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TAIL MUTUAL EXCLUSIVITY AND TAIL-VAR LOWER BOUNDS

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Abstract

In this paper, we extend the concept of mutual exclusivity proposed by Dhaene and Denuit (1999) to its tail counterpart and baptise this new dependency structure as tail mutual exclusivity. Probability levels are first specified for each component of the random vector. Under this dependency structure, at most one exceedance over the corresponding VaRs is possible, the other components being zero in such a case. No condition is imposed when all components stay below the VaRs. Several properties of this new negative dependence concept are derived. We show that this dependence structure gives rise to the smallest value of Tail-VaR of a sum of risks within a given Fréchet space, provided that the probability level of the Tail-VaR is close enough to one.

Keywords: Mutual exclusivity, stop-loss transform, tail convex order, risk measures.

1 Introduction and motivation

Numerous concepts of positive dependency have appeared in the literature to express the notion that "large" (or "small") values of the random variables tend to occur together. Contrarily to positive dependence, negative dependence has attracted relatively less interest so far, especially in the multivariate case. The bivariate countermonotonicity has been thoroughly studied; see, e.g., Cheung et al. (2014). Negative dependence properties express the notion that "large" values of one variable tend to occur together with "small" values of the others. In general, a negative dependence results in more predictable aggregate losses for the insurance company than mutual independence. The independence assumption is thus conservative in such a case. Moreover, assuming independence is mathematically convenient, and also obviates the need for elaborate models to be devised and statistics to be kept on mutual dependence of claims. If the individual risks are known to be positively associated (see, e.g., Denuit (2005, Section 7.2.3) for a precise definition) then they dominate their independent version in the supermodular order and their sum is larger than the sum of independent random variables with the same univariate marginals in the convex sense. Therefore, independence provides a lower bound on the Tail-VaR in this case. The same occurs if the risks are positively cumulative dependent, as defined in Denuit, Dhaene and Ribas (2001).

By generalizing the results in Hu and Wu (1999) in the case of two-point distributions, Dhaene and Denuit (1999) introduced and studied systematically an extreme case of negative dependence, called mutual exclusivity. Considering non-negative random variables with probability masses at the origin, mutual exclusivity corresponds to the case where at most one of them can be different from zero. This can be considered as a sort of dual notion of comonotonicity. Indeed, the knowledge that one risk assumes a positive value directly implies that all the other ones vanish. Dhaene and Denuit (1999) proved that mutual exclusivity is the safest dependence structure between risks with given marginals, in the sense that the corresponding sum is minimal in the stop-loss order. They showed that the dependence structure of the safest portfolio is described by the so-called Fréchet-Höffding lower bound, although some mathematical conditions are involved.

The total probability mass at the origin must indeed be high enough to ensure that mutual exclusivity can occur with given marginals. This condition considerably reduces the potential applications of mutual exclusivity. This is precisely why we relax it in the present paper, concentrating on the tail of the joint distribution. Specifically, we fix high enough probability levels and we consider the corresponding quantiles, or VaRs. Tail mutual exclusivity then restricts the behavior of the exceedances over these VaRs. More precisely, only one excess is allowed so that either all components of a tail mutually exclusive random vector are below these VaRs or only one of the component exceeds its VaR while all others vanish. This particular dependence structure can be seen as safe in the sense that it achieves the lower bound for Tail-VaRs derived in Cheung and Lo (2013b).

The remainder of this paper is organized as follows. Section 2 recalls some results about VaRs, TVaRs and mutual exclusivity. In Section 3, we introduce the new negative dependence structure proposed in this paper, called tail mutual exclusivity. Existence conditions are also obtained there. Section 4 derives the expression for the stop-loss transforms in case of tail mutual exclusivity. It is shown that this negative dependence structure corresponds to minimal sums in the tail convex order. In Section 5, we establish that lower bounds on TVaRs are attained under tail mutual exclusivity, provided the probability level is high enough.

