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Private Road Networks with Uncertain Demand

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Abstract

There has been wide interest in private supply of roads as a solution to traffic congestion. We study its efficiency under demand uncertainty: we solve for equilibrium and optimum as benchmarks, and evaluate the efficiency of possible regulatory policies for private road operators. We obtain analytic solutions for simple networks and numerical simulation results for more complex ones. For two serial links and two parallel links, self-financing still holds in expected terms for the first-best case, even though the capacity is higher than the capacity for the deterministic demand equal to the expected value. When forced to apply the second-best optimal pricing, the private supplier makes an expected loss (profit) if there is an untolled substitute (complement) in the network. In contrast to the deterministic counterpart of the problem we study, regulation by competitive auction cannot replicate the second-best

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zero-profit result. For more complex networks, when private firms adds capacity one link at a time, entry by competitive auctions performs better than free entry. For the parameter range considered in the numerical simulation, entry by generalized auction performs better than entry by patronage auction.

Keywords: Traffic Congestion, Road Pricing, Uncertain Demand, Road Network, Private Supply

JEL codes:

1 Introduction

There has been wide interest in private supply of roads as a solution to increasing traffic congestion around the world. Many countries already have private tolled roads and the trend is increasing. This is because many governments have insufficient public funds to finance new road projects, and private firms are believed to manage the tolled roads more efficiently. However, there is one disadvantage: suppliers have market power and tend to maximize their profits, resulting in a loss of social welfare compared to optimal pricing (Edelson (1971), Verhoef et al. (1996) and de Palma and Lindsey (2000)).

Previous studies have examined the effects of private supply of roads on social welfare. Following Moring and Harwitz(1962), Yang and Meng (2002) shows that, if both the toll and the capacity are set optimally for every link of the network and neutral scale economies prevail, the private road is self-financing, meaning that the collected tolls can cover the capacity costs. In that case, the social optimum can be offered by a private firm without making the firm running to losses. Obviously, not all roads are priced optimally in reality. For example, free public roads are common in every country, even if a toll could reduce congestion and improve social welfare. Verhoef (2007) demonstrates that a private road supplier who is forced to price and invest second-best optimally makes a loss when there is an untolled substitute road in the network, so a subsidy from the government is needed to achieve the desired social welfare. If such subsidy is ruled out due to political or economic reasons, Verhoef (2008) derives the highest social welfare under the condition that the private firm makes at least a zero profit. This is a natural benchmark to compare the efficiency of various ways of regulating private supply of roads. Verhoef (2008) found that among many possible regulation tools, two competitive auctions, namely the patronage auction and the generalized price auction, are preferred, because they make the private firm choose the socially optimum tolls and capacity under the zero profit condition.

The study of private supply of road can be further complicated by demand uncertainty. Demand fluctuations are common in road transportation and have been studied by numerous scholars. Most scholars focus on a single link and find demand uncertainty has substantial influence on the capacity and toll decisions. Kraus (1982), D'Ouville and McDonald (1990), Arnott et al. (1996) and De Borger and Van Dender (2006) all find that for a single link the optimal capacity with demand uncertainty is larger than the optimal capacity for a deterministic demand equal to the expected value. Lindsey and de Palma (2014) proves that the cost recovery theorem holds also with uncertain demand and cost in expected terms. To date, however, models about demand uncertainty have not been applied to examine private supply of road in a mixed network.

The aim of the present paper is to study, under demand uncertainty, the impact of private supply of road in a public network and of alternative ways to regulate it. Demand uncertainty plays a role, because the capacity is set when demand is unknown, but tolls can be adjusted according to the realized demand. For simple networks, we distinguish between parallel links and serial links, and derive analytically the expected social welfare under various scenarios, which are used as objectives to assess the efficiency of private supply of road. We also examine various ways of regulation aiming to achieve the highest social welfare possible. For more complex networks, we are interested in the development of private supply of road networks through competition. We run simulations to compare the outcomes of free entry and entry by regulation.

Our research brings new insights into private supply of roads and demand uncertainty. Consistent with the findings for a single link with demand uncertainty, the optimal capacity is larger than the optimal capacity for a deterministic demand equal to the expected value. This is because if the capacity is set on the basis of expected demand, the benefit of increasing the capacity when demand is high outweighs the cost when demand is low. In addition, the self-financing results still holds in expectation for optimal toll and capacity: also the expected toll is higher under uncertainty. However, the following results are unique due to demand uncertainty. When the private firm needs at least zero profit, the toll generally differs from the case without demand uncertainty. An especially important finding for policy makers is that regulation by simple competitive auctions can no longer make the private suppliers adopt the socially desirable tolls and capacities under the zero profit constraint. In other words, demand uncertainty makes regulating the private supply of roads more difficult. With the help of competition, the numerical simulation suggests that entry by regulation still works better than free entry.

The remaining paper is organized as follows. Section 2 introduces the model for simple networks and discusses the analytical results. Section 3 contains the simulation results for more complex networks and section 4 concludes.

2 Analytical Model

In this section, we study two serial links and two parallel links. They are the basic components of any complex network, and are common in real life.

They have, however, quite different implications for policy maker. We can get clear analytical results for the two basic network, and the insights from these two basic networks carry over to more complex networks.

2.1 Model Setup

This paper attempts to model demand uncertainty in a general way to represent a variety of uncertainty distributions and inverse demand functions. Denote the probability of state i as p^i, the total traffic flow in state i as N^i, and the inverse demand in state i as $D^i(N^i)$. We assume the inverse demand function decreases with traffic flow. For example, if the highest willingness to pay varies with the realized demand state and the probability of each state is one half, we can denoted the demand function as $D^i(N^i) = d_0^i - d_1 N^i$ and $i \in \{h, l\}, p^h = \frac{1}{2}, p^l = \frac{1}{2}$. Due to demand uncertainty, the timing of the game is as follows. Capacity is decided before the demand state is known, because there are long lead times to adapt capacity. Tolls are decided after the demand uncertainty is resolved, because prices are easy to adjust.

The other assumptions are standard in traffic congestion models. There is a single market with one origin and one destination, and the users are homogeneous. The congestion cost is increasing and homogeneous of degree zero in the ratio of the traffic flow and the capacity. This includes the widely used BPR function. We also assume that the marginal capacity cost is a constant. Let j denotes link j, the capacity, congestion cost and marginal capacity costs are denoted as $K_j, c(N_j^i, K_j)$ and γ respectively.

2.2 Serial Links

We study a network of two serial links in this section. A traveler must use both links to get from the origin to the destination. The two links are complementary to each other. As a result, the total traffic flow equals the traffic flow on each link, i.e. $N^i = N_0^i = N_1^i$.

The main findings are summarized here for four different cases. In the first-best case, a social planner can design the capacity and tolls on both links to maximize expected social welfare. The results are consistent with previous studies. The toll equals the sum of the marginal external cost of congestion on the full trip(De Borger and Van Dender (2006)), and the capacity is larger than the case without demand uncertainty (Kraus (1982), D'Ouville and McDonald (1990) and Arnott et al. (1996)). In the second-best case, there exists an untolled road and the social planner chooses the capacity and tolls on the other road. An expected profit is gained on the tolled link. Similar to Verhoef (2007), the intuition is that the toll on the priced link also includes the marginal external cost on the unpriced link, so that an efficient investment policy for the tolled link would not exhaust all toll revenues. In the second-best zero-profit case, the social planner faces an additional requirement that the expected toll revenue covers the capacity cost of the toll link. Contrary to existing literature on private supply of road without demand uncertainty, Verhoef (2008) for example, the toll is not equal to the marginal external cost of congestion of both links. Finally, in the regulation by auction case, we study how to implement the second-best zero-profit case through competitive auctions. Two auctions are examined, namely the patronage auction and the generalized price auction. However, neither can implement the second-best zero-profit result. We will discuss each case in more detail in the remainder of this section.

