \in \{1, \ldots, t-1\}, \text{ and } h_t = i \end{array} \right\}. \tag{2.3}$$

We also define $\overline{P}_D^{-1}(i) = \{j \in \widehat{P}_D^{-1}(i) \mid i \in \overline{P}_D(j)\}$.

Weak structural monotonicity For every $(N, v, D) \in \mathcal{GP}$ with $(N, v) \in \mathcal{G}_M$, if $i \in N$ and $j \in \overline{P}_D^{-1}(i)$ then $f_i(N, v, D) \geq f_j(N, v, D)$.

[2]We refer to van den Brink and Gilles (1996) and van den Brink (1997, 1999) for a discussion of these properties.

[3]The axioms that are defined before for the class of all games with a permission structure can be straightforwardly defined on any subclass of games with a permission structure.

Further, the disjunctive permission value satisfies *disjunctive fairness* which states that deleting the arc between two players h and $j \in P_D^{-1}(h)$ (with $|P_D(j)| \geq 2$) changes the payoffs of players h and j by the same amount. Moreover, also the payoffs of all players i that completely dominate player h change by this same amount. The conjunctive permission value does not satisfy this disjunctive fairness. However, it satisfies the alternative *conjunctive fairness* which states that deleting the arc between two players h and $j \in P_D^{-1}(h)$ changes the payoffs of player j and any *other* predecessor $k \in P_D(j) \setminus \{h\}$ by the same amount. Moreover, also the payoffs of all players that completely dominate the other predecessor k change by this same amount.

For $D \in \mathcal{D}_A^N$, $h \in N$ and $j \in P_D^{-1}(h)$ we denote the permission structure that is left after deleting the arc between h and j by

$$D_{-(h,j)} = D \setminus \{(h, j)\}.$$

Disjunctive fairness For every $(N, v, D) \in \mathcal{GP}$ with $D \in \mathcal{D}_A^N$, if $h \in N$ and $j \in P_D^{-1}(h)$ with $|P_D(j)| \geq 2$ then $f_j(N, v, D) - f_j(N, v, D_{-(h,j)}) = f_i(N, v, D) - f_i(N, v, D_{-(h,j)})$ for all $i \in \{h\} \cup \overline{P}_D(h)$.

Conjunctive fairness For every $(N, v, D) \in \mathcal{GP}$ with $D \in \mathcal{D}_A^N$, if $h, j, k \in N$ are such that $h \neq k$ and $h, k \in P_D(j)$, then $f_j(N, v, D) - f_j(N, v, D_{-(h,j)}) = f_i(N, v, D) - f_i(N, v, D_{-(h,j)})$ for all $i \in \{k\} \cup \overline{P}_D(k)$.

Theorem 2.2. [4]

(i) (van den Brink (1997)) *A solution f on the class of games with an acyclic permission structure is equal to the disjunctive Shapley permission value φ^d if and only if it satisfies efficiency, additivity, the inessential player property, the necessary player property, weak structural monotonicity and disjunctive fairness.*

(ii) (van den Brink (1999)) *A solution f on the class of games with an acyclic permission structure is equal to the conjunctive Shapley permission value φ^c if and only if it satisfies efficiency, additivity, the inessential player property, the necessary player property, weak structural monotonicity and conjunctive fairness.*

2.3 Polluted river problems

Consider the *cost sharing problem on a river network*, shortly called *polluted river problem*, on rivers with multiple springs (sink tree structures) as introduced by Dong, Ni, and Wang (2012), generalizing Ni and Wang (2007). Such a polluted river problem is given by a triple (N, D, c),

[4] In the mentioned articles in Theorem 2.2, these axiomatizations are shown for games with an acyclic and quasi-strongly connected permission structure. A digraph D is *quasi-strongly connected* if there exists an $i \in N$ such that $\widehat{P}_D^{-1}(i) = N \setminus \{i\}$. These results can straightforwardly be extended to games with an acyclic permission structure.

where $N \subset \mathbb{N}$ is a finite set of agents located along a river, $D \subset N \times N$ is a sink tree that represents the river structure, and $c \in \mathbb{R}^N_+$ is an $|N|$-dimensional cost vector.[5] The river structure D is such that the river water flows along the arcs in D such that if $(i, j) \in D$ then river water flows from agent i to its downstream neighbour j. So, the arcs in D are the river segments. The sink is denoted by $L \in N$. From here the river flows into a sea or lake. The cost vector $c \in \mathbb{R}^N_+$ is such that c_i is the cost of cleaning the river segment between agent i and its unique downstream neighbour. For the sink c_L is the cost of cleaning the river before it flows into the sea or lake. We denote by $\mathcal{R}$ the class of all polluted river problems (N, D, c). Note that the agents in $P_D(i)$ are the upstream neighbours, and $P^{-1}_D(i)$ consists of the unique downstream neighbour of $i \in N$ in the river structure (N, D) where $|P^{-1}_D(i)| = 1$ for $\forall i \neq L$.

A *cost allocation* for a polluted river problem $(N, D, c) \in \mathcal{R}$ is a vector $y \in \mathbb{R}^N_+$, where y_i is the cost to be paid by agent $i \in N$ in the total joint cleaning cost of the river $\sum_{i \in N} c_i$. A *cost sharing method* $g : \mathcal{R} \to \mathbb{R}^N_+$ is a mapping that assigns a cost allocation to every polluted river problem.

The following three cost sharing methods are introduced and axiomatized by Dong, Ni, and Wang (2012). First, the *Local Responsibility Sharing method*, shortly LRS method, assigns to every agent its own cost, and thus is given by

$$g^{LRS}_i(N, D, c) = c_i \text{ for all } i \in N.$$

The *Upstream Equal Sharing method*, shortly UES method, equally shares the cost of cleaning a certain river segment over all agents that are located upstream of that segment and thus is given by

$$g^{UES}_i(N, D, c) = \sum_{j \in \{i\} \cup \widehat{P}^{-1}_D(i)} \frac{c_j}{\left|\{j\} \cup \widehat{P}_D(j)\right|} \text{ for all } i \in N. \tag{2.4}$$

Finally, the *Downstream Equal Sharing method*, shortly DES method, equally shares the cost of a certain river segment over all agents that are located downstream of that segment and thus is given by

$$g^{DES}_i(N, D, c) = \sum_{j \in \{i\} \cup \widehat{P}_D(i)} \frac{c_j}{\left|\{j\} \cup \widehat{P}^{-1}_D(j)\right|} \text{ for all } i \in N.$$

The LRS method reflects that the cleaning cost of each river segment is paid by the corresponding local agent, while the UES method reflects that the cost is (equally) shared by the upstream agents and the DES method reflects that the cost is (equally) shared by the downstream agents of the river segment. For a discussion on these solutions and water allocation principles, we refer to Dong, Ni, and Wang (2012).

[5] We remark that our notation is slightly different from that of Dong, Ni, and Wang (2012) but the models are equivalent.

Dong, Ni, and Wang (2012) also associate three TU-games to polluted river problems $(N, D, c) \in \mathcal{R}$. The first one is the (additive) *stand-alone game* $L^{sa}_{(N,D,c)}$ given by

$$L^{sa}_{(N,D,c)}(\emptyset) = 0 \text{ and } L^{sa}_{(N,D,c)}(S) = \sum_{i \in S} c_i \text{ for all } S \subseteq N.$$

The second is the *Upstream-oriented game* $L^{U}_{(N,D,c)}$ given by

$$L^{U}_{(N,D,c)}(\emptyset) = 0 \text{ and } L^{U}_{(N,D,c)}(S) = \sum_{i \in S \cup \widehat{P}^{-1}_D(S)} c_i \text{ for all } S \subseteq N.$$

The third is the *Downstream-oriented game* $L^{D}_{(N,D,c)}$ given by

$$L^{D}_{(N,D,c)}(\emptyset) = 0 \text{ and } L^{D}_{(N,D,c)}(S) = \sum_{i \in S \cup \widehat{P}_D(S)} c_i \text{ for all } S \subseteq N.$$

They show that the LRS-, UES- and DES methods can be obtained by applying the Shapley value to the stand-alone, Upstream-oriented, respectively Downstream-oriented game.

Further, Dong, Ni, and Wang (2012) provide axiomatizations using the following axioms. (We refer to their article for a discussion relating these to water allocation principles.)

Efficiency For every $(N, D, c) \in \mathcal{R}$, it holds that $\sum_{i \in N} g_i(N, D, c) = \sum_{i \in N} c_i$.

Additivity For any $(N, D, c'), (N, D, c'') \in \mathcal{R}$, we have $g(N, D, c' + c'') = g(N, D, c') + g(N, D, c'')$.

Independence of Irrelevant Costs For every $(N, D, c) \in \mathcal{R}$, and $i, j \in N$ such that $j \in N \setminus (\widehat{P}_D(i) \cup \{i\} \cup \widehat{P}^{-1}_D(i))$, we have that $g_j(N, D, c) = 0$ whenever $c_h = 0$ for all $h \in N \setminus \{i\}$.

Independence of Upstream Costs For every $(N, D, c), (N, D, c') \in \mathcal{R}$ and $i \in N$ such that $c_h = c'_h$ for all $h \in \widehat{P}^{-1}_D(i)$, we have that $g_j(N, D, c) = g_j(N, D, c')$ for all $j \in \widehat{P}^{-1}_D(i)$.

Upstream Symmetry For every $(N, D, c) \in \mathcal{R}$ and $i \in N$, it holds that $g_j(N, D, c) = g_k(N, D, c)$ for all $j, k \in \{i\} \cup \widehat{P}_D(i)$, whenever $c_h = 0$ for all $h \in N \setminus \{i\}$.

Independence of Downstream Costs For every $(N, D, c), (N, D, c') \in \mathcal{R}$ and $i \in N$ such that $c_h = c'_h$ for all $h \in \widehat{P}_D(i)$, we have that $g_j(N, D, c) = g_j(N, D, c')$ for all $j \in \widehat{P}_D(i)$.

Downstream Symmetry For every $(N, D, c) \in \mathcal{R}$ and $i \in N$, it holds that $g_j(N, D, c) = g_k(N, D, c)$ for all $j, k \in \{i\} \cup \widehat{P}^{-1}_D(i)$, whenever $c_h = 0$ for all $h \in N \setminus \{i\}$.

Theorem 2.3 (Dong, Ni, and Wang (2012))**.** [6]

[6]Besides these axiomatizations, Dong, Ni, and Wang (2012) axiomatize the LRS method by Efficiency, Additivity and No Blind Cost, the last axiom requiring that for every $(N, D, c) \in \mathcal{R}$ and $i \in N$ such that $c_i = 0$, we have $g_i(N, D, c) = 0$.

(i) *The UES method is the unique cost allocation method satisfying efficiency, additivity, independence of upstream costs, upstream symmetry and independence of irrelevant costs.*

(ii) *The DES method is the unique cost allocation method satisfying efficiency, additivity, independence of downstream costs, downstream symmetry and independence of irrelevant costs.*

3 The UES method and the permission values

3.1 An Axiomatization

In van den Brink, van der Laan, and Vasil'ev (2014) it is mentioned that, in case the river has a single spring (as in Ni and Wang (2007)), the Upstream-oriented game $L^U_{(N,D,c)}$ associated to a polluted river problem (N, D, c) equals the dual game of the conjunctive restricted game of the game with permission structure $(N, L^{sa}_{(N,D,c)}, D)$ of the stand-alone game $L^{sa}_{(N,D,c)}$ on the permission structure D associated to the river structure with the arcs oriented from upstream to downstream. This can easily be extended to rivers with multiple springs. Since the conjunctive permission value of a game with a permission structure is obtained as the Shapley value of the corresponding conjunctive restricted game, and the Shapley value of a game is equal to the Shapley value of its dual game (see Kalai and Samet (1987)), it follows that the UES method can be obtained by applying the conjunctive permission value to the game with permission structure $(N, L^{sa}_{(N,D,c)}, D)$.

