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From Hierarchies to Levels: New Solutions for Games

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From Hierarchies to Levels: New Solutions for Games with Hierarchical Structure

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Abstract

Recently, applications of cooperative game theory to economic allocation problems have gained popularity. In many of these problems, players are organized according to either a hierarchical structure or a levels structure that restrict players' possibilities to cooperate. In this paper, we propose three new solutions for games with hierarchical structure and characterize them by properties that relate a player's payoff to the payoffs of other players located in specific positions in the structure relative to that player. To define each of these solutions, we consider a certain mapping that transforms any hierarchical structure into a levels structure, and then we apply the standard generalization of the Shapley Value to the class of games with levels structure. The transformations that map the set of hierarchical structures to the set of levels structures are also studied from an axiomatic viewpoint by means of properties that relate a player's position in both types of structure.

Keywords: TU-game; hierarchical structure; levels structure; Shapley Value; axiomatization.

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1 Introduction

Many economic allocation problems are modeled by a TU-game. A *cooperative game with transferable utility*, or simply a *TU-game*, is a pair consisting of a finite set of players and a mapping that associates a real number with each subset (or coalition) of players. Each of these numbers represents the aggregate benefit that players can obtain by cooperating within the corresponding coalition. In many of these problems there is a natural "structure" according to which players are arranged, which needs to be taken into account together with the information contained in the TU-game. For instance, in water allocation problems (Ambec and Sprumont, 2002) or in polluted river cost allocation problems (Ni and Wang, 2007), a graph is used to describe the agents' location along the riverside. Examples in which there is an implicit order of the players include Littlechild and Owen (1973), Curiel et al. (1989), Graham et al. (1990), and Maniquet (2003).

In this paper, we deal with *games with hierarchical structure*. They consist of a TU-game together with a directed graph in the form of a tree that describes the organizational design of the set of players. Most political, economic or military organizations are organized in a hierarchical structure, which turns the study of such structures a relevant topic in economic theory, both from a normative and a positive viewpoint. In particular, the study of the properties that certain solutions for games with hierarchical structure satisfy has received great attention in the literature, and many important contributions to this model—and to more general, closely related models—have been made to date.

The class of *games with permission structure* consists of all pairs made up of a TU-game plus a directed graph, and hence it contains the class of games with hierarchical structure. In the *conjunctive* approach, Gilles et al. (1992) considered a certain subclass of games with permission structure containing all TU-games with cycle-free graphs, and they assumed that every player needs permission from all her predecessors in the graph before she can cooperate with other players. Therefore, a coalition of players can only cooperate if for every player who is in the coalition, all her predecessors also belong to the coalition (see also van den Brink and Gilles, 1996). In contrast, in the *disjunctive* approach of Gilles and Owen (1994), the subclass of games with permission structure where the digraph is acyclic and quasi-strongly connected was considered, and it was assumed that every player needs permission from at least one of her predecessors in the graph (see also van den Brink, 1997). According to this approach, a coalition can cooperate if for every player, at least one of her predecessors—if any—is also in the coalition. The two approaches yield the conjunctive, respectively disjunctive, permission restricted games. To each game, the *Conjunctive (Disjunctive) Permission Value* assigns the Shapley Value of the conjunctive (disjunctive) permission restricted game.[1] More recently, van den Brink et al. (2014) introduced another value for the class of games with permission structure where the TU-game is totally positive by restricting the allocation of dividends. In van den Brink et al. (2015), a value for games with hierarchical structure has been proposed, the so-called *Average Tree permission value*. This value is obtained by taking the average of all hierarchical outcomes (see Demange, 2004) of the permission restricted game on the underlying undirected graph. Lastly, in Faigle and Kern (1992), restrictions on the coalition formation process induced by a so-called precedence constraint, which includes directed trees, are considered instead of direct restrictions on coalitions.

Another class of games with restricted cooperation that has long been studied is that of *games with a coalition structure* introduced by Aumann and Drèze (1974) and studied later on in Owen (1977) and Hart and Kurz (1983). In this model, the TU-game is enriched with a partition of the player set that aims at describing situations where some players are more closely related to each other than to other players. The class of *games with levels structure of cooperation* introduced by Winter (1989) constitutes a natural generalization of the former model. In a game with levels structure, there is an ordered set of partitions, with the first level a partition of the player set consisting at least of two non-empty subsets, each level (except the last one) being coarser than the next level, and the last level being equal to the partition of all singletons. Hence, the latter games describe situations where also within each coalition in some level, some players are closer to each other than to other players within that coalition. This occurs when the former players are still together in one coalition in the next thinner level, while the latter are not.

In this paper, we propose and characterize three new solutions for games with hierarchical

[1] When the graph is a directed tree, the conjunctive and disjunctive approaches coincide. In that case, we simply refer to the Permission Value.

structure. Our approach differs from the previous solutions proposed in the literature in that, first, we convert the hierarchical structure (i.e., the tree) into a certain levels structure and, second, we apply the Shapley Levels Value (Winter, 1989) to the game with levels structure, with the levels structure induced by the hierarchical structure. Each of the three solutions is obtained by choosing a specific conversion mapping that associates a levels structure to every directed tree. Before defining and characterizing the new solutions, we propose several properties that might be satisfied by a mapping that converts a hierarchical structure into a levels structure and we show that there is a single class of mappings satisfying the required properties.

By construction, our procedure mapping directed trees into levels structure of cooperation may entail a certain loss of information when one leaf of the tree has no *sibling*, i.e., there are no other players having the same predecessor in the hierarchy. When restricted to the set of trees where each leaf has at least one sibling, our procedure establishes nonetheless a bijection between the latter set and the set of levels structures. In any case, our procedure will enable us to define three values that exhibit interesting features. First, they satisfy some properties that relate a player's payoff to the payoffs of certain groups of other players. To any player in a hierarchical structure, we associate a *team* of players consisting of the player herself, together with players who are below her in the hierarchy. Demange (2004) argues that in a hierarchical structure, teams are the relevant units in the decision-making process. More specifically, the properties satisfied by the new values relate the payoff of a player to the payoffs of the other players in her team and/or the teams of her siblings in the hierarchy. To the best of our knowledge, these properties have not been introduced before in the literature. Second, compared to other values examined in the literature, our three solutions seem to be more responsive to changes in the players' position in the directed tree, at least when restricting to unanimity games. The importance of how responsive should values be to changes in the hierarchy seems to have been underestimated in the literature on games with hierarchical structure. When the TU-game is the unanimity game of the grand coalition, the payoff assigned by a value to a player in a game with hierarchical structure can be interpreted as the player's importance in the hierarchy. Our paper then adds to the knowledge regarding how important each player is in a given directed tree.

Both mathematical objects, directed trees and levels structures, are useful when capturing the restrictions to players' cooperation possibilities that may exist in an organization due to its structure. On the one hand, a directed tree expresses a hierarchical configuration in a set of players. In a firm, for instance, one employee might not be allowed to take some decision without the approval of her superior. According to Demange (2004), a hierarchical structure is the organizational form that maximizes stability. On the other hand, a levels structure organizes a set of players into a series of nested partitions. For instance, a firm might be organized in different divisions, which in turn might be divided in different departments, in which employees work. To take a decision it might be that, first, all workers of each department have to reach a consensus, second, all departments of each division have to reach a consensus, and, third and last, all divisions have to reach an agreement so that the firm as a whole takes the ultimate decision.

Any mapping from the set of directed trees to the set of levels structures can thus be interpreted as a particular way to translate the hierarchical relation given by a directed tree into a nested relation given by the corresponding levels structure. These mappings may be useful when comparing organizations with different, yet non exclusive power structures.

The rest of the paper is organized as follows. Section 2 is a preliminary section on cooperative TU-games, directed graphs, and levels structures (including coalition structures). In Section 3 we introduce and characterize a mapping that assigns a levels structure to every hierarchical structure represented by a directed tree. In Section 4 we introduce and axiomatize our first new solution for games with hierarchical structure. This solution uses all the information contained in the levels structure obtained from the conversion mapping of Section 3. In Section 5 we define and axiomatize two alternative solutions for these games. These solutions use proper subsets of the aforesaid transformed information. In Section 6 we compare our three solutions with other existing values from the literature. Section 7 contains the concluding remarks. Lastly, there are appendices containing some proofs of Sections 3 and 4 together with the logical independence of the axioms used in each characterization.

2 Preliminaries

2.1 TU-games

Let a finite non-empty set $\Omega \subset \mathbb{N}$ of potential players be given. Then, a *cooperative game with transferable utility*, or simply a TU-game, is a pair (N, v), where $\emptyset \neq N \subseteq \Omega$ is a finite set of players and $v \colon 2^N = \{S : S \subseteq N\} \to \mathbb{R}$ is a *characteristic function*, with $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ represents the worth of coalition S, i.e., the total payoff that members of the coalition can obtain by agreeing to cooperate. We denote the collection of all TU-games by $\mathcal{G}$. For the sake of readability, we henceforth abuse notation slightly and write $T \cup i$ and $T \setminus i$ instead of $T \cup \{i\}$ and $T \setminus \{i\}$ for $T \subseteq N$ and $i \in N$, respectively. We use the $|\cdot|$ operator to denote the cardinality of a finite set.

Given $(N, v) \in \mathcal{G}$ and $i \in N$, we say that i is a *null player* in (N, v) if $v(T \cup i) - v(T) = 0$ for all $T \subseteq N \setminus i$, and we say that i is a *necessary player* in (N, v) if $v(T) = 0$ for all $T \subseteq N \setminus i$. A game $(N, v) \in \mathcal{G}$ is *monotone* if $v(T) \leq v(R)$ for all $T \subseteq R \subseteq N$. We denote by $\mathcal{G}_M \subset \mathcal{G}$ the class of all monotone TU-games. For each nonempty $T \subseteq N$, the *unanimity game* (N, u_T) is given by $u_T(R) = 1$ if $T \subseteq R$, and $u_T(R) = 0$ otherwise. It is well-known that every game $(N, v) \in \mathcal{G}$ can be written in a unique way as a linear combination of unanimity games (N, u_T) by $v = \sum_{\emptyset \neq T \subseteq N} \Delta_T(v) u_T$, where $\Delta_T(v) = \sum_{R \subseteq T} (-1)^{|T|-|R|} v(R)$ are the *Harsanyi dividends* (Harsanyi, 1959).

A *solution on* $\mathcal{G}$ is a map, f, that assigns a unique vector $f(N, v) \in \mathbb{R}^N$ to every $(N, v) \in \mathcal{G}$, where $f_i(N, v)$ represents the payoff to player $i \in N$. A *permutation* of N is a bijective map $\pi : N \to N$. Let $\Pi(N)$ denote the set of permutations of N. Given $\pi \in \Pi(N)$ and $i \in N$, let $\pi^{-1}[i]$ indicate the set of players ordered before i in permutation π, i.e., $\pi^{-1}[i] = \{j \in N : \pi(j) < \pi(i)\}$. The best-known solution on $\mathcal{G}$ is the Shapley Value (Shapley, 1953), which is defined for every $(N, v) \in \mathcal{G}$ and $i \in N$ by

$$Sh_i(N, v) = \frac{1}{|\Pi(N)|} \sum_{\pi \in \Pi(N)} \left(v\left(\pi^{-1}[i] \cup \{i\}\right) - v\left(\pi^{-1}[i]\right) \right).$$

2.2 Games with hierarchical structure

Given the set Ω of potential players, a directed graph or *digraph* is a pair (N, D), where $\emptyset \neq N \subseteq \Omega$ is a finite set of nodes (representing the players) and $D \subseteq N \times N$ is a binary relation on N (representing the hierarchy). Given (N, D) and $T \subseteq N$, the digraph (T, D_T) is the induced subgraph on T given by $D_T = \{(i,j) \in D : i, j \in T\}$. For a given digraph (N, D) and $i, j \in N$, a *(directed) path* from i to j is a sequence of distinct nodes $(i_1, \ldots, i_m)$ such that $i_1 = i$, $i_m = j$, and $(i_k, i_{k+1}) \in D$ for $k = 1, \ldots, m-1$. A digraph (N, D) is a *(rooted) directed tree* with root $i_0 \in N$ if there does not exist a player $j \in N$ with $(j, i_0) \in D$ and if there is exactly one directed path from the top-node i_0 to every other node. Note that, in particular, $(i,i) \notin D$ for all $i \in N$ if (N, D) is a directed tree. We denote the set of all directed trees by $\mathcal{D}$.

For every $(N, D) \in \mathcal{D}$ and $i \in N$, the nodes in $S_D(i) = \{j \in N : (i,j) \in D\}$ are called the *successors* of i, and the nodes in $P_D(i) = \{j \in N : (j,i) \in D\}$ are called the *predecessors* of i. For a directed tree $(N, D) \in \mathcal{D}$ with root i_0, it holds that $P_D(i_0) = \emptyset$ and $|P_D(j)| = 1$ for every $j \in N \setminus \{i_0\}$ and, accordingly, we denote by $p_D(j)$ the unique predecessor of $j \neq i_0$. Let $\left(N, \widehat{D}\right)$ be the *transitive closure* of a digraph (N, D), i.e., $(i,j) \in \widehat{D}$ if and only if there is a directed path form i to j. The players in $\widehat{S}_D(i) = S_{\widehat{D}}(i)$ are called the *subordinates* of i, and the players in $\widehat{P}_D(i) = P_{\widehat{D}}(i)$ are called the *superiors* of i.

For a directed tree, we follow Demange (2004) and call the set $\widehat{S}_D(i) \cup i$ the *team* of player i, i.e. the team of i consists of i herself plus all her subordinates. We stress that the set $\widehat{P}_D(i)$, with $i \neq i_0$, is the set of nodes on the unique path from i_0 to i, excluding i herself. The *rank* of i in the hierarchy is the length of this path and is denoted by $l^D(i)$, i.e., for every $i \in N$, $l^D(i) = \left|\widehat{P}_D(i)\right|$. When there is no possible confusion regarding (N, D), we write $l(i)$ instead of $l^D(i)$. For every $(N, D) \in \mathcal{D}$, the *depth* of the hierarchy is given by $l(D) = \max_{i \in N} l(i)$. Further, given $(N, D) \in \mathcal{D}$ and $i \in N \setminus i_0$, the set of *siblings* of i is denoted by $A(i)$ and is the set of players with the same predecessor as i, including i herself, i.e., $A(i) = \{j \in N : (p_D(i), j) \in D\}$. Finally, for $T \subseteq N$ we denote $S_D(T) = \cup_{i \in T} S_D(i)$.

A *game with hierarchical structure* is a triple (N, v, D), where $(N, v) \in \mathcal{G}$ and $(N, D) \in \mathcal{D}$. We denote by $\mathcal{GD}$ the set of all games with hierarchical structure. A *solution on* $\mathcal{GD}$ is a map, f, that assigns a vector $f(N, v, D) \in \mathbb{R}^N$ to every triple $(N, v, D) \in \mathcal{GD}$.