2.2.1 First-Best for Serial Links

To assess the efficiency of private supply of road, a natural benchmark is the first best case, which generates the highest expected social welfare by optimizing the capacity and toll of both links. The optimization problem can be stated as maximize the expected social welfare, which equals the expected consumer benefit minus the expected congestion cost and the capacity cost, under the constraint that the generalized price of any active route equals the marginal consumer benefit. Mathematically, let us denote the sum of the tolls as τ^i, the traffic volume as N^i, the capacity on link j as K_j, the inverse demand function as $D^i(N^i)$, and the congestion cost as $c(N^i, K_j)$. The resulting Lagrangian is as follows:

$$
\begin{aligned}
L = &\sum_i p^i(\int_0^{N^i} D^i(n)dn - N^i(c(N^i, K_0) + c(N^i, K_1))) - \gamma(K_0 + K_1) \\
&+ \sum_i \lambda^i(c(N^i, K_0) + c(N^i, K_1) + \tau^i - D^i(N^i))
\end{aligned} \tag{1}
$$

Solving the FOCs yields the familiar solution:

$$
\begin{aligned}
&D^i(N^i) - c(N^i, K_0) - c(N^i, K_1) - N^i(c_{N^i}(N^i, K_0) + c_{N^i}(N^i, K_1)) = 0 \\
&\sum_i p^i(-N^i c_{K_0}(N^i, K_0)) = \gamma \\
&\sum_i p^i(-N^i c_{K_1}(N^i, K_1)) = \gamma \\
&\tau^i = N^i(c_{N^i}(N^i, K_0) + c_{N^i}(N^i, K_1))
\end{aligned} \tag{2}
$$

The results are consistent with previous studies on uncertain demand for a single link (Kraus (1982), D'Ouville and McDonald (1990), Arnott et al. (1996), De Borger and Van Dender (2006) and Lindsey and de Palma (2014)). The toll is equal to the marginal external congestion cost over the full trip in each state. The two links are self-financing in expectation, because $\sum_i p^i N^i \tau^i = \sum_i p^i N^i N^i(c_{N^i}(N^i, K_0) + c_{N^i}(N^i, K_1)) = \sum_i p^i N^i(-K_0 c_{K_0}(N^i, K_0) - K_1 N^i c_{K_1}(N^i, K_1)) = \gamma(K_0 + K_1)$. For a linear inverse demand function and a BPR congestion function, we show in Appendix I that the optimal capacity is larger than in the case without uncertainty. In other words: both the expected toll revenue and the total capacity cost are higher with uncertainty than without, but they are so in equal amounts, so that self-financing still prevails. The intuition why both are higher is that due to the convexity of the user cost function, the expected value of the marginal external cost under uncertainty exceeds the deterministic marginal external cost for a traffic flow equal to the expected value of the flow under uncertainty. This raises the expected value of the toll, but also the optimal capacity of the road.

2.2.2 Second-Best for Serial Links

The comparison of the private supply of a road with the first-best case can be less accurate if some untolled initial roads already exist in a network. Because a social planner can only design and hence optimise the capacity and tolls of a new road, the resulting expected social welfare is generally lower for this second best case. Mathematically, the optimization problem looks similar to the fist best one, except that the choice variables are reduced to K_1, τ^i and N^i. The Lagrangian is:

$$\begin{aligned} L = \sum_i p^i \Big(\int_0^{N^i} D^i(n)dn - N^i(c(N^i, K_0) + c(N^i, K_1))\Big) - \gamma(K_0 + K_1) \\ + \sum_i \lambda^i (c(N^i, K_0) + c(N^i, K_1) + \tau^i - D^i(N^i)) \end{aligned} \quad (3)$$

The solution is:

$$\begin{aligned} &D^i(N^i) - c(N^i, K_0) - c(N^i, K_1) - N^i(c_{N^i}(N^i, K_0) + c_{N^i}(N^i, K_1)) = 0 \\ &\sum_i p^i(-N^i c_{K_1}(N^i, K_1)) = \gamma \\ &\tau^i = N^i(c_{N^i}(N^i, K_0) + c_{N^i}(N^i, K_1)) \end{aligned} \quad (4)$$

The results are in line with the research on private supply of roads without uncertain demand (Verhoef et al. (1996)). The toll in each demand state equals the marginal external congestion cost of the full trip, so the expected toll revenues more than compensate the capacity cost of the tolled link. In fact, the toll revenue would be sufficient to cover the cost of supplying both links in an optimal fashion, if the capacity of the unpriced links happens to be optimal.

2.2.3 Second-Best Zero-Profit for Serial Links

We study the second best zero profit case in this subsection, where the provider earns zero profit on the tolled link. This is a good benchmark for competing private firm, because under perfect competition, profit are driven down to zero. It is also a good benchmark for regulation, because the toll revenue must cover capacity cost if no subsidy is provided by the government. The problem has the following Lagrangian:

$$
\begin{aligned}
L = &\sum_i p^i\Big(\int_0^{N^i} D^i(n)dn - N^i(c(N^i, K_0) + c(N^i, K_1))\Big) - \gamma(K_0 + K_1) \\
&+ \sum_i \lambda^i(c(N^i, K_0) + c(N^i, K_1) + \tau^i - D^i(N^i)) + \lambda^{zp}\Big(\sum_i p^i N^i \tau^i - \gamma K_1\Big)
\end{aligned}
\tag{5}
$$

The FOCs are:

$$
\begin{aligned}
\frac{\partial L}{\partial N^i} &= p^i(D^i(N^i) - c(N^i, K_0) - c(N^i, K_1) - N^i c_{N^i}(N^i, K_0) - N^i c_{N^i}(N^i, K_1)) \\
&\quad + \lambda^i(c_{N^i}(N^i, K_0) + c_{N^i}(N^i, K_1) - D^i_{N^i}(N^i)) + \lambda^{zp} p^i \tau^i = 0 \\
\frac{\partial L}{\partial \tau_1^i} &= \lambda^i + \lambda^{zp} p^i N^i = 0 \\
\frac{\partial L}{\partial K_1} &= -\sum_i p^i N^i c_{K_1}(N^i, K_1) - \gamma + \sum_i \lambda^i c_{K_1}(N^i, K_1) - \lambda^{zp}\gamma = 0 \\
\frac{\partial L}{\partial \lambda^i} &= c(N^i, K_0) + c(N^i, K_1) + \tau^i - D^i(N^i) = 0 \\
\frac{\partial L}{\partial \lambda^{zp}} &= \sum_i p^i N^i \tau^i - \gamma K_1 = 0 \qquad (6)
\end{aligned}
$$

Substitute the Lagrangian multipliers out and we have the following expression:

$$
\lambda^{zp} = \frac{\tau^i - N^i(c_{N^i}(N^i, K_0) + c_{N^i}(N^i, K_1))}{-\tau^i - N^i(D^i_{N^i}(.) - c_{N^i}(N^i, K_0) - c_{N^i}(N^i, K_1))} \qquad (7)
$$

The Lagrangian multiplier for the zero profit constraint, λ^{zp}, reflects how much the expected social welfare changes if we allow for a small expected deficit. The numerator is the derivative of the social welfare in state i with respect to N^i. It equals the height of the Harberger triangle, which measures the deadweight loss due to inefficient tolling. The denominator is the derivative of the deficit in state i with respect to N^i. To derive it, consider the effect of a marginal increase in traffic volume. On the one hand, the deficit decreases, because we get the toll payment from the new marginal traveler. On the other hand, the deficit also increases, because we can collect less toll from every original traveler [1]. In sum, λ^{zp} measures how much a change in

[1]The toll is less because the congestion cost increases and the willingness to pay de-

the expected deficit leads to a change in the expected social welfare in equilibrium. Its value must be equal across states, because the marginal effect is the same whether it is realized through through a change of the traffic flow in state i or j.

Using λ^{zp}, we can see why, contrary to cases without demand uncertainty Verhoef (2008), the toll in the second best zero profit case is not the marginal external congestion cost of the link. For example, if inverse demand function is $D^i(N^i) = d_0^i - d_1 N^i$ and $d_0^i > d_0^j$, apply the Pigouvian toll and we will have $N^i > N^j$ and $\frac{-c_{N^i}(N^i,K_0)}{c_{N^i}(N^i,K_0)+d_1} < \frac{-c_{N^j}(N^j,K_0)}{c_{N^j}(N^j,K_0)+d_1}$. We prove that Pigouvian is not optimal by contradiction. For state i, the traffic volume is set optimally, so $\frac{\partial L}{\partial N^i} = 0$ and $\lambda^{zp} = \frac{-c_{N^i}(N^i,K_0)}{c_{N^i}(N^i,K_0)+d_1}$. For state j, because $\lambda^{zp} < \frac{-c_{N^j}(N^j,K_0)}{c_{N^j}(N^j,K_0)+d_1}$, we can show that $\frac{\partial L}{\partial N^j} > 0$ and the traffic volume in state j is too low.