2 Preliminaries

Throughout this paper, we consider non-negative random variables, also called risks, $X_1, \ldots, X_n$ with finite means and respective distribution functions $F_1, \ldots, F_n$, i.e. $F_i(t) = \Pr(X_i \leq t)$. Given a distribution function F, we denote as F^{-1+} its right-continuous inverse, which is defined as

$$F^{-1+}(\alpha) := \sup\{t \in \mathbb{R} | F(t) \leq \alpha\}, \quad \alpha \in [0,1],$$

and as F^{-1} its left-continuous inverse, or quantile function, defined as

$$F^{-1}(\alpha) := \inf\{t \in \mathbb{R} | F(t) \geq \alpha\}, \quad \alpha \in [0,1].$$

In these definitions, we adopt the convention that $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.

If the non-negative random variable X has distribution function F, its Value-at-Risk (VaR) at probability level α is defined by $\mathrm{VaR}_\alpha(X) = F^{-1}(\alpha)$ and its Tail-VaR (TVaR) at probability level α is defined by

$$\mathrm{TVaR}_\alpha(X) := \frac{1}{1-\alpha} \int_\alpha^1 F^{-1}(t)\, dt, \quad \alpha \in (0,1).$$

Sometimes, we may also write $\mathrm{TVaR}_\alpha(F)$ to stress that TVaR depends only on the distribution function F.

Very often, actuaries are interested in evaluating the TVaR of the aggregate risk $S = X_1 + \cdots + X_n$. Obviously, calculating $\mathrm{TVaR}_\alpha(S)$ requires not only information about the marginal distributions of the risks but also their joint distribution. Therefore, (tight) upper and lower bounds for $\text{TVaR}_\alpha(S)$ appear to be very useful so that we can have a better idea about the possible magnitude of this quantity. Moreover, it is also instructive to know what kind of dependence structures among the risks with given marginals would attain the upper and lower bounds, which correspond to the most dangerous and the safest scenarios respectively.

As for the problem of upper bound, the solution is well known in the literature:

$$\text{TVaR}_\alpha(S) \leq \text{TVaR}_\alpha(X_1) + \cdots + \text{TVaR}_\alpha(X_n) \text{ for all } \alpha \in (0,1).$$

For a proof and more discussion on this important result, see, for instance, Denuit et al. (2005). Moreover, the upper bound on the right hand side is attained for all $\alpha \in (0,1)$ if, and only if, the variables $X_1, \dots, X_n$ are comonotonic, and the upper bound is attained for α close enough to 1 if, and only if, the random variables $X_1, \dots, X_n$ are upper comonotonic in the sense of Cheung and Lo (2013a).

As for the lower bound problem, Cheung and Lo (2013b, Theorem 4.1) proved that if $X_1, \dots, X_n$ are all positive with $F_i^{-1+}(0) = 0$, then for any $\varepsilon \in (0,1)$,

$$\text{TVaR}_\varepsilon(X_1 + \cdots + X_n) \geq \max_{\sum \varepsilon_i = 1-\varepsilon} \sum_{i=1}^{n} \frac{\varepsilon_i}{1-\varepsilon} \text{TVaR}_{1-\varepsilon_i}(X_i). \tag{2.1}$$

Furthermore, the solution set of the maximum on the right hand side is given by

$$\left\{ (\varepsilon_1, \dots, \varepsilon_n) \mid \sum_{i=1}^{n} \varepsilon_i = 1 - \varepsilon, \bigcap_{i=1}^{n} \left[F_i^{-1}(1-\varepsilon_i), F_i^{-1+}(1-\varepsilon_i) \right] \neq \emptyset \right\}. \tag{2.2}$$

Throughout this paper, implicit in the constraint "$\sum \varepsilon_i = 1 - \varepsilon$" is that each ε_i is non-negative. We refer to Jakobsons et al. (2015) and the references therein for alternative results on TVaR lower bounds.