2.3 Games with levels structure

A *partition* of a finite set N is a collection of subsets, $P \subseteq 2^N$, such that $T \neq \emptyset$ for every $T \in P$, $\cup_{T \in P} T = N$, and for every $T, R \in P$ with $T \neq R$, $T \cap R = \emptyset$. Given Ω, a *levels structure* is a pair $(N, \underline{B})$ with $\emptyset \neq N \subseteq \Omega$, $|N| \geq 2$ and, for some integer $m \geq 0$, $\underline{B} = (B_1, \ldots, B_{m+1})$ is a sequence of $m+1$ partitions of N such that *(i)* B_1 is a proper partition, i.e. $|B_1| \geq 2$, *(ii)* for each $r \in \{1, \ldots, m\}$, B_r is coarser than B_{r+1}, i.e., for each $T \in B_r$, there is $U \subseteq B_{r+1}$ such that $T = \cup_{R \in U} R$, and *(iii)* $B_{m+1} = \{\{i\} : i \in N\}$.[2] The partition B_{m+1} consisting of all singleton coalitions is added for notational convenience. For each $r \in \{1, \ldots, m+1\}$, the partition B_r is called the *r-th level of* $\underline{B}$ and each $T \in B_r$ is called a *union* (of the r-th level). We denote by $(N, \underline{B}_0)$ the trivial level structure with $\underline{B}_0 = (\{\{i\} : i \in N\})$. The collection of all levels structures $(N, \underline{B})$, with $\emptyset \neq N \subseteq \Omega$, is denoted by $\mathcal{L}$.

Example 2.1. *Let* $N = \{1, 2, 3, 4, 5\}$ *and* $\underline{B} = (B_1, B_2, B_3)$ *be given by* $B_1 = \{\{1, 2\}, \{3, 4, 5\}\}$, $B_2 = \{\{1, 2\}, \{3\}, \{4, 5\}\}$, *and* $B_3 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$. *Then, the pair* $(N, \underline{B})$ *is a levels structure, i.e.,* $(N, \underline{B}) \in \mathcal{L}$.

A *game with levels structure* is a triple $(N, v, \underline{B})$, where $(N, v) \in \mathcal{G}$ and $(N, \underline{B}) \in \mathcal{L}$. We denote by $\mathcal{GL}$ the set of all games with levels structure. A *solution* on $\mathcal{GL}$ is a map, f, that assigns to every $(N, v, \underline{B}) \in \mathcal{GL}$ a vector $f(N, v, \underline{B}) \in \mathbb{R}^N$. The best-known solution on $\mathcal{GL}$ is the *Shapley Levels Value*, introduced by Winter (1989).[3] This solution is based on the assumption that the levels structure imposes restrictions on the order in which players enter when forming a coalition. For a levels structure $(N, \underline{B}) \in \mathcal{L}$ with $\underline{B} = (B_1, \ldots, B_{m+1})$, define the sets of permutations $\Omega_r(\underline{B})$, with $r \in \{0, \ldots, m\}$, starting with $\Omega_0(\underline{B}) = \Pi(N)$, and then recursively for $r = 1, \ldots, m$,

$$\Omega_r(\underline{B}) = \{\pi \in \Omega_{r-1}(\underline{B}) : \forall T \in B_r, \forall i, j \in T \text{ and } k \in N, \text{ if } \pi(i) < \pi(k) < \pi(j) \text{ then } k \in T\}.$$

Therefore, $\Omega_r(\underline{B})$, with $r > 0$, is the subset of permutations of $\Omega_{r-1}(\underline{B})$ such that the elements of each union of B_r are consecutive. We let $\Omega(\underline{B}) = \Omega_m(\underline{B})$ be the set of permutations that keep the agents of every union of every level consecutive. Then, the Shapley Levels Value Sh^L

[2] Note that we do not exclude that $B_r = B_{r+1}$ for some $r \in \{1, \ldots, m\}$.

[3] Other solutions for games with levels structure can be found in Álvarez-Mozos and Tejada (2011).

is the solution on $\mathcal{GL}$ defined for every $(N, v, \underline{B}) \in \mathcal{GL}$ and $i \in N$ by

$$Sh_i^L(N, v, \underline{B}) = \frac{1}{|\Omega(\underline{B})|} \sum_{\pi \in \Omega(\underline{B})} \left(v\left(\pi^{-1}[i] \cup i\right) - v\left(\pi^{-1}[i]\right)\right). \tag{2.1}$$

Note that the trivial levels structure $(N, \underline{B}_0)$ does not put any restriction on the order of players. Therefore, for every $(N, v) \in \mathcal{G}$, $Sh^L(N, v, \underline{B}_0) = Sh(N, v)$. A game with a levels structure where $r = 1$ corresponds to a *game with a coalition structure* as introduced in Aumann and Drèze (1974). It is easy to verify that in this case, the Shapley Levels Value is equal to the Owen Value (Owen, 1977).

3 From Directed Trees to Levels Structures

Next, we turn to the first relevant question of this paper: How to convert a given hierarchical structure into a levels structure? Accordingly, we abstract for the moment from the analysis of how payoffs should be allocated in a game with hierarchical structure, and we only focus on the relation between directed trees and levels structures. First, we propose a procedure to convert a directed tree into a collection of nested partitions. Second, we show that a mapping that converts any directed tree into a levels structure satisfies some required properties if and only if any directed tree is mapped into a levels structure that is a subset of the collection of nested partitions obtained from our procedure.

Accordingly, we next propose a specific way to map any directed tree into a levels structure. Let $\mathcal{H} : \mathcal{D} \to \mathcal{L}$ be such that for every $(N, D) \in \mathcal{D}$, $\mathcal{H}(N, D) = (N, B^D)$ with

$$B^D = \left(B_{1,1}^D, B_{1,2}^D, B_{2,1}^D, B_{2,2}^D, \ldots, B_{l(D),1}^D, B_{l(D),2}^D\right), \tag{3.2}$$

defined for $r \in \{1, \ldots, l(D)\}$ by

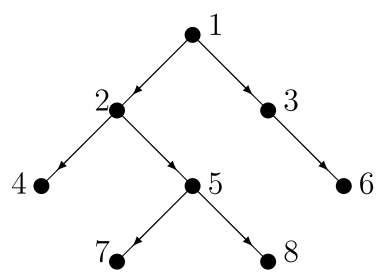
$$B_{r,1}^D = \Big\{\{i\} : l(i) < r\Big\} \cup \Big\{\widehat{S}_D(j) : l(j) = r - 1\Big\} \quad \text{and} \tag{3.3}$$

$$B_{r,2}^D = \Big\{\{i\} : l(i) < r\Big\} \cup \Big\{\widehat{S}_D(j) \cup j : l(j) = r\Big\}. \tag{3.4}$$

By definition, *(i)* $B_{1,1}^D$ is a proper partition of N, *(ii)* $B_{1,2}^D$ is a refinement of $B_{1,1}^D$, and *(iii)* for every $r \in \{2, \ldots, l(D)\}$ the partition $B_{r,1}^D$ is a refinement of $B_{r-1,2}^D$, and partition $B_{r,2}^D$ is a refinement of $B^D_{r,1}$. Further, note that $B^D_{l(D),2} = \{\{i\} : i \in N\}$, as $\widehat{S}_D(j) = \emptyset$ for every $j \in N$ with $l(j) = l(D)$.[4] As a consequence, $(N, B^D) \in \mathcal{L}$.

We illustrate the above definitions by means of an example which will be used throughout the paper.

Example 3.1. *Consider the directed tree* $(N, D) \in \mathcal{D}$ *with* $N = \{1, \ldots, 8\}$ *and* $D = \{(1,2), (1,3), (2,4), (2,5), (3,6), (5,7), (5,8)\}$*—see the graph below. Then,* $(N, B^D) \in \mathcal{L}$ *is given by* $B^D = (B^D_{1,1}, \ldots, B^D_{3,2})$*, where*



$$B^D_{1,1} = \{\{1\}, \{2,3,4,5,6,7,8\}\},$$
$$B^D_{1,2} = \{\{1\}, \{2,4,5,7,8\}, \{3,6\}\},$$
$$B^D_{2,1} = \{\{1\}, \{2\}, \{3\}, \{4,5,7,8\}, \{6\}\},$$
$$B^D_{2,2} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5,7,8\}, \{6\}\},$$
$$B^D_{3,1} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7,8\}\},$$
$$B^D_{3,2} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}$$

In the remaining part of this section we provide a justification for the mapping $\mathcal{H}$ defined in Eqs. (3.2)—(3.4). First note that the conversion mapping $\mathcal{H}$ respects the important role of the teams as highlighted by Demange (2004) in the following sense. At each even level $B^D_{r,2}$, every player i with $l(i) < r$ is a singleton in the partition, while each player with rank $l(i) = r$ in the hierarchy forms a union with all other players in his team. Furthermore, at each odd level $B^D_{r,1}$, every player i with $l(i) < r$ is a singleton in the partition, while the team of each player with rank $l(i) = r$ in the hierarchy forms a union together with the teams of all her siblings. Accordingly, we shall call every even level a *team level* and every odd level a *sibling level*.

In the following, we show that the mapping $\mathcal{H}$ can essentially be characterized by five properties. These properties connect the position of agents in a directed tree with their participation in a given union of a partition of the player set. Although we only consider properties that apply to a single partition of the player set, these properties can be generalized to a levels structure in a straightforward way.

[4]Note that two consecutive partitions of B^D might coincide.

The properties considered obey two principles. The first principle considers for a given player in the directed tree her most natural companions in the partition. In order of preference, these are: *(i)* her team, *(ii)* her siblings, and *(iii)* all other players. The second principle takes a certain equality notion into account by requiring that under some additional conditions, two agents with the same rank in the directed tree are treated equally in the partition. Given a partition P of N and $i \in N$, let P_i denote the element of P to which player i belongs.

Definition 3.1. *Given a directed tree $(N, D) \in \mathcal{D}$, a partition P of N is said to respect (N, D) when the following five properties hold:*

P1 *If $(i, j) \in D$, then $[P_i \neq \{i\}] \Longrightarrow [j \in P_i]$.*

P2 *If $j \in A(i)$, then $\left[P_i \setminus (\widehat{S}_D(i) \cup i) \neq \emptyset\right] \Longrightarrow [j \in P_i]$.*

P3 *If $l(i) = l(j)$, then $[S_D(i) \neq \emptyset$ and $P_i = \{i\}] \Longrightarrow [P_j = \{j\}]$.*

P4 *If $(i, j) \in D$, then $[P_i = \{i\}] \Longrightarrow \left[P_j \subseteq \widehat{S}_D(i)\right]$.*

P5 *If $l(i) = l(j)$, $A(i)\backslash i \neq \emptyset$, and $A(j)\backslash j \neq \emptyset$, then $\left[P_i \subseteq (\widehat{S}_D(i) \cup i)\right] \Longleftrightarrow \left[P_j \subseteq (\widehat{S}_D(j) \cup j\right]$.*

The first property, P1, states that if in the partition, a player depends on some other player in the sense that she is in the same union with this other player, then she and all other members of her team must be in the same union of the partition. This holds regardless of whether a player depends within the partition on another player who is in her team or on a player who is not in her team. P2 states that if a player i depends within the partition on some player who is not in her team, then all i's siblings must belong to the same union as i does. P3 requires that if a player is a singleton in the partition while having at least one successor in the directed tree, then any other player with the same rank should also be a singleton. According to P4, when a player is a singleton in the partition, then none of the members of her team depends on a player who does not belong to this team. Finally, P5 states that for any two players with the same rank each having one sibling at least, it holds that either each of them only depends within the partition on her team or each of them depends on a player who is not in her team.

Next, we characterize the mappings that convert any directed tree $(N, D) \in \mathcal{D}$ into a levels structure $(N, \underline{B})$ where any partition $P \in \underline{B}$ respects (N, D), i.e., it satisfies the five properties P1—P5. Recall that $(N, B^D) = \mathcal{H}(N, D)$ denotes the levels structure defined by Eqs. (3.2)—(3.4).

Theorem 3.2. *Consider a directed tree $(N, D) \in \mathcal{D}$. Then, a proper partition P of N respects (N, D) if and only if $P \in B^D$. Moreover, the five properties of Definition 3.1 are independent.*

Proof. See Appendix A. □

Note that the mapping assigning the trivial partition $\{N\}$ with N as its unique element to every directed tree also satisfies the five properties. Since some properties become meaningless for this partition, in the theorem we have only considered proper partitions. The next corollary follows immediately from Theorem 3.2.

Corollary 3.1. *Any mapping $\mathcal{I} : \mathcal{D} \to \mathcal{L}$, such that every partition in $\mathcal{I}(N, D)$ respects (N, D) and has a maximal number of different partitions, can be built from the partitions in $\left(N, B^D\right)$.*

To sum up, in this section we have focused on a particular way to map a directed tree into a levels structure, namely that every partition of the levels structure respects the tree in the sense of Definition 3.1. The properties P1—P5 and Theorem 3.2 imply that in such a levels structure, a player that is *higher in the hierarchy* is *more independent from the other players* in the collection of nested partitions in the sense that such a player becomes a singleton coalition at a lower-ranked partition in the levels structure than another player who is lower in the hierarchy.[5] While the proposed mapping between the set of directed trees and the set of levels structures is reasonable both from the way it is constructed and from the properties it satisfies, the values for games with hierarchical structure introduced in the remaining part of the paper—all of which use such a mapping—add to its relevance.

4 A New Solution for Games with Hierarchical Structure

We are now ready to introduce and axiomatize a new type of solution for the class of games with hierarchical structure $\mathcal{GD}$. For each solution of this type, we follow a two-stage procedure.

[5]We stress that more independent does not mean better off, not even for monotone games. We elaborate on this in Section 7.

First, the hierarchical structure (N, D) is transformed into a levels structure $(N, \underline{B})$, with $\underline{B}$ being a subset of the collection B^D of partitions generated by the conversion mapping $\mathcal{H}$. Second, the Shapley Levels Value is applied to the game with levels structure obtained in the first stage. The first solution of this type is obtained by taking the levels structure $(N, B^D) = \mathcal{H}(N, D)$, i.e., by considering the entire collection of partitions that satisfy the five properties of Theorem 3.2.

Definition 4.1. *The solution φ^{SL} on the class of games with hierarchical structure $\mathcal{GD}$ is the solution given by*

$$\varphi^{SL}(N, v, D) = Sh^L\left(N, v, B^D\right), \text{ with } (N, v, D) \in \mathcal{GD}.$$

By definition, the first level of (N, B^D) consists of two unions: the root player as a singleton and the set of all other players, consisting of all teams of the players having the root player as their predecessor. Consequently, because φ^{SL} is obtained by applying Sh^L to (N, B^D), besides receiving her singleton worth (dividend), the root player earns half the dividends of all coalitions she belongs to. Any other player i having subordinates belongs to the same union as the latter in any partition $B^D_{r,l}$, with $(r, l) \leq_{lex} (l(i), 1)$, while i's team is a union in $B^D_{l(i),2}$.[6] In the next level, namely the one corresponding to $B^D_{l(i)+1,1}$, the player then becomes a singleton and her subordinates form a union in the next level. Consequently, each player obtains, according to φ^{SL}, a share equal to the sum of all shares of all her subordinates in the dividends of any coalition to which she belongs together with at least one of her subordinates.

We illustrate the new value with an example.