Proposition 3.1. *For every polluted river problem $(N, D, c) \in \mathcal{R}$, the Upstream-oriented game $L^U_{(N,D,c)}$ is equivalent to the dual game of $r^c_{L^{sa}_{(N,D,c)},D}$.*[7]

Proof. Recall that the dual game of a game v, denoted by $\widetilde{v}$, on player set N is given by

$$\widetilde{v}(S) = v(N) - v(N \setminus S) \quad \text{for each } S \subseteq N.$$

From this definition, one has $\widetilde{r}^c_{L^{sa}_{(N,D,c)},D}(\emptyset) = 0$, which coincides with $L^U_{(N,D,c)}(\emptyset) = 0$. For any nonempty subset $S \subseteq N$, define $\sigma^c_D(S) = \bigcup_{\{T\in\Phi^c_D | T\subseteq S\}} T$, and thus $r^c_{v,D}(S) = v(\sigma^c_D(S))$ for all $S \subseteq N$. Since, for any $S \subseteq N$, $\sigma^c_D(N \setminus S) = \{i \in N \setminus S \mid \widehat{P}_D(i) \subseteq N \setminus S\} = \{i \in N \setminus S \mid \widehat{P}_D(i) \cap S = \emptyset\} = (N \setminus S) \setminus \widehat{P}^{-1}_D(S)$, we have

$$\begin{aligned}
\widetilde{r}^c_{L^{sa}_{(N,D,c)},D}(S) &= r^c_{L^{sa}_{(N,D,c)},D}(N) - r^c_{L^{sa}_{(N,D,c)},D}(N \setminus S) \\
&= \sum_{i\in N} c_i - L^{sa}_{(N,D,c)}(\sigma^c_D(N \setminus S)) \quad = \sum_{i\in N} c_i - L^{sa}_{(N,D,c)}((N \setminus S) \setminus \widehat{P}^{-1}_D(S)) \\
&= \sum_{i\in N} c_i - \sum_{i\in(N\setminus S)\setminus\widehat{P}^{-1}_D(S)} c_i \quad = \sum_{i\in S} c_i + \sum_{i\in(N\setminus S)\cap\widehat{P}^{-1}_D(S)} c_i \quad = \sum_{i\in S\cup\widehat{P}^{-1}_D(S)} c_i \\
&= L^U_{(N,D,c)}(S).
\end{aligned}$$

[7]This proposition holds under the more general condition that D is acyclic. The proof here does not require that D is a sink tree.

□

Since the Shapley value is self-dual, i.e. $Sh(v) = Sh(\widetilde{v})$ for all $v \in \mathcal{G}^N$, and the UES method is obtained as the Shapleuy value of $L^U_{(N,D,c)}$, we have the following corollary.

Corollary 3.2. *Let $(N, D, c) \in \mathcal{R}$ be a polluted river problem. Then*

$$g^{UES}(N, D, c) = \varphi^c(N, L^{sa}_{(N,D,c)}, D).$$

Since Corollary 3.2 shows that the UES method can be obtained by applying the conjunctive permission value to the stand-alone game on the up-downstream oriented permission structure D, we can verify the implication of the axioms underlying the conjunctive permission value for polluted river problems, and investigate if axioms that characterize the conjunctive permission value also give uniqueness on the class of Upstream-oriented games $\mathcal{GPR} = \{(N, v, D) \in \mathcal{GP} \mid v = L^{sa}_{(N,D,c)}$ for some $(N, D, c) \in \mathcal{R}\} \subset \mathcal{GP}$. Instead of considering this class of games with a permission structure, we directly interpret and apply the axioms in terms of polluted river problems.[8] It turns out that these axioms do not only provide uniqueness, but also are a good reflection of established water allocation principles.

To show equivalence between properties of solutions for games with a permission structure and cost allocation methods for polluted river problems, we say that a cost allocation method g is an *Upstream-oriented game method* if there is a solution f for games with a permission structure such that $g(N, D, c) = f(N, L^{sa}_{(N,D,c)}, D)$ for all $(N, D, c) \in \mathcal{R}$. Now, we can first state that efficiency for permission values on the class $\mathcal{GPR}$ is equivalent to efficiency for polluted river cost allocation methods in the sense that cost allocation method g given by $g(N, D, c) = f(N, L^{sa}_{(N,D,c)}, D)$ satisfies efficiency on $\mathcal{R}$ if and only if solution f satisfies efficiency on $\mathcal{GPR}$. In this sense also additivity for permission values on the class $\mathcal{GPR}$ is equivalent to additivity for polluted river cost allocation methods. The obvious proofs are omitted.

Next, we interpret the other axioms of Theorem 2.1. Since a player is an inessential player in game with permission structure $(N, L^{sa}_{(N,D,c)}, D)$ for some $(N, D, c) \in \mathcal{R}$, if and only if its own cost as well as the cost of all its subordinates is zero, the inessential player property for polluted river games with a permission structure is equivalent to requiring zero contributions for such agents.

Inessential agent property For every $(N, D, c) \in \mathcal{R}$ and $i \in N$ such that $c_j = 0$ for all $j \in \widehat{P}^{-1}_D(i) \cup \{i\}$, it holds that $g_i(N, D, c) = 0$.

[8] Note that there is a one-to-one correspondence between games with permission structure (N, v, D) with v an inessential game and D a sink tree, and polluted river problems. Above we saw that every polluted river problem (N, D, c) yields a game with a permission structure (N, v, D) with the permission structure D and the inessential game v determined by c. On the other hand, given an inessential game v with a sink tree permission structure D, the corresponding polluted river problem is determined by the permission structure D with costs equal to $c_i = v(\{i\})$ for all $i \in N$.

The inessential agent property is stronger than independence of irrelevant costs since it also states requirements for the payoffs in polluted river problems where more than one agent has a positive cleaning cost. Moreover, independence of irrelevant costs only considers cases where costs are zero for an agent, all its superiors and all its subordinates, while the inessential agent property can apply when superiors have a positive cost.

Proposition 3.3. *Every cost allocation method that satisfies the inessential agent property also satisfies independence of irrelevant costs.*

Proof. Suppose that cost allocation method g satisfies the inessential agent property, and let river problem $(N, D, c) \in \mathcal{R}$ be such that there is an $i \in N$ with $c_h = 0$ for all $h \in N \setminus \{i\}$. For $j \in N \setminus (\widehat{P}_D(i) \cup \{i\} \cup \widehat{P}_D^{-1}(i))$, we have that $c_k = 0$ for all $k \in \widehat{P}_D^{-1}(j) \cup \{j\}$, and thus $g_j(N, D, c) = 0$ by the inessential agent property. Thus, g satisfies independence of irrelevant costs. □

Since a player is a necessary player in a game with permission structure $(N, L^{sa}_{(N,D,c)}, D)$ for some $(N, D, c) \in \mathcal{R}$ if and only if the costs of all other agents is zero, and stand-alone games are monotone, the necessary agent property for polluted river games with a permission structure is equivalent to requiring that such an agent contributes at least as much as any other agent.

Necessary agent property For every $(N, D, c) \in \mathcal{R}$ and $i \in N$, with $c_j = 0$ for all $j \in N \setminus \{i\}$, it holds that $g_i(N, D, c) \geq g_j(N, D, c)$ for all $j \in N \setminus \{i\}$.

Finally, structural monotonicity for permission values is equivalent to requiring that upstream agents contribute at least as much as downstream agents.

Structural monotonicity For every $(N, D, c) \in \mathcal{R}$ and $i, j \in N$ with $i \in P_D(j)$, it holds that $g_i(N, D, c) \geq g_j(N, D, c)$.

Note that the structural monotonicity implies that $g_i(N, D, c) \geq g_j(N, D, c)$ for all $i \in \widehat{P}_D(j)$. The necessary agent property and structural monotonicity together are stronger than upstream symmetry.

Proposition 3.4. *Every cost allocation method that satisfies the necessary agent property and structural monotonicity also satisfies upstream symmetry.*[9]

Proof. Suppose that cost allocation method g satisfies the necessary agent property and structural monotonicity, and let polluted river problem $(N, D, c) \in \mathcal{R}$ be such that there is an $i \in N$ with $c_h = 0$ for all $h \in N \setminus \{i\}$. The necessary agent property implies that $g_i(N, D, c) \geq g_j(N, D, c)$

[9]In fact, neither the necessary agent property nor structural monotonicity on its own implies upstream symmetry, and upstream symmetry implies neither the necessary agent property nor structural monotonicity. We show these in Appendix A.

for all $j \in \widehat{P}_D(i)$. Structural monotonicity implies that $g_i(N, D, c) \leq g_j(N, D, c)$ for all $j \in \widehat{P}_D(i)$. Together these imply that $g_i(N, D, c) = g_j(N, D, c)$ for all $j \in \widehat{P}_D(i)$, and thus g satisfies upstream symmetry. □

It turns out that replacing independence of irrelevant costs, upstream symmetry and independence of upstream costs in Theorem 2.3 by the inessential agent property, the necessary agent property and structural monotonicity characterizes the UES method.

Theorem 3.5. *The UES method is the unique method that satisfies efficiency, additivity, the inessential agent property, the necessary agent property and structural monotonicity.*

Proof. It is straightforward that the UES method satisfies the five axioms. To show uniqueness, suppose that cost allocation method g satisfies the five axioms, and consider polluted river problem $(N, D, c) \in \mathcal{R}$. For every $i \in N$, define $c^i \in \mathbb{R}^N_+$ by $c^i_i = c_i$ and $c^i_j = 0$ for all $j \in N \setminus \{i\}$. The inessential agent property implies that $g_j(N, D, c^i) = 0$ for all $j \in N \setminus (\{i\} \cup \widehat{P}_D(i))$. By Proposition 3.4, g satisfies upstream symmetry, and thus $g_i(N, D, c^i) = g_j(N, D, c^i)$ for all $j \in \widehat{P}_D(i)$. Efficiency then determines that $g_i(N, D, c^i) = g_j(N, D, c^i) = c_i/(|\widehat{P}_D(\{i\})| + 1)$ for all $j \in \widehat{P}_D(i)$, which equals the payoffs assigned by the UES method. Finally, additivity determines the payoffs according to the UES method for any polluted river problem $(N, D, c) \in \mathcal{R}$ since $c = \sum_{i \in N} c^i$. □

The logical independence of the five axioms in Theorem 3.5 is shown in Appendix B.

The axioms in Theorem 3.5 have three main advantages. First, since the axioms of Theorem 3.5 are direct applications of the axioms for the conjunctive permission value in van den Brink and Gilles (1996), this also shows that these axioms characterize the conjunctive permission value on the smaller class of games with a permission structure that are obtained from polluted river problems. Moreover, we have put the UES method for polluted river problems in the broader context of games with a permission structure. Second, although the inessential agent property is stronger than independence of irrelevant costs, and the necessary agent property and structural monotonicity together are stronger than upstream symmetry, a main advantage is that these axioms together allow us to drop the independence of upstream costs used by Dong, Ni, and Wang (2012). Third, 'splitting' upstream symmetry in the necessary agent property and structural monotonicity, is in line with the main motivation for axiomatizing a solution, that is, to break up one method into 'smaller' principles or axioms. This also makes it more easy to generate a link between river cost sharing problem and international water resources sharing principles.

Turning to water allocation principles, water resources sharing and water pollution cost sharing methods have in common that they provide rules for upstream and downstream agents to reach agreement on the allocation or cleaning of river water. The axioms of Theorem 3.5 reflect such water allocation principles. *Efficiency* and *additivity* are discussed by Dong, Ni, and Wang (2012). The other axioms of Theorem 3.5 can be related to the following principles.

- *Absolute Territorial Sovereignty (ATS)* requires that a state has absolute sovereignty over the area of the river basin within it. This principle emphasis the local right. It implies that an agent has absolute responsibility over the cost at its local river segment. *The necessary agent property* reflects this principle weakly, as it only requires the local agent to share no less than others of its own cost. Or, in other words, the local agent is always most responsible for the costs generated within its river basin.

- *Unlimited Territorial Integrity* suggests that a riparian state has the right to demand the natural flow of an international watercourse into its territory by the upper riparian states. It implies responsibility of upstream agents to downstream agents. Both *inessential agent property* and *structural monotonicity* reflect the spirit of this principle. The former property assigns no responsibility of an upstream agent if there is no costs generated at its downstream; the latter always assigns higher responsibility to the upstream agents compared to its downstream ones.