If the marginals F_i allow mutual exclusivity, that is, if the condition

$$\sum_{i=1}^{n} F_i(0) \geq n - 1 \tag{2.3}$$

is fulfilled, then equality holds in (2.1) if and only if the risks are mutually exclusive, i.e. $X_1, \dots, X_n$ are such that

$$\Pr(X_i > 0, X_j > 0) = 0 \text{ for all } i \neq j. \tag{2.4}$$

Roughly speaking, condition (2.3) requires that each marginal F_i has a sufficient large point mass at the origin, which is rather stringent. If the marginals do not satisfy condition (2.3), the lower bound in (2.1) is not tight in the sense that for some $\varepsilon \in (0,1)$, the lower bound is never attainable by any random vector with marginals $F_1, \ldots, F_n$. On the other hand, TVaR as a measurement of risk is mostly relevant when the probability level is close to one. This leads us to consider the possibility of having a tight lower bound in (2.1) *only for* ε close enough to zero, by some suitable relaxation of the stringent condition (2.3). The main results of this paper are to show that as long as

(i) each $F_i(0)$ is non-zero, and

(ii) $F_i^{-1}(1) = +\infty$ for at least one i,

the lower bound in (2.1) remains tight for all ε close enough to zero, and is attained by a specific dependence structure that can be interpreted as "mutual exclusivity in the tail". Furthermore, we also prove that assuming condition (ii) alone, that is, without imposing any point mass requirement at the origin on the marginals, the lower bound in (2.1) is asymptotically tight.

In the remainder of this paper, $F_1, \ldots, F_n$ are fixed marginals with $F_i^{-1+}(0) = 0$ for all i. For any univariate cdf's $G_1, \ldots, G_n$, $\mathcal{R}(G_1, \ldots, G_n)$ denotes the Fréchet space of all random vectors with marginals $G_1, \ldots, G_n$.

3 Tail mutual exclusivity

The following definition, which is the main subject of this paper, is a generalization of the mutual exclusivity concept (2.4) proposed by Dhaene and Denuit (1999).

Definition 3.1. *A random vector* $\boldsymbol{X} = (X_1, \ldots, X_n) \in \mathcal{R}(F_1, \ldots, F_n)$ *is said to be mutually exclusive in the tail, abbreviated as MET, if there exists some probability vector* $\boldsymbol{p} = (p_1, \ldots, p_n) \in (0,1)^n$ *such that*

$$\Pr(X_i > F_i^{-1}(p_i), X_j > 0) = 0 \quad \textit{for any } i \neq j.$$

Such a random vector is also called $\boldsymbol{p}$*-mutually exclusive, abbreviated as* $\boldsymbol{p}$*-ME.*