2.2.4 Auctions for Serial Links

When a social planner is not sure about the optimal toll and capacity due to lack of information on the congestion cost function or the capacity cost, a competitive auction can improve the efficiency of private supply of the road. In a competitive auction, all sellers have the same marginal capacity cost γ and full information of the congestion cost function, and they will bid until the profit is exhausted. We already know that in the deterministic case, two auctions can implement the second best zero profit outcome (Verhoef, 2007). There are the patronage auction, where firms bid for the highest traffic flow on the tolled road, and the generalized price auction, where they bid for the lowest generalized price. So we will study how theses two auctions perform under uncertain demand.

The patronage auction maximizes the expected traffic flow and the Lagrangian is:

creases.

$$
\begin{aligned}
L &= \sum_i p^i N^i + \sum_i \lambda^i (c(N^i, K_0) + c(N^i, K_1) + \tau^i - D^i(N^i)) + \lambda^{zp}(\sum_i p^i N^i \tau^i - \gamma K_1) \\
\frac{\partial L}{\partial N^i} &= p^i + \lambda^i (c_{N^i}(N^i, K_0) + c_{N^i}(N^i, K_1) - D^i_{N^i}(N^i)) + \lambda^{zp} p^i \tau^i = 0 \\
\frac{\partial L}{\partial \tau_1^i} &= \lambda^i + \lambda^{zp} p^i N^i = 0 \\
\frac{\partial L}{\partial K_1} &= \sum_i \lambda^i c_{K_1}(N^i, K_1) - \lambda^{zp}\gamma = 0 \\
\frac{\partial L}{\partial \lambda^i} &= c(N^i, K_0) + c(N^i, K_1) + \tau^i - D^i(N^i) = 0 \\
\frac{\partial L}{\partial \lambda^{zp}} &= \sum_i p^i N^i \tau^i - \gamma K_1 = 0
\end{aligned} \tag{8}
$$

If we simplify the first order conditions, we have

$$
\lambda^{zp} = \frac{1}{-\tau^i - N^i(D^i_{N^i}(.) - c_{N^i}(N^i, K_0) - c_{N^i}(N^i, K_1))} \tag{9}
$$

The Lagrangian multiplier for the zero profit constraint, λ^{zp}, reflects how much the expected patronage changes if we allow for a small expected deficit. The numerator equals the derivative of the traffic volume in state i with respect to itself, and is therefore 1 now (as apposed to the numerator of (7)). The denominator equals the derivative of the expected deficit with respect to the traffic volume, which is of course the same as in (7). To see why the toll cannot be the marginal congestion cost of the tolled road, consider again a linear inverse demand function with $D^i(N^i) = d_0^i - d_1 N^i$. If $d_0^i > d_0^j$, we can derive that $N^i > N^j$ and $\frac{1}{N^i(d_1 + c_{N^i}(N^i, K_0))} < \frac{1}{N^j(d_1 + c_{N^j}(N^j, K_0))}$.

The generalized price auction minimizes the generalized price, so the Lagrangian is:

$$L = \sum_i p^i D^i(N^i) + \sum_i \lambda^i (c(N^i, K_0) + c(N^i, K_1) + \tau^i - D^i(N^i)) + \lambda^{zp}(\sum_i p^i N^i \tau^i - \gamma K_1)$$

$$\begin{aligned}
\frac{\partial L}{\partial N^i} &= p^i D^i_{N^i}(N^i) + \lambda^i (c_{N^i}(N^i, K_0) + c_{N^i}(N^i, K_1) - D^i_{N^i}(N^i)) + \lambda^{zp} p^i \tau^i = 0 \\
\frac{\partial L}{\partial \tau^i_1} &= \lambda^i + \lambda^{zp} p^i N^i = 0 \\
\frac{\partial L}{\partial K_1} &= \sum_i \lambda^i c_{K_1}(N^i, K_1) - \lambda^{zp}\gamma = 0 \\
\frac{\partial L}{\partial \lambda^i} &= c(N^i, K_0) + c(N^i, K_1) + \tau^i - D^i(N^i) = 0 \\
\frac{\partial L}{\partial \lambda^{zp}} &= \sum_i p^i N^i \tau^i - \gamma K_1 = 0 \qquad (10)
\end{aligned}$$

If we substitute out the Lagrangian multipliers, we have

$$\lambda^{zp} = \frac{D^i_{N^i}(.)}{-\tau^i - N^i(D^i_{N^i}(.) - c_{N^i}(N^i, K_0) - c_{N^i}(N^i, K_1))} \qquad (11)$$

λ^{zp} reflects how much the expected generalized price changes as a result of allowing a small expected deficit. The numerator equals the derivative of the expected generalized price with respect to the traffic volume in state i. The denominator again equals the derivative of the expected deficit with respect to the traffic volume. For a linear inverse demand function, where $D^i_{N^i}(.)$ is a constant, the expression of λ^{zp} suggests the same relation between τ^i, K_i and N^i in both auctions, so the solutions are also the same for both auctions. This is because, for linear demand function and serial links, the maximum expected total traffic flow corresponds to the minimum expected generalized price. For non-linear demands, the outcomes of the auctions will be different.

In sum, the patronage auction and the generalized price auction cannot replicate the result for the second best zero profit case with serial links if there is demand uncertainty. There are two ways to explain it. Firstly, the auctions and the second best zero profit case have different *expected* optimization objectives, which is the sum of the optimization objectives in each state weighted by the probability of that state. Note that the expected social welfare equals the expected consumer surplus under the zero profit constraint. For linear inverse demand functions, in each state, the consumer surplus is quadratic in the traffic flow while the patronage and the generalized price are linear in the traffic flow, so the highest *expected* consumer surplus corresponds to neither the highest *expected* patronage nor the lowest *expected* generalized price. Secondly, and as a result of the first point, the last four sets of FOCs, which are the same for the two auctions and the second best zero profit case, cannot determine a unique solution. The solution depends also on the first three sets of FOCs, which are in general different across the auctions and the second best zero profit case. On the contrary, when there is no demand uncertainty, in terms of the optimization objective, maximization of the social welfare under the zero profit constraint is equivalent to maximization of the patronage and minimization of the generalized price. In terms of the resulting FOCs, the last four sets of FOCs determine a unique combination of the traffic flows, capacity and toll (Verhoef 2007).

2.3 Parallel Links

In this section, we consider two parallel links, where two roads both connect the same origin and destination and a traveler can use either of them. As a result, the total traffic flow is the sum of the traffic flow of both links, i.e. $N^i = N_0^i + N_1^i$.

Here is a summary of the key results for the parallel links for four cases. In the first best case, where the expected social welfare is maximized, the toll is the marginal external cost of congestion and both roads are self-financing in expectation. In the second best case, however, when one road is untolled, the social planner expects a loss when maximizing the expected social welfare by setting capacity and tolls on the other road. In the second best zero profit case, we study the highest social welfare attainable under the condition that the toll road generates at least a zero profit, because private firms will exit the market if they make a loss. This results in the same toll as in the first best. Finally in the auctions case, we show that, contrary to the deterministic case Verhoef (2007), neither the patronage auction nor the least price auction can implement the second best zero profit result when demand uncertainty is a concern.

2.3.1 First-Best for Parallel Links

As a benchmark for efficiency loss due to private supply of road, we discuss the first best case, where a social planner can design the capacities and tolls of the two parallel links to obtain the highest expected social welfare. The Lagrangian is the following:

$$
\begin{aligned}
L = &\sum_i p^i[\int_0^{N_0^i+N_1^i} D^i(n)dn - N_0^i c(N_0^i, K_0) - N_1^i c(N_1^i, K_1)] - \gamma(K_0 + K_1) \\
&+ \sum_i \lambda_0^i[c(N_0^i, K_0) + \tau_0^i - D^i(N_0^i + N_1^i)] + \sum_i \lambda_1^i[c(N_1^i, K_1) + \tau_1^i - D^i(N_0^i + N_1^i)]
\end{aligned}
\tag{12}
$$

Solve the first order conditions of $K_j, \tau_j^i, N_j^i, \lambda_j^i$ and we have the following:

$$
\begin{aligned}
&D^i(N_0^i + N_1^i) - c(N_0^i, K_0) - \tau_j^i = 0 \\
&\sum_i p^i(-N_j^i c_{K_j}(N_j^i, K_j)) = \gamma \\
&\tau_j^i = N_j^i c_{N_j^i}(N_j^i, K_j)
\end{aligned}
\tag{13}
$$

In the first best case, the Pigouvian toll is levied on each link in each state, so the externality of congestion is internalized. Consistent with Lindsey and de Palma (2014), the roads are self-financing in expectation. For linear inverse demand function and BPR congestion function, we show in Appendix I that the optimal capacity is larger than the case without uncertainty. Due to the convexity of the user cost function, the expected value of the marginal external cost under uncertainty exceeds the deterministic marginal external cost for a traffic flow equal to the expected value of the flow under uncertainty. This raises the expected value of the toll, but also the optimal capacity of the road.