3.2 A new axiomatization: externality fairness

Suppose that an agent with all its upstream agents stop being part of the pollution cleaning agreement. If we model this by deleting the link between this agent and its downstream agent then this results in two different river structures that act as if not connected to each other. Although the river structure itself does not change, the cooperation structure, which initially is the same as the river structure, might 'break up' in different components. Thus, the cooperation structure which reflects the participated agents in the agreement, is now a subgraph of the river structure.

Note that in conjunctive fairness, deleting an arc (i, j) means that j does not need permission anymore from i to cooperate with other players. In the polluted river problem, we want that when i stops participation in an agreement with $j \in P_D^{-1}(i)$, i and all its superiors will make a new agreement on their own, and similarly for j with the rest of the agents. This brings up the axiom of externality fairness. Suppose that the sub-river consisting of i and all its superiors retreat from the agreement and only pay their own cost and do not contribute anymore in the cleaning cost of the others, in particular not of j and its subordinates. Of course, then those other agents will not contribute to the cleaning cost of i and its superiors, and the complement should pay its own cost. Externality fairness states that in this case the change (increase) of the contribution of j in the cost of its component (in the new cooperation structure) should be equal to the change in the contribution of any of its other predecessors. So, the refusal of an upstream neighbour of j to contribute to the cleaning cost in the river component with j affects the contributions of the other upstream neighbours of j by the same amount as j. This partly reflects who has to pay extra in the cleaning cost of j when its upstream neighbour i stops contributing. According to this principle, the repsonsibility that was taken by upstream neighbour i is equally taken over by

j and each other upstream neighbour.

Before formally stating the axiom we introduce some notation. For river structure D, let $K^j_{ij}(D)$, $(i, j) \in D$, be the component containing j that is created after the deletion of the arc (i, j), i.e. $K^j_{ij}(D) = N \setminus (\{i\} \cup \widehat{P}_D(i))$. To simplify, we denote $K^j_{ij}(D)$ by K^j_{ij}. Note that L remains the sink in the polluted river problem $(K^j_{ij}, D|_{K^j_{ij}}, c|_{K^j_{ij}})$, where $D|_{K^j_{ij}} = \{(h, k) \in D \mid \{h, k\} \subseteq K^j_{ij}\}$ is the river structure restricted to K^j_{ij} (note that this is again a sink tree), and $c|_{K^j_{ij}}$ is the projection of the cost vector c on K^j_{ij}.

Externality fairness For any polluted river problem $(N, D, c) \in \mathcal{R}$ and $i, j \in N$ with $(i, j) \in D$, it holds that

$$g_j(K^j_{ij}, D|_{K^j_{ij}}, c|_{K^j_{ij}}) - g_j(N, D, c) = g_h(K^j_{ij}, D|_{K^j_{ij}}, c|_{K^j_{ij}}) - g_h(N, D, c)$$

for every $h \in P_D(j) \setminus \{i\}$.

Note that, besides a difference in interpretation, another difference with conjunctive fairness is that we only require equal change in payoffs for j and its upstream neighbours, while conjunctive fairness also requires this for the 'complete superiors' of the upstream neighbors of j. It turns out that the sink tree structure of the river allows this weakening of the axiom.

Using externality fairness, we can weaken structural monotonicity by requiring it only for an agent and its unique upstream neighbor. For sink trees, weak structural monotonicity can be restated as follows.

Weak structural monotonicity For any polluted river problem $(N, D, c) \in \mathcal{R}$ and any $j \in N$, if $P_D(j) = \{i\}$, then $g_i(N, D, c) \geq g_j(N, D, c)$.

For polluted river problems this is a considerable weakening of structural monotonicity since it only requires monotonicity with respect to an agent and its upstream neighbour in case it is its unique upstream neighbour, whereas structural monotonicity requires this between any pair of agents such that one is upstream of the other. Obviously, all pollution that enters an agent from the upstream river must pass through such an upstream neighbour. Structural monotonicity also applies to an upstream agent $i \in \widehat{P}_D(j) \setminus \overline{P}_D(j)$ is not on every path from a source to j. Such an upstream agent can argue that the pollution of the river segment at j is created by another flow of upstream agents, so it should not contribute to the cleaning cost at j. However, when i is the unique upstream neighbour of j then, although the pollution at j might not be created by agent i, in any case it is created by the river flow that goes through i and j.

It turns out that when a cost allocation method satisfies externality fairness then weak structural monotonicity does imply structural monotonicity.

Proposition 3.6. *Every cost allocation method that satisfies externality fairness and weak structural monotonicity also satisfies structural monotonicity.*

Proof. Suppose that cost allocation method g satisfies externality fairness and weak structural monotonicity, and consider polluted river problem $(N, D, c) \in \mathcal{R}$. It is obvious that the claim holds for line-rivers, i.e. with $|T(D)| = 1$, since in that case weak structural monotonicity is equivalent to structural monotonicity. We show that the claim also holds for general river structures by induction on $|T(D)|$. Assume that the claim holds for all rivers with $|T(D)| \leq m$ for some $m > 1$. Now for rivers with $|T(D)| = m + 1$, for any $j \in N$, if $|P_D(j)| = 1$, from weak structural monotonicity it follows that $g_i(N, D, c) \geq g_j(N, D, c)$ for $i \in P_D(j)$. If $|P_D(j)| > 1$, then for any $i \in P_D(j)$, externality fairness implies that

$$g_j(K_{ij}^j, D|_{K_{ij}^j}, c|_{K_{ij}^j}) - g_j(N, D, c) = g_h(K_{ij}^j, D|_{K_{ij}^j}, c|_{K_{ij}^j}) - g_h(N, D, c) \tag{3.5}$$

for all $h \in P_D(j) \setminus \{i\}$. Note that the number of springs of $D|_{K_{ij}^j}$ is less than $m + 1$. From the induction hypothesis we have $g_h(K_{ij}^j, D|_{K_{ij}^j}, c|_{K_{ij}^j}) \geq g_j(K_{ij}^j, D|_{K_{ij}^j}, c|_{K_{ij}^j})$, which with (3.5) implies that $g_h(N, D, c) \geq g_j(N, D, c)$. Thus g satisfies structural monotonicity. □

Note that structural monotonicity implies weak structural monotonicity, but does not lead to externality fairness. This is illustrated by, for example, the method $g^{\overline{UES}}$ defined in Appendix B, which satisfies structural monotonicity but does not satisfy externality fairness.

Theorem 3.7. *The UES method is the unique method that satisfies efficiency, additivity, the inessential agent property, the necessary agent property, weak structural monotonicity and externality fairness.*

Proof. It is straightforward that the UES method satisfies the first five axioms. For any polluted river problem $(N, D, c) \in \mathcal{R}$ and any $i, j \in N$ such that $i \in P_D(j)$ and $|P_D(j)| \geq 2$, it holds that

$$\begin{aligned} & g_h^{UES}(K_{ij}^j, D|_{K_{ij}^j}, c|_{K_{ij}^j}) - g_j^{UES}(K_{ij}^j, D|_{K_{ij}^j}, c|_{K_{ij}^j}) \\ = & \sum_{k \in \{h\} \cup \widehat{P}^{-1}_{D|_{K_{ij}^j}}(h)} \frac{c_k}{\left|\{k\} \cup \widehat{P}_{D|_{K_{ij}^j}}(k)\right|} - \sum_{k \in \{j\} \cup \widehat{P}^{-1}_{D|_{K_{ij}^j}}(j)} \frac{c_k}{\left|\{k\} \cup \widehat{P}_{D|_{K_{ij}^j}}(k)\right|} \\ = & \frac{c_h}{\left|\{h\} \cup \widehat{P}_{D|_{K_{ij}^j}}(h)\right|}, \end{aligned}$$

and

$$\begin{aligned} g_h^{UES}(N, D, c) - g_j^{UES}(N, D, c) &= \sum_{k \in \{h\} \cup \widehat{P}_D^{-1}(h)} \frac{c_k}{\left|\{k\} \cup \widehat{P}_D(k)\right|} - \sum_{k \in \{j\} \cup \widehat{P}_D^{-1}(j)} \frac{c_k}{\left|\{k\} \cup \widehat{P}_D(k)\right|} \\ &= \frac{c_h}{\left|\{h\} \cup \widehat{P}_D(h)\right|} \end{aligned}$$

for any $h \in P_D(j) \setminus \{i\}$. Since the number of superiors of h in D is equal to that in $D|_{K_{ij}^j}$, one has

$$g_h^{UES}(K_{ij}^j, D|_{K_{ij}^j}, c|_{K_{ij}^j}) - g_j^{UES}(K_{ij}^j, D|_{K_{ij}^j}, c|_{K_{ij}^j}) = g_h^{UES}(N, D, c) - g_j^{UES}(N, D, c),$$

implying that the UES method also satisfies externality fairness.

The uniqueness follows from Proposition 3.6 and Theorem 3.5. □

The logical independence of the six axioms in Theorem 3.7 is also shown in Appendix B.

Compared to the previous section we replaced structural monotonicity by weak structural monotonicity and externality fairness. Considering water allocation principles resulting from international water resources sharing disputes, similar to structural monotonicity, weak structural monotonicity reflects UTI but in a weaker form. Additionally, externality fairness requires that when the downstream agent stops the agreement with another upstream agent, the responsibility of the additional contribution to be made by an agent and its upstream neighbor is equal. This also reflects UTI as 'relational equal treatment' principle in the sense that it equalizes the changes of payoffs of different agents in case the (polluted river) situation changes in the same way from the perspective of these agents. Besides that, externality fairness can be related to the following water allocation principles.

- *Equitable Utilization of River Water* requires that each state can use the river water unless this use negatively affects other states. Since water use almost always has an effect on downstream countries, this principle has little direct implication except efficiency: agents in the river basin should take responsibility of the full cost to not negatively affect other states out of this basin. However, the principle becomes relevant in combination with the next principle.

- *The Mutual Use Principle* requires that a state may object to another state's use of river water, unless it receives reasonable direct compensation. In contrast to ATS, which emphasizes the local responsibility, the mutual use principle favors the downstream agents by allowing them to demand compensation from the upstream agents for their cleaning costs. Question is what is reasonable amount of compensation. This question is addressed by UES using a simple equal sharing rule between the upstream and downstream agents. Externality fairness is a dynamic equal treatment principle that requires equal changes in payoffs when the situation changes in some sense symmetrically for certain agents.

4 The ULS method and the permission values

Considering polluted river problems as games with a permission structure, we can define a new cost allocation method for polluted river problems by applying the disjunctive permission value to any polluted river problem. For sink trees, the conjunctive and disjunctive permission value differ except for directed line-graphs, i.e. single-spring rivers. Therefore, for all rivers with a sink tree structure with at least two springs, applying the disjunctive permission value yields a new allocation rule for polluted river problems.

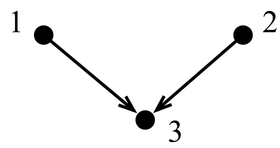

Figure 1: A river with 3 agents.