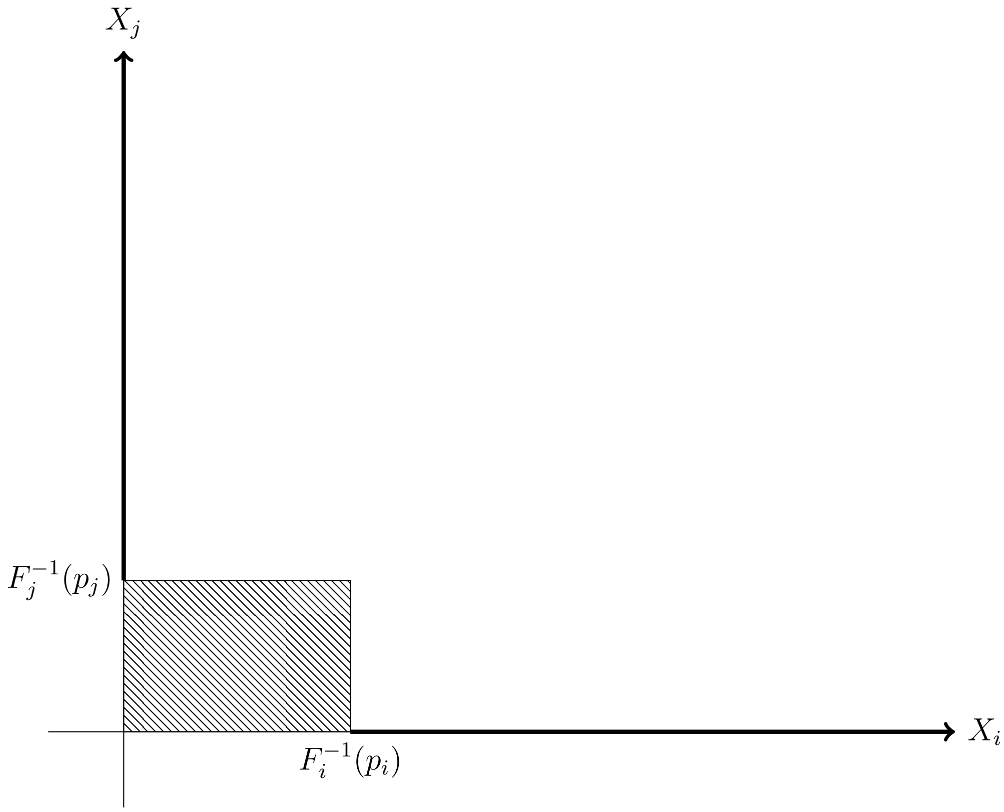
In words, a random vector is mutual exclusivity in the tail means if whenever one of the components of a random vector exceeds a certain quantile, all other components must be zero. Hence, only one component can be large, i.e., exceeds the VaR at the specified probability level, and no simultaneous extremes can occur. See Figure 1 for an illustration. It is clear that the random vector $\boldsymbol{X} \in \mathcal{R}(F_1, \dots, F_n)$ is $\boldsymbol{p}$-ME if and only if

$$\Pr(\boldsymbol{X} \in B \cup A_1 \cup \cdots \cup A_n) = 1,$$

where $A_1, \dots, A_n$ and B are subsets of $\mathbb{R}^n$ defined by

$$\begin{aligned} A_i &:= \{\boldsymbol{x} \in \mathbb{R}^n | x_i > F_i^{-1}(p_i), x_j = 0 \text{ for } j \neq i\}, \quad i = 1, \dots, n, \\ B &:= \{\boldsymbol{x} \in \mathbb{R}^n | 0 \leq x_i \leq F_i^{-1}(p_i) \text{ for all } i.\} \end{aligned} \tag{3.1}$$

Figure 1: Support of a $\boldsymbol{p}$-ME random vector.



From Definition 3.1, it is clear that if $\boldsymbol{X}$ is $\boldsymbol{p}$-ME and if $\boldsymbol{p}' \in (0,1)^n$ satisfies $p_i' \geq p_i$ for all i, then $\boldsymbol{X}$ is $\boldsymbol{p}'$-ME too. Therefore, one can always increase the probability levels p_i without destroying the MET structure. Moreover, we can assume that the probability levels defining the MET structure indeed belong to the range of the marginal distribution functions, as precisely stated next.

Property 3.2. *A random vector* $\boldsymbol{X} = (X_1, \dots, X_n) \in \mathcal{R}(F_1, \dots, F_n)$ *is* $\boldsymbol{p}$*-ME if and only if it is* $\boldsymbol{p}'$*-ME, where* $p_i' = F_i(F_i^{-1}(p_i))$.

Proof. This simply follows from the fact that $F_i^{-1}(p_i) = F_i^{-1}(F_i(F_i^{-1}(p_i))$. $\square$

In view of Property 3.2, we can assume without loss of generality that the transformation from p_i to $p_i' = F_i(F_i^{-1}(p_i))$ has always been performed, so we adopt the following convention in the sequel:

> **Convention** Whenever we talk about $\boldsymbol{p}$-mutual exclusivity, each p_i satisfies the equation $p_i = F_i(F_i^{-1}(p_i))$.