2.3.2 Second-Best for Parallel Links

In the second best case, an untolled initial network already exists, and the best a social planner can do is to optimize the capacity and tolls of a new road. It is common to find the situation that, to get from one city to another, there are a fast highway with tolls and a slow but free road. The optimization problem looks similar to the fist best one, except the choice variables are reduced to K_1, τ_1^i and N_j^i. Let us consider the case where the new road runs parallel to an existing untolled one. The Lagrangian is:

$$\begin{aligned} L = &\sum_i p^i [\int_0^{N_0^i+N_1^i} D^i(n)dn - N_0^i c(N_0^i, K_0) - N_1^i c(N_1^i, K_1)] - \gamma(K_0 + K_1) \\ &+ \sum_i \lambda_0^i [c(N_0^i, K_0) - D^i(N_0^i + N_1^i)] + \sum_i \lambda_1^i [c(N_1^i, K_1) + \tau_1^i - D^i(N_0^i + N_1^i)] \end{aligned} \tag{14}$$

The solution is:

$$
\begin{aligned}
&D^i(N_0^i + N_1^i) - c(N_0^i, K_0) = 0 \\
&D^i(N_0^i + N_1^i) - c(N_1^i, K_1) - \tau_1^i = 0 \\
&\sum_i p^i(-N_1^i c_{K_1}(N_1^i, K_1)) = \gamma \\
&\tau_1^i = N_1^i c_{N_1^i}(N_1^i, K_1) + N_0^i c_{N_0^i}(N_0^i, K_0) \frac{D^i_{N_1^i}(N_0^i + N_1^i)}{c_{N_0^i}(N_0^i, K_0) - D^i_{N_0^i}(N_0^i + N_1^i)}
\end{aligned} \tag{15}
$$

Similar to the case without uncertainty Verhoef et al. (1996), the toll in each demand state equals the sum of the marginal external cost of congestion and a negative network spillover effect. The social planner expects a loss on the tolled link, because of a downward adjustment of the toll from the Pigouvian toll.

2.3.3 Second-Best Zero-Profit for Parallel Links

As shown in the previous section, in the second-best case with parallel links, the operator of the tolled road makes a loss. What is the highest expected social welfare if this road has to be self-financing? This is a relevant question because it is a natural benchmark for private supply (with or without regulation) when there is an untolled alternative parallel road: we cannot expect the private road to make a loss, so the best we can hope for is the setting where welfare is maximized while the firm makes (at least) a zero profit. Mathematically, we optimize over the following Lagrangian funciton:

$$L = \sum_i p^i[\int_0^{N_0^i+N_1^i} D^i(n)dn - N_0^i c(N_0^i, K_0) - N_1^i c(N_1^i, K_1)] - \gamma(K_0 + K_1)$$
$$+ \sum_i \lambda_0^i[c(N_0^i, K_0) - D^i(N_0^i + N_1^i)] + \sum_i \lambda_1^i[c(N_1^i, K_1) + \tau_1^i - D^i(N_0^i + N_1^i)]$$
$$+ \lambda^{zp}[\sum_i p^i \tau_1^i N_1^i - \gamma K_1] \tag{16}$$

The FOCs are:

$$\frac{\partial L}{\partial N_0^i} = p^i[D^i(N_0^i + N_1^i) - c(N_0^i, K_0) - N_0^i c_{N_0^i}(N_0^i, K_0)] + \lambda_0^i[c_{N_0^i}(N_0^i, K_0) - D^i_{N_0^i}(N_0^i + N_1^i)]$$
$$- \lambda_1^i D^i_{N_0^i}(N_0^i + N_1^i) = 0$$
$$\frac{\partial L}{\partial N_1^i} = p^i[D^i(N_0^i + N_1^i) - c(N_1^i, K_1) - N_1^i c_{N_1^i}(N_1^i, K_1)] - \lambda_0^i D^i_{N_1^i}(N_0^i + N_1^i)$$
$$+ \lambda_1^i[c_{N_1^i}(N_1^i, K_1) - D^i_{N_1^i}(N_0^i + N_1^i)] + \lambda^{zp} p^i \tau_1^i = 0$$
$$\frac{\partial L}{\partial \tau_1^i} = \lambda_1^i + \lambda^{zp} p^i N_1^i = 0$$
$$\frac{\partial L}{\partial K_1} = \sum_i p^i(-N_1^i c_{K_1}(N_1^i, K_1)) - \gamma + \sum_i \lambda_1^i c_{K_1}(N_1^i, K_1) - \lambda^{zp}\gamma = 0$$
$$\frac{\partial L}{\partial \lambda_0^i} = c(N_0^i, K_0) - D^i(N_0^i + N_1^i) = 0$$
$$\frac{\partial L}{\partial \lambda_1^i} = c(N_1^i, K_1) + \tau_1^i - D^i(N_0^i + N_1^i) = 0$$
$$\frac{\partial L}{\partial \lambda^{zp}} = \sum_i p^i \tau_1^i N_1^i - \gamma K_1 = 0 \tag{17}$$

The Lagrangian multiplier for the zero profit condition is simplified to:

$$\lambda^{zp} = \frac{\tau_1^i - N_1^i c_{N_1^i}(.) - N_0^i c_{N_0^i}(.) \frac{D^i_{N_1^i}(.)}{c_{N_0^i}(.) - D^i_{N_0^i}(.)}}{-\tau_1^i - N_1^i (D^i_{N_1^i}(.) + D^i_{N_0^i}(.) \frac{D^i_{N_1^i}(.)}{c_{N_0^i}(.) - D^i_{N_0^i}(.)} - c_{N_1^i}(.))} \tag{18}$$

λ^{zp} again shows how much expected social welfare changes when we allow a small expected deficit on the tolled road. The numerator is the derivative of the social welfare in state i with respect to N_1^i, taking into account of the induced changes in N_0^i[2]. It equals the sum of the height of the Harberger triangle of both links, where that of untold link is weighted to reflect the substitution between equilibrium use of the two links. The denominator is the derivative of the deficit in state i with respect to N_1^i, taking into account of the induced changes in N_0^i. The deficit decreases directly, because we get the toll payment from the new traveler on the tolled road. It also increases indirectly, because we can collect less toll from every original traveler due to the diversion of traffic to the untolled link [3]. In sum, λ^{zp} measures how much a change in the expected deficit leads to a change in the expected social welfare in equilibrium and its value must be equal across states.

Using the expression for λ^{zp} together with the last four equations, we find the solution for the equilibrium:

---

[2]If N_1^i increases by a small amount Δ, N_0^i will decrease by $\frac{D^i_{N_1^i}(.)}{c_{N_0^i}(.) - D^i_{N_0^i}(.)}\Delta$, because on the untolled road, user cost must equal inverse demand.

[3]The willingness to pay decreases with an increase in N_1^i, and decreases with an induced decrease in N_0^i, while the congestion on the tolled link increases with an increase in N_1^i, so in total the toll decreases.

$$
\begin{aligned}
&\lambda_0^i = p^i N_0^i \\
&\lambda_1^i = -p^i N_0^i \\
&\lambda_{zp} = \frac{N_0^i}{N_1^i} \\
&D^i(N_0^i + N_1^i) - c(N_0^i, K_0) = 0 \\
&D^i(N_0^i + N_1^i) - c(N_1^i, K_1) - \tau_1^i = 0 \\
&\sum_i p^i(-N_1^i c_{K_1}(N_1^i, K_1)) = \gamma \\
&\tau^i = N_1^i c_{N_1^i}(N_1^i, K_1)
\end{aligned}
\tag{19}
$$

The toll in each state equals the marginal external cost of congestion of the tolled link, even for a general structure of demand uncertainty. If the social planner allows an expected deficit on the tolled road, the positive λ^{zp} shows that the expected social welfare will increase. This is because, the Pigouvian toll on the tolled road does not account for the crowding out effect on the untolled road. In addition, we can achieve an equivalent welfare increase if we give every user on the tolled road a small subsidy[4]. Such effect on welfare is larger when there are more travelers on the untolled road and less travelers on the tolled road. In the extreme case of zero traffic flow on the untolled road, the effect is zero because we already achieve the highest social welfare by optimum tolling.