Example 4.1. *Consider $(N, u_N, D) \in \mathcal{GD}$ with (N, D) the hierarchy given in Example 3.1. Then,*

$$\varphi^{SL}(N, u_N, D) = \left(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{8}, \frac{1}{64}, \frac{1}{64}\right).$$

Note that the root of the tree obtains half the worth of the grand coalition. Then, the proposed value assigns half the remainder to Player 2 and her subordinates, and half the remainder to Player 3 and her subordinates. And so on, until the last level of the levels structure is reached.

[6] For $(r, l), (r, l) \in \mathbb{N} \times \mathbb{N}$, we write $(r, l) \leq_{lex} (r', l')$ if either $r < r'$ or $r = r'$ and $l \leq l'$.

Next, we provide a set of axioms that will characterize the value φ^{SL} on the class $\mathcal{GD}$. The first three are natural generalizations of the classical efficiency, additivity, and null player properties.

EFF A solution f on $\mathcal{GD}$ satisfies *efficiency* if for every $(N, v, D) \in \mathcal{GD}$,

$$\sum_{i \in N} f_i(N, v, D) = v(N).$$

ADD A solution f on $\mathcal{GD}$ satisfies *additivity* if for every $(N, v, D), (N, w, D) \in \mathcal{GD}$,

$$f(N, v + w, D) = f(N, v, D) + f(N, w, D).$$

NPP A solution f on $\mathcal{GD}$ satisfies the *null player property* if for every $(N, v, D) \in \mathcal{GD}$ and $i \in N$ null player in (N, v),

$$f_i(N, v, D) = 0.$$

NPP is stronger than the Inessential Player property used to axiomatize the Permission Value in van den Brink and Gilles (1996) and van den Brink (1997). The latter property demands that a null player in (N, v) whose subordinates in (N, D) are also all null players in (N, v) earns zero payoff.

The following axioms describe how the solution behaves with respect to the hierarchical structure. The first property states that merging all players in a team does not affect the payoffs of the remaining players. Merging player j with the rest of her team $\widehat{S}_D(j)$ in game (N, v, D) yields a new game $\left(N \setminus \widehat{S}_D(j), v^j, D_{N \setminus \widehat{S}_D(j)}\right)$ with v^j defined for every $T \subseteq N \setminus \widehat{S}_D(j)$ by

$$v^j(T) = \begin{cases} v(T) & \text{if } j \notin T, \\ v\left(T \cup \widehat{S}_D(j)\right) & \text{if } j \in T. \end{cases}$$

We are now in a position to formally present this property.

IMT A solution f on $\mathcal{GD}$ satisfies *independence of merging teams* if for every $(N, v, D) \in \mathcal{GD}$, $j \in N$, and $i \in N \setminus \left(\widehat{S}_D(j) \cup j\right)$,

$$f_i(N, v, D) = f_i\left(N \setminus \widehat{S}_D(j), v^j, D_{N \setminus \widehat{S}_D(j)}\right).$$

Note that together with EFF, IMT implies that when all players in a team are merged into the *boss* of the team, i.e. the player of the team who has the highest position in the hierarchy, the payoff to this player in the new game has to be equal to the sum of the payoffs of all members in the team in the original game.

Now, we present two lemmas that will prove to be helpful in the proofs of our characterization results. Given $(N, D) \in \mathcal{D}$ and $i \in N$, let N_i be given by

$$N_i = \widehat{S}_D(i) \cup \widehat{P}_D(i) \cup S_D\left(\widehat{P}_D(i)\right), \tag{4.5}$$

Note that $i \in S_D\left(\widehat{P}_D(i)\right)$ and thus $i \in N_i$, and that N_i further consists of all subordinates and superiors of i together with all successors of her superiors. We call the players in $N_i \setminus i$ the *relatives* of i.[7] We next define the game $(N_i, v_i) \in \mathcal{G}$ by, $v_i(T) = v\left(T_i^D\right)$ for every $T \subseteq N_i$, with

$$T_i^D = T \cup \left(\bigcup_{j \in T \setminus \left(\widehat{S}_D(i) \cup \widehat{P}_D(i) \cup i\right)} \widehat{S}_D(j) \right). \tag{4.6}$$

Lemma 4.2. *If a solution on $\mathcal{GD}$, f, satisfies independence of merging teams, then for every $(N, v, D) \in \mathcal{GD}$ and $i \in N$, $f_i(N, v, D) = f_i(N_i, v_i, D_{N_i})$.*

Proof. See Appendix B. □

When all players in $N \setminus i$ are relatives of i, we have $N_i = N$ and thus $(N_i, v_i, D_{N_i}) = (N, v, D)$. In that case, the statement of the lemma is trivial. When they are not relatives, the lemma provides us with a property implied by IMT requiring the payoff of a player not be affected by changes in the position in the hierarchy of players who are not her relatives. As the proof of Lemma 4.2 shows, (N_i, v_i, D_{N_i}) is obtained from the repeated application of IMT. Indeed, when $N_i \neq N$, there is an integer $m > 0$ and a sequence of games with hierarchical structure $\left(N^{(k)}, v^{(k)}, D^{(k)}\right)$, with $k = 0, \ldots, m$, such that *(i)* $\left(N^{(0)}, v^{(0)}, D^{(0)}\right) = (N, v, D)$, *(ii)* for every $k \in \{1, \ldots m\}$, $N^{(k)} = N^{(k-1)} \setminus \widehat{S}_D(j)$ for some $j \in N^{(k-1)}$, $v^{(k)} = \left(v^{(k-1)}\right)^j$, and $D^{(k)} = \left(D^{(k-1)}\right)_{N^{(k)}}$, and *(iii)* $\left(N^{(m)}, v^{(m)}, D^{(m)}\right) = (N_i, v_i, D_{N_i})$.

The next example illustrates the consequences of IMT as stated in Lemma 4.2.

[7] Note that "being a relative" is not a symmetric relation.

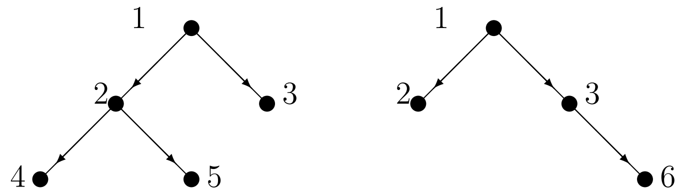


Figure 1: Directed trees D_{N_4} and D_{N_3} of Example 3.1

Example 4.3. *Consider* $(N, u_N, D) \in \mathcal{GD}$ *with* D *as given in Example 3.1. For* $i = 4$*, we have* $N_4 = \{1, 2, 3, 4, 5\}$ *and* (N_4, D_{N_4}) *the tree given by the left side of Figure 1. If a solution* f *satisfies* IMT*, then Lemma 4.2 implies that the payoff to Player 4 in* (N, v, D) *is equal to the payoff of Player 4 in* (N_4, v_4, D_{N_4})*. Hence,* $\varphi_4^{SL}(N_4, (u_N)_4, D_{N_4}) = \varphi_4^{SL}(N, u_N, D) = \frac{1}{16}$*. Taking* $i = 3$ *yields* $N_3 = \{1, 2, 3, 6\}$ *and* (N_3, D_{N_3})*, given by the right side of Figure 1. By Lemma 4.2,* $\varphi_3^{SL}(N_3, (u_N)_3, D_{N_3}) = \varphi_3^{SL}(N, u_N, D) = \frac{1}{8}$*.*

The next lemma shows that for a unanimity game (N, u_T), the repeated application of IMT results in a new unanimity game.

Lemma 4.4. *Consider* $(N, cu_T, D) \in \mathcal{GD}$*, where* $c > 0$ *and* $T \subseteq N$*. Then, for every* $i \in N$*,* $(N_i, (cu_T)_i) = (N_i, cu_{T(i)})$ *with*

$$T(i) = (T \cap N_i) \cup \left\{ j \in N_i \setminus \left(\widehat{S}_D(i) \cup \widehat{P}_D(i) \cup i \right) : \widehat{S}_D(j) \cap T \neq \emptyset \right\}.$$

Proof. See Appendix B. □

We stress that $T(i) = T$ when $N_i = N$. The remaining three axioms apply to the subclass of monotone games and impose bounds on players' payoffs depending on the position of necessary players in the tree.

NSP A solution f on $\mathcal{GD}$ satisfies the *necessary sibling property* if for every $(N, v, D) \in \mathcal{GD}$ with $(N, v) \in \mathcal{G}_M$ and for every two players $i, j \in N$ such that $S_D(i) = S_D(j) = \emptyset$, $i \in A(j)$, and i is necessary player in (N, v), then

$$f_i(N, v, D) \geq f_j(N, v, D).$$

NSP requires that given two players who are siblings and endpoints in the tree, if one of them is a necessary player in a monotone game, then she must earn at least as much as the other agent. The next two axioms impose upper and lower bounds to the payoffs of a player compared to the payoffs of her subordinates.

SLB A solution f on $\mathcal{GD}$ satisfies *superior lower bound* if for every $(N,v,D) \in \mathcal{GD}$ with $(N,v) \in \mathcal{G}_M$, if $i \in N$ is necessary in (N,v), then

$$f_i(N,v,D) \geq \sum_{j \in \widehat{S}_D(i)} f_j(N,v,D).$$

SUB A solution f on $\mathcal{GD}$ satisfies *superior upper bound* if for every $(N,v,D) \in \mathcal{GD}$ with $(N,v) \in \mathcal{G}_M$, if some $j \in \widehat{S}_D(i)$ is necessary in (N,v) for some $i \in N$, then

$$f_i(N,v,D) \leq \sum_{h \in \widehat{S}_D(i)} f_h(N,v,D).$$

SLB requires that a player i earns at least as much as all her subordinates together in the case where i is a necessary player in a monotone game, and thus vetoes all her subordinates. In the following, we state and prove a characterization of the solution φ^{SL} on the class $\mathcal{GD}$ of games with hierarchical structure.

Theorem 4.5. *A solution f on $\mathcal{GD}$ satisfies* EFF, ADD, NPP, IMT, NSP, SLB, *and* SUB, *if and only if $f = \varphi^{SL}$. Moreover, the seven axioms are independent.*

Proof. We first prove that there is at most one solution on $\mathcal{GD}$ that satisfies all properties. After that, we will prove that φ^{SL} satisfies them. The logical independence of the axioms is shown in Appendix C.

Uniqueness: Suppose that f satisfies the seven axioms. For every (N,v_0,D), with (N,v_0) the null game given by $v_0(S) = 0$ for all $S \subseteq N$, NPP implies that $f_i(N,v_0,D) = 0$ for all $i \in N$.

Next, let $(N,cu_T,D) \in \mathcal{GD}$, where $\emptyset \neq T \subseteq N$ and $c > 0$. We prove uniqueness of $f(N,cu_T,D)$ by induction on the depth $l(D)$ of the directed tree (N,D). If $l(D) = 0$, we have that $N = T = \{i_0\}$. Then EFF implies that $f_{i_0}(N,cu_T,D) = c$. Proceeding by induction, assume that for every $N' \subseteq N$, every $T' \subseteq N'$, and every (N',D'), $f(N',cu_{T'},D')$ is uniquely determined whenever $l(D') < l(D)$.[8] For every $i \in N$, we define the set of subordinates of i with rank equal to the depth of the directed tree, i.e.,

$$H^D(i) = \left\{ j \in \widehat{S}_D(i) : l(j) = l(D) \right\}.$$

We distinguish five cases with respect to $i \in N$.

Case I: $i \in N \setminus T$. By NPP, $f_i(N, cu_T, D) = 0$.

Case II: $i \in T$, $S_D(i) \neq \emptyset$, and $H^D(i) = \emptyset$. By IMT, $f_i(N, cu_T, D) = f_i(N_i, (cu_T)_i, D_{N_i})$.[9] Due to Lemma 4.4, $(N_i, (cu_T)_i)$ is a scaled unanimity game and, by definition of N_i, $l(D_{N_i}) < l(D)$. Then, it follows from the induction hypothesis that $f_i(N_i, (cu_T)_i, D_{N_i})$, and thus $f_i(N, cu_T, D)$, is uniquely determined.

Case III: $i \in T$, $S_D(i) = \emptyset$, and $l(i) < l(D)$. Following exactly the same argumentation as in Case II, it can be shown that $f_i(N, cu_T, D)$ is uniquely determined.

Case IV: $i \in T$ and $H_D(i) \neq \emptyset$. Note that $H_D(i) \neq \emptyset$ implies $S_D(i) \neq \emptyset$. By IMT, we have $f_i(N, cu_T, D) = f_i(N_i, (cu_T)_i, D_{N_i})$. However, unlike in Case II, we have $l(D_{N_i}) = l(D)$. Define $Q_i = S_D\left(\widehat{P}_D(i)\right) \setminus \left(\widehat{P}_D(i) \cup i\right)$ as the set of subordinates of superiors of i, except i and all her superiors. Note that $\left\{\widehat{S}_D(i), \{i\}, \widehat{P}_D(i), Q_i\right\}$ is a partition of N_i. Therefore, by EFF,

$$\begin{aligned} & f_i(N_i, (cu_T)_i, D_{N_i}) \\ & = c - \sum_{j \in \widehat{S}_D(i)} f_j(N_i, (cu_T)_i, D_{N_i}) - \sum_{j \in \widehat{P}_D(i)} f_j(N_i, (cu_T)_i, D_{N_i}) - \sum_{j \in Q_i} f_j(N_i, (cu_T)_i, D_{N_i}). \end{aligned} \tag{4.7}$$

For each $j \in Q_i$, by Lemma 4.4 we have that $\left((N_i)_j, ((cu_T)_i)_j, D_{(N_i)_j}\right)$ is also a unanimity game and, by construction of $(N_i)_j$, we obtain that $l\left(D_{(N_i)_j}\right) < l(D_{N_i}) = l(D)$. The latter holds since $S_D(i) \neq \emptyset$ and $l^D(j) \leq l^D(i)$ for any $j \in Q_i \subseteq N_i$. Then, by applying IMT to $(N_i, (cu_T)_i, D_{N_i})$ we obtain that $f_j\left(N_i, (cu_T)_i, D_{N_i}\right) = f_j\left((N_i)_j, ((cu_T)_i)_j, D_{(N_i)_j}\right)$ for each $j \in Q_i$. From the induction hypothesis, it then follows that

$$f_j\left(N_i, (cu_T)_i, D_{N_i}\right) \text{ is uniquely determined for } j \in Q_i. \tag{4.8}$$

[8] Note that we assume the induction hypothesis for every subset N' of N and the (scaled) unanimity game on every subset T' of N'.

[9] Throughout the proof, every time we apply IMT, we are actually using the result from Lemma 4.2.

We note that $(N_i)_j \cap \widehat{S}_D(i) = \emptyset$ for $j \in Q_i$. By NPP, SLB, and SUB, we have that

$$\sum_{j\in\widehat{S}_D(i)} f_j(N_i,(cu_T)_i,D_{N_i}) = \begin{cases} f_i(N_i,(cu_T)_i,D_{N_i}) & \text{if } \widehat{S}_D(i)\cap T\neq\emptyset,\\ 0 & \text{otherwise.}\end{cases} \tag{4.9}$$

Let $\mathcal{X}_i = 1$ if $\widehat{S}_D(i)\cap T \neq \emptyset$ and $\mathcal{X}_i = 0$ otherwise. Next, we show uniqueness of $f_i(N_i,(cu_T)_i,D_{N_i})$ by a second induction on $\left|\widehat{P}_D(i)\cap T\right|$. First, assume that $\left|\widehat{P}_D(i)\cap T\right| = 0$. Then, Eq. (4.7) reduces to

$$(1+\mathcal{X}_i)\cdot f_i(N_i,(cu_T)_i,D_{N_i}) = c - \sum_{j\in Q_i} f_j(N_i,(cu_T)_i,D_{N_i}).$$

Hence, $f_i(N_i,(cu_T)_i,D_{N_i})$ is uniquely determined.