In particular, under this convention, we have $p_i \geq F_i(0)$ for all i. Furthermore, if $p_i = F_i(0)$ for all i, a random vector is $\boldsymbol{p}$-ME if and only if it is mutually exclusive in the classical sense as defined in Denuit and Dhaene (1999). In fact, the structures mutual exclusivity in the tail and mutual exclusivity are also related in another way, as indicated in the following lemma.

Lemma 3.3. *If the random vector $\boldsymbol{X} = (X_1, \ldots, X_n) \in \mathcal{R}(F_1, \ldots, F_n)$ is $\boldsymbol{p}$-ME, then the random vector*

$$((X_1 - F_1^{-1}(p_1))_+, \ldots, (X_n - F_n^{-1}(p_n))_+)$$

is mutually exclusive.

Proof. Let $Y_i := (X_i - F_i^{-1}(p_i))_+$ for $i = 1, \ldots, n$. Then

$$\{Y_i > 0, Y_j > 0\} = \{X_i > F_i^{-1}(p_i), X_j > F_j^{-1}(p_j)\} \subset \{X_i > F_i^{-1}(p_i), X_j > 0\}. \qquad (3.2)$$

If $\boldsymbol{X} \in \mathcal{R}(F_1, \ldots, F_n)$ is $\boldsymbol{p}$-ME, then event on the right most has zero probability. Therefore, $\Pr(Y_i > 0, Y_j > 0) = 0$. $\square$

Notice that the converse implication of Lemma 3.3 may not hold true because the set inclusion "$\subset$" in (3.2) is strict in general.

Another closely related result is that tail mutual exclusivity is closed under increasing transformation.

Lemma 3.4. *Let $(X_1, \ldots, X_n) \in \mathcal{R}(F_1, \ldots, F_n)$ be $\boldsymbol{p}$-ME. For $i = 1, \ldots, n$, let $f_i : [0, \infty) \to [0, \infty)$ be increasing, left-continuous, and satisfies $f_i(0) = 0$. Then $(f_1(X_1), \ldots, f_n(X_n))$ is also $\boldsymbol{p}$-ME.*

Proof. For $i = 1, \ldots, n$, let $\tilde{F}_i^{-1}$ be the left-continuous inverse of the distribution function of $f_i(X_i)$. Since f_i is increasing and left-continuous, $\tilde{F}_i^{-1}(p_i) = f_i(F_i^{-1}(p_i))$. The result follows from the observation that

$$\{f_i(X_i) > \tilde{F}_i^{-1}(p_i), f_j(X_j) > 0\} \subset \{X_i > F_i^{-1}(p_i), X_j > 0\}.$$

□

The following result gives a necessary and sufficient condition to guarantee the existence of a $\boldsymbol{p}$-ME random vector in a given Fréchet space $\mathcal{R}(F_1, \ldots, F_n)$.

Proposition 3.5. *(i) Suppose that $p_i \geq F_i(0)$ for all i. There exists a $\boldsymbol{p}$-ME random vector in $\mathcal{R}(F_1, \ldots, F_n)$ if and only if*

$$\sum_{i=1}^{n}(1 - p_i) + \max_{1 \leq i \leq n} \big(p_i - F_i(0)\big) \leq 1. \tag{3.3}$$

In this case, $F_i(0) > 0$ for all i.

(ii) There exists a MET random vector in $\mathcal{R}(F_1, \ldots, F_n)$ if and only if $F_i(0) > 0$ for all i.