2.3.4 Auctions for Parallel Links

The second best zero profit case is the best result we can expect for private supply of road in a mixed network, but private firms will not automatically

[4]To achieve the same increase in expected social welfare, the total subsidy must equal the deficit, because $\lambda_1^i = -\lambda^{zp} p^i N_1^i$.

achieve it. This subsection again examines how the social planner can use the patronage auction and the generalized price auction to regulate private supply of roads. Both auctions can produce the same result as the second-best zero-profit case for parallel links without demand uncertainty (Verhoef 2007), but as we have seen in the serial links, this is no longer true with demand uncertainty.

Patronage Auction for Parallel Links For the patronage auction, a firm maximizes the expected traffic flow on its link. Due to competitive bidding, the firm earns zero profit. The Lagrangian and FOCs for the problem are:

$$\begin{aligned}
&L = \sum_i p^i N_1^i + \sum_i \lambda_0^i [c(N_0^i, K_0) - D^i(N_0^i + N_1^i)] + \sum_i \lambda_1^i [c(N_1^i, K_1) + \tau_1^i - D^i(N_0^i + N_1^i)] \\
&\quad + \lambda^{zp}[\sum_i p^i \tau_1^i N_1^i - \gamma K_1] \\
&\frac{\partial L}{\partial N_0^i} = \lambda_0^i [c_{N_0^i}(N_0^i, K_0) - D^i_{N^i}(N_0^i + N_1^i)] - \lambda_1^i D^i_{N^i}(N_0^i + N_1^i) = 0 \\
&\frac{\partial L}{\partial N_1^i} = p^i - \lambda_0^i D^i_{N^i}(N_0^i + N_1^i) + \lambda_1^i [c_{N_1^i}(N_1^i, K_1) - D^i_{N^i}(N_0^i + N_1^i)] + \lambda^{zp} p^i \tau_1^i = 0 \\
&\frac{\partial L}{\partial \tau_1^i} = \lambda_1^i + \lambda^{zp} p^i N_1^i = 0 \\
&\frac{\partial L}{\partial K_1} = \sum_i \lambda_1^i c_{K_1}(N_1^i, K_1) - \lambda^{zp}\gamma = 0 \\
&\frac{\partial L}{\partial \lambda_0^i} = c(N_0^i, K_0) - D^i(N_0^i + N_1^i) = 0 \\
&\frac{\partial L}{\partial \lambda_1^i} = c(N_1^i, K_1) + \tau_1^i - D^i(N_0^i + N_1^i) = 0 \\
&\frac{\partial L}{\partial \lambda^{zp}} = \sum_i p^i \tau_1^i N_1^i - \gamma K_1 = 0
\end{aligned} \tag{20}$$

The last four sets of equations, which specify the investment rule, the Wardropian user equilibrium conditions for the untolled and tolled roads and the zero profit constraint, are the same as those of the second best zero profit case. However, unlike the case without uncertainty Wu et al. (2011), those four sets of equations cannot determine a unique solution, because demand uncertainty offers more choice variables now. To find the solution, we need to substitute out the three Lagrangian multipliers from the first three sets of equations. λ^{zp} for any state i can be expressed as follows:

$$\lambda^{zp} = \frac{1}{-\tau_1^i - N_1^i(D^i_{N_1^i}(.) + D^i_{N_0^i}(.)\frac{D^i_{N_1^i}(.)}{c_{N_0^i}(.) - D^i_{N_0^i}(.)} - c_{N_1^i}(.))} \tag{21}$$

λ^{zp} shows how much the expected traffic volume changes if there is a small expected deficit, as in (9) for serial links. The numerator is the derivative of the traffic volume on the tolled road in state i with respect to itself, thus its value is 1. The denominator is the derivative of the deficit with respect to the traffic volume on the tolled road, which is of course the same as the denominator in (7).

Contrary to the case without demand uncertainty, the resulting toll cannot be equal to the marginal external congestion cost of the tolled road. If it was so, λ^{zp} would be different for state i from state j. For example, for a linear inverse demand function and a BPR congestion cost function, we can show that if $d_0^i > d_0^j$, we have $\frac{1}{N_1^i d \frac{c_{N_0^i}(.)}{c_{N_0^i}(.)+d}} < \frac{1}{N_1^j d \frac{c_{N_0^j}(.)}{c_{N_0^j}(.)+d}}$.

Generalized Price Auction for Parallel Links For the generalized price auction, a firm minimizes the expected generalized price, such that the tolled road beaks even. The Lagrangian for the problem is:

$$
\begin{aligned}
L = &\sum_i p^i D^i(N_0^i + N_1^i) + \lambda^{zp}(\sum_i p^i \tau_1^i N_1^i - \gamma K_1) \\
&+ \sum_i \lambda_0^i (c(N_0^i, K_0) - D^i(N_0^i + N_1^i)) + \sum_i \lambda_1^i (c(N_1^i, K_1) + \tau_1^i - D^i(N_0^i + N_1^i))
\end{aligned}
\tag{22}
$$

The FOCs are:

$$
\begin{aligned}
\frac{\partial L}{\partial N_0^i} &= p^i D^i_{N_0^i}(N_0^i + N_1^i) + \lambda_0^i(c_{N_0^i}(N_0^i, K_0) - D^i_{N_0^i}(N_0^i + N_1^i)) - \lambda_1^i D^i_{N_0^i}(N_0^i + N_1^i) = 0 \\
\frac{\partial L}{\partial N_1^i} &= p^i D^i_{N_1^i}(N_0^i + N_1^i) - \lambda_0^i D^i_{N_1^i}(N_0^i + N_1^i) + \lambda_1^i(c_{N_1^i}(N_1^i, K_1) - D^i_{N_1^i}(N_0^i + N_1^i)) + \lambda^{zp} p^i \tau_1^i = 0 \\
\frac{\partial L}{\partial \tau_1^i} &= \lambda_1^i + \lambda^{zp} p^i N_1^i = 0 \\
\frac{\partial L}{\partial K_1} &= \sum_i \lambda_1^i c_{K_1}(N_1^i, K_1) - \lambda^{zp}\gamma = 0 \\
\frac{\partial L}{\partial \lambda_0^i} &= c(N_0^i, K_0) - D^i(N_0^i + N_1^i) = 0 \\
\frac{\partial L}{\partial \lambda_1^i} &= c(N_1^i, K_1) + \tau_1^i - D^i(N_0^i + N_1^i) = 0 \\
\frac{\partial L}{\partial \lambda^{zp}} &= \sum_i p^i \tau_1^i N_1^i - \gamma K_1 = 0
\end{aligned}
\tag{23}
$$

After substitution, λ^{zp} is as follows:

$$
\lambda^{zp} = \frac{D^i_{N_1^i}(.) + D^i_{N_0^i}(.)\frac{D^i_{N_1^i}(.)}{c_{N_0^i}(.) - D^i_{N_0^i}(.)}}{-\tau_1^i - N_1^i(D^i_{N_1^i}(.) + D^i_{N_0^i}(.)\frac{D^i_{N_1^i}(.)}{c_{N_0^i}(.) - D^i_{N_0^i}(.)} - c_{N_1^i}(.))}
\tag{24}
$$

λ^{zp} now shows how much the expected generalized price changes if there is a small expected deficit, as in (10) for serial links. The numerator is the derivative of the generalized price in state i with respect to the traffic volume on the tolled road, taking into consideration the induced traffic volume on the untolled link. The denominator is the derivative of the deficit in state i with respect to the traffic volume on the tolled road, which is the same in the second best zero profit case.

The tolls again cannot be Pigouvian in the generalized price auction, because then λ^{zp} would be different across states. For a linear demand and a BPR congestion functions, if $d_0^i > d_0^j$, we can show that $\frac{-1}{N_1^i} > \frac{-1}{N_1^j}$.

