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# A Note on "Continuous Invertibility and Stable QML Estimation of the EGARCH(1,1) Model"*

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**Abstract**

We revisit Wintenberger (2013) on the continuous invertibility of the EGARCH(1,1) model. We note that the definition of continuous invertibility adopted in Wintenberger (2013) may not always be sufficient to deliver strong consistency of the QMLE. We also take the opportunity to provide other small clarifications and additions.

*Some Key words:* invertibility, quasi-maximum likelihood estimator, volatility models

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# 1 Introduction

Wintenberger (2013) introduces the important notion of *continuous invertibility* for GARCH-type models. Wintenberger (2013) builds on the stochastic recurrence equations approach of Straumann (2005) and Straumann and Mikosch (2006), and originally develops the asymptotic theory for the Quasi Maximum Likelihood Estimator (QMLE) of the EGARCH(1,1) model of Nelson (1991). In this note on Wintenberger (2013), we discuss the adopted definition of continuous invertibility and argue that it does not necessarily ensure the desired consistency result for the QMLE. In particular, we show that the desired asymptotic properties of the QMLE can be obtained through a small adjustment of the definition of continuous invertibility. Finally, we clarify some minor issues and discuss smaller remarks and additions to Wintenberger (2013). Throughout, we adopt the same notation as in Wintenberger (2013).

# 2 On the definition of continuous invertibility

Theorem 1 of Wintenberger (2013) is proved through the intermediate results (a)-(c). The result (a) is $\|\hat{L}_n - L_n\|_\Theta \xrightarrow{\text{a.s.}} 0$ as $n \to \infty$, where $\hat{L}_n$ is the quasi-likelihood function and $n$ denotes the sample size. To prove that (a) holds it is claimed that, as the functions $1/\ell(\cdot)$ and $\log(\ell(\cdot))$ are Lipschitz continuous on $\mathcal{K}$ by the lower bound condition (**LB**), there exists a constant $C > 0$ such that

$$|\hat{L}_n(\theta) - L_n(\theta)| \leq C\frac{1}{n}\sum_{t=1}^{n} |\hat{g}_t(\theta) - g_t(\theta)|, \tag{1}$$

for any $\theta \in \Theta$, where $\hat{g}_t$ denotes the functional time varying parameter. The continuous invertibility $\|\hat{g}_t - g_t\|_\Theta \xrightarrow{\text{a.s.}} 0$ is then used to obtain the almost sure uniform convergence to zero of the right hand side of (1) and thus the desired convergence result $\|\hat{L}_n - L_n\|_\Theta \xrightarrow{\text{a.s.}} 0$. However, in general, the inequality in (1) does not hold as the quasi-likelihood function $\hat{L}_n$ depends also on the observations $\{X_1, \ldots, X_n\}$ and therefore the Lipschitz coefficient $C$ cannot be constant. Indeed, the condition (**LB**) implies that there exists a constant $C > 0$

such that $|\ell(\hat{g}_t(\theta))^{-1} - \ell(g_t(\theta))^{-1}| \leq C|\hat{g}_t(\theta) - g_t(\theta)|$ and $|\log \ell(\hat{g}_t(\theta)) - \log \ell(g_t(\theta))| \leq C|\hat{g}_t(\theta) - g_t(\theta)|$ for any $\theta \in \Theta$. Therefore, the resulting inequality should be

$$\|\hat{L}_n - L_n\|_\Theta \leq \frac{1}{n}\sum_{t=1}^{n} C_t \|\hat{g}_t - g_t\|_\Theta, \tag{2}$$

where $C_t = C(1 + X_t^2)$. In this situation, the notion of continuous invertibility adopted in Definition 2 of Wintenberger (2013) is not enough to ensure the almost sure convergence to zero of the right hand side of (2). Instead of the almost sure (a.s.) convergence of $\|\hat{g}_t - g_t\|_\Theta$ on a compact $\Theta$, we need the stronger exponential almost sure (e.a.s.) convergence of $\|\hat{g}_t - g_t\|_\Theta$ on the compact $\Theta$; see Lemma 2.1 of Straumann and Mikosch (2006) and the discussion in page 339 of Davidson (1994). An appropriate definition of continuous invertibility could thus be as follows.