Second, suppose that for some integer $t>0$, $f_i(N_i,(cu_T)_i,D_{N_i})$ is uniquely determined for every $i'\in T$ with $H_D(i')\neq\emptyset$ and $\left|\widehat{P}_D(i')\cap T\right| < t$, and let $i\in T$ be such that $H_D(i)\neq\emptyset$ and $\left|\widehat{P}_D(i)\cap T\right| = t$. Take $k\in\widehat{P}_D(i)\cap T$ such that for every $j\in\widehat{P}_D(i)\cap T$ we have $l(k)\geq l(j)$, i.e., k is the predecessor of i who is closest to the latter in the tree among those superiors of i that belong to T. By SLB and SUB, and the fact that $i\in T\cap\widehat{S}_D(k)$, we obtain

$$\begin{aligned} f_k(N_i,(cu_T)_i,D_{N_i}) = f_i(N_i,(cu_T)_i,D_{N_i}) &+ \sum_{j\in\widehat{S}_D(i)} f_j(N_i,(cu_T)_i,D_{N_i}) \\ + \sum_{j\in Q_i\cap\widehat{S}_D(k)} f_j(N_i,(cu_T)_i,D_{N_i}) &+ \sum_{j\in\widehat{P}_D(i)\cap\widehat{S}_D(k)} f_j(N_i,(cu_T)_i,D_{N_i}). \end{aligned} \tag{4.10}$$

Moreover, due to NPP and the definition of k, Eq. (4.10) reduces to

$$\begin{aligned} f_i(N_i,(cu_T)_i,D_{N_i}) = f_k(N_i,(cu_T)_i,D_{N_i}) &- \sum_{j\in\widehat{S}_D(i)} f_j(N_i,(cu_T)_i,D_{N_i}) \\ &- \sum_{j\in Q_i\cap\widehat{S}_D(k)} f_j(N_i,(cu_T)_i,D_{N_i}). \end{aligned} \tag{4.11}$$

By definition of k, we also have $\left|\widehat{P}_D(k)\cap T\right| < \left|\widehat{P}_D(i)\cap T\right| = t$. Hence, by the second induction hypothesis, $f_k(N_i,(cu_T)_i,D_{N_i})$ is uniquely determined. Then, using Eq. (4.9), we can rewrite Eq. (4.11) as

$$(1+\mathcal{X}_i)\cdot f_i(N_i,(cu_T)_i,D_{N_i}) = f_k(N_i,(cu_T)_i,D_{N_i}) - \sum_{j\in Q_i\cap\widehat{S}_D(k)} f_j(N_i,(cu_T)_i,D_{N_i}).$$

Therefore, $f_i(N_i, (cu_T)_i, D_{N_i})$ is uniquely determined.

Case V: $i \in T$, $S_D(i) = \emptyset$, and $l(i) = l(D)$. Note that $S_D(i) = \emptyset$ implies $H_D(i) = \emptyset$. By IMT,

$$f_i(N, cu_T, D) = f_i(N_i, (cu_T)_i, D_{N_i}). \tag{4.12}$$

Note that every $j \in N_i \backslash A(i)$ belongs to one of the four cases above with respect to $(N_i, (cu_T)_i, D_{N_i})$. Then, from the previous cases, the fact that $l(D_{N_i}) = l(D)$, and the induction hypothesis

$$f_j\left(N_i, (cu_T)_i, D_{N_i}\right) \text{ is uniquely determined for } j \in N_i \setminus A(i). \tag{4.13}$$

Next, by EFF,

$$\sum_{j \in A(i)} f_j(N_i, (cu_T)_i, D_{N_i}) = c - \sum_{j \in N_i \backslash A(i)} f_j(N_i, (cu_T)_i, D_{N_i}).$$

Further, by NPP and NSP, for each $j \in A(i)$

$$f_j(N_i, (cu_T)_i, D_{N_i}) = \begin{cases} f_i(N_i, (cu_T)_i, D_{N_i}) & \text{if } j \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{Y}_j = 1$ if $j \in T$ and $\mathcal{Y}_j = 0$ otherwise. Then, from the two above equations, it follows that

$$f_i(N_i, (cu_T)_i, D_{N_i}) \cdot \sum_{j \in A(i)} \mathcal{Y}_j = c - \sum_{j \in N_i \backslash A(i)} f_j(N_i, (cu_T)_i, D_{N_i}).$$

Uniqueness of $f_i(N, cu_T, D)$ is obtained straightforwardly by applying Eq. (4.13) to the above equation, as $\mathcal{Y}_i = 1$.

We conclude from Cases I–V that $f(N, cu_T, D)$ is uniquely determined if $c > 0$. Now, consider (N, cu_T, D) with $c < 0$.[10] We have already mentioned that NPP implies that $f_i(N, v_0, D) = 0$ for all $i \in N$, where (N, v_0) is the null game. Since $cu_T + (-cu_T) = v_0$, ADD implies that

$$f(N, cu_T, D) = f(N, v_0, D) - f(N, -cu_T, D) = -f(N, -cu_T, D).$$

Since $-c > 0$, $f(N, -cu_T, D)$ is uniquely determined. Thus, $f(N, cu_T, S) = -f(N, -cu_T, S)$ is also uniquely determined if $c < 0$.

[10] Note that we cannot apply the superior upper and lower bound properties, since cu_T is not monotone if $c < 0$.

Finally, by ADD we have that $f(N, v, D) = \sum_{\emptyset \neq T \subseteq N} f(N, \Delta_v(T) u_T, D)$ is uniquely determined for all $(N, v) \in \mathcal{G}$.

Existence: First, solution φ^{SL} satisfying EFF, ADD, and NPP follows from the fact that for a given $(N, v, D) \in \mathcal{GD}$, the levels structure (N, B^D) does not depend on the TU-game (N, v), and from the properties Efficiency, Additivity, and Dummy Player Property satisfied by the Shapley Levels Value (see Winter, 1989).

To show that φ^{SL} also satisfies IMT, let $(N, v, D) \in \mathcal{GD}$ and $i \in N$. Let also $U^i_{r,s} \in B^D_{r,s}$ be such that $i \in U^i_{r,s}$, with $(r, s) \in \{1, \ldots, l(D)\} \times \{1, 2\}$. Then,

$$U^i_{r,l} = \begin{cases} U \supseteq \widehat{S}_D(i) \cup i & \text{if } r < l(i) \text{ or } (r, s) = (l(i), 1), \\ \widehat{S}_D(i) \cup i & \text{if } (r, l) = (l(i), 2), \\ \{i\} & \text{if } r > l(i). \end{cases} \tag{4.14}$$

Moreover, $\widehat{S}_D(i) \in B^D_{l(i)+1,1}$. That is, i and her subordinates belong to the same union in all levels of (N, B^D) prior to level $(l(i), 2)$, at which point the union that contains i is exactly $\widehat{S}_D(i) \cup i$, with $\widehat{S}_D(i)$ and $\{i\}$ being unions of the next level. From then on, $\{i\}$ is always a union herself.

In Álvarez-Mozos et al. (2013), a Multiplication Property satisfied by the Shapley Levels Value is identified. Here, we only provide an informal description of the property to avoid the introduction of further notation.[11] The property requires the share of $v(N)$—prescribed by the Shapley Levels Value—that a player obtains in a game with a levels structure be obtained by multiplying each of the shares received according to the Shapley Value by each of the unions $U^i_{r,l}$ in certain internal games defined for each player and each level of the levels structure. The internal game that corresponds to level (r, l) only uses the information contained in the unions of the coarser levels to which the unions of the game belong, i.e. $U^i_{k,h}$, with $(k, h) \leq_{lex} (r, l)$, and disregards any other information. For example, with only one level of cooperation there is an internal game played by all unions and then, there are as many internal games as unions in the partition, and the player set of each of these games is the corresponding union. For every union, the internal game then describes the prospects of a coalition that defects from

[11]For a formal description, we refer to Álvarez-Mozos et al. (2013).

the union to form a union itself. From the fact that the Shapley Levels Value satisfies the Multiplication Property, it follows from Eq. (4.14) that, due to the way it is constructed, φ^{SL} satisfies IMT.

To show that φ^{SL} satisfies NSP, let $(N, v, D) \in \mathcal{GD}$ and $i \in N$ with $S_D(i) = \emptyset$. Then, it can be easily verified that every $j \in A(i)$ with $S_D(j) = \emptyset$ has a completely symmetric position in the structure (N, B^D) w.r.t. player i, meaning that for every level, both i and j either belong to the same union or each of them forms a union as a singleton. From this observation, it follows that in exactly half of the permutations in $\Omega(B^D)$, i comes before j and vice versa. Then, whenever j is a necessary player and $(N, v) \in \mathcal{G}_M$, we obtain from Eq. (2.1) and the definition of φ^{SL} that $f_i(N, v, D) \leq f_j(N, v, D)$.

Finally, to show that φ^{SL} satisfies SLB and SUB, we use the Level game property fulfilled by the Shapley Levels Value (Álvarez-Mozos and Tejada, 2011). Let $(N, v, \underline{B})$ be a game with levels structure of cooperation and let $T \in B_r \in \underline{B}$. This property states that the joint payoff to the members of T according to the Shapley Levels Value is precisely the payoff to the union T in a game with levels structure where the players are the unions at level r and the structure is obtained from $\underline{B}$ by truncating its levels at level r. That is, $\sum_{i \in T} Sh^L_i(N, v, \underline{B}) = Sh^L_T(B_r, v^r, (B_1, \ldots, B_r))$, where for every $Q \subseteq B_r$, $v^r(Q) = v(\cup_{R \in Q} R)$.

Let $(N, v, D) \in \mathcal{GD}$ and $i \in N$. Then, note that Eq. (4.14) implies that among the admissible permutations of $\Omega(B^D)$, i comes before her subordinates in half of them and in the other half, all her subordinates come before player i. By the Level game property described above, it is enough to study the payoffs to the unions up to level $(l(i) + 1, 1)$. Following a reasoning similar to the one used for NSP, we can conclude that if i is a necessary player in a monotone game, then she earns as much as all her subordinates together and vice versa. Thus, φ^{SL} satisfies both SLB and SUB. □

5 Two Alternative Solutions for Games with Hierarchical Structure

The solution we proposed in the previous section uses the most refined levels structure characterized in Section 3, and therefore uses the most information in relation to a directed tree that can be modeled by a levels structure obtained by our procedure. In this section, we propose two alternative solutions for games with hierarchical structure. Both alternatives are obtained by the same procedure used to construct φ^{SL}. First the hierarchical structure (N, D) is transformed into a levels structure $(N, \underline{B})$, with $\underline{B}$ being a subset of the collection B^D of partitions generated by the conversion mapping $\mathcal{H}$. Second, the Shapley Levels Value is applied to the game with levels structure obtained in the first stage. The difference is that for the two alternatives, a certain proper subset of the collection of partitions B^D defined in Eq. (3.2) is considered instead of the entire collection.

5.1 First alternative solution

For the first alternative solution, we consider the levels structure $\left(N, \overline{B}^D\right)$ given by $\overline{B}^D = \left(\overline{B}_1^D, \overline{B}_2^D, \ldots, \overline{B}_{l(D)}^D\right)$, where

$$\overline{B}_k^D = B_{k,2}^D \text{ for } k \in \{1, \ldots, l(D)\},$$

with $B_{k,2}^D$ as given in Section 3—see Eq. (3.4). Compared with the original levels structure (N, B^D), we now disregard all partitions $B_{k,1}^D$, with $k \in \{1, \ldots, l(D)\}$. Hence, we only take all the team levels and disregard all the sibling levels. Note that $\overline{B}_{l(D)}^D$ is the trivial partition of singletons. To every $(N, v, D) \in \mathcal{GD}$, the first alternative solution assigns the Shapley Levels Value of the game with levels structure $\left(N, v, \overline{B}^D\right)$.

Definition 5.1. *The solution $\overline{\varphi}^{SL}$ on the class of games with hierarchical structure $\mathcal{GD}$ is the solution given by*

$$\overline{\varphi}^{SL}(N, v, D) = Sh^L\left(N, v, \overline{B}^D\right), \text{ with } (N, v, D) \in \mathcal{GD}.$$

Example 5.1. *Consider $(N, u_N, D) \in \mathcal{GD}$, with (N, D) being the hierarchy given in Example 3.1. Then, the level structure consists of the three even levels given in Example 3.1. Note that compared to the level structure used for solution φ^{SL}, the first level's partition of this structure immediately consists of the top player as a singleton union and of the teams of each of her two successors. This yields the payoff vector given by*

$$\overline{\varphi}^{SL}(N, u_N, D) = \left(\frac{1}{3}, \frac{1}{9}, \frac{1}{6}, \frac{1}{9}, \frac{1}{27}, \frac{1}{6}, \frac{1}{27}, \frac{1}{27}\right).$$

From an inspection of the payoffs in the example above, we conclude that $\overline{\varphi}^{SL}$ does not satisfy SLB because although player 1 is necessary in (N, u_N), she earns less that her subordinates jointly. For necessary players and when the game is monotone, the properties SLB and SUB satisfied by the first solution φ^{SL} establish a relation between the payoff of a player and the sum of the payoffs of all other players in her team or, equivalently, the sum of the payoffs of all her successors' teams. Thus, all these teams are considered at once. Nevertheless, from the perspective that teams are autonomous, it may be equally interesting to relate the payoff of a player to the sum of the payoffs of only one of her successors' teams. We do so in the following two properties.

TLB A solution on $\mathcal{GD}$, f, satisfies *team lower bound* if for every $(N, v, D) \in \mathcal{GD}$ with $(N, v) \in \mathcal{G}_M$, and if $i \in N$ is necessary in (N, v), then for every $j \in S_D(i)$,

$$f_i(N, v, D) \geq \sum_{k \in \widehat{S}_D(j) \cup j} f_k(N, v, D).$$

TUB A solution on $\mathcal{GD}$, f, satisfies *team upper bound* if for every $(N, v, D) \in \mathcal{GD}$ with $(N, v) \in \mathcal{G}_M$, and if some $j \in \widehat{S}_D(i)$ is necessary in (N, v) for some $i \in N$, then for $h \in S_D(i) \cap \left(\widehat{P}_D(j) \cup j\right)$,[12]

$$f_i(N, v, D) \leq \sum_{k \in \widehat{S}_D(h) \cup h} f_k(N, v, D).$$

Compared to SLB and SUB, the bounds for the payoff of player i are now determined by the team of any successor of i instead of all her subordinates together. Moreover, it is easy to verify that SLB $\Rightarrow$ TLB and SUB $\Leftarrow$ TUB. Next, we show that substituting SLB and SUB with TLB and TUB in Theorem 4.5 singles out solution $\overline{\varphi}^{SL}$ as the unique solution for games with hierarchical structure.