In sum, the patronage auction and the generalized price auction cannot replicate the result of the second best zero profit case, even for a linear demand function. To see it from the perspective of the FOCs, the conditions for λ^{zp} are not equivalent among the three cases, because many combinations of tolls and capacity can satisfy the last four sets of equations. Which combination is optimal depends on the first three sets of equations. To see it from the perspective of the optimization target, the traffic volume that maximizes the **expected** patronage on the tolled road does not minimize the **expected** generalized price.

3 Numerical Analysis

The analytical results show that the impact of demand uncertainty on the efficiency of the network is different for parallel and serial links. For a more general network, as well as comparative statistics, clear-cut analytical results are hard to obtain, so we rely on numerical simulations to gain some insights. We are not only interested in how private roads perform in a given mixed network, but also in how private provision of roads influence the network development. So in the simulation we will model how competition and regulation can influence the network formation under demand uncertainty.

3.1 Setup

Similar to Verhoef (2008), we assume there are two serial segments a and b in a network connecting one origin and one destination. The initial links on the two segments, denoted as $a0$ and $b0$ respectively, are untolled, which represent a free public road network. Private firms can add capacities on each segment one at a time, and then charge tolls. For example, if the first firm adds a link in section a, we denote the new link as $a1$. $a1$ is now parallel to the existing link on section a, denoted by $a0$, and serial to the existing link on section b, denoted by $b0$. In this way, we can model both parallel and serial competition and the development of the network.

The timing of the game is as follows: since construction takes time, at the beginning of each round, there is uncertainty about the future demand. Firms compete to add a capacity to one section of the network without knowing the realized demand. After the demand uncertainty is resolved, the firms can no longer change their capacities, but they can decide on tolls to charge on their own links. When setting capacities, firms are myopic in expecting no more new entry. They only consider the resulted toll setting game in the second stage. However, to their surprise, a new round begins and a new firm may enter. There is again demand uncertainty and capacity setting of the new firm in the first stage, and toll setting of all firms in the second stage. The sequential game continues until there is no profit for a new entry.

Some assumptions are made for the simplicity of the simulation and to avoid unsolvable dynamic games, but we think they are also reasonable. We assume that within each round, firms are forward looking and rational, so the capacity decision takes into account of the toll setting in the next stage. But between rounds, firms are assumed myopic, in the sense that they take every round as repeating itself forever, until they are surprised by new comers who change the network structure. We make this assumption, thinking of the slow and lump-sum development in infrastructure we usually observe in reality. Because it usually takes considerate time to assess the expected cost and benefit and finally allow new developments in roads, a firm can focus on competing with the existing firms for now and not worry too much about possible new competitors in the future. But we admit it is a simplification. Another assumption is the uncertain demand in each round. Firms can learn to predict the demand over time, but in general, since the road service period is long, it is hard to know about the demand far into the future. In addition, the economy usually fluctuates between booms and busts during the long service years of a road. We think the model is best suited for development in road networks where capacity addition is slow in pace and once it's added, it remains there for a long time while prices are relatively easy to adjust for demand changes. We could of course make other ad hoc assumptions on how demand uncertainty would change between investment rounds, but this would in fact complicate the interpretation of the results.

We compare and contrast two regimes, the unregulated free-entry regime and the regulated entry-by-auction regime. In the free-entry regime, firm compete freely. We assume the firm with the highest expected profit adds a capacity on its desired section, because this firm is the most motivated to lobby the government for it. After the capacity is built, the demand is known and all firms in the network play a Bertrand price setting game, i.e. every firm sets its own toll simultaneously while taking the tolls of the others as given. In the entry-by-auction regime, the winner of an auction can add a capacity. Due to the perfect competition in auctions, any firm that adds a capacity earns zero profit in expectation. The auction can be on either the expected patronage of the new road or the expected generalized price. To be comparable to Verhoef 2008, when demand is known, all existing firms charge tolls as promised in the auction, thus no direct toll competition in the entry-by-auction regime.

The parameters of the numerical simulation is as follows. To be comparable with Verhoef (2008), the inverse demand function is linear and $D^i(N^i) = \delta_0 - \delta_1 N^i$. The demand uncertainty is about δ_0, where $\delta_0 = \delta_0^h$ with probability p^h and $\delta_0 = \delta_0^l$ with probability $1 - p^h$. We set $\delta_1 = 0.01167, \delta_0^h = 74.11, \delta_0^l = 49.41, p^h = 0.5$, which means compared with Verhoef 2008, the highest willingness to pay can go up or down by 20% with equal probability. According to Flyvbjerg et al. (2006), actual traffic deviates from the forecasted ones by more than 20% for half of roads projects. The congestion cost function is of the familiar BPR form, i.e. $c(N_j^i, K_j) = \alpha t_f(1 + \beta(\frac{N_j^i}{K_j})^x)$. The parameter for the value of time, α, is set at 7.5. t_f is the free-flow travel time and is set at 0.25, implying a total trip length of 60 kilometers for a highway with 120 kilometer/hour speed. β and x take their conventional values of 0.15 and 4 respectively. The marginal capacity cost is set at 3.5 for both segments, which represents the hourly capital cost. We assume the initial capacities are $K_{a0} = K_{b0} = 1500$.

3.2 Benchmark

As a benchmark, Table 1 summarizes the characteristics of the base equilibrium, the first best case, the second best case and the second best zero profit case. The expected social welfare, efficiency, expected profits, capacity, tolls, generalized prices and traffic volumes are denoted by $S, \omega, \pi, K, \tau, P$ and N respectively. The superscript $h(l)$ denotes the high(low) demand state. And the subscript $a(b)$ is for section a(b), while 0(1) is for the initial (newly-added) link. E is for expectation. The results are consistent with the theory in the previous section.

The base equilibrium with the two untolled links is quite congested as its expected social welfare, i.e. $E(S) = 60189$, is only half of the first-best case's. Since no toll is charged, the government expects a loss of 5250 on the two initial links. In both states, congestion cost is much higher than that of the first best case in both low and high demand states. There is a large room for improvement from the base equilibrium.

In the first best case, where the capacities and tolls on the two initial segments are set by a social planner to maximize the social welfare, the capacity more than doubles from the base case and the congestion costs in both states decrease. The expected profit is zero, because the profit in the high state and the loss in the low state cancel out. Therefore, first-best pricing and capacity setting is self-financing in expected terms, as in Lindsey and de Palma (2014). Compared with the deterministic case discussed in Verhoef (2008), the first-best capacity is larger under uncertainty, because of the convexity of the congestion cost function, as we also explained above.

The second-best case, where the initial segments are not toll but the capacity and tolls on the two new segments are set to maximize the expected social welfare, can achieve 97.1% of the increase in the expected social welfare from the base equilibrium to the first-best case. However, this generates a considerable loss for the newly-added link in either realized demand state, because the capacity expansion is too large, and the toll revenues cannot cover the capacity cost.

As a benchmark for private supply of roads, we consider the second-best zero-profit case, where the tolls must cover the capacity cost of the new segments in expectation. As predicted by the theoretical result for the parallel links, the second-best zero-profit case have the same toll, total traffic flow and generalized price, as in the first best case, in both demand states. It can achieve 80.4% of the increase in social welfare. The expected profit of the tolled link is zero, because the profit in the good state is the same as the loss in the bad state.

3.3 Entry Game

Free Entry In the free entry regime, the firm with the highest expected profit sets a capacity on a segment of his choice. Then demand uncertainty is resolved and firms sets tolls simultaneously by Bertrand competition. We allow both old and new firms to add capacity, so the double marginalization problem is mitigated, in the sense that one may expect competition between firms on parallel segments to drive down tolls.

The result of the free entry regime is shown in Table 2. We assume without loss of generality that capacity is first added on section a. In equilibrium, the firm that has added capacity on section a will, in the next round, add capacity on section b. Then a new firm adds capacity on section a, then section b, and so on. This pattern emerges because (1) the same firm can better coordinate the tolls on both sections; (2) when the capacity on one section is expanded, it is more profitable to add capacity on the complementary section. In addition, the capacity addition in section b is always larger than that in section a in the previous round because of increased demand. A new firm always adds capacity in section a in the next round because if an old firm does so, it will end up competing with its own capacities in section a. After eights rounds of capacity building, the total capacity on both section are above that of the first-best case in the numerical example, which means competition will cause over investment in capacity in the end. The expected profit of the entrant decreases as more firms join the network, and so does the expected profit of the incumbent firms. The generalized price in each state falls over time due to increased competition, but it remains higher than the second-best zero-profit case's level owing to the market power of the firms. The expected social welfare is 98433 after eight rounds, which means that $\omega = 0.71$, where ω gives the improvement in social welfare from the base equilibrium to the first best case. The qualitative patterns match those described in Verhoef(2008) for deterministic demand.