**Definition 1.** *The model is continuously invertible in a compact set* $\Theta$ *if* $\|\hat{g}_t - g_t\|_\Theta \xrightarrow{e.a.s.} 0$ *as* $t \to \infty$.

Note that e.a.s. convergence is stronger than a.s. convergence as it imposes an exponential rate of convergence. Considering this definition of continuous invertibility, the a.s. convergence to zero of the right hand side of (2) can be obtained by an application of Lemma 2.1 of Straumann and Mikosch (2006). To apply this lemma, besides $\|\hat{g}_t - g_t\|_\Theta \xrightarrow{\text{e.a.s.}} 0$, we also need the integrability condition $E\log^+(X_t^2 + 1) < \infty$. This condition is satisfied as $E\log^+(X_t^2 + 1) \leq E\log^+(X_t^2) + 1$ and the conditions $EZ_t^2 = 1$ and $E\log^+\sigma_t^2 < \infty$ in (**SE**) implies $E\log^+ X_t^2 < \infty$.

## 3 On the continuous invertibility of the EGARCH(1,1) model

Theorem 2 of Wintenberger (2013) obtains the continuous invertibility of the model under a set of sufficient conditions labeled (**ST**) and (**CI**). We highlight here that the continuous invertibility condition used in Theorem 2 differs from the pointwise condition for the continuous invertibility considered in Section 4. The former is more restrictive than the latter.

In any case, we highlight that the conclusions drawn in Wintenberger (2013) concerning the EGARCH(1,1) model are still correct and valid.

First, we note the distinction between the functional Stochastic Recurrent Equation (SRE) $g_{t+1} = \Phi_t(g_t)$ and the SRE $g_{t+1}(\theta) = \phi_t(g_t, \theta)$ for a given $\theta \in \Theta$. The former lies in the space of continuous functions $\mathcal{C}_\Theta$ with values in $\mathcal{K}_\Theta \subseteq \mathbb{R}$, whereas the latter lies in the set $\mathcal{K}_\theta \subseteq \mathbb{R}$. In general, the set $\mathcal{K}_\Theta$ is not the same as the set $\mathcal{K}_\theta$. For the EGARCH(1,1) model, we have that $\mathcal{K}_\Theta = [\inf_{\theta \in \Theta} \alpha/(1-\beta), +\infty)$ and $\mathcal{K}_\theta = [\alpha/(1-\beta), +\infty)$, see Straumann (2005) and Straumann and Mikosch (2006). For simplicity, Wintenberger (2013) does not make an explicit distinction between these two sets, and the set denoted by $\mathcal{K}$ is indistinctly used to refer to both $\mathcal{K}_\Theta$ and $\mathcal{K}_\theta$.

As stated in Section 3.3, the set $\mathcal{K}$ used to define the condition (**CI**) is the set $\mathcal{K}_\Theta$. Note that this has to be true otherwise the proof of Theorem 2 does not hold. Therefore, the restriction to ensure that (**CI**) is satisfied for the EGARCH(1,1) model should be $E \log \Lambda_0^*(\theta) < 0$, where

$$\Lambda_0^*(\theta) \quad := \quad \max\left\{\beta, 2^{-1}(\gamma X_0 + \delta|X_0|)\exp\left(-2^{-1}\inf_{\theta^* \in \Theta} \alpha^*/(1-\beta^*)\right) - \beta\right\}.$$

In Section 4 of Wintenberger (2013), it is stated that the EGARCH(1,1) model satisfies the condition (**CI**) as long as $E \log \Lambda_0(\theta) < 0$ for any $\theta \in \Theta$, where

$$\Lambda_0(\theta) \quad := \quad \max\left\{\beta, 2^{-1}(\gamma X_0 + \delta|X_0|)\exp\left(-2^{-1}\alpha/(1-\beta)\right) - \beta\right\}.$$

However, in certain cases, this may fail to be true since $E \log \Lambda_0^*(\theta) < 0$ is more restrictive than $E \log \Lambda_0(\theta) < 0$. Note also that the uniform condition derived in Straumann and Mikosch (2006) to apply Theorem 3.1 of Bougerol (1993) in the space $\mathcal{C}_\Theta$ is $E \sup_{\theta \in \Theta} \log \Lambda_0^*(\theta) < 0$ and not $E \sup_{\theta \in \Theta} \log \Lambda_0(\theta) < 0$.