Definition 4.1. *The* Upstream Limited Sharing *method (ULS method) is given by*

$$g^{ULS}(N,D,c) = \varphi^d(N, L^{sa}_{(N,D,c)}, D) \textit{ for every } (N,D,c) \in \mathcal{R}.$$

The idea behind this ULS method is that agents who are predecessor, but not the only predecessor, of a downstream agent feel less responsible for cleaning the river at their downstream agent than according to the UES method. Consider, for example, the river (N, D, c) with $N = \{1,2,3\}$, $D = \{(1,3),(2,3)\}$ (and thus $L = 3$) and $c = (c_1, c_2, c_3) = (0, 0, c_3)$ with $c_3 > 0$, see Figure 1. According to the UES method, the cost c_3 is equally shared by the agents 1, 2 and 3, i.e. $g^{UES}(N,D,c) = (c_3/3, c_3/3, c_3/3)$. According to the ULS method the cost shares are $g^{ULS}(N,D,c) = (c_3/6,\ c_3/6,\ 2c_3/3)$ which are obtained as the Shapley value of the restricted game $r^d_{L^{sa}_{(N,D,c)},D}$ given by $r^d_{L^{sa}_{(N,D,c)},D}(S) = c_3$ if $S \in \{\{1,3\},\{2,3\},\{1,2,3\}\}$, and $r^d_{L^{sa}_{(N,D,c)},D}(S) = 0$ otherwise. Since agent 1 can argue that it is not responsible for the pollution at agent 3 (since it claims that the pollution comes from agent 2), the contribution of agent 1, $c_3/6$, is less than when agents 1, 2 and 3 are held equally responsible for the pollution at agent 3 (as in the UES method where agent 1 contributes $c_3/3$). The same argument holds for agent 2, yielding a cost allocation where the upstream agents 1 and 2 pay less in the cleaning cost at 3 than in the UES method. Although agent 3 might argue that the pollution comes from 1 or 2, the uncertainty about which agent is responsible yields a smaller responsibility and contribution of the upstream neighbours 1 and 2. Note that the ULS method yields some kind of compromise between the UES method and LRS method in the sense that according to the LRS method agent 3 has to pay its cost fully with no contribution from other agents, while according to the UES method c_3 is equally shared among agent 3 and its upstream agents. According to the ULS method the upstream agents 1 and 2 do contribute in the cleaning cost of agent 3, but less than agent 3.

Definition 4.1 is an indirect one in the sense that it is based on a disjunctive restricted game defined on another game generated from a polluted river problem. This two step definition increases the difficulty of understanding and evaluating the ULS solution. Here we provide an alternative direct definition by introducing a new game generated from a polluted river problem.

Define the Limited Upstream-oriented coalition $Q(S)$ for $S \subseteq N$ as

$$Q(S) = \bigcap \Big\{ F \;\Big|\; S \subseteq F \subseteq N, \emptyset \neq P_D(i) \subseteq F \Rightarrow i \in F \Big\}. \tag{4.6}$$

This definition says if $i \in T(D)$, then $i \in S$ is necessary and sufficient to have $i \in Q(S)$; if $i \notin T(D)$, then $i \in Q(S)$ if and only if $i \in S$ or $P_D(i) \subseteq Q(S)$. Obviously it holds that $Q(\emptyset) = \emptyset$ and $S \subseteq Q(S)$. Then the Limited Upstream-oriented game $L^{LU}_{(N,D,c)}$ associated to the polluted river problem $(N, D, c) \in \mathcal{R}$ is defined by

$$L^{LU}_{(N,D,c)}(\emptyset) = 0 \quad \text{and} \quad L^{LU}_{(N,D,c)}(S) = \sum_{i \in Q(S)} c_i \quad \text{for all} \quad S \subseteq N.$$

In the following proposition we show that the Limited Upstream-oriented game $L^{LU}_{(N,D,c)}$ associated to a polluted river problem (N, D, c) equals the dual game of the disjunctive restricted game of the game with permission structure $(N, L^{sa}_{(N,D,c)}, D)$ of the stand-alone game $L^{sa}_{(N,D,c)}$ on the permission structure D associated to the river structure with the arcs oriented from upstream to downstream.

Proposition 4.2. *For any polluted river problem $(N, D, c) \in \mathcal{R}$, $L^{LU}_{(N,D,c)}$ is the dual game of $r^d_{L^{sa}_{(N,D,c)},D}$.*

Proof. Denote by $\widetilde{L}^{LU}_{(N,D,c)}$ the dual game of $L^{LU}_{(N,D,c)}$. Thus,

$$\widetilde{L}^{LU}_{(N,D,c)} = L^{LU}_{(N,D,c)}(N) - L^{LU}_{(N,D,c)}(N \setminus S) = \sum_{i \in N} c_i - \sum_{i \in Q(N \setminus S)} c_i = \sum_{i \notin Q(N \setminus S)} c_i.$$

From the definition of $Q(S)$ it holds that $i \notin Q(N \setminus S)$ if and only if

$$\begin{cases} i \notin N \setminus S & \text{if } i \in T(D), \\ i \notin N \setminus S \text{ and } \exists j \in P_D(i) \text{ such that } j \notin Q(N \setminus S) & \text{if } i \notin T(D). \end{cases}$$

Define $Q^*(S) := N \setminus Q(N \setminus S)$. The fact above can be rewritten as $i \in Q^*(S)$ if and only if

$$\begin{cases} i \in S & \text{if } i \in T(D), \\ i \in S \text{ and } \exists j \in P_D(i) \text{ such that } j \in Q^*(S) & \text{if } i \notin T(D). \end{cases}$$

It is obvious that $Q^*(S) \subseteq S \subseteq Q(S)$. Therefore, it can be seen that $Q^*(S)$ is the largest disjunctive feasible subset of coalition S. Consequently, one has

$$\widetilde{L}^{LU}_{(N,D,c)} = \sum_{i \in Q^*(S)} c_i = r^d_{L^{sa}_{(N,D,c)},D}(S),$$

completing the proof. □

Since the Shapley value of a TU-game equals the Shapley value of its dual game, we have the following proposition.

Proposition 4.3.

$$g^{ULS}_i(N, D, c) = \sum_{S \subseteq N : i \in S} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} \left(\sum_{j \in Q(S)} c_j - \sum_{j \in Q(S \setminus \{i\})} c_j \right) \text{ for all } i \in N. \tag{4.7}$$

Equation (4.7) can be used as a definition of the ULS method.

4.1 An axiomatization: participation fairness

If the cost allocation method reflects this lower responsibility for upstream agents in case it is not sure where the pollution comes from, the question becomes to what extent this uncertainty should be reflected in the cost allocation method. Here disjunctive fairness plays a role which, in case of polluted river problems, states that when agent i stops participation in an agreement with its downstream neighbor j (and i with all its upstream agents will make a new agreement on their own without the other players, and the same for the component containing j), the change of the contribution of i (and each of its complete dominating superiors) and j after breaking the agreement should be equal. So, the refusal of an upstream neighbour of j to contribute to the cleaning cost in the river component with j affects j and the upstream neighbour by the same amount.

Before stating the axiom we need to introduce some notation. Recall from the previous section that for river structure D, $K^j_{ij}(D)$ with $(i,j) \in D$ such that $i \in P_D(j)$ is the component that is created after the deletion of the arc (i,j) and contains j. Next, we denote by $K^i_{ij}(D) = N \backslash K^j_{ij}(D) = \{i\} \cup \widehat{P}_D(i)$ the component that is created after the deletion of the arc (i,j) and contains i. Again, if there is no confusion about the river structure we denote $K^i_{ij}(D)$ just by K^i_{ij}. Note that i is the sink in the polluted river problem $(K^i_{ij}, D|_{K^i_{ij}}, c|_{K^i_{ij}})$ where $D|_{K^i_{ij}} = \{(h,k) \in D \mid \{h,k\} \subseteq K^i_{ij}\}$ is the river structure restricted to K^i_{ij}, and $c|_{K^i_{ij}}$ is the projection of the cost vector c on K^i_{ij}.

Participation fairness For any polluted river problem $(N, D, c) \in \mathcal{R}$ and $i, j \in N$ with $(i, j) \in D$ such that $|P_D(j)| \geq 2$, it holds that

$$g_j(K^j_{ij}, D|_{K^j_{ij}}, c|_{K^j_{ij}}) - g_j(N, D, c) = g_h(K^i_{ij}, D|_{K^i_{ij}}, c|_{K^i_{ij}}) - g_h(N, D, c), \tag{4.8}$$

for all $h \in \{i\} \cup \overline{P}_D(i)$.

Replacing externality fairness in Theorem 3.7 by participation fairness, characterizes the ULS method.

Lemma 4.4. *If $F \subseteq S \subseteq N$, then $Q(F) \subseteq Q(S)$.*

Proof. Let $F \subseteq S \subseteq N$. If $F = \emptyset$, it is clear that $Q(F) = \emptyset \subset Q(S)$. Assume $F \neq \emptyset$, and assume there exists some $i \in Q(F)$ such that $i \notin Q(S)$. $i \notin Q(S)$ implies $[i \notin S$ and $\emptyset \neq P_D(i) \nsubseteq Q(S)]$, or $[i \notin S$ and $P_D(i) = \emptyset]$. Since $i \notin S \Rightarrow i \notin F$, one has $\emptyset \neq P_D(i) \subseteq Q(F)$ from the assumption $i \in Q(F)$. Then there exists some $j \in P_D(i)$ such that $j \in Q(F) \setminus Q(S)$. Applying the same argument to j implies that there exists some $k \in P_D(j)$ such that $k \in Q(F) \setminus Q(S)$. One can repeat this argument infinitely many times, which then contradicts the fact that N is finite, and the fact that $k \in \widehat{P}_D(j)$ and $j \in P_D(i)$ implies that $i \notin P_D(k)$. Therefore, for any $i \in Q(F)$, it holds that $i \in Q(S)$, completing the proof. ◻

Proposition 4.5. *The ULS method satisfies efficiency, additivity, the inessential agent property, the necessary agent property, weak structural monotonicity and participation fairness.*

Proof. Efficiency, additivity, the inessential agent property, the necessary agent property and weak structural monotonicity, follows from Algaba et al. (2003, Theorem 1), the fact that the set of disjunctive feasible coalitions in an acyclic digraph is an antimatroid[10], and the definition of the ULS method as the disjunctive permission value of a game on a sink tree.[11]

To show participation fairness, note that the Shapley value also can be written using the Harsanyi dividends (Harsanyi (1959)) as

$$Sh_i(N, v) = \sum_{S \subseteq N: i \in S} \frac{\Delta_v(S)}{|S|},$$

where the Harsanyi dividend of coalition $S \subseteq N$ is given by $\Delta_v(\emptyset) = 0$ and $\Delta_v(S) = v(S) - \sum_{T \subset S: T \neq S} \Delta_v(T)$ for $S \neq \emptyset$, which can be seen as the extra value that is generated by cooperation of the players in S that was not yet generated by the proper subsets of S.

Now, for any polluted river problem $(N, D, c) \in \mathcal{R}$ and $i, j \in N$ with $(i, j) \in D$ such that $|P_D(j)| \geq 2$, letting $w = r^d_{L^{sa}_{(N,D,c)}}$ and $w|_T(S) = w(S)$ for all $S \subseteq T$, we can write

$$\begin{aligned}
& g_j^{ULS}(K^j_{ij}, D|_{K^j_{ij}}, c|_{K^j_{ij}}) - g_j^{ULS}(N, D, c) \\
&= \varphi^d(K^j_{ij}, L^{sa}_{(K^j_{ij}, D|_{K^j_{ij}}, c|_{K^j_{ij}})}, D|_{K^j_{ij}}) - \varphi^d(N, L^{sa}_{(N,D,c)}, D) \\
&= Sh_j(K^j_{ij}, r^d_{L^{sa}_{(K^j_{ij}, D|_{K^j_{ij}}, c|_{K^j_{ij}})}}) - Sh_j(N, r^d_{L^{sa}_{(N,D,c)}}) \\
&= \sum_{S \subseteq K^j_{ij}: j \in S} \frac{\Delta_{w|_{K^j_{ij}}}(S)}{|S|} - \sum_{S \subseteq N: j \in S} \frac{\Delta_w(S)}{|S|} \quad = \sum_{S \subseteq K^j_{ij}: j \in S} \frac{\Delta_w(S)}{|S|} - \sum_{S \subseteq N: j \in S} \frac{\Delta_w(S)}{|S|} \\
&= - \sum_{S \subseteq N: S \not\subseteq K^j_{ij}, j \in S} \frac{\Delta_w(S)}{|S|} \quad = - \sum_{S \subseteq N: S \cap K^i_{ij} \neq \emptyset, j \in S} \frac{\Delta_w(S)}{|S|},
\end{aligned}$$

where the fourth equality follows since $w(S) = w|_{K^j_{ij}}(S)$ for all $S \subseteq K^j_{ij}$.