Entry by Patronage Auction In the entry by patronage auction regime, the firm which offers the highest expected traffic flow on the new link is allowed to add the link. Every time a new firm enters, it makes zero expected profit due to the competitive nature of the auction. Afterwards, it keeps the toll scheme (conditional on the realized demand) unchanged. It may not collect enough tolls to cover the capacity cost if later too many firms enter with low tolls.

The key characteristics of the resulting equilibrium are shown in Table 3. After four rounds already, the expected social welfare is higher than that after eight rounds in the free entry regime. Capacity addition is rapid initially, and after five rounds the capacity addition is negligible, which means a stable network is formed. $\omega = 0.79$, which is much higher than that of the free entry regime and close to that of the second-best zero-profit case (0.80). Judging from these criteria, the entry by patronage auction performs well.

Entry by Generalized-Price Auction In the entry by generalized-price auction regime, the firm which offers the lowest expected generalized price can add the link. Afterwards it keeps the toll scheme (conditional on the realized demand) unchanged. As shown in Table 4, the characteristics of the equilibrium are similar to those in the entry by patronage auction regime. The expected social welfare is even higher and $\omega = 80\%$. Both auctions are quick to achieve a good result, thus an improvement on the free entry regime.

Entry by Social Welfare Auction Finally, we can create a benchmark by considering an entry by social welfare auction, where the firm that generates the highest expected social welfare is allowed add the capacity. The result will be as in Table 5. We shall not confuse it with the second-best zero-profit case in the benchmark, where the capacities on *both* sections a and b are selected at the same time, whereas here capacity is added one link at a time. For this auction, $\omega = 80\%$. Compared with the patronage and the generalized-price auctions, this auction has a lower(higher) generalized price, and a higher(lower) traffic volume in the high(low) demand state. This is in line with the theoretical analysis for the parallel networks, where the tolls in the other two auctions are adjusted upward from the Pigouvian toll in the high-demand state.

In sum, the simulation suggests that the two regulated auctions perform better than the unregulated free entry.

3.4 Sensitivity Analysis

The numerical simulation seems to suggest that entry by either auction performs better than free entry. In addition, entry by the generalized price auction seems to generate higher efficiency than the patronage auction when demand is uncertain. In this section we test the sensitivity of such results with respect to the degree of uncertainty and the price elasticity of demand.

In the numerical simulation, the inverse demand function is represented by $D^i(N^i) = \delta_0^i - \delta_1 N^i$ and the degree of uncertainty is represented by a, where $\delta_0^h = (1+a)\delta_0$ for the high demand state and $\delta_0^l = (1-a)\delta_0$ for the low demand state. $a = 0$ means demand is completely certain and as a increases, demand becomes more uncertain. $a = 0.2$ is used in the numerical simulation in the previous section. Figure 1 shows the relative efficiency of the three regimes after five rounds of entry corresponding with different degrees of demand uncertainty. For $0 \leq a \leq 0.35$, the two auctions clearly perform better than the free entry, because capacity addition is quicker with auctions. The efficiencies of the two auctions are similar, which is consistent with the case without demand uncertainty. As the degree of uncertainty increases, the relative efficiency of all three regulatory regimes increases. The main reason is that when the degree of uncertainty increases, the optimal capacity in the first-best case increases, so the social welfare of the base equilibrium decreases.

To study the robustness of the results with respect to demand elasticity, we vary the demand elasticity by changing the intercept and slope of the inverse demand function, but keep the base equilibrium unchanged. In other words, we change the value of b, where the new intercept is $\bar{\delta}_0^i = \delta_0^i + b\delta_1 N^i$ and the new slope is $\bar{\delta}_1 = (1+b)\delta_1$. For the simulation in the previous section, we set $b = 0$ and the resulting demand elasticity is 0.50. Figure 2 shows the relative efficiency of the three regimes after five rounds of entry corresponding with different demand elasticity. When b changes from -0.5 to 0.5, the demand elasticity changes from -1.01 to -0.34. For the parameter range in

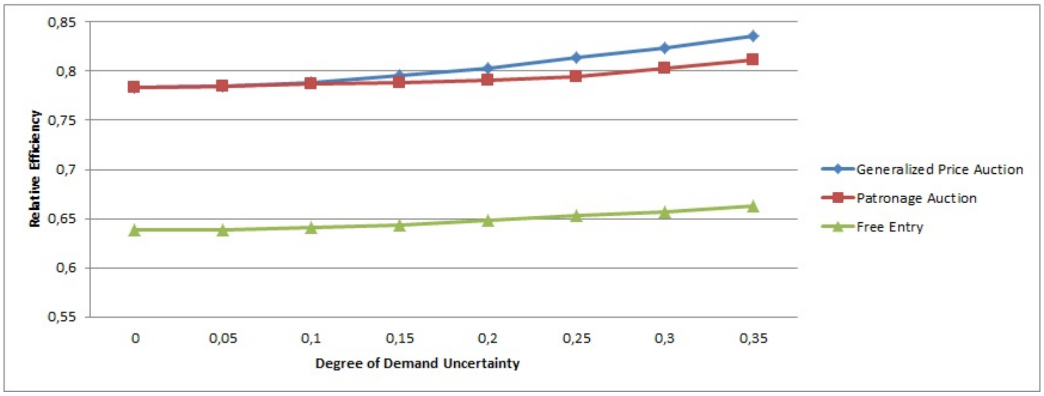

Figure 1: Sensitivity Analysis: Degree of Uncertainty

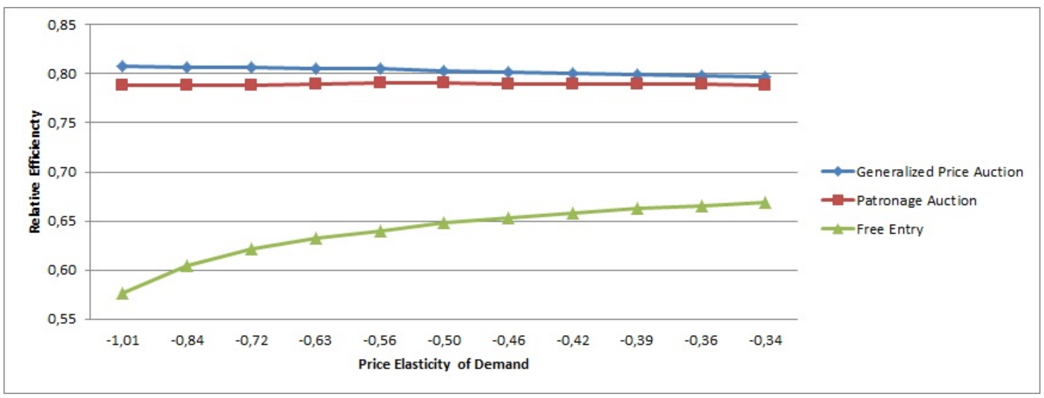

Figure 2: Sensitivity Analysis: Demand Elasticity

the simulation, it seems that the two auctions generate similar social welfare, with the generalized price auction performing only slightly better. They both perform much better than free entry, due to quick capacity addition. As the demand elasticity increases, due to larger capacity adjustment under the first-best case and the auctions, the expected social welfare of the three cases increases less significantly than that of the base equilibrium and the free-entry case. As a result, the relative efficiency of the auctions decreases and that of the free-entry case increases.

4 Concluding Remarks

This paper investigates how demand uncertainty influences the efficiency of private supply of road in a mixed network. The results allows us to compare different benchmarks and evaluate the efficiency of regulation policies for both simple networks and more complex ones.

For simple networks, such as two serial links and two parallel links, we have clear analytical results for four difference cases, namely the first-best regime, the second-best regime, the second-best zero-profit regime and the regulation-by-auction regime. On the one hand, some results are similar to previous studies. For example, in the first-best case, the toll is Pigouvian in every state and the roads are self-financing in expected terms; the optimal capacity is higher than that for a deterministic demand with the same expected value. In the second-best case, when there is an untolled substitute (complement), the tolled road makes a loss (profit) in expectation. In the second-best zero-profit case, the the toll for parallel links are Pigouvian. On the other hand, we gain new insights into the complexity added by demand uncertainty. For instance, in the second-best zero-profit case, the toll for serial links are no longer Pigouvian. The patronage and the generalized price auction can no longer implement the second-best zero-profit result because the highest *expected* maximum social welfare implies neither the highest *expected* patronage nor the lowest *expected* generalized price under the zero profit constraint.