Theorem 5.2. *A solution f on $\mathcal{GD}$ satisfies* EFF, ADD, NPP, IMT, NSP, TLB, *and* TUB *if and only if $f = \overline{\varphi}^{SL}$. Moreover, the seven axioms are independent.*

Proof. We first show uniqueness and then existence. The logical independence of the axioms is shown in Appendix C.

[12]Note that h is uniquely determined.

Uniqueness: The proof follows the same steps as in the proof of Theorem 4.5, except for Case IV, where we use TLB and TUB instead of SLB and SUB. Therefore, we only show how to adapt Case IV. Accordingly, let $i \in T$ be such that $H_D(i) \neq \emptyset$. Proceeding as in the proof of Theorem 4.5, Eqs. (4.7) and (4.8) follow. Then, by applying NPP, TLB, and TUB, instead of Eq. (4.9) we obtain for all $j \in S_D(i)$

$$\sum_{h \in \widehat{S}_D(j) \cup j} f_h(N_i, (cu_T)_i, D_{N_i}) = \begin{cases} f_i(N_i, (cu_T)_i, D_{N_i}) & \text{if } (\widehat{S}_D(j) \cup j) \cap T \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{5.15}$$

Now, for each $j \in S_D(i)$, let $\mathcal{Z}_j = 1$ if $\left(\widehat{S}_D(j) \cup j\right) \cap T \neq \emptyset$ and $\mathcal{Z}_j = 0$ otherwise. Next, we conduct a second induction on $\left|\widehat{P}_D(i) \cap T\right|$ as in the proof of Theorem 4.5. When $\left|\widehat{P}_D(i) \cap T\right| = 0$, Eq. (4.7) now reduces to

$$\left(1 + \sum_{j \in S_D(i)} \mathcal{Z}_j\right) \cdot f_i(N_i, (cu_T)_i, D_{N_i}) = c - \sum_{j \in Q_i} f_j(N_i, (cu_T)_i, D_{N_i}),$$

so $f_i(N_i, (cu_T)_i, D_{N_i})$ is uniquely determined.

Now, suppose that for some integer $t > 0$, $f_i(N_i, (cu_T)_i, D_{N_i})$ is uniquely determined for every $i' \in T$ with $H_D(i') \neq \emptyset$ and $\left|\widehat{P}_D(i') \cap T\right| < t$, and assume that $\left|\widehat{P}_D(i) \cap T\right| = t$. Take $k \in \widehat{P}_D(i) \cap T$ such that for every $j \in \widehat{P}_D(i) \cap T$, we have $l(k) \geq l(j)$, i.e., k is the superior of i who is closest to the latter in the tree among those superiors of i that belong to T. Let also $p \in S_D(k) \cap \left(\widehat{P}_D(i) \cup i\right)$, and note that p is uniquely defined. By TLB and TUB, and the fact that $i \in T$, we obtain

$$\begin{aligned} f_k(N_i, (cu_T)_i, D_{N_i}) = {} & f_i(N_i, (cu_T)_i, D_{N_i}) + \sum_{j \in \widehat{S}_D(i)} f_j(N_i, (cu_T)_i, D_{N_i}) \\ & + \sum_{j \in Q_i \cap \widehat{S}_D(p)} f_j(N_i, (cu_T)_i, D_{N_i}) + \sum_{j \in \widehat{P}_D(i) \cap (\widehat{S}_D(p) \cup p)} f_j(N_i, (cu_T)_i, D_{N_i}). \end{aligned} \tag{5.16}$$

Moreover, due to NPP, Eq. (5.16) reduces to

$$\begin{aligned}
&f_i(N_i,(cu_T)_i,D_{N_i})\\
=&f_k(N_i,(cu_T)_i,D_{N_i}) - \sum_{j\in \widehat{S}_D(i)} f_j(N_i,(cu_T)_i,D_{N_i}) - \sum_{j\in Q_i\cap \widehat{S}_D(k)} f_j(N_i,(cu_T)_i,D_{N_i})\\
=&f_k(N_i,(cu_T)_i,D_{N_i}) - \sum_{j\in S_D(i)} \sum_{h\in\left(\widehat{S}_D(j)\cup j\right)} f_h(N_i,(cu_T)_i,D_{N_i}) - \sum_{j\in Q_i\cap \widehat{S}_D(k)} f_j(N_i,(cu_T)_i,D_{N_i}).
\end{aligned} \tag{5.17}$$

By definition of k, we have $\left|\widehat{P}_D(k)\cap T\right| < \left|\widehat{P}_D(i)\cap T\right| = t$. Hence, by the second induction hypothesis, $f_k(N_i,(cu_T)_i,D_{N_i})$ is uniquely determined. By using Eq. (5.15), we can rewrite Eq. (5.17) as

$$\left(1+\sum_{j\in S_D(i)} \mathcal{Z}_j\right)\cdot f_i(N_i,(cu_T)_i,D_{N_i}) = f_k(N_i,(cu_T)_i,D_{N_i}) - \sum_{j\in Q_i\cap \widehat{S}_D(k)} f_j(N_i,(cu_T)_i,D_{N_i}).$$

Therefore, $f_i(N_i,(cu_T)_i,D_{N_i})$ is uniquely determined. This completes Case IV. The rest of the uniqueness part of the proof is done in the same way as the proof of Theorem 4.5.

Existence: Solution $\overline{\varphi}^{SL}$ satisfying EFF, ADD, and NPP can be shown as in the proof of Theorem 4.5. To prove that $\overline{\varphi}^{SL}$ also satisfies IMT, let $(N,v,D)\in\mathcal{GD}$ and $i\in N$. Note that when $l(i)<l(D)$, it holds that

$$\begin{aligned}
&\widehat{S}_D(i)\cup i\in B^D_{l(i),2} \text{ and for every } j\in S_D(i),\\
&\widehat{S}_D(j)\cup j\in B^D_{l(i)+1,2}.
\end{aligned} \tag{5.18}$$

That is, i and her subordinates belong to the same union in all levels of $\left(N,\overline{B}^D\right)$ prior to level $l(i)$, at which point i and her successors' teams form separate unions. From then on, i is always a singleton. From the fact that the Shapley Levels Value satisfies the multiplication property (Álvarez-Mozos et al., 2013), it can be verified due to Eq. (5.18) that because of the way it is constructed, $\overline{\varphi}^{SL}$ satisfies IMT. To show that $\overline{\varphi}^{SL}$ satisfies NSP, we can repeat the argument used in the proof of Theorem 4.5, since two siblings with no successors also have a symmetric position in the levels structure $\left(N,\overline{B}^D\right)$. Finally, to show that $\overline{\varphi}^{SL}$ satisfies TLB and TUB we can replicate the argument used in Theorem 4.5 to show the subordinate bounds using Eq. (5.18) instead of Eq. (4.14). □

5.2 Second alternative solution

Both solutions φ^{SL} and $\overline{\varphi}^{SL}$ are obtained in a similar way. First, the hierarchical structure is mapped into a levels structure and, second, the Shapley Levels Value is applied. While for φ^{SL}, all levels in the set B^D obtained by the mapping $\mathcal{H}$ are used, the levels structure used for $\overline{\varphi}^{SL}$ only contains the even levels of B^D, i.e. the team levels.

A third solution for games with hierarchical structure reached by applying the Shapley Levels Value is next obtained by considering the odd levels of B^D, i.e. the sibling levels. More precisely, for this solution, we delete from B^D all partitions $B^D_{k,2}$ except the trivial partition of singletons $B^D_{l(D),2}$. Hence, this third solution can be considered as the "complement" of $\overline{\varphi}^{SL}$, in the sense that both solutions use disjoint subsets of the information contained in the levels structure obtained by applying $\mathcal{H}$ to the original hierarchical structure: While the second solution uses the team levels, this third solution is obtained by using the sibling levels. By so doing, we obtain the levels structure $(N, \widetilde{B}^D)$ with $\widetilde{B}^D = \left(\widetilde{B}^D_1, \widetilde{B}^D_{l(D)}, \widetilde{B}^D_{l(D)+1}\right)$, where $\widetilde{B}^D_k = B^D_{k,1}$ for $k \in \{1, \ldots, l(D)\}$ and $\widetilde{B}^D_{l(D)+1} = B^D_{l(D),2} = \{\{i\} : i \in N\}$.

Definition 5.2. *The solution $\widetilde{\varphi}^{SL}$ on the class of games with hierarchical structure $\mathcal{GD}$ is the solution given by*

$$\widetilde{\varphi}^{SL}(N, v, D) = Sh^L\left(N, v, \widetilde{B}^D\right), \text{ with } (N, v, D) \in \mathcal{GD}.$$

Example 5.3. *Consider again $(N, u_N, D) \in \mathcal{GD}$, with (N, D) the hierarchy given in Example 3.1. Then, the level structure consists of the three odd levels given in Example 3.1 and the level $B^D_{3,2}$ of singletons. In that case, the first level consists of two unions, the top player being a singleton union and all other players, i.e., all her successors together with all their team members, being in the other union. Applying the Shapley Levels Value yields the payoff vector*

$$\widetilde{\varphi}^{SL}(N, u_N, D) = \left(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{24}, \frac{1}{24}, \frac{1}{8}, \frac{1}{48}, \frac{1}{48}\right).$$

From an inspection of the payoffs in the example above, it is easy to verify that $\widetilde{\varphi}^{SL}$ does not satisfy TUB. Take $i = 2$ and $j = h = 4$, for instance. Then, $\{4\}$ is a team that contains a necessary player in (N, u_N), but the payoff to 2 is greater than the payoff to 4. It can be checked that among the properties considered so far, solution $\widetilde{\varphi}^{SL}$ does not satisfy IMT either. However, it satisfies stronger versions of NSP, SUB, and SLB, which are introduced next.

First, we consider a property that relates, in monotone games, the payoff of two players—some of them being necessary in the game—who are siblings but who may have successors.

SNSP A solution on $\mathcal{GD}$, f, satisfies the *strong necessary sibling property* if for every $(N, v, D) \in \mathcal{GD}$ with $(N, v) \in \mathcal{G}_M$ and every $i \in N$, and if i is necessary, then for each $j \in A(i)$

$$f_i(N, v, D) \geq f_j(N, v, D).$$

Note that SNSP $\Rightarrow$ NSP. Second, we consider two properties that relate the payoff of a player to the joint payoff of the subordinates of one of her siblings in a monotone game where either i is a necessary player or the set of subordinates contains necessary players.

SSLB A solution on $\mathcal{GD}$, f, satisfies *strong superior lower bound* if for every $(N, v, D) \in \mathcal{GD}$ with $(N, v) \in \mathcal{G}_M$, and if $i \in N$ is necessary in (N, v), then for each $j \in A(i)$

$$f_i(N, v, D) \geq \sum_{h \in \widehat{S}_D(j)} f_h(N, v, D).$$

SSUB A solution on $\mathcal{GD}$, f, satisfies *strong superior upper bound* if for every $(N, v, D) \in \mathcal{GD}$ with $(N, v) \in \mathcal{G}_M$ and every $i \in N$, and if $h \in A(i)$ and some $j \in \widehat{S}_D(h)$ is necessary in (N, v), then

$$f_i(N, v, D) \leq \sum_{p \in \widehat{S}_D(h)} f_p(N, v, D).$$

We note that SSLB $\Rightarrow$ SLB and SSUB $\Rightarrow$ SUB. Third, we also consider a property that connects the payoff of the subordinates of two players when the former contain necessary players. Together with EFF, ADD, NPP and the latter three properties, this additional property is needed to single out a unique solution for games with hierarchical structure.

NOP A solution on $\mathcal{GD}$, f, satisfies the *necessary offspring property* if for every $(N, v, D) \in \mathcal{GD}$ and every $i \in N$ such that $\widehat{S}_D(i)$ contains necessary players, it holds that for every $j \in A(i)$

$$\sum_{h \in \widehat{S}_D(i)} f_h(N, v, D) \geq \sum_{h \in \widehat{S}_D(j)} f_h(N, v, D).$$

We are now in a position to characterize $\widetilde{\varphi}^{SL}$.

Theorem 5.4. *A solution f on $\mathcal{GD}$ satisfies* EFF*,* ADD*,* NPP*,* SNSP*,* SSLB*,* SSUB*, and* NOP *if and only if $f = \widetilde{\varphi}^{SL}$. Moreover, the seven properties are independent.*

Proof. We first show uniqueness and then existence. The logical independence of the axioms is shown in Appendix C.

Uniqueness: Suppose that f satisfies the seven axioms. For every (N, v_0, D), with (N, v_0) being the null game, NPP implies that $f_i(N, v_0, D) = 0$ for all $i \in N$. Next, let $(N, cu_T, D) \in \mathcal{GD}$, where $\emptyset \neq T \subseteq N$ and $c > 0$. For every $i \in N \setminus T$, by NPP, $f_i(N, cu_T, D) = 0$.

Let $i \in T$ henceforth. If $|T| = 1$, NPP and EFF imply that $f_i(N, v, D) = c$. Hence, assume that $|T| \geq 2$. We show uniqueness of $f_i(N, cu_T, D)$ by induction on her rank, $l(i)$. If $l(i) = 0$, by using EFF, SSLB, and SSUB, we easily obtain that $f_i(N, cu_T, D) = \frac{c}{2}$. Then, assume that for every $j \in T$ with $l(j) < l$, $f_j(N, cu_T, D)$ is uniquely determined, and consider that $l(i) = l$ for some integer $l \in \{1, \ldots, l(D)\}$. For each $j \in N$, let $\mathcal{X}_j = 1$ if $\widehat{S}_D(j) \cap T \neq \emptyset$ and $\mathcal{X}_j = 0$ otherwise. Also, for each $j \in N$, let $\mathcal{Y}_j = 1$ if $j \in T$ and $\mathcal{Y}_j = 0$ otherwise.

By NPP, SSLB, and SSUB, we have that for each $j \in A(i)$

$$\sum_{h \in \widehat{S}_D(j)} f_h(N, v, D) = \mathcal{X}_j \cdot f_i(N, v, D). \tag{5.19}$$

Similarly, by NPP and SNSP, we have that

$$\sum_{j \in A(i)} f_j(N, v, D) = \sum_{j \in A(i)} \mathcal{Y}_j \cdot f_i(N, v, D). \tag{5.20}$$

Next, we distinguish two cases, depending on whether $l = 1$ or $l > 1$.

Case I: $l = 1$. Note that we have $i_0 = p_D(i)$. By EFF, we obtain

$$c = f_{i_0}(N, v, D) + \sum_{j \in A(i)} \left(f_j(N, v, D) + \sum_{h \in \widehat{S}_D(j)} f_h(N, v, D) \right).$$

By using Eqs. (5.19) and (5.20), the above equation reduces to

$$c = f_{i_0}(N, v, D) + \sum_{j \in A(i)} (\mathcal{Y}_j + \mathcal{X}_j) \cdot f_i(N, v, D).$$

By the induction hypothesis, $f_{i_0}(N, v, D)$ is determined. Moreover, the coefficient of $f_i(N, v, D)$ in the above equation is strictly positive. Thus, $f_i(N, v, D)$ is determined.