Similarly it can be shown that

$$g_i^{ULS}(K^i_{ij}, D|_{K^i_{ij}}, c|_{K^i_{ij}}) - g_i^{ULS}(N, D, c) = - \sum_{S \subseteq N: S \cap K^j_{ij} \neq \emptyset, i \in S} \frac{\Delta_w(S)}{|S|}.$$

[10] A set of feasible coalitions $\mathcal{A} \subseteq 2^N$ is an antimatroid (see Edelman and Jamison (1985) and Korte, Lovász, and Schrader (1991)) if it satisfies the following three properties: (i) $\emptyset \in \mathcal{A}$ (feasible empty set), (ii) $S, T \in \mathcal{A}$ implies that $S \cup T \in \mathcal{A}$ (union closedness), and (iii) $S \in \mathcal{A}$ with $S \neq \emptyset$, implies that there exists $i \in S$ such that $S \setminus \{i\} \in \mathcal{A}$ (accessibility).

[11] To be self-contained we also give direct proofs in Appendix C.

Next, we define a coalition S to be *connected* if for all $i, j \in S$, it holds that one of the following three conditions is satisfied:

(i) $i \in \widehat{P}_D(j)$, or

(ii) $i \in \widehat{P}_D^{-1}(j)$, or

(iii) there is an $h \in S$ such that $h \in \widehat{P}_D^{-1}(i) \cap \widehat{P}_D^{-1}(j)$.

A coalition that is not connected is called *disconnected*.

To proceed with the proof we need the following lemma, whose proof can be found in Appendix C.

Lemma. *For any game $(N, v, D) \in \mathcal{GPR}$, the Harsanyi dividend $\Delta_{r^d_{v,D}}(S) = 0$ if $S \subseteq N$ is disconnected.*

Since (i) $S \notin \Phi^d_D$ implies $\Delta_w(S) = 0$ (see Algaba et al. (2003)), (ii) [$S \in \Phi^d_D$, $S \cap K^i_{ij} \neq \emptyset, j \in S$, and S is connected] implies that $\{i, j\} \subseteq S$, (iii) [$S \in \Phi^d_D$, $S \cap K^j_{ij} \neq \emptyset, i \in S$, and S is connected] implies that $\{i, j\} \subseteq S$, and (iv) S is disconnected implies $\Delta_w(S) = 0$ (see the lemma above), we have that

$$g^{ULS}_j(K^j_{ij}, D|_{K^j_{ij}}, c|_{K^j_{ij}}) - g^{ULS}_j(N, D, c) = g^{ULS}_i(K^i_{ij}, D|_{K^i_{ij}}, c|_{K^i_{ij}}) - g^{ULS}_i(N, D, c).$$

Since [$S \in \Phi^d_D$ and $i \in S$] implies that $h \in S$ for all $h \in \overline{P}_D(i)$, we get that also

$$g^{ULS}_j(K^j_{ij}, D|_{K^j_{ij}}, c|_{K^j_{ij}}) - g^{ULS}_j(N, D, c) = g^{ULS}_h(K^i_{ij}, D|_{K^i_{ij}}, c|_{K^i_{ij}}) - g^{ULS}_h(N, D, c)$$

for all $h \in \overline{P}_D(i)$, showing that participation fairness is satisfied. □

Next we state the axiomatization of the ULS method.

Theorem 4.6. *The ULS method is the unique method that satisfies efficiency, additivity, the inessential agent property, the necessary agent property, weak structural monotonicity and participation fairness.*

Proof. Proposition 4.5 shows that the ULS method satisfies all the axioms.

To show uniqueness, suppose that cost allocation method g satisfies the six axioms, and consider polluted river problem (N, D, c). We prove the uniqueness of allocation method g for rivers with one sink by induction on the number of sources. We first show that for line-rivers the ULS method is uniquely determined by all axioms (except participation fairness). A line-river has only one spring i_0 and satisfies $|P_D(i)| = 1$ for $i \in N \setminus \{i_0\}$. For any $i \in N$, let $c^i \in \mathbb{R}^N_+$ be given by by $c^i_i = c_i$ and $c^i_j = 0$ for all $j \in N \setminus \{i\}$. Similar to the proof of Theorem 3.5, the inessential agent property implies that $g_j(N, D, c^i) = 0$ for all $j \in N \setminus (\{i\} \cup \widehat{P}_D(i))$. By the necessary agent property, there is an $a \in \mathbb{R}$ such that

$$g_i(N, D, c^i) = a \text{ and } g_j(N, D, c^i) \leq a \text{ for all } j \in \widehat{P}_D(i). \tag{4.9}$$

By repeated application of weak structural monotonicity, it holds that

$$g_j(N, D, c^i) \geq g_i(N, D, c^i) \text{ for all } j \in \widehat{P}_D(i). \tag{4.10}$$

Equation (4.9) and (4.10) imply $g_j(N, D, c^i) = a$ for all $j \in \{i\} \cup \widehat{P}_D(i)$. Efficiency then determines that $a = \frac{c_i}{|\widehat{P}_D(i)|+1}$ for all $j \in \{i\} \cup \widehat{P}_D(i)$. Finally, additivity determines the payoffs according to the ULS method for any $c \in \mathbb{R}^N_+$.

Proceeding by induction, assume that $g(N, D, c^i)$ is uniquely determined under the six axioms for all rivers with $|T(D)| \leq m$. For polluted river problems (N, D, c) with $|T(D)| = m + 1$, we will show that there are $|N|$ independent linear equations of $|N|$ unknown variables $g_i(N, D, c^\ell)$, $i \in N$, for each $\ell \in N$, which means $g(N, D, c^\ell)$ is uniquely determined. Then $g(N, D, c)$ is obtained by additivity. Note that $|D| = |N| - 1$. We establish one equation associated with each arc in D. Since the river structure is a sink tree, every arc falls into one of the following cases:

(1) Suppose that $(i, j) \in D$ is such that $|P_D(j)| \geq 2$. Then from participation fairness we have

$$g_j(K^j_{ij}, D|_{K^j_{ij}}, c^\ell|_{K^j_{ij}}) - g_j(N, D, c^\ell) = g_i(K^i_{ij}, D|_{K^i_{ij}}, c^\ell|_{K^i_{ij}}) - g_i(N, D, c^\ell), \tag{4.11}$$

where $g_j(K^j_{ij}, D|_{K^j_{ij}}, c^\ell|_{K^j_{ij}})$ and $g_i(K^i_{ij}, D|_{K^i_{ij}}, c^\ell|_{K^i_{ij}})$ are already determined by the induction hypothesis because both river $(K^j_{ij}, D|_{K^j_{ij}})$ and $(K^i_{ij}, D|_{K^i_{ij}})$ have at most m springs.

(2) Suppose that $(i, j) \in D$ is such that $|P_D(j)| = 1$. This case further splits into two sub-cases:

(2-1) Suppose that there is an $h \in \overline{P}^{-1}_D(i)$ such that $|P_D(P^{-1}_D(h))| \geq 2$. Let $P^{-1}_D(h) = \{k\}$. Then from participation fairness we have

$$g_k(K^k_{hk}, D|_{K^k_{hk}}, c^\ell|_{K^k_{hk}}) - g_k(N, D, c^\ell) = g_i(K^h_{hk}, h, D|_{K^h_{hk}}, c^\ell|_{K^h_{hk}}) - g_i(N, D, c^\ell), \tag{4.12}$$

where $g_k(K^k_{hk}, D|_{K^k_{hk}}, c^\ell|_{K^k_{hk}})$ and $g_i(K^h_{hk}, h, D|_{K^h_{hk}}, c^\ell|_{K^h_{hk}})$ are already determined by the induction hypothesis.

(2-2) The sink $L \in \overline{P}^{-1}_D(i)$. In this case the equation depends on the location of agent ℓ. We consider again two subcases.

(2-2-1) If $\ell \in \widehat{P}_D(i) \cup \{i\}$, then by the inessential agent property it holds that

$$g_j(N, D, c^\ell) = 0; \tag{4.13}$$

(2-2-2) if $\ell \in \widehat{P}^{-1}_D(i)$, then by the necessary agent property and weak structural monotonicity, one has

$$g_i(N, D, c^\ell) = g_\ell(N, D, c^\ell). \tag{4.14}$$

The equations (4.11), (4.12) and (4.13) or (4.14)) yield $|D| = |N| - 1$ linear independent equations in the $|N|$ unknown variables $g_i(N, D, c^\ell)$, $i \in N$. Together with the last linear equation

$$\sum_{i \in N} g_i(N, D, c^\ell) = \sum_{i \in N} c_i^\ell = c_\ell,$$

which follows from efficiency, we can uniquely determine $g(N, D, c^\ell)$ for each $\ell \in N$. Additivity then uniquely determines $g(N, D, c)$. Since the ULS method satisfies these six axioms, g is the ULS method. □

The logical independence of the six axioms in Theorem 4.6 is again shown in Appendix B.

Compared with externality fairness, participation fairness equalizes the change in contribution between two agents if cooperation stops along the river segment between them. It is an expression of fairness where two agents are equally responsible when cooperation between them stops. Put differently, when two agents decide to let their components cooperate then they benefit equally from that. In contrast, externality fairness expresses a fairness property between an agent and its remaining upstream neighbours when cooperation with one of its upstream neighbours stops. This reflects that the agent and its remaining upstream neighbours are equally responsible for the additional cost caused by the withdrawal of one of its upstream neighbours.

Similarly, participation fairness can be also related to the same water allocation principles as externality fairness (UTI, Equitable Utilization of River Water, and The Mutual Use Principle). However, it generates a different effect on the responsibility of an upstream agent to its downstream neighbors.

4.2 Comparison of UES and ULS: an example

We demonstrate the difference between the ULS method and UES method by applying the ULS method to the example discussed in Section 3.4 of Dong, Ni, and Wang (2012), where the UES solution is evaluated. This example models the Baiyangdian Lake Catchment in Northern China, see Dong, Ni, and Wang (2012) for details. The river structure and costs are depicted in Figure 2, which is reproduced from Figure 3 of Dong, Ni, and Wang (2012). The solutions are summarized in Table 1.[12]

From Table 1, we can see that the ULS method allocates less costs to all the top agents compared to the UES method. In contrast, it allocates much higher costs to the agent at the bottom. This shows that the ULS method favors the upstream agents by emphasizing the local responsibility. From the table, we can also see that for agents with middle position (agents with both upstream and downstream neighbours), the difference of these two methods depends on

[12] It should be noted that the final calculation of UES method given in Dong, Ni, and Wang (2012) contains some mistakes. Specifically, the UES solution of agent 4, 5, 6, 7, 9, 10, 11, 12 and 13 on page 385 of their paper are incorrect.

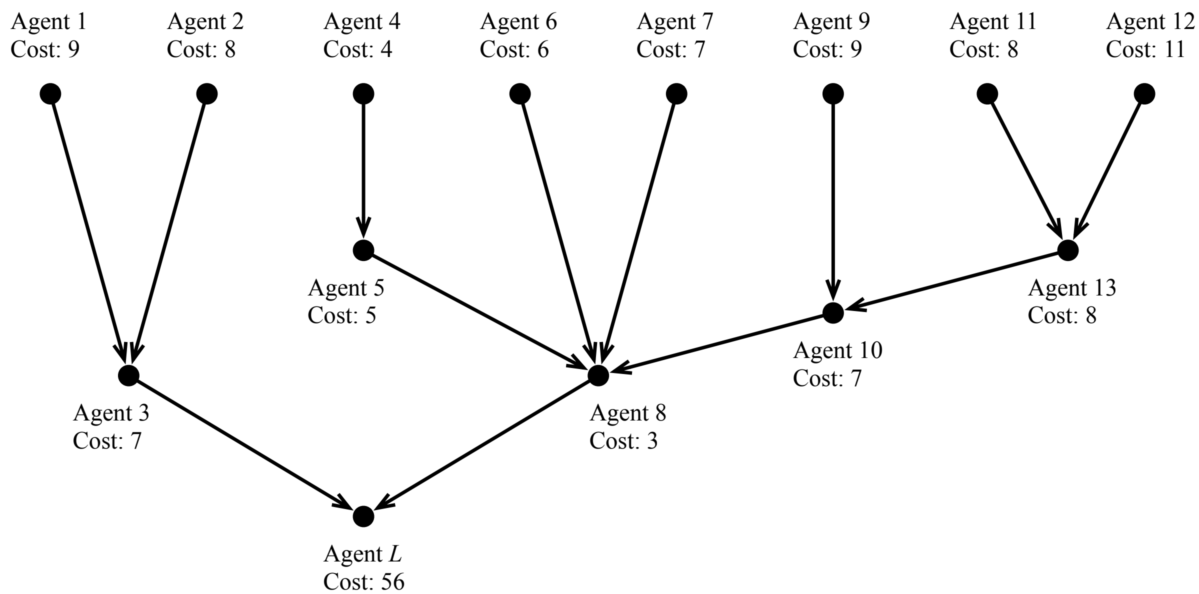

Figure 2: The river structure and costs reproduced from Figure 3 of Dong, Ni, and Wang (2012).