Let us show how the conclusions drawn in Wintenberger (2013) concerning the EGARCH(1,1) model are still correct and valid. In particular, the condition $E \log \Lambda_0(\theta) < 0$ is sufficient for the continuous invertibility of the EGARCH(1,1) model and therefore Theorem 4 and the subsequent results remain valid. This can be easily established by using the same continuity argument as in the proof of Theorem 2. First, we define the following function

$$\Lambda_0(\theta,\theta^*) := \max\left\{\beta, 2^{-1}(\gamma X_0 + \delta|X_0|)\exp\left(-2^{-1}\alpha^*/(1-\beta^*)\right) - \beta\right\}.$$

Clearly, $E\sup_{(\theta,\theta^*)\in\Theta\times\Theta}\log\Lambda_0(\theta,\theta^*)$ is equivalent to $E\sup_{\theta\in\Theta}\log\Lambda_0^*(\theta)$ for any compact set $\Theta$. This means that $E\sup_{(\theta,\theta^*)\in\Theta\times\Theta}\log\Lambda_0(\theta,\theta^*) < 0$ is sufficient to apply Theorem 3.1 of Bougerol (1993) in the space of continuous functions $\mathcal{C}_\Theta$ with values in $\mathcal{I}_\Theta$. By continuity of $\Lambda_0(\theta,\theta^*)$ and $E\sup_{(\theta,\theta^*)\in\Theta\times\Theta}\log^+\Lambda_0(\theta,\theta^*) < \infty$, we obtain that any $\theta_k \in \Theta$ such that $E\log\Lambda_0(\theta_k) < 0$ has a neighborhood $V(\theta_k)$ such that $E\sup_{(\theta,\theta^*)\in V(\theta_k)\times V(\theta_k)}\log\Lambda_0(\theta,\theta^*) < 0$. Therefore, Bougerol's theorem applies in the space of continuous functions $\mathcal{C}_{V(\theta_k)}$ with values in $\mathcal{I}_{V(\theta_k)}$ and we obtain $\|\hat{g}_t - g_t\|_{V(\theta_k)} \xrightarrow{\text{e.a.s.}} 0$. Finally, by compactness of $\Theta$, the result $\|\hat{g}_t - g_t\|_\Theta \xrightarrow{\text{e.a.s.}} 0$ follows as in the proof of Theorem 2.

## 4 On the consistency of the stable QML estimator

In this section, we clarify two issues related to the consistency of the stable QML estimator in Section 4 of Wintenberger (2013).

The first clarification is related to a lack of continuity in $\Theta$. The stable QMLE $\hat{\theta}_n^S$ is defined as the minimizer of the quasi-likelihood on the set $\Theta^S$, see Definition 3 of Wintenberger (2013). In the definition of $\Theta^S$, the set $\Theta$ is allowed to be any compact set. In the proof of Theorem 5 it is stated that $\log\Lambda_t$ is an element in the Banach space $\mathcal{C}_\Theta$ and the desired uniform convergence result $\|n^{-1}\sum_{t=1}^n \log\Lambda_t(\theta) - E\log\Lambda_0(\theta)\|_\Theta \xrightarrow{a.s.} 0$ follows by an application of the ergodic theorem as $E\|\log\Lambda_0\|_\Theta < \infty$. However, strictly speaking, this may not true as, for instance, the point $(\alpha,\beta,\gamma,\delta)^T = (0,0,0,0)^T$ can be in $\Theta$ and $\log\Lambda_t(\theta)$ is not continuous at this point. Therefore, the function $\log\Lambda_t$ is not an element of $\mathcal{C}_\Theta$. Note also that in this case we have $E\|\log\Lambda_0\|_\Theta = \infty$. Defining $\Theta$ as a compact set such that $\beta > 0$, $\gamma \geq 0$ and $\delta \geq 0$ seems enough to solve this small issue.