Case II: $l > 1$. For every $r \in \{0, \ldots, l\}$, let $i_r \in \left(\widehat{P}_D(i) \cup i\right)$ be such that $l(i_r) = r$. Note that, in particular, we have $i_l = i$. Also note that $\mathcal{Y}_{i_l} = 1$ and $\mathcal{X}_{i_r} = 1$ for all $r \in \{0, \ldots, l-1\}$.

First, by NPP and NOP, for each $r \in \{1, \ldots, l-1\}$,

$$\sum_{j \in A(i_r)} \sum_{h \in \widehat{S}_D(j)} f_h(N, v, D) = \sum_{j \in A(i_r)} \mathcal{X}_j \cdot \sum_{j \in \widehat{S}_D(i_r)} f_j(N, v, D). \tag{5.21}$$

Second, we claim that for $r \in \{0, \ldots, l\}$,[13]

$$c = \sum_{s=0}^{r} \left[\prod_{p=0}^{s-1} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \sum_{j \in A(i_s)} f_j(N, v, D) \right] + \prod_{p=0}^{r} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \sum_{j \in \widehat{S}_D(i_r)} f_j(N, v, D). \tag{5.22}$$

We prove Eq. (5.22) by induction on r. The case $r = 0$ is straightforward, since due to EFF, $c = f_{i_0}(N, v, D) + \sum_{j \in \widehat{S}_D(i_0)} f_j(N, v, D)$. Hence, assume that Eq. (5.22) holds if we substitute r by $r - 1$. To facilitate the presentation of the calculations, we denote $f_j = f_j(N, v, D)$ for all $j \in N$. Then,

$$\begin{aligned}
&\sum_{s=0}^{r} \left[\prod_{p=0}^{s-1} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \sum_{j \in A(i_s)} f_j \right] + \prod_{p=0}^{r} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \sum_{j \in \widehat{S}_D(i_r)} f_j \\
=&\sum_{s=0}^{r-1} \left[\prod_{p=0}^{s-1} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \sum_{j \in A(i_s)} f_j \right] + \prod_{p=0}^{r-1} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \sum_{j \in \widehat{S}_D(i_{r-1})} f_j \\
&+ \prod_{p=0}^{r-1} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \sum_{j \in A(i_r)} f_j + \prod_{p=0}^{r} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \sum_{j \in \widehat{S}_D(i_r)} f_j - \prod_{p=0}^{r-1} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \sum_{j \in \widehat{S}_D(i_{r-1})} f_j \\
=&c + \prod_{p=0}^{r-1} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \left(\sum_{j \in A(i_r)} f_j + \sum_{j \in A(i_r)} \mathcal{X}_j \cdot \sum_{j \in \widehat{S}_D(i_r)} f_j - \sum_{j \in \widehat{S}_D(i_{r-1})} f_j \right) = c,
\end{aligned}$$

where the second equality holds from the second induction hypothesis and the last equality is explained as follows. Indeed, note that for every $r \in \{1, \ldots, l-1\}$

$$\begin{aligned}
&\sum_{j \in A(i_r)} f_j + \sum_{j \in A(i_r)} \mathcal{X}_j \cdot \sum_{j \in \widehat{S}_D(i_r)} f_j - \sum_{j \in \widehat{S}_D(i_{r-1})} f_j \\
=& \sum_{j \in A(i_r)} f_j + \sum_{j \in A(i_r)} \mathcal{X}_j \cdot \sum_{j \in \widehat{S}_D(i_r)} f_j - \sum_{j \in A(i_r)} f_j - \sum_{j \in A(i_r)} \sum_{h \in \widehat{S}_D(j)} f_h = 0,
\end{aligned}$$

where the last equality holds by Eq. (5.21).

[13] The multiplication over an empty set is 1.

Third, by using Eq. (5.22) when $r = l$, Eq. (5.22) reduces to

$$c = \sum_{s=0}^{l-1} \left[\prod_{p=0}^{s-1} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \sum_{j \in A(i_s)} f_j \right] + \prod_{p=0}^{l-1} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \sum_{j \in A(i)} f_j + \prod_{p=0}^{l} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \sum_{j \in \widehat{S}_D(i)} f_j. \quad (5.23)$$

Fourth, applying Eqs. (5.19) and (5.20) to Eq. (5.23) yields

$$c = \sum_{s=0}^{l-1} \left[\prod_{p=0}^{s-1} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \sum_{j \in A(i_s)} f_j \right] + \left[\prod_{p=0}^{l-1} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \sum_{j \in A(i)} \mathcal{Y}_j + \prod_{p=0}^{l} \sum_{j \in A(i_p)} \mathcal{X}_j \cdot \mathcal{X}_i \right] \cdot f_i(N, v, D). \quad (5.24)$$

By the first induction hypothesis, the first term in the right-hand side of Eq. (5.24) is determined. Moreover, the coefficient of $f_i(N, v, D)$ in Eq. (5.24) is strictly positive. Therefore, $f_i(N, v, D)$ is unique.

Thus, we have shown that $f(N, cu_T, D)$ is uniquely determined. Following the lines of the proof of Theorem 4.5, it can also be verified that $f(N, cu_T, D)$ is uniquely determined if $c < 0$, hence implying by ADD that $f(N, v, D) = \sum_{\emptyset \neq T \subseteq N} f(N, \Delta_v(T) u_T, D)$ is uniquely determined for all $(N, v) \in \mathcal{G}$.

Existence: First, solution $\widetilde{\varphi}^{SL}$ satisfying EFF, ADD, and NPP can be shown as in the proof of Theorem 4.5. To prove that $\widetilde{\varphi}^{SL}$ also satisfies SNSP, SSLB, SSUB, and NOP, let $(N, v, D) \in \mathcal{GD}$, $i \in N$, and $j \in A(i)$. Then for every $l < l(i)$,

$$\begin{aligned} &\widehat{S}_D(i) \cup \widehat{S}_D(j) \cup \{i, j\} \subseteq T \in B^D_{l,1} \text{ and} \\ &\widehat{S}_D(i), \widehat{S}_D(j), \{i\}, \{j\} \in B^D_{l(i),1}. \end{aligned} \quad (5.25)$$

That is, i and her siblings (as well as all their subordinates) belong to the same union in all levels of $\left(N, \widetilde{B}^D\right)$ prior to level $l(i)$, at which point i and her siblings are all singletons, while the subordinates of each sibling of i (including i herself) constitute a union of that level.

To show that $\widetilde{\varphi}^{SL}$ satisfies SNSP, we can repeat the argument used in the proof of Theorem 4.5 since, by Eq. (5.25), two siblings have a symmetric position in the structure $\left(N, \widetilde{B}^D\right)$, even if they have successors. A similar reasoning shows that $\widetilde{\varphi}^{SL}$ satisfies SSLB, SSUB, and NOP. Indeed, from Eq. (5.25), it is easy to verify that for every $i \in N$ and $j \in A(i)$, coalitions $\{i\}$ and $\widehat{S}_D(j)$ have a symmetric position in the structure $\left(N, \widetilde{B}^D\right)$ because they are in the same union up to certain level at which both coalitions became a union of the partition. Consequently, there are as many admissible permutations in $\Omega(\widetilde{B}^D)$ in which i comes before coalition $\widehat{S}_D(j)$ than admissible permutations in which i comes after coalition $\widehat{S}_D(j)$. From this observation, it follows that $\widetilde{\varphi}^{SL}$ satisfies SSLB and SSUB. Finally, to show that $\widetilde{\varphi}^{SL}$ satisfies NOP, we note that the reasoning above also applies to the coalitions $\widehat{S}_D(i)$ and $\widehat{S}_D(j)$, given that they also have a symmetric position in the structure $\left(N, \widetilde{B}^D\right)$. □

6 Comparison to Other Values

As mentioned in the Introduction, the literature on games with hierarchical structure offers a series of values, including the Precedence Shapley Value (PSV) of Faigle and Kern (1992), the Permission Value (PV) of van den Brink and Gilles (1996), the Hierachical Outcome (HO) of Demange (2004), and the Average Tree Permission Value ($ATPV$) of van den Brink et al. (2015). In this paper, we have proposed three new values and characterized them by means of certain properties. Many of these properties are new in the literature. In particular, we compared the new solutions based on bound properties that specify lower and upper bounds for the payoffs of players with respect to certain of their relatives. The purpose of this section is to study which of our new properties do satisfy the aforesaid classic values. By doing so, we will be able to compare the different solutions from an axiomatic perspective and then to sort out the features that they share and those in which they differ. It can be easily verified that all the solutions considered in this section, as well as our three values, satisfy EFF, ADD, and NSP. For the remaining properties, we present Table 1, which summarizes the situation.

In particular, our three values satisfy NPP. Several further comments are in order. First, compared to the first solution φ^{SL}, which uses the most refined levels structure among the three solutions, the alternative solution $\overline{\varphi}^{SL}$ only considers the team levels and satisfies IMT and SUB, but does not satisfy SLB, while the alternative $\widetilde{\varphi}^{SL}$ only considers the sibling levels and satisfies SLB and SUB, but does not satisfy IMT. To axiomatize $\overline{\varphi}^{SL}$ and $\widetilde{\varphi}^{SL}$, we have considered TLB and TUB on the one hand, and SSLB, SSUB, SNSP, and NOP on the other hand.

Second, the properties introduced in this paper are satisfied by at least one of the values on $\mathcal{GD}$ that can be found in the literature. Second, among the latter, the HO seems to be the value which has more features in common with the solutions based on levels structures. Third, we point out that except for EFF, ADD, and NSP, the $ATPV$ solution does not satisfy

	NPP	IMT	SLB	SUB	TLB	TUB	SSLB	SSUB	SNSP	NOP
φ^{SL}	$*$	$*$	$*$	$*$	$+$	$-$	$-$	$-$	$-$	$-$
$\overline{\varphi}^{SL}$	$*$	$*$	$-$	$+$	$*$	$*$	$-$	$-$	$-$	$-$
$\widetilde{\varphi}^{SL}$	$*$	$-$	$+$	$+$	$+$	$-$	$*$	$*$	$*$	$*$
Sh	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$
PSV	$+$	$-$	$+$	$-$	$+$	$-$	$-$	$-$	$-$	$-$
PV	$-$	$-$	$-$	$+$	$-$	$+$	$-$	$+$	$+$	$-$
HO	$-$	$+$	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$+$
$ATPV$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$

Table 1: The $+$ sign means that the property is satisfied, the $-$ sign that it is not satisfied, and $*$ indicates each of the properties used to characterize the corresponding value.

any of the other properties considered in this paper.

Finally, it is worth studying the payoffs that the five values considered in the table propose in the case of the example used throughout the paper (see Example 3.1). There are two polar cases: either they divide the spoils equally or they allocate everything to the root of the tree. Indeed,

$$Sh(N,u_N) = PV(N,u_N,D) = ATPV(N,u_N,D) = \left(\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{1}{8}\right),$$
$$PSV(N,u_N,D) = HO(N,u_N,D) = (1,0,0,0,0,0,0,0).$$

Accordingly, the three values proposed in the paper, namely φ^{SL}, $\overline{\varphi}^{SL}$, and $\widetilde{\varphi}^{SL}$, can all be considered to be a compromise between the two extreme proposals above. The reason is that the values based on the Shapley levels value are more sensitive to the agents' position in the hierarchy. Depending on the context, one of the values might be more suitable than the others. Our characterization results, together with Table 1, allow us to select the most appropriate value.

7 Concluding Remarks

In this paper, we have defined and characterized three new solutions for games with hierarchical structure based on *(i)* certain mappings that transform hierarchical structures into levels structures; *(ii)* the Shapley Levels Value for games with levels structure of cooperation. We have also studied the transformations between the set of hierarchical structures and the set of levels structures from an axiomatic viewpoint.

Regarding the latter issue, when mapping a directed tree into a levels structure, we have required that every partition of the levels structure respect the directed tree in the sense of properties P1–P5. Theorem 3.2 implies that in such a levels structure, a player that is *higher in the hierarchy* is *more independent from the other players* in the collection of nested partitions, in the sense that such a player becomes a singleton coalition at a lower-ranked partition than a player that is lower in the hierarchy. We stress that we do not claim that, when coupling either a directed tree or a levels structure with a cooperative game, "higher in the hierarchy" or "being more independent" is beneficial for a player. To verify whether such a statement is true or not, we would need to specify *(i)* how to modify the original cooperative game (possibly into a number of different games), with the information contained in either the directed tree or the levels structure and *(ii)* to specify which solution concept to apply to the resulting modified game(s). For instance, take $N = \{1, 2, 3\}$, $(N, \underline{B}_0)$ being the trivial levels structure and $(N, \underline{B}')$ the level structure with the non-trivial level given by $B'_1 = \{\{1\}, \{2, 3\}\}$. For the trivial levels structure, the Shapley Levels Value reduces to the Shapley Value, and for $(N, \underline{B}')$ the Shapley levels Value reduces to the Owen Value. From these two facts, it is easy to derive that $Sh^L_1(N, v, \underline{B}') - Sh^L_1(N, v, \underline{B}_0) = \frac{1}{6}\left(v(N) + \sum_{i=1}^{3}(v(i) - v(N \setminus i)\right)$. Since the difference can be either positive or negative, it cannot be unambiguously predicted whether players 2 and 3 benefit from forming the union $\{2, 3\}$ or not.

To sum up, we have advanced a new way to exploit the information contained in a hierarchical structure by mapping it into a levels structure, without any loss of information in many settings. By so doing, we have defined three new values for games with hierarchical information which seem to be more sensitive to the agents' position in the hierarchy than other values in the literature in various interesting circumstances. Of course, a full understanding of the usefulness of our procedure will require further examination. For instance, it seems interesting to answer a research question that can be seen as the reversal of what we do in this paper: Which are all values that can be obtained according to our procedure?

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Appendix A Proofs of Section 3

Proof of Theorem 3.2 We note that $|N| \geq 2$. We first prove that for each $(N, D) \in \mathcal{D}$, every partition $P \in B^D$ satisfies P1–P5. Second, we prove that if a partition $P \neq \{N\}$ respects a given directed tree $(N, D) \in \mathcal{D}$ (i.e., the partition satisfies P1–P5 with respect to (N, D)), then P must belong to B^D.

Existence: Let $(N, D) \in \mathcal{D}$, $i \in N$ and P be any proper partition of the levels structure $\mathcal{H}(N, D)$. We assume that $P \neq \{\{j\}\}_{j\in N}$, for if not it is immediate to verify that P satisfies all properties with respect to any D. We distinguish two cases, depending on whether $P = B^D_{s,1}$ as defined in Eq. (3.3) or $P = B^D_{s,2}$ as defined in Eq. (3.4), where $s \geq 0$ is an integer.