Table 1: The ULS and UES solutions of the polluted river problem in Figure 2.

Agent	ULS	UES	Agent	ULS	UES
1	12.2310	15.3333	8	11.0259	4.3000
2	11.2310	14.3333	9	10.6796	14.7000
3	11.7660	6.3333	10	4.8870	5.7000
4	6.8745	10.8000	11	9.6171	16.3667
5	2.8745	6.8000	12	12.6171	19.3667
6	7.3685	10.3000	13	6.2953	8.3667
7	8.3685	11.3000	*L*	32.1639	4.0000

the river structure and the very position of the agent. For example, agent 3 shares higher costs in ULS than in UES, the intuition is that since agent 3 has two direct upstream neighbours, the costs generated at the bottom agent L is not clearly contributed by agent 1 or agent 2, but certainly passed through agent 3. Therefore, agent 3 is more responsible than its upstream agents regarding its downstream costs.

5 The DES method and the permission values

Besides the UES method, it is straightforward to see that the DES method can be obtained as the conjunctive permission value of the game with permission structure $(N, L^{sa}_{(N,D,c)}, D^-)$ where $L^{sa}_{(N,D,c)}$ is the stand-alone game and the permission structure $D^- = \{(i, j) \in N \times N \mid (j, i) \in D\}$ is the downstream oriented digraph.

Moreover, since D^- is a rooted tree, and for rooted trees the conjunctive and disjunctive permission values coincide, the DES method is also obtained as the disjunctive permission value for the above mentioned game with permission structure.

Proposition 5.1. *Let* $(N, D, c) \in \mathcal{R}$ *be a polluted river problem. Then*

$$g^{DES}(N, D, c) = \varphi^c(N, L^{sa}_{(N,D,c)}, D^-) = \varphi^d(N, L^{sa}_{(N,D,c)}, D^-).$$

So, whereas applied to D the conjunctive and disjunctive permission value yield different cost allocation methods, applied to D^- both permission values yield the same cost allocation method, being the DES method.

Also in this case the axioms underlying the conjunctive (and disjunctive) permission value on rooted trees yield an axiomatization of the DES method.

We say that a cost allocation method g is a *downstream oriented game method* if there is a solution f for games with a permission structure such that $g(N, D, c) = f(N, L^{sa}_{(N,D,c)}, D^-)$ for all $(N, D, c) \in \mathcal{R}$. Again, (i) efficiency for permission values on the class $\mathcal{GPR}^- = \{(N, v, D^-) \in \mathcal{GP} \mid v = L^{sa}_{(N,D,c)}$ for some $(N, D, c) \in \mathcal{R}\} \subset \mathcal{GP}$ is equivalent to efficiency for polluted river cost allocation methods, and (ii) additvity for permission values on the class $\mathcal{GPR}^-$ is equivalent to additivity for polluted river cost allocation methods.

Since a player is an inessential player in a polluted river game with permission structure $(N, L^{sa}_{(N,D,c)}, D^-)$ if and only if its own cost as well as the cost of all its superiors is zero, the inessential player property for games with permission structure $(N, L^{sa}_{(N,D,c)}, D^-)$ is equivalent to requiring zero contributions for such agents.

D^--inessential agent property For every $(N, D, c) \in \mathcal{R}$ and $i \in N$ such that $c_j = 0$ for all $j \in \{i\} \cup \widehat{P}^{-1}_{D^-}(i) = \{i\} \cup \widehat{P}_D(i)$, it holds that $g_i(N, D, c) = 0$.

Again, this property implies independence of irrelevant costs.

Proposition 5.2. *Every cost allocation method that satisfies the D^--inessential agent property also satisfies independence of irrelevant costs.*

Since the proof goes similar as that of Proposition 3.3, the proof is omitted.[13]

Again, the D^--inessential agent property is stronger than independence of irrelevant costs since it also states requirements for the payoffs in polluted river problems where more than one agent has a positive cleaning cost, and allows an inessential agent to have subordinates with positive cost.

Since the necessary player property for games with a permission structure does not relate to the permission structure, also for polluted river games with permission structure $(N, L^{sa}_{(N,D,c)}, D^-)$ the necessary player property is equivalent to the necessary agent property of Section 3.

Since stand-alone games are monotone, structural monotonicity on D^- is equivalent to requiring that downstream agents contribute at least as much as upstream agents.

D^--structural monotonicity For every $(N, D, c) \in \mathcal{R}$ and $i, j \in N$ with $j \in P^{-1}_{D^-}(i) = P_D(i)$, it holds that $g_i(N, D, c) \geq g_j(N, D, c)$.

Proposition 5.3. *Every cost allocation method that satisfies the necessary agent property and D^--structural monotonicity also satisfies downstream symmetry.*

Again, since the proof goes similar as that of Proposition 3.4, it is omitted.[14]

Compared to Theorem 3.5, replacing the inessential agent property and structural monotonicity by the D^--inessential agent property and D^--structural monotonicity (or replacing independence of irrelevant costs, downstream symmetry and independence of downstream costs in Theorem 2.3 by the D^--inessential agent property, the necessary agent property and D^--structural monotonicity) characterizes the DES method. Similar as with Theorem 3.5 we do not need independence of downstream costs which is a rather strong axiom.

Theorem 5.4. *The DES method is the unique method that satisfies efficiency, additivity, the D^--inessential agent property, the necessary agent property and D^--structural monotonicity.*

The proof goes similar to that of Theorem 3.5, and is therefore omitted.[15]

6 Concluding remarks

In this paper we considered polluted river problems as games with a permission structures and showed how the UES and DES methods of Dong, Ni, and Wang (2012) can be obtained by applying the conjunctive permission value to an appropriate game with a permission structure. We

[13]The proof can be obtained from the authors on request.
[14]The proof can be obtained from the authors on request.
[15]The proof can be obtained from the authors on request.

also showed that axiomatizations of the conjunctive permission value yield new axiomatizations og the UES and DES methods that have a good interpretation in terms of International Water Law. Also, we applied the disjunctive permission value to obtain a new cost allocation method, the ULS method, for polluted river problems.

Although our goal was to stay within the framework of Dong, Ni, and Wang (2012) in the sense that we considered single sink rivers, we mention that the axiomatizations discussed in this paper hold for all strongly acyclic digraphs, being connected digraphs that might have multiple springs as well as multiple sinks, but from every agent there is a unique directed path to any of its downstream agents. The axioms can be defined as they are, and the proofs follow more or less the same argument. Uniqueness follows similar as in the proofs of Theorems 3.5, 3.7 and 4.6 by considering the cost vectors c^i, $i \in N$, where only one agent has a positive cost. Considering that all agents $j \neq i$ that are not upstream of i pay zero in c^i by the inessential agent property, considering the river structure on i and all its upstream agents is, in fact, a sink tree and we can apply the axioms similar as in the proofs of Theorems 3.5, 3.7 and 4.6.

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Appendix A

The following cost allocation methods show that neither the necessary agent property nor structural monotonicity on its own implies upstream symmetry, and upstream symmetry implies neither the necessary agent property nor structural monotonicity.

1. The DES method satisfies the necessary agent property but does not satisfy upstream symmetry.

2. Consider the method

$$g_i(N, D, c) = \begin{cases} \frac{\sum_{h \in N} c_h}{|T(D)|} & \text{if } i \in T(D) \\ 0 & \text{otherwise,} \end{cases}$$

   which equally allocates the full cleaning cost in the river over the most upstream agents. This method satisfies structural monotonicity but does not satisfy upstream symmetry.

3. Consider the modified DES method given by

$$g_i^{\overline{DES}}(N, D, c) = \sum_{j \in \widehat{P}_D(i)} \frac{c_j}{\left|\widehat{P}_D^{-1}(j)\right|} + \frac{c_L}{|N|}$$

   where the cost of every river segment is equally shared among all agents downstream of the segment (so compared to the DES method the upstream agent on a river segment does not contribute to the cleaning costs), and the cost of the sink is equally shared among all agents. This method satisfies upstream symmetry, but it does not satisfy the necessary agent property nor structural monotonicity.

Appendix B

The logical independence of the five axioms in Theorem 3.5 can be seen from the following alternative cost allocation methods:

1. The Local Responsibility method satisfies all axioms except structural monotonicity.

2. Consider the method

$$g_i(N, D, c) = \frac{\sum_{h \in N} c_h}{|N|} \text{ for all } i \in N$$

   where the full cleaning cost is equally shared among all agents. This method satisfies all axioms except the inessential player property.

3. Consider the method given by

$$g_i(N, D, c) = 0 \text{ for all } i \in N$$

This method satisfies all axioms except efficiency.

4. Consider the modified UES method given by

$$g_i^{\overline{UES}}(N, D, c) = \begin{cases} \sum_{j \in \widehat{P}_D^{-1}(i)} \frac{c_j}{\left|\widehat{P}_D(j)\right|} & \text{if } P_D(i) \neq \emptyset \\ \sum_{j \in \widehat{P}_D^{-1}(i)} \frac{c_j}{\left|\widehat{P}_D(j)\right|} + c_i & \text{if } P_D(i) = \emptyset \end{cases}$$

where the cost of every river segment is equally shared among all agents upstream of the upstream agent on the segment (so compared to the UES method the upstream agent on a river segment does not contribute to the cleaning costs)[16]. In the case that the upstream agent of a river segment is a top agent, the cost of this agent is allocated to itself. This method satisfies all axioms except the necessary agent property.

5. Consider the method which allocates as the UES method in the case that there is a necessary agent (that is, when a single agent has non-zero cleaning cost), and allocates as $\overline{\text{UES}}$ otherwise (that is, when there is more than one agent having non-zero cleaning cost, i.e., no necessary agent exist). This method satisfies all axioms except additivity.[17]

The logical independence of the six axioms in Theorem 3.7 can be seen from the following alternative cost allocation methods:

1. The Upstream Limited Sharing (see Section 4) method satisfies all axioms except externality fairness.

2. The Local Responsibility method satisfies all axioms except weak structural monotonicity.

3. The method that equally assigns the full cleaning cost among all agents satisfies all axioms except the inessential agent property.

4. The method that assigns zero costs to all agents satisfies all axioms except efficiency.

5. Consider the method given by

$$g_i(N, D, c) = \begin{cases} \sum_{j \in \widehat{P}_D^{-1}(i)} \frac{c_j(1+\frac{1}{|N|})}{\left|\widehat{P}_D(j)\right|} + c_i\left(\frac{1+\frac{1}{|N|}}{\left|\widehat{P}_D(i)\right|} - \frac{1}{|N|}\right) & \text{if } P_D(i) \neq \emptyset \\ \sum_{j \in \widehat{P}_D^{-1}(i)} \frac{c_j(1+\frac{1}{|N|})}{\left|\widehat{P}_D(j)\right|} + c_i & \text{if } P_D(i) = \emptyset \end{cases}$$

[16]This is a modification of the UES method in a similar way as the DES method is modified to $g^{\overline{DES}}$.

[17]Note that the necessary agent property only states a requirement if there is a single agent who has non-zero cleaning cost at its downstream river segment. So by allocating the cost in a different way from UES when a necessary agent is absent, additivity is violated.

where the cost of every river segment is unequally shared among all agents that are located upstream of that segment, such that each agent upstream of the local agent always share fixed portion more than the local agent. In the case that the local (upstream) agent of a river segment is a top agent, the cost of this agent is allocated to itself. This method satisfies all axioms except the necessary agent property.