The second clarification concerns the limit behavior of the set $\Theta^S$. The empirical invertibility constraint ($\widehat{INV(\theta)}$) is defined as

$$(\widehat{INV(\theta)}) \ : \ \delta \geq |\gamma| \quad \text{and} \quad \sum_{t=1}^{n} \log \Lambda_t(\theta) \leq -\epsilon.$$

In the proof of Theorem 5, it is stated that $\Theta^S$ coincides asymptotically a.s. with a compact continuously invertible domain. However, this seems not to be true in general. Even in the limit, we can have $\theta \in \Theta^S$ and $E \log \Lambda_0(\theta) = 0$ with positive probability. We provide a counterexample below.

For any given $n \in \mathbb{N}$, a $\theta \in \Theta$ is contained in $\Theta^S$ if $\sum_{t=1}^{n} \log \Lambda_t(\theta) \leq -\epsilon$. This condition is the same as having $\sum_{t=1}^{n} \log \Lambda_t(\theta)/\sqrt{n} + \epsilon/\sqrt{n} \leq 0$. Now, assume that there is a $\theta^* \in \Theta$ such that $E \log \Lambda_0(\theta^*) = 0$ and $E|\log \Lambda_0(\theta^*)|^2 < \infty$. An application of a central limit theorem yields that $\sum_{t=1}^{n} \log \Lambda_t(\theta^*)/\sqrt{n} + \epsilon/\sqrt{n}$ is asymptotically distributed as a normal random variable with zero mean and finite variance. Therefore, we obtain that the limit probability of having $\theta^* \in \Theta^S$ is given by

$$\lim_{n\to\infty} P\left(\theta^* \in \Theta^S\right) = \lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \log \Lambda_t(\theta^*) + \frac{\epsilon}{\sqrt{n}} \leq 0\right) = \frac{1}{2}.$$

This limit probability is non-zero and therefore the probability that $\Theta^S$ is a continuous invertible domain does not go to 1 as $n \to \infty$. Anyway, any $\theta^* \in \Theta$ lying in $\Theta^S$ a.s. belongs to a domain of continuous invertibility. Thus, the procedure should always been associated with the check that the constraint $(\widehat{INV(\theta)})$ is not saturated.

## 5 Minor typos

Finally, we take the opportunity to correct three minor typos.

In the definition of the quasi-likelihood function, equation (9) of Wintenberger (2013), the quasi-likelihood function $\hat{L}_n$ should be defined as

$$2n\hat{L}_n(\theta) = \sum_{t=1}^{n} X_t^2/\ell(\hat{g}_t(\theta)) + \log(\ell(\hat{g}_t(\theta))).$$

In the definition of the stable QMLE, Definition 3 of Wintenberger (2013), the stable QML estimator $\hat{\theta}_n^S$ should be defined as

$$\hat{\theta}_n^S = \underset{\theta \in \Theta^S}{arg\,min} \sum_{t=1}^{n} 2^{-1} \left( X_t^2 \exp(-\hat{g}_t(\theta)) + \hat{g}_t(\theta) \right).$$

In the proof of Theorem 1, the equality

$$\lim_{n \to \infty} \inf_{\theta^* \in V(\theta)} L_n(\theta^*) \wedge K = E \left[ \inf_{\theta^* \in V(\theta)} l_0(\theta^*) \wedge K \right] \quad \text{a.s.}$$

does not hold in general. This because the infimum in the left hand side of the equality is not within the summation $L_n(\theta) = (2n)^{-1} \sum_{t=1}^{n} l_t(\theta)$. Therefore, we should have the following inequality

$$\lim_{n \to \infty} \inf_{\theta^* \in V(\theta)} L_n(\theta^*) \wedge K \geq E \left[ \inf_{\theta^* \in V(\theta)} l_0(\theta^*) \wedge K \right] \quad \text{a.s.,}$$

see the proof of Lemma 3.11 of Pfanzagl (1969) for more details.

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