For more complex networks, we resolve to simulation. We consider a network with both parallel and serial links and study also the network formation. When the firms with the highest expected profit can add capacity one link at a time, the capacity addition to the network is slow and will generally lead to over-investment, similar to van den Berg and Verhoef (2012). When we control the process by the patronage or the generalized price auctions, the expected social welfare increases much quicker and reaches a steady state rather closely after only five rounds. The results appear robust to changes in the degree of uncertainty and demand elasticity.

In sum, demand uncertainty complicates the evaluation and regulation of private supply of roads in mixed networks. For future research, we will consider more general networks, dynamic games of capacity addition, user heterogeneity and optimal auction design.

5 Acknowledgments

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Appendix I

Consider the simplest case with only one road. Let N, K, λ denote the flow, the capacity and the unit capacity cost. The congestion cost and demand functions are as follows:

$$
\begin{aligned}
&c(N, K) = \alpha t_f(1 + \beta(\frac{N}{K})^x) \\
&D(N) = \delta_0 - \delta_1 N
\end{aligned}
$$

The demand uncertainty concerns δ_0, which can be δ_0^h with probability p_h and δ_0^l with probability $1-p_h$. A social planner commits to a capacity before knowing the demand but can adjust tolls according the realized demand.

If the social planner chooses K units of capacity and the realized demand parameter is δ_0, then the optimal toll (denoted as τ) should solve the following maximization problem.

$$\max_{\tau,N} W = \delta_0 N - \frac{\delta_1 N^2}{2} - c(N,K)N - \lambda K$$
$$s.t. D(N) = c(N,K) + \tau \tag{25}$$

The FOCs for N is:

$$\frac{\partial W}{\partial N} = \delta_0 - \delta_1 N - \alpha t_f (1 + \beta (\frac{N}{K})^x (1+x)) = 0 \tag{26}$$

Solving the implicit function and the resulting flow and social welfare are denoted as $N^*(K,\delta_0)$ and $W^*(K,\delta_0)$ respectively.

To determine the optimal capacity, the social planner needs to consider the marginal effect of the capacity on social welfare, as in $\frac{\partial W^*(K,\delta_0)}{\partial K} = \alpha t_f \beta x (N^*(K,\delta_0)/K)^{1+x} - \lambda$. The marginal cost is simply λ. The realized marginal benefit, depending on the demand state δ_0, is $\alpha t_f \beta x (N^*(K,\delta_0)/K)^{1+x}$. The expected marginal benefit is then the probability weighted average of the realized ones.

Note that for any chosen capacity K, the realized marginal benefit is convex in δ_0 for $x > 0$, as can be shown by the derivatives below:

$$\frac{\partial N^*(K,\delta_0)^{1+x}}{\partial \delta_0} = \frac{(1+x)N^{*1+x}}{\delta_1 N^* + \alpha\beta t_f (1+x) x (\frac{N^*}{K})^x} > 0$$
$$\frac{\partial^2 N^*(K,\delta_0)^{1+x}}{\partial \delta_0^2} = \frac{N^{*1+x} x (1+x)(\delta_1 N^* + (1+x)\alpha t_f \beta (\frac{N^*}{K})^x)}{(\delta_1 N^* + (1+x) x \alpha t_f \beta (\frac{N^*}{K})^x)^3} > 0 \tag{27}$$

The convexity implies that $p_h \frac{\partial W^*(K,\delta_0^h)}{\partial K} + (1-p_h)\frac{\partial W^*(K,\delta_0^l)}{\partial K} > \frac{\partial W^*(K, p_h\delta_0^h + (1-p_h)\delta_0^l)}{\partial K}$. In other words, for any chosen capacity, the expected marginal benefit is higher with demand uncertainty. So the social planner should choose higher capacity under demand uncertainty.

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Table 1: Benchmark Equilibria

Variables	Base	First-best	Second-best	Second-best zero-profit
S^h	76513	167573	166325	146383
S^l	43864	59908	58061	60099
$E[S]$	60189	113741	112193	103241
ω	0	1	0.971	0.804
π^h_{a0}, π^h_{b0}	-5250	8685	-5250	-5250
π^h_{a1}, π^h_{b1}	-	-	-7461	3340
π^l_{a0}, π^l_{b0}	-5250	-8685	-5250	-5250
π^l_{a1}, π^l_{b1}	-	-	-8625	-3340
$E[\pi_{a0}], E[\pi_{b0}]$	-5250	0	-5250	-5250
$E[\pi_{a1}], E[\pi_{b1}]$	-	-	-8043	0
K_{a0}, K_{b0}	1500	3644	1500	1500
K_{a1}, K_{b1}	-	-	2479	1401
τ^h_{a0}, τ^h_{b0}	0	4.226	0	0
τ^h_{a1}, τ^h_{b1}	-	-	0.350	4.226
τ^l_{a0}, τ^l_{b0}	0	1.119	0	0
τ^l_{a1}, τ^l_{b1}	-	-	0.022	1.119
c^h_{a0}, c^h_{b0}	14.230	2.931	3.313	7.157
c^h_{a1}, c^h_{b1}	-	-	2.963	2.931
c^l_{a0}, c^l_{b0}	6.697	2.155	2.131	3.273
c^l_{a1}, c^l_{b1}	-	-	2.110	2.155
$D^h = P^h$	28.459	14.315	6.626	14.315
$D^l = P^l$	13.395	6.546	4.263	6.546
N^h_{a0}, N^h_{b0}	3862	5074	2256	3123
N^h_{a1}, N^h_{b1}	-	-	3477	1951
N^h	3862	5074	5733	5074
N^l_{a0}, N^l_{b0}	3052	3639	1466	2240
N^l_{a1}, N^l_{b1}	-	-	2369	1399
N^l	3052	3639	3835	3639

Table 2: Free Entry

Round	K_a	K_b	$E(S)$	$E(\pi)$	$D^h = p^h$	$D^l = p^l$
0	1500	1500	60189	-	-	-
1	1862	1500	66436	1145	26.678	12.364
2	1862	1960	77972	1837	23.606	10.761
3	2176	1960	82920	721	22.043	9.953
4	2176	2283	88639	759	20.231	9.049
5	2426	2283	91887	380	19.073	8.511
6	2426	2518	94892	321	17.963	8.017
7	2607	2518	96824	172	17.194	7.689
8	2607	2677	98448	128	16.523	7.412

Table 3: Entry by Patronage Auction

Round	K_a	K_b	$E(S)$	$D^h = p^h$	$D^l = p^l$
0	1500	1500	60189	-	-
1	2275	1500	69744	25.315	11.311
2	2275	2764	96109	17.419	6.802
3	2886	2764	101567	15.501	6.006
4	2886	3154	102353	14.946	5.753
5	2930	3154	102503	14.870	5.725
6	2930	3159	102503	14.865	5.723
7	2930	3159	102503	14.864	5.723
8	2930	3159	102503	14.864	5.723

Table 4: Entry by Generalized Price Auction

Round	K_a	K_b	$E(S)$	$D^h = p^h$	$D^l = p^l$
0	1500	1500	60189	-	-
1	2277	1500	69818	25.223	11.382
2	2277	2767	96906	16.802	7.174
3	2888	2767	102517	14.793	6.447
4	2888	2916	103137	14.502	6.346
5	2907	2916	103203	14.468	6.335
6	2907	2918	103210	14.464	6.334
7	2908	2918	103211	14.464	6.333
8	2908	2918	103211	14.464	6.333

Table 5: Entry by Auction: Social Welfare

Round	K_a	K_b	$E(S)$	$D^h = p^h$	$D^l = p^l$
0	1500	1500	60189	-	-
1	2278	1500	69827	25.186	11.424
2	2278	2771	96969	16.636	7.355
3	2889	2771	102603	14.595	6.667
4	2889	2902	103172	14.329	6.582
5	2906	2902	103229	14.299	6.582
6	2906	2904	103236	14.296	6.573
7	2906	2904	103236	14.296	6.572
8	2906	2904	103236	14.296	6.572