6. Consider the method that allocates as UES when there is necessary agent and allocates as method stated in 5 otherwise. This method satisfies all axioms except additivity.

The logical independence of the six axioms in Theorem 4.6 can be seen from the following alternative cost allocation methods:

1. The UES method satisfies all axioms except participation fairness.

2. The LRS method satisfies all axioms except weak structural monotonicity.

3. Consider the method that equally assigns the full cleaning costs among all agents with line rivers. And with non-line rivers, given the allocation results in the linear case, it restricts one more condition in addition to efficiency, additivity, necessary agent property and participation fairness. This condition requires that for any agent with only one direct upstream neighbor, it always pay the same as its upstream neighbor. This method provides unique sharing outcome that satisfies all axioms except the inessential agent property.

4. The method that assigns zero costs to all agents satisfies all axioms except efficiency.

5. Consider the method that allocates all the costs to its top agent with line rivers. And with non-line rivers, given the allocation results in the linear case, it restricts one more condition in addition to efficiency, additivity, necessary agent property and participation fairness. This condition requires for any non-inessential agents who has only one direct upstream neighbor, it always pay the same as its upstream neighbor. This method provides unique sharing outcome that satisfies all axioms except the necessary agent property.

6. Consider the method that allocates as ULS when there is necessary agent and allocates as method stated in 5 otherwise. This method satisfies all axioms except additivity.

Appendix C

We give a direct proof that the ULS method satisfies efficiency, additivity, the inessential agent property, the necessary agent property, and weak structural monotonicity (see Proposition 4.5).

Proof. Efficiency of the ULS method of $(N, D, c) \in \mathcal{R}$ follows from the efficiency of the disjunctive permission value $\phi^d(N, L^{sa}_{(N,D,c)}, D)$ (see van den Brink (1997)) and the fact that $L^{sa}_{(N,D,c)}(N) = \sum_{i \in N} c_i$.

Additivity follows from the linearity of equation (4.7).

Since $(S \setminus \{i\}) \subseteq S$, it holds that $Q(S \setminus \{i\}) \subseteq Q(S)$. Therefore, one has

$$\sum_{j \in Q(S)} c_j - \sum_{j \in Q(S \setminus \{i\})} c_j = \sum_{j \in Q(S) \setminus Q(S \setminus \{i\})} c_j.$$

If $k \notin \{i\} \cup \widehat{P}_D^{-1}(i)$, then it holds that $k \in Q(S) \Rightarrow k \in Q(S \setminus \{i\})$, which is equivalent to $k \notin Q(S) \setminus Q(S \setminus \{i\})$. Taking the contraposition, one has if $k \in Q(S) \setminus Q(S \setminus \{i\})$ then $k \in \{i\} \cup \widehat{P}_D^{-1}(i)$. Therefore, if $c_j = 0$ for all $j \in \{i\} \cup \widehat{P}_D^{-1}(i)$, then $\sum_{j \in Q(S) \setminus Q(S \setminus \{i\})} c_j = 0$ for any $S \subseteq N$ with $i \in S$. This implies the inessential agent property.

Equation (4.7) can be alternatively written as

$$\begin{aligned}
g_i^{ULS}(N, D, c) &= \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \left[\sum_{k \in Q(S \cup \{i\})} c_k - \sum_{k \in Q(S)} c_k \right] \\
&= \sum_{S \subseteq N \setminus \{i,j\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \left[\sum_{k \in Q(S \cup \{i\})} c_k - \sum_{k \in Q(S)} c_k \right] \\
&\quad + \sum_{S \subseteq N \setminus \{i,j\}} \frac{(|S| + 1)!(|N| - |S| - 2)!}{|N|!} \left[\sum_{k \in Q(S \cup \{i,j\})} c_k - \sum_{k \in Q(S \cup \{j\})} c_k \right]
\end{aligned}$$

for any $i, j \in N$. It then follows that

$$\begin{aligned}
g_i^{ULS}(N, D, c) - g_j^{ULS}(N, D, c) &= \sum_{S \subseteq N \setminus \{i,j\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \left[\sum_{k \in Q(S \cup \{i\})} c_k - \sum_{k \in Q(S \cup \{j\})} c_k \right] \\
&\quad + \sum_{S \subseteq N \setminus \{i,j\}} \frac{(|S| + 1)!(|N| - |S| - 2)!}{|N|!} \left[\sum_{k \in Q(S \cup \{i\})} c_k - \sum_{k \in Q(S \cup \{j\})} c_k \right].
\end{aligned} \tag{6.15}$$

If $c_j = 0$ for all $j \in N \setminus \{i\}$, taking into account that $i \in Q\{S \cup \{i\}\}$ for any $S \subseteq N \setminus \{i, j\}$, one has $\sum_{k \in Q(S \cup \{i\})} c_k - \sum_{k \in Q(S \cup \{j\})} c_k \geq 0$ which in turn leads to $g_i^{ULS}(N, D, c) \geq g_j^{ULS}(N, D, c)$, proving that the ULS method satisfies the necessary agent property.

If $P_D(j) = \{i\}$, it holds that $j \in Q(S \cup \{i\})$ for any $S \subseteq N \setminus \{i, j\}$ since $P_D(j) = i$ and $i \in Q(S \cup \{i\})$. Thus $Q(S \cup \{i\}) = Q(S \cup \{i, j\})$. From Lemma 4.4, one has $Q(S \cup \{j\}) \subseteq Q(S \cup \{i, j\}) = Q(S \cup \{i\})$. Equation (6.15) then implies $g_i^{ULS}(N, D, c) \geq g_j^{ULS}(N, D, c)$, meaning that the ULS method satisfies weak structural monotonicity. □

The following lemma is used in the proof of Proposition 4.5 in Section 4.

Lemma. *For any game $(N, v, D) \in \mathcal{GPR}$, the Harsanyi dividend $\Delta_{r^d_{v,D}}(S) = 0$ if $S \subseteq N$ is disconnected.*

Proof. It follows from Algaba et al. (2003) that $S \notin \Phi^d_D$ implies $\Delta_{r^d_{v,D}}(S) = 0$, therefore we only need to consider coalitions $S \in \Phi^d_D$. Here we say $R \subseteq S$ is a maximal connected part of S if there exists no other connected $R' \subseteq S$ such that $R \subset R'$ and $R \neq R'$. Let $H(S) = \{R \subseteq S \mid R \text{ is a maximal connected part of } S\}$. Obviously $H(S)$ is a partition of S. If $S \in \Phi^d_D$, then $r^d_{v,D}(S) = \sum_{i\in S} v(\{i\})$. For any $R \in H(S)$, it holds that $R \in \Phi^d_D$ and thus $r^d_{v,D}(R) = \sum_{i\in R} v(\{i\})$. It is easy to see that $\Delta_{r^d_{v,D}}(S) = 0$ for disconnected $S \in \Phi^d_D$ with $|S| = 2$, since $H(S)$ contains two singletons and $\Delta_{r^d_{v,D}}(\{i\}) = v(\{i\})$ for any $i \in N$.[18] Assume for some $m > 2$ that $\Delta_{r^d_{v,D}}(S) = 0$ holds true for disconnected $S \in \Phi^d_D$ with $|S| \leq m$, then the Harsanyi dividend $\Delta_{r^d_{v,D}}(S)$ of disconnected $S \in \Phi^d_D$ with $|S| = m + 1$ can be written as

$$\begin{aligned}
\Delta_{r^d_{v,D}}(S) &= r^d_{v,D}(S) - \sum_{T\subset S:T\neq S} \Delta_{r^d_{v,D}}(T) \\
&= \sum_{i\in S} v(\{i\}) - \sum_{T\subset S:T\neq S} \Delta_{r^d_{v,D}}(T) \\
&= \sum_{R\in H(S)} \Big[\sum_{i\in R} v(\{i\}) - \sum_{T\subset R:T\neq R} \Delta_{r^d_{v,D}}(T) - \Delta_{r^d_{v,D}}(R)\Big] \\
&= \sum_{R\in H(S)} \Big[\Delta_{r^d_{v,D}}(R) - \Delta_{r^d_{v,D}}(R)\Big] \quad = 0,
\end{aligned}$$

where the third equality of the above equation follows from $|R| < |S|$ for $R \in H(S)$ and the induction hypothesis. □

An alternative proof of that the ULS method satisfies participation fairness without using Harsanyi dividends is also given below.

Proof. To show participation fairness, we consider cost vectors c^i such that $c^i_i = c_i$ and $c^i_j = 0$ for all $i, j \in N$, $j \neq i$. It is clear that $c = \sum_{i\in N} c^i$. Choose an arbitrary pair (i, j) such that $(i, j) \in D$ and $|P_D(j)| \geq 2$. We discuss the values of ULS method of (N, D, c^ℓ), $(K^i_{ij}, D|_{K^i_{ij}}, c^\ell|_{K^i_{ij}})$ and $(K^j_{ij}, D|_{K^j_{ij}}, c^\ell|_{K^j_{ij}})$ for different $\ell \in N$. For any $h \in \{i\} \cup \overline{P}_D(i)$:

1. If $\ell \notin \{h\} \cup \widehat{P}^{-1}_D(h)$, by the inessential agent property, it holds that $g^{ULS}_j(K^j_{ij}, D|_{K^j_{ij}}, c^\ell|_{K^j_{ij}}) = 0$, $g^{ULS}_h(K^i_{ij}, D|_{K^i_{ij}}, c^\ell|_{K^i_{ij}}) = 0$, and $g^{ULS}_j(N, D, c^\ell) = g^{ULS}_h(N, D, c^\ell) = 0$. Thus Equation (4.8) holds true.

2. If $\ell \in \{h\} \cup \widehat{P}^{-1}_D(h)$ but $\ell \notin \{j\} \cup \widehat{P}^{-1}_D(j)$, then one has $g^{ULS}_j(K^j_{ij}, D|_{K^j_{ij}}, c^\ell|_{K^j_{ij}}) = 0$ and $g^{ULS}_j(N, D, c^\ell) = 0$ from the inessential agent property. From the necessary agent property one has

$$g^{ULS}_\ell(N, D, c^\ell) \geq g^{ULS}_h(N, D, c^\ell), \quad g^{ULS}_\ell(K^i_{ij}, D|_{K^i_{ij}}, c^\ell|_{K^i_{ij}}) \geq g^{ULS}_h(K^i_{ij}, D|_{K^i_{ij}}, c^\ell|_{K^i_{ij}}),$$

[18] This holds only for inessential games but not for general games.

and from weak structural monotonicity it holds that

$$g_\ell^{ULS}(N,D,c^\ell) \le g_h^{ULS}(N,D,c^\ell), \quad g_\ell^{ULS}(K_{ij}^i, D|_{K_{ij}^i}, c^\ell|_{K_{ij}^i}) \le g_h^{ULS}(K_{ij}^i, D|_{K_{ij}^i}, c^\ell|_{K_{ij}^i}).$$

It then follows that

$$g_\ell^{ULS}(N,D,c^\ell) = g_h^{ULS}(N,D,c^\ell), \quad g_\ell^{ULS}(K_{ij}^i, D|_{K_{ij}^i}, c^\ell|_{K_{ij}^i}) = g_h^{ULS}(K_{ij}^i, D|_{K_{ij}^i}, c^\ell|_{K_{ij}^i}).$$

In order to show participation fairness, it is then sufficient and necessary to show $g_\ell^{ULS}(N,D,c^\ell) = g_\ell^{ULS}(K_{ij}^i, D|_{K_{ij}^i}, c^\ell|_{K_{ij}^i})$. If $\ell \in Q(S\setminus\{\ell\})$ for some $S \subseteq N$, then $\ell \in Q(S)$. One has

$$\begin{aligned} g_\ell^{ULS}(N,D,c^\ell) &= \sum_{S\subseteq N:\ell\in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!}\Big[\sum_{k\in Q(S)} c_k^\ell - \sum_{k\in Q(S\setminus\{\ell\})} c_k^\ell\Big] \\ &= \sum_{S\subseteq N:\ell\in S,\ell\notin Q(S\setminus\{\ell\})} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} c_\ell \\ &= c_\ell \times \sum_{S\subseteq N\setminus\{\ell\}:\ell\notin Q(S)} \frac{|S|!(|N|-|S|-1)!}{|N|!} \end{aligned}$$