Case 1: $T = B^D_{s,1}$. In this case, we have

$$P = \Big\{\{i\} : l(i) \leq s-1\Big\} \cup \Big\{\widehat{S}_D(i) : l(i) = s-1\Big\}. \tag{A.26}$$

First, let $i, j \in N$ be such that $(i, j) \in D$ and $P_i \neq \{i\}$.[14] Then, by Eq. (A.26), we obtain $l(i) \geq s$ and then $\widehat{S}_D(i) \subseteq P_i$. In particular, P1 is satisfied. Second, let $i, j, k \in N$ be such that $(k, i), (k, j) \in D$ and $P_i \setminus \widehat{S}_D(i) \neq \emptyset$. By Eq. (A.26), we have $l(i) > s$. Since $l(j) = l(i)$, $P_i = P_j$ and P2 holds. Third, let $i, j \in N$ be such that $l(i) = l(j) \geq 1$, $S_D(i) \neq \emptyset$, and $P_i = \{i\}$. From Eq. (A.26), it follows that $l(i) \leq s-1$, so $P_j = \{j\}$ and P3 is met. Fourth, let $i, j \in N$ be such that $(i, j) \in D$ and $P_i = \{i\}$. Note in particular that $S_D(i) \neq \emptyset$. Then, $l(i) \leq s-1$ and $l(j) \leq s$. If $l(i) < s-1$, we have $P_j = \{j\}$. If $l(i) = s-1$, we have $P_j = \widehat{S}_D(i)$. In any case, P4 is satisfied. Fifth, let $i, j \in N$ be such that $l(i) = l(j)$, $A(i) \setminus i \neq \emptyset$, and $A(j) \setminus j \neq \emptyset$. If $l(i) < s$, we have $P_i = \{i\}$ and $P_j = \{j\}$, whereas if $l(i) \geq s$, we have $A(i) \subseteq P_i$ and $A(j) \subseteq P_j$. In any case, P5 is satisfied.

Case 2: $P = B^D_{s,2}$. In this case, we have

$$P = \Big\{\{i\} : l(i) \leq s-1\Big\} \cup \Big\{\widehat{S}_D(j) \cup j : l(j) = s\Big\}. \tag{A.27}$$

First, let $i, j \in N$ be such that $(i, j) \in D$ and $P_i \neq \{i\}$. Then, by Eq. (A.27), we have $l(i) \geq s+1$ and $P_i = P_j$, so P1 is satisfied. Second, let $i, j \in N$ be such that $j \in A(i)$ and $P_i \setminus \Big(\widehat{S}_D(i) \cup i\Big) \neq \emptyset$. By Eq. (A.26), we have $l(i) > s+1$. Then, $P_i = P_j$ and P2 holds. Third, let $i, j \in N$ be such that $l(i) = l(j)$, $S_D(i) \neq \emptyset$, and $P_i = \{i\}$. Then, Eq. (A.27) implies that $l(i) \leq s+1$ and hence $P_j = \{j\}$. Therefore, P3 is met. Fourth, let $i, j \in N$ be such that $(i, j) \in D$ and $P_i = \{i\}$. Note in particular that $S_D(i) \neq \emptyset$ and then, $l(i) \leq s+1$. If $l(i) = s+1$, we have $P_j = \widehat{S}_D(i)$. If $l(i) < s+1$, we

[14] Recall that for a given partition of N, P and $i \in N$, we denote by P_i the element of P that contains player i.

have $P_j = \{j\}$. In any case, P4 is satisfied. Fifth, let $i, j \in N$, with $i \neq j$, be such that $l(i) = l(j)$, $A(i) \setminus i \neq \emptyset$, and $A(j) \setminus j \neq \emptyset$. If $l(i) \leq s$, we obtain $P_i = \{i\}$ and $P_j = \{j\}$. Otherwise, if $l(i) = s+1$, we obtain $P_i = \widehat{S}_D(i) \cup i$ and $P_j = \widehat{S}_D(j) \cup j$, and if $l(i) < s+1$, we obtain $P_i \nsubseteq \widehat{S}_D(i)$ and $P_j \nsubseteq \widehat{S}_D(j)$. In any case, P5 holds.

Uniqueness: Let P be a proper partition of N that satisfies P1—P5. First, let $i \in N$ be such that $P_i \neq \{i\}$. By P1, $P_i = P_j$ for every $j \in S_D(i)$. Thus, also $P_{-j} \neq \emptyset$, and applying P1 again yields $P_k = P_j$ for every $k \in S_D(j)$. By repeating the same procedure until there is no more agent with successors, we obtain that $P_l = P_i$ for every $l \in \widehat{S}_D(i)$. Hence the next implication holds for every $i \in N$:

$$[P_i \neq \{i\}] \Longrightarrow \left[\widehat{S}_D(i) \subseteq P_i\right]. \tag{A.28}$$

Next, for given i and $j \in \widehat{P}_D(i)$, suppose that $P_j \neq \{j\}$. Then by Eq. (A.28) it holds that $i \in P_j$, and thus $P_i \neq \{i\}$. From this, we obtain the implication below for every $i \in N$

$$[P_i = \{i\}] \Longrightarrow \left[\forall j \in \widehat{P}_D(i), P_j = \{j\}\right]. \tag{A.29}$$

We can assume that there is at least one player who is a singleton in P. Conversely, suppose that for every $l \in N$, $P_l \neq \{l\}$. In particular, $P_{i_0} \neq \{i_0\}$. Since $\widehat{S}_D(i_0) = N \setminus \{i_0\}$, from Eq. (A.28) it follows that $P = \{N\}$, and thus P is not a proper partition of N.

Define the set $\mathcal{K}^D(P) \subseteq N$ by

$$\mathcal{K}^D(P) = \{i \in N : P_i = \{i\}, S_D(i) \neq \emptyset, \text{ and } \exists j \in S_D(i) \text{ such that } P_j \neq \{j\}\}. \tag{A.30}$$

We show that either $\mathcal{K}^D(P) \neq \emptyset$ or

$$P = \{\{i\} : i \in N\}. \tag{A.31}$$

In the latter case, $P = B^D_{l(D),2}$ and thus $P \in B^D$, which concludes the proof.

If $\mathcal{K}^D(P) = \emptyset$, suppose there is a player i with $P_i = \{i\}$ and $S_D(i) \neq \emptyset$. Then P3 implies that $P_j = \{j\}$ for every $j \in N$, where $l(j) = l(i)$. By Eq. (A.29), it then follows that $P_k = \{k\}$ for every $k \in N$ where $l(k) < l(i)$. Further, for every $j \in N$ such that $l(j) = l(i)$ and $S_D(j) \neq \emptyset$, we have that $P_k = \{k\}$ for every $k \in S_D(j)$, since otherwise $j \in \mathcal{K}^D(P)$, which contradicts that $\mathcal{K}^D(P) = \emptyset$. Hence, $P_k = \{k\}$ for every $k \in N$ with $l(k) = l(i) + 1$. By repeating the latter step iteratively, it follows that Eq. (A.31) holds. It remains to consider the case where $S_D(i) = \emptyset$ for all $i \in N$, with $P_i = \{i\}$. Since there is at least one player i with $P_i = \{i\}$, from Eq. (A.29) it follows that $P_{i_0} = \{i_0\}$. As $|N| \geq 2$ and $\mathcal{K}^D(P) = \emptyset$, we obtain again following an iterative argument that Eq. (A.31) holds, hence completing the proof.

In the remaining, we consider the case where $\mathcal{K}^D(P)$ is nonempty. We first prove that

$$l(i) = l(j) \text{ for } i, j \in \mathcal{K}^D(P), \tag{A.32}$$

i.e., all players of $\mathcal{K}^D(P)$ are at the same distance from the root. Suppose this is not the case, and let $i, j \in \mathcal{K}^D(P)$ be such that $l(i) < l(j)$. Let $k \in S_D(i)$ be a successor of i such that $P_k \neq \{k\}$. The existence of such a player is guaranteed, as $i \in \mathcal{K}^D(P)$. When $l(i) + 1 = l(j)$, we obtain $l(k) = l(j)$. Since $j \in \mathcal{K}^D(P)$, and so $S_D(j) \neq \emptyset$ and $P_j = \{j\}$, P3 implies that $P_k = \{k\}$, which is a contradiction. When $l(i) + 1 < l(j)$, let $h \in \widehat{P}_D(j)$ be such that $l(h) = l(i) + 1$. Since $j \in \mathcal{K}^T_D$, $P_h = \{h\}$ by Eq. (A.29). By P3, also $P_k = \{k\}$, again contradicting the hypothesis above. Therefore, Eq. (A.32) holds.

Next, we consider the set of all players who are subordinates of some player in $\mathcal{K}^D(P)$, i.e.,

$$S(\mathcal{K}^D(P)) = \left\{ i \in N : \exists j \in \mathcal{K}^D(P) \text{ such that } i \in \widehat{S}_D(j) \right\}.$$

We deal with the players in this set and the players in its complement separately.

First, we consider the set $N \setminus S(\mathcal{K}^D(P))$. Since $\mathcal{K}^D(P) \neq \emptyset$, it follows by P3 that $P_i = \{i\}$ for every $i \in N \setminus \mathcal{K}^D(P)$ with $l(i) = s$, and consequently, $P_i = \{i\}$ for every $i \in N$ with $l(i) = s$. Further, by Eq. (A.29), we obtain that $P_i = \{i\}$ for every $i \in N$ with $l(i) < s$. Next, consider a player $i \in N \setminus \mathcal{K}^D(P)$ such that $l(i) = s$ and $S_D(i) \neq \emptyset$ and take $j \in S_D(i)$. Since, $i \notin \mathcal{K}^D(P)$, it holds that $P_j = \{j\}$. By repeating the latter argument for every $k \in S_D(j)$ and so forth, it follows that $P_k = \{k\}$ for all $k \in \widehat{S}_D(i)$. Taking everything together, we have shown that for every $j \in N \setminus S(\mathcal{K}^D(P))$,

$$P_j = \{j\}. \tag{A.33}$$

Second, we consider the set $S(\mathcal{K}^D(P))$. Take some $i \in \mathcal{K}^D(P)$ and suppose that for some $j \in S_D(i)$, it holds that $P_j \backslash \left(\widehat{S}_D(j) \cup j \right) \neq \emptyset$, and thus $P_j \neq \{j\}$. By Eq. (A.28), it holds that $\widehat{S}_D(j) \subseteq P_j$. Then, since $P_i \neq \{i\}$, by P4, we have

$$P_j \subseteq \widehat{S}_D(i).$$

On the other hand, when $A(j) = \{j\}$, it holds trivially that

$$\bigcup_{k \in S_D(i)} \left(\widehat{S}_D(k) \cup k \right) = \widehat{S}_D(i) \subseteq P_j,$$

and thus $P_j = \widehat{S}_D(i)$. Also, if $A(j) \setminus j \neq \emptyset$, then, by P2, $k \in P_j$ for all $k \in A(j) \setminus j$ and Eq. (A.28) implies that $\widehat{S}_D(l) \subseteq P_j$ for all $k \in A(j) \setminus j$. Therefore,

$$\bigcup_{k \in S_D(i)} \left(\widehat{S}_D(k) \cup k \right) = \widehat{S}_D(i) \subseteq P_j$$

and then, $P_j = \widehat{S}_D(i)$. Hence, when there exists $j \in S_D(i)$ such that $P_j \setminus \left(\widehat{S}_D(j) \cup j\right) \neq \emptyset$, it holds that P_j includes all subordinates of i, and thus for every $k \in S_D(i)$,

$$P_k = \widehat{S}_D(i). \tag{A.34}$$

Next, suppose that for some $i \in \mathcal{K}^D(P)$, it holds that $P_j \subseteq \widehat{S}_D(j)$ for every $j \in S_D(i)$. If $P_j \neq \{j\}$, then Eq. (A.28) implies that $\widehat{S}_D(j) \subseteq P_j$ and thus $P_j = \widehat{S}_D(j)$. When $P_j = \{j\}$ and $S_D(j) \neq \emptyset$, we obtain by P3 that $P_k = \{k\}$ for every $k \in S_D(i)$, which contradicts that $i \in \mathcal{K}^D(P)$. Hence, when $P_j = \{j\}$, $S_D(j) = \emptyset$. Therefore, $P_j = \widehat{S}_D(j) \cup j$. In both cases, we have that $P_j = \widehat{S}_D(j) \cup j$. It follows that for every $i \in \mathcal{K}^D(P)$, either Eq. (A.34) holds, or for all $j \in S_D(i)$,

$$P_j = \widehat{S}_D(j) \cup j. \tag{A.35}$$

Finally, let $i_1, i_2 \in \mathcal{K}^D(P)$ be two different players and suppose that Eq. (A.35) holds when we take $i = i_1$, but it does not hold when we take $i = i_2$, that Eq. (A.34) holds when we take $i = i_2$, but that it does not hold when we take $i = i_1$. It follows, in particular, that $A(i_1) \setminus \{i_1\} \neq \emptyset$ and $A(i_2) \setminus \{i_2\} \neq \emptyset$. However, this leads to a contradiction with P5.

As a consequence of all the above steps, we have proved that either

$$P = \bigcup_{i \in N: l(i)=s} \left[\{\{i\}\} \cup \left(\bigcup_{j \in \widehat{P}_D(i)} \{\{j\}\} \right) \cup \{\widehat{S}_D(i)\} \right], \tag{A.36}$$

or

$$P = \bigcup_{i \in N: l(i)=s} \left[\{\{i\}\} \cup \left(\bigcup_{j \in \widehat{P}_D(i)} \{\{j\}\} \right) \cup \left(\bigcup_{j \in S_D(i)} \{\widehat{S}_D(j) \cup \{j\}\} \right) \right]. \tag{A.37}$$

Note that Eqs. (A.36) and (A.37) are equivalent to $P = B^D_{1,s}$ and $P = B^D_{2,s}$ respectively, so that $P \in B^D$. □

Independence of the properties used in Theorem 3.2

Consider the following triples composed of a finite set N, a rooted tree $D \in \mathcal{G}^N$, and a partition $T \in \mathcal{P}^N$:

1. $N = \{1, 2, 3\}$, $D = \{(1, 2), (2, 3)\}$ and $T = \{\{1, 2\}, \{3\}\}$.

   In this case, D and T satisfy P2, P3, P4, and P5 but fail to satisfy P1.

2. $N = \{1, 2, 3, 4, 5\}$, $D = \{(1, 2), (1, 3), (1, 4), (1, 5)\}$ and $T = \{\{1\}, \{2, 3\}, \{4, 5\}\}$.

   In this case, D and T satisfy P1, P3, P4, and P5 but fail to satisfy P2.

3. Case 3: $N = \{1,2,3,4,5\}$, $D = \{(1,2),(1,3),(2,4),(3,5)\}$ and $T = \{\{1\},\{2,4\},\{3\},\{5\}\}$.

   In this case, D and T satisfy P1, P2, P4, and P5 but fail to satisfy P3.

4. $N = \{1,2,3,4,5\}$, $D = \{(1,2),(1,3),(2,4),(3,5)\}$ and $T = \{\{1\},\{2\},\{3\},\{4,5\}\}$.

   In this case, D and T satisfy P1, P2, P3, and P5 but fail to satisfy P4.

5. $N = \{1,2,3,4,5,6,7\}$, $D = \{(1,2),(2,4),(2,5),(1,3),(3,6),(3,7)\}$ and $T = \{\{1\},\{2\},\{3\},\{4,5\},\{6\},\{7\}\}$.

   In this case, D and T satisfy P1, P2, P3, and P4 but fail to satisfy P5.