Now we consider the partition $S = S^\wedge \cup S^\vee$ where $S^\wedge = S \cap K_{ij}^i$ and $S^\vee = S \cap K_{ij}^j$. Obviously $S^\wedge \cap S^\vee = \emptyset$. It then holds that

$$\begin{aligned} &\sum_{S\subseteq N\setminus\{\ell\}:\ell\notin Q(S)} \frac{|S|!(|N|-|S|-1)!}{|N|!} \\ =& \sum_{S^\wedge\subseteq K_{ij}^i\setminus\{\ell\}:\ell\notin Q|_{K_{ij}^i}(S^\wedge)} \sum_{S^\vee\subseteq K_{ij}^j} \frac{(|S^\wedge|+|S^\vee|)!\,(|N|-|S^\wedge|-|S^\vee|-1)!}{|N|!} \\ =& \sum_{S^\wedge\subseteq K_{ij}^i\setminus\{\ell\}:\ell\notin Q|_{K_{ij}^i}(S^\wedge)} \frac{|S^\wedge|!(|K_{ij}^i|-|S^\wedge|-1)!}{|K_{ij}^i|!} \sum_{S^\vee\subseteq K_{ij}^j} \frac{(|S^\wedge|+|S^\vee|)!\,(|N|-|S^\wedge|-|S^\vee|-1)!\,|K_{ij}^i|!}{|S^\wedge|!\,(|K_{ij}^i|-|S^\wedge|-1)!\,|N|!}. \end{aligned}$$

For any $S^\wedge \subseteq K_{ij}^i \setminus \{\ell\}$,

$$\begin{aligned} &\sum_{S^\vee\subseteq K_{ij}^j} \frac{(|S^\wedge|+|S^\vee|)!\,(|N|-|S^\wedge|-|S^\vee|-1)!\,|K_{ij}^i|!}{|S^\wedge|!\,(|K_{ij}^i|-|S^\wedge|-1)!\,|N|!} \\ =& \sum_{s=0}^{|K_{ij}^j|} \frac{(|S^\wedge|+s)!\,(|N|-|S^\wedge|-s-1)!\,|K_{ij}^i|!}{|S^\wedge|!\,(|K_{ij}^i|-|S^\wedge|-1)!\,|N|!} \frac{|K_{ij}^j|!}{s!\,(|K_{ij}^j|-s)!} \\ =& \sum_{s=0}^{|K_{ij}^j|} \frac{(|S^\wedge|+s)!}{|S^\wedge|!\,s!} \frac{(|N|-|S^\wedge|-s-1)!}{(|K_{ij}^i|-|S^\wedge|-1)!\,(|K_{ij}^j|-s)!} \frac{|K_{ij}^i|!\,|K_{ij}^j|!}{|N|!} \\ =& \frac{|K_{ij}^i|!\,|K_{ij}^j|!}{|N|!} \sum_{s=0}^{|K_{ij}^j|} \binom{|S^\wedge|+s}{s}\binom{|N|-|S^\wedge|-s-1}{|K_{ij}^j|-s} = \frac{|K_{ij}^i|!\,|K_{ij}^j|!}{|N|!}\binom{|N|}{|K_{ij}^j|} \qquad (6.16) \\ =& 1, \end{aligned}$$

where (6.16) follows from Vandermonde's convolution, see Gould (1956, Equation (3)). Therefore, it holds that

$$
\begin{aligned}
g_\ell^{ULS}(N, D, c^\ell) &= c_\ell \times \sum_{S \subseteq K_{ij}^i \setminus \{\ell\} : \ell \notin Q|_{K_{ij}^i}(S)} \frac{|S|!(|K_{ij}^i| - |S| - 1)!}{|K_{ij}^i|!} \\
&= \sum_{S \subseteq K_{ij}^i \setminus \{\ell\}} \frac{|S|!(|K_{ij}^i| - |S| - 1)!}{|K_{ij}^i|!} \left[\sum_{k \in Q|_{K_{ij}^i}(S \cup \{\ell\})} c_k^\ell - \sum_{k \in Q|_{K_{ij}^i}(S)} c_k^\ell \right] \\
&= g_\ell^{ULS}(K_{ij}^i, D|_{K_{ij}^i}, c^\ell|_{K_{ij}^i}).
\end{aligned}
$$

3. If $\ell \in \{j\} \cup \widehat{P}_D^{-1}(j)$, then $g_h^{ULS}(K_{ij}^i, D|_{K_{ij}^i}, c^\ell|_{K_{ij}^i}) = 0$ from the inessential agent property. Similar to Equation (6.15), it holds that

$$
\begin{aligned}
& g_j^{ULS}(N, D, c^\ell) - g_h^{ULS}(N, D, c^\ell) \\
&= \sum_{S \subseteq N \setminus \{j,h\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \left[\sum_{k \in Q(S \cup \{j\})} c_k^\ell - \sum_{k \in Q(S \cup \{h\})} c_k^\ell \right] \\
&\quad + \sum_{S \subseteq N \setminus \{j,h\}} \frac{(|S| + 1)!(|N| - |S| - 2)!}{|N|!} \left[\sum_{k \in Q(S \cup \{j\})} c_k^\ell - \sum_{k \in Q(S \cup \{h\})} c_k^\ell \right] \\
&= c_\ell \times \sum_{S \subseteq N \setminus \{j,h\} : \ell \in Q(S \cup \{j\}), \ell \notin Q(S \cup \{h\})} \left(\frac{|S|!(|N| - |S| - 1)!}{|N|!} + \frac{(|S| + 1)!(|N| - |S| - 2)!}{|N|!} \right) \\
&\quad + c_\ell \times \sum_{S \subseteq N \setminus \{j,h\} : \ell \notin Q(S \cup \{j\}), \ell \in Q(S \cup \{h\})} \left(\frac{|S|!(|N| - |S| - 1)!}{|N|!} + \frac{(|S| + 1)!(|N| - |S| - 2)!}{|N|!} \right)
\end{aligned}
$$

Since $\ell \notin T(D)$, $\ell \notin Q(S \cup \{j\})$ implies $\ell \notin S$ and $\ell \notin \overline{P}_D^{-1}(j)$. Furthermore, $\ell \notin \overline{P}_D^{-1}(j) \Rightarrow (P_D(\ell) \setminus \{j\}) \nsubseteq Q(S \cup \{j\}) \Rightarrow (P_D(\ell) \setminus \{j\}) \nsubseteq Q(S \cup \{h\})$. Therefore, one has $\ell \notin Q(S \cup \{h\})$. Hence,

$$
\begin{aligned}
& g_j^{ULS}(N, D, c^\ell) - g_h^{ULS}(N, D, c^\ell) \\
&= c_\ell \times \sum_{S \subseteq N \setminus \{j,h\} : \ell \in Q(S \cup \{j\}), \ell \notin Q(S \cup \{h\})} \left(\frac{|S|!(|N| - |S| - 1)!}{|N|!} + \frac{(|S| + 1)!(|N| - |S| - 2)!}{|N|!} \right) \\
&= c_\ell \times \sum_{S \subseteq N \setminus \{j,h\} : \ell \in Q(S \cup \{j\}), \ell \notin Q(S \cup \{h\})} \frac{|S|!\,(|N| - 1 - |S| - 1)!}{(|N| - 1)!}
\end{aligned}
$$

Here we consider $S = S^\wedge \cup S^\vee$ again where $S^\wedge = S \cap K_{ij}^i$ and $S^\vee = S \cap K_{ij}^j$. The above equation becomes

$$
\begin{aligned}
& g_j^{ULS}(N, D, c^\ell) - g_h^{ULS}(N, D, c^\ell) \\
&= c_\ell \times \sum_{S^\wedge \subseteq K_{ij}^i \setminus \{h\}} \ \sum_{S^\vee \subseteq K_{ij}^j \setminus \{j\} : \ell \in Q|_{K_{ij}^j}(S^\vee \cup \{j\}), \ell \notin Q|_{K_{ij}^j}(S^\vee)} \frac{(|S^\wedge| + |S^\vee|)!\,(|N| - 1 - |S^\wedge| - |S^\vee| - 1)!}{(|N| - 1)!} \\
&= c_\ell \times \sum_{S^\vee \subseteq K_{ij}^j \setminus \{j\} : \ell \in Q|_{K_{ij}^j}(S^\vee \cup \{j\}), \ell \notin Q|_{K_{ij}^j}(S^\vee)} \frac{|S^\vee|!(|K_{ij}^j| - |S^\vee| - 1)!}{|K_{ij}^j|!} \cdot A(S^\vee),
\end{aligned}
$$

where

$$
\begin{aligned}
A(S^\vee) &= \sum_{S^\wedge \subseteq K_{ij}^i \setminus \{h\}} \frac{|K_{ij}^j|!}{|S^\vee|!(|K_{ij}^j| - |S^\vee| - 1)!} \frac{(|S^\wedge| + |S^\vee|)!\,(|N| - 1 - |S^\wedge| - |S^\vee| - 1)!}{(|N| - 1)!} \\
&= \sum_{s=0}^{|K_{ij}^i|-1} \frac{|K_{ij}^j|!\,(s + |S^\vee|)!\,(|N| - 1 - s - |S^\vee| - 1)!}{|S^\vee|!\,(|K_{ij}^j| - |S^\vee| - 1)!\,(|N| - 1)!} \frac{(|K_{ij}^i| - 1)!}{s!(|K_{ij}^i| - s - 1)!} \\
&= \frac{(|K_{ij}^i| - 1)!\,|K_{ij}^j|!}{(|N| - 1)!} \sum_{s=0}^{|K_{ij}^i|-1} \frac{(s + |S^\vee|)!}{s!|S^\vee|!} \frac{(|N| - 1 - s - |S^\vee| - 1)!}{(|K_{ij}^i| - s - 1)!(|K_{ij}^j| - |S^\vee| - 1)!} \\
&= \frac{(|K_{ij}^i| - 1)!\,|K_{ij}^j|!}{(|N| - 1)!} \sum_{s=0}^{|K_{ij}^i|-1} \binom{s + |S^\vee|}{s} \binom{|N| - 1 - s - |S^\vee| - 1}{|K_{ij}^i| - 1 - s} \\
&= \frac{(|K_{ij}^i| - 1)!\,|K_{ij}^j|!}{(|N| - 1)!} \binom{|N| - 1}{|K_{ij}^i| - 1} = 1.
\end{aligned}
$$

Therefore, one has

$$
\begin{aligned}
& g_j^{ULS}(N, D, c^\ell) - g_h^{ULS}(N, D, c^\ell) \\
&= c_\ell \times \sum_{S \subseteq K_{ij}^j \setminus \{j\} : \ell \in Q|_{K_{ij}^j}(S \cup \{j\}), \ell \notin Q|_{K_{ij}^j}(S)} \frac{|S|!(|K_{ij}^j| - |S| - 1)!}{|K_{ij}^j|!} \\
&= \sum_{S \subseteq K_{ij}^j \setminus \{j\}} \frac{|S|!(|K_{ij}^j| - |S| - 1)!}{|K_{ij}^j|!} \Bigg[\sum_{k \in Q|_{K_{ij}^j}(S \cup \{j\})} c_k^\ell - \sum_{k \in Q|_{K_{ij}^j}(S)} c_k^\ell \Bigg] \\
&= g_j^{ULS}(K_{ij}^j, D|_{K_{ij}^j}, c^\ell|_{K_{ij}^j}) \\
&= g_j^{ULS}(K_{ij}^j, D|_{K_{ij}^j}, c^\ell|_{K_{ij}^j}) - g_h^{ULS}(K_{ij}^i, D|_{K_{ij}^i}, c^\ell|_{K_{ij}^i}).
\end{aligned}
$$

Cases 1, 2 and 3 covers all $\ell \in N$. Participation fairness of the ULS method then follows from additivity. □