Appendix B Proofs of Section 4

Proof of Lemma 4.2

Let $(N,v,D) \in \mathcal{GD}$ and $i \in N$, and consider (N_i, v_i, D_{N_i}) as defined in Eqs. (4.5) and (4.6). If $N_i = N$, then $D_{N_i} = D$ and $T_i^D = T$ for all T, so that $v_i = v$. Hence $f_i(N_i, v_i, D_{N_i}) = f_i(N,v,D)$. Hence, consider the case in which $N_i \neq N$. Then, for some integer $m > 0$, there is a sequence of games $(N^{(k)}, v^{(k)}, D^{(k)})$, $k = 0,\dots,m$, such that (i) $(N^{(0)}, v^{(0)}, D^{(0)}) = (N,v,D)$, (ii) for $k = 1,\dots m$, $N^{(k)} = N^{(k-1)} \setminus \widehat{S}_D(j)$ for some $j \in N^{(k-1)}$, $v^{(k)} = \left(v^{(k-1)}\right)^k$ and $D^{(k)} = \left(D^{(k-1)}\right)_{N^{(k)}}$, and (iii) $N^{(m)} = N_i$ and so $D^{(m)} = D_{N_i}$. The existence of such a sequence is guaranteed by Eq. (4.5). Then, by IMT, for all $k \in \{1,\dots,m\}$,

$$f_i(N^{(k)}, v^{(k)}, D^{(k)}) = f_i(N^{(k-1)}, v^{(k-1)}, D^{(k-1)}) \text{ for all } i \in N^{(m)} = N_i.$$

Moreover, for each $T \subseteq N_i$,

$$v^{(m)}(S) = v^{(0)}\left(\bigcup_{j \in T \setminus \left(\widehat{S}_D(i) \cup \widehat{P}_D(i) \cup i\right)} \widehat{S}_D(j)\right).$$

Since $v^{(0)} = v$, it follows that $v^{(m)} = v_i$, which completes the proof. □

Proof of Lemma 4.4

Let $(N, cu_T, D) \in \mathcal{GD}$, with $c > 0$ and $T \subseteq N$, and let $i \in N$. For each $R \subseteq N_i$, we have

$$(u_T)_i(R) = u_T(R_i^D) = \begin{cases} 1 & \text{if } T \subseteq R_i^D, \\ 0 & \text{otherwise,} \end{cases}$$

where R_i^D is defined according to Eq. (4.6). Note that

$$T \subseteq R_i^D \iff T \subseteq R \cup \left(\bigcup_{j \in R \setminus \left(\widehat{S}_D(i) \cup \widehat{P}_D(i) \cup i\right)} \widehat{S}_D(j) \right)$$

$$\iff [T \cap N_i \subseteq R] \text{ and } \left[T \setminus N_i \subseteq \bigcup_{j \in R \setminus \left(\widehat{S}_D(i) \cup \widehat{P}_D(i) \cup i\right)} \widehat{S}_D(j) \right].$$

We claim that

$$\left[T \setminus N_i \subseteq \bigcup_{j \in R \setminus \left(\widehat{S}_D(i) \cup \widehat{P}_D(i) \cup i\right)} \widehat{S}_D(j) \right] \iff \left[\left\{ j \in N_i \setminus \left(\widehat{S}_D(i) \cup \widehat{P}_D(i) \cup i \right) : \widehat{S}_D(j) \cap T \neq \emptyset \right\} \subseteq R \right]. \tag{B.38}$$

Then, $(cu_T)_i = cu_{T(i)}$ with

$$T(i) = (T \cap N_i) \cup \left\{ j \in N_i \setminus \left(\widehat{S}_D(i) \cup \widehat{P}_D(i) \cup i \right) : \widehat{S}_D(j) \cap T \neq \emptyset \right\}.$$

Therefore, it only remains to prove the claim in Eq. (B.38). On the one hand, let $j \in N_i \setminus \left(\widehat{S}_D(i) \cup \widehat{P}_D(i) \cup i \right)$ be such that $\widehat{S}_D(j) \cap T \neq \emptyset$ and $j \notin R$. Take $k \in \widehat{S}_D(j) \cap T$. Then, by construction of N_i—note that $j \in N_i \setminus \left(\widehat{S}_D(i) \cup \widehat{P}_D(i) \cup i \right)$—, we have $k \in T$ and $k \notin N_i$, but

$$k \notin \bigcup_{h \in R \setminus \left(\widehat{S}_D(i) \cup \widehat{P}_D(i) \cup i\right)} \widehat{S}_D(h). \tag{B.39}$$

On the other hand, let $k \in T \setminus N_i$ such that Eq. (B.39) holds. Then, let $j \in N_i \setminus \left(\widehat{S}_D(i) \cup \widehat{P}_D(i) \cup i \right)$ be such that $k \in \widehat{S}_D(j)$. The existence of j is guaranteed by construction of N_i. Hence,

$$j \in \left\{ h \in N_i \setminus \left(\widehat{S}_D(i) \cup \widehat{P}_D(i) \cup i \right) : \widehat{S}_D(h) \cap T \neq \emptyset \right\} \setminus R.$$

Appendix C Logical independence of the axioms

Independence of the axioms of Theorem 4.5

1. The solution $f_i(N, v, D) = 0$ for all $(N, v, D) \in \mathcal{GD}$ and $i \in N$ satisfies ADD, NPP, IMT, NSP, SLB, and SUB. It does not satisfy EFF.

2. The solution $f(N, v, D) = v(N)\varphi^{SL}(N, u_{E(N,v)}, D)$ for all $(N, v, D) \in \mathcal{GD}$, with $E(N, v)$ being the set of all non-null players in (N, v), satisfies EFF, NPP, IMT, NSP, SLB, and SUB. It does not satisfy ADD.

3. Let $\omega \in \mathbb{R}^{\Omega}_{++}$ be an exogenous vector. For a given $(N, v, D) \in \mathcal{GD}$, let

$$R = \{i \in N : i \text{ is not a null player and } S_D(A(i)) = \emptyset\}.$$

Then, consider the solution f defined for each $(N, v, D) \in \mathcal{GD}$ and each $i \in N$ as follows. First, if $v = u_T$ for some $T \subseteq N$,

$$f_i(N, v, D) = \begin{cases} \frac{\omega_i}{\sum_{j \in A(i) \cap R} \omega_j} \sum_{j \in A(i)} \varphi_i^{SL}(N, v, D) & \text{if } N = \Omega \text{ and } i \in R, \\ \varphi_i^{SL}(N, v, D) & \text{otherwise.} \end{cases}$$

Second, for an arbitrary $(N, v) \in \mathcal{G}$, $f(N, v, D) = \sum_{\emptyset \neq T \subseteq N} f(N, \Delta_v(T) u_T, D)$. Then f satisfies ADD, EFF, NPP, IMT, SLB, and SUB. Moreover, f does not satisfy NSP.

4. The solution $f(N, v, D) = v(N)\varphi^{SL}(N, u_N, D)$ for all $(N, v, D) \in \mathcal{GD}$ satisfies ADD, EFF, IMT, NSP, SLB, and SUB. It does not satisfy NPP.

5. The solution $\widetilde{\varphi}^{SL}$ introduced in Definition 5.2 satisfies ADD, EFF, NPP, NSP, SLB, and SUB. It does not satisfy IMT.

6. Recall that i_0 denotes the root of the tree. The solution given by $f_{i_0}(N, v, D) = v(N) - v(N \setminus \{i_0\})$, and $f_i(N, v, D) = \varphi^{SL}(N, v \mid_{N \setminus \{i_0\}}, D)$ if $i \in N \setminus \{i_0\}$, satisfies EFF, ADD, NPP, IMT, NSP, and SLB. It does not satisfy SUB.

7. The solution $\overline{\varphi}^{SL}$ defined Definition 5.1 satisfies EFF, ADD, NPP, IMT, NSP, and SUB. It does not satisfy SLB.

Independence of the axioms of Theorem 5.2

1. The solution $f_i(N, v, D) = 0$ for all $(N, v, D) \in \mathcal{GD}$ and $i \in N$ satisfies ADD, NPP, IMT, NSP, TLB, and TUB. It does not satisfy EFF.

2. The solution $f(N, v, D) = v(N)\overline{\varphi}^{SL}(N, u_{E(N,v)}, D)$ for all $(N, v, D) \in \mathcal{GD}$, with $E(N, v)$ being the set of all non-null players in (N, v), satisfies EFF, NPP, IMT, NSP, TLB, and TUB. It does not satisfy ADD.

3. The solution $f(N, v, D) = v(N)\overline{\varphi}^{SL}(N, u_N, D)$ for all $(N, v, D) \in \mathcal{GD}$ satisfies ADD, EFF, IMT, NSP, TLB, and TUB. It does not satisfy NPP.

4. Let $\omega \in \mathbb{R}^{\Omega}_{++}$ be an exogenous vector. For a given $(N, v, D) \in \mathcal{GD}$, let

$$R = \{i \in N : i \text{ is not a null player and } S_D(A(i)) = \emptyset\}.$$

Then, consider the solution f defined for each $(N, v, D) \in \mathcal{GD}$ and each $i \in N$ as follows. First, if $v = u_T$ for some $T \subseteq N$,

$$f_i(N, v, D) = \begin{cases} \frac{\omega_i}{\sum_{j \in A(i) \cap R} \omega_j} \sum_{j \in A(i)} \overline{\varphi}^{SL}_i(N, v, D) & \text{if } N = \Omega \text{ and } i \in R, \\ \overline{\varphi}^{SL}_i(N, v, D) & \text{otherwise.} \end{cases}$$

Second, for an arbitrary $(N, v) \in \mathcal{G}$, $f(N, v, D) = \sum_{\emptyset \neq T \subseteq N} f(N, \Delta_v(T) u_T, D)$. Then f satisfies ADD, EFF, NPP, IMT, TLB, and TUB. Moreover, f does not satisfy NSP.

5. Let $N^* = \{1, 2, 3, 4, 5\}$ and $D^* = \{(1, 2), (1, 3), (2, 4), (3, 5)\}$. Let also $\alpha \in [0, 1] \setminus \{0.5\}$. Then, consider the solution f defined for each $(N, u_T, D) \in \mathcal{GD}$, with $T \subseteq N$, and $i \in N$ as follows:

$$f_i(N, u_{\{3,4\}}, D) = \begin{cases} \alpha & \text{if } (N, v, D) = (N^*, u_{\{3,4\}}, D^*) \text{ and } i = 3, \\ 1 - \alpha & \text{if } (N, v, D) = (N^*, u_{\{3,4\}}, D^*) \text{ and } i = 4, \\ \overline{\varphi}^{SL}_i(N, u_T, D) & \text{otherwise.} \end{cases}$$

The solution f on $\mathcal{GD}$ is then simply obtained as the additive extension on the whole class of games with hierarchical structure, and it satisfies ADD, EFF, NPP, NSP, TLB, and TUB. Moreover, f does not satisfy IMT.

6. The solution $f(N, v, D) = Sh(N, v)$ satisfies ADD, EFF, NPP, IMT, NSP, and TUB. It does not satisfy TLB.

7. The solution φ^{SL} satisfies ADD, EFF, NPP, IMT, NSP, and TLB. It does not satisfy TUB.

Independence of the axioms of Theorem 5.4

1. The solution $f_i(N, v, D) = 0$ for all $(N, v, D) \in \mathcal{GD}$ and $i \in N$ satisfies ADD, NPP, SNSP, SSLB, SSUB, and NOP. It does not satisfy EFF.

2. The solution $f(N, v, D) = v(N)\widetilde{\varphi}^{SL}(N, u_{E(N,v)}, D)$ for all $(N, v, D) \in \mathcal{GD}$, with $E(N, v)$ being the set of all non-null players in (N, v), satisfies EFF, NPP, SNSP, SSLB, SSUB, and NOP. It does not satisfy ADD.

3. The solution $f(N,v,D) = v(N)\widetilde{\varphi}^{SL}(N,u_N,D)$ for all $(N,v,D) \in \mathcal{GD}$ satisfies EFF, ADD, SNSP, SSLB, SSUB, and NOP. It does not satisfy NPP.

4. Let $N^* = \{1,2\}$ and $D^* = \{(1,2)\}$, and consider the solution f defined for each $(N,u_T,D) \in \mathcal{GD}$, with $T \subseteq N$, and $i \in N$ as follows:

$$f_i(N,u_T,D) = \begin{cases} \frac{i}{3} & \text{if } (N,v,D) = (N^*,u_{\{1,2\}},D^*) \text{ and } i \in \{1,2\}, \\ \widetilde{\varphi}_i^{SL}(N,u_T,D) & \text{otherwise.} \end{cases}$$

The solution f on $\mathcal{GD}$ is then simply obtained as the additive extension on the entire class of games with hierarchical structure and it satisfies EFF, ADD, NPP, SNSP, SSUB, and NOP. It does not satisfy SSLB.

5. Let $N^* = \{1,2\}$ and $D^* = \{(1,2)\}$, and consider the solution f defined for each $(N,u_T,D) \in \mathcal{GD}$, with $T \subseteq N$, and $i \in N$ as follows:

$$f_i(N,u_t,D) = \begin{cases} \frac{3-i}{3} & \text{if } (N,v,D) = (N^*,u_{\{1,2\}},D^*) \text{ and } i \in \{1,2\}, \\ \widetilde{\varphi}_i^{SL}(N,u_T,D) & \text{otherwise.} \end{cases}$$

The solution f on $\mathcal{GD}$ is then simply obtained as the additive extension on the whole class of games with hierarchical structure and it satisfies EFF, ADD, NPP, SNSP, SSLB, and NOP. It does not satisfy SSUB.

6. Let $N^* = \{1,2,3\}$ and $D^* = \{(1,2),(1,3)\}$, and consider the solution f defined for each $(N,u_T,D) \in \mathcal{GD}$, with $T \subseteq N$, and $i \in N$ as follows:

$$f_i(N,v,D) = \begin{cases} \frac{i}{5} & \text{if } (N,v,D) = (N^*,u_{\{2,3\}},D^*) \text{ and } i \in \{2,3\}, \\ \widetilde{\varphi}_i^{SL}(N,u_T,D) & \text{otherwise.} \end{cases}$$

Then, solution f on $\mathcal{GD}$ is then simply obtained as the additive extension on the whole class of games with hierarchical structure and it satisfies EFF, ADD, NPP, SSUB, SSLB, and NOP. It does not satisfy SNSP.

7. Let $N^* = \{1,2,3,4,5\}$ and $D^* = \{(1,2),(1,3),(2,4),(3,5)\}$, and consider the solution f defined for each $(N,u_T,D) \in \mathcal{GD}$, with $T \subseteq N$, and $i \in N$ as follows:

$$f_i(N,v,D) = \begin{cases} \frac{i}{9} & \text{if } (N,v,D) = (N^*,u_{\{4,5\}},D^*) \text{ and } i \in \{4,5\}, \\ \widetilde{\varphi}_i^{SL}(N,u_T,D) & \text{otherwise.} \end{cases}$$

Then, solution f on $\mathcal{GD}$ is then simply obtained as the additive extension on the whole class of games with hierarchical structure and it satisfies EFF, ADD, NPP, SNSP, SSUB, and SSLB. It does not satisfy NOP.