Proof. Define

$$\Delta := \max_i \big(p_i - F_i(0)\big) \geq 0 \quad \text{and} \quad R := 1 - \sum_{i=1}^{n}(1 - p_i) - \Delta. \tag{3.4}$$

To prove statement (i), suppose that condition (3.3) holds true for some $\boldsymbol{p} \in (0, 1)^n$ with $p_i \geq F_i(0)$, then $R \geq 0$ and

$$0 < \sum_{i \neq j}^{n}(1 - p_i) + R = p_j - \Delta \leq p_j - \big(p_j - F_j(0)\big) = F_j(0).$$

To construct a $\boldsymbol{p}$-ME random vector with marginals $F_1, \ldots, F_n$, let $L, K, K_1, \ldots, K_n$ be disjoint open intervals contained in $(0, 1)$ with length $|L| = \Delta$, $|K| = R$, and $|K_i| = 1 - p_i$ for $i = 1, \ldots, n$. The exact positions of these intervals are irrelevant. Let U be a random variable uniformly distributed over $(0, 1)$. For $i = 1, \ldots, n$, let U_i be uniformly distributed over $(0, 1)$ obtained from a shuffling of U that satisfies the following conditions:

$$\begin{cases} U_i \in (p_i, 1) & \Leftrightarrow U \in K_i \\ U_i \in (p_i - \Delta, p_i) & \Leftrightarrow U \in L \\ U_i \in (0, p_i - \Delta) & \Leftrightarrow U \in \Big((\cup_{j \neq i} K_j) \cup K \Big). \end{cases} \tag{3.5}$$

The relationship between U and U_i at the end points of the intervals is not important as there are only finitely many of them. Now we define $X_i := F_i^{-1}(U_i)$ for all i. By construction, $X_i > F_i^{-1}(p_i)$ implies that $U_i \in (p_i, 1)$, which is equivalent to $U \in K_i$ and hence for any $j \neq i$, $U_j \in (0, p_j - \Delta)$. Since $p_j - \Delta \leq F_j(0)$, we have $X_j = 0$. This proves that $(X_1, \ldots, X_n)$ is $\boldsymbol{p}$-ME.

Next, suppose that there exists a $\boldsymbol{p}$-ME random vector $\boldsymbol{X}$ with marginals $F_1, \ldots, F_n$, for some $\boldsymbol{p} \in (0,1)^n$ with $p_i \geq F_i(0)$ for all i. For any $j = 1, \ldots, n$, the events

$$\{X_1 > F_1^{-1}(p_1)\}, \cdots, \{X_n > F_n^{-1}(p_n)\}, \{0 < X_j \leq F_j^{-1}(p_j)\}$$

are mutually disjoint, and hence

$$\sum_{i=1}^n (1 - F_i(F_i^{-1}(p_i))) + \big(F_j(F_j^{-1}(p_j)) - F_j(0)\big) \leq 1.$$

Under our convention that $p_i = F_i(F_i^{-1}(p_i))$, this condition becomes

$$\sum_{i=1}^n (1 - p_i) + \big(p_j - F_j(0)\big) \leq 1 \quad \text{for all } j = 1, \ldots, n,$$

which is equivalent to condition (3.3).

To prove statement (ii), it is enough to show that a MET random vector with marginals $F_1, \ldots, F_n$ can be constructed when $F_i(0) > 0$ for all i. In fact, if each $F_i(0)$ is strictly positive, it is always possible to find some $\boldsymbol{p} \in (0,1)^n$ with $p_i \geq F_i(0)$ for all i such that condition (3.3) holds true. One such example is given by

$$p_1 = \cdots = p_n = 1 - \varepsilon$$

with

$$0 < \varepsilon \leq \min \left\{ 1 - F_1(0), \ldots, 1 - F_n(0), \frac{\min_i F_i(0)}{n-1} \right\}.$$

By statement (i) of the proposition, one can then construct a $\boldsymbol{p}$-ME random vector with marginals $F_1, \ldots, F_n$. □

Recall that classical mutual exclusivity is equivalent to $\boldsymbol{p}$-mutual exclusivity with $p_i = F_i(0)$ for all i. In such case, condition (3.3) in Proposition 3.5 becomes $\sum_i (1 - F_i(0)) \leq 1$, which is equivalent to condition (2.3).

The following result presents one particular shuffling rule that is consistent with (3.5) and indicates how one can simulate a MET random vector with given marginals.

Corollary 3.6. *Suppose that* $F_1, \ldots, F_n$ *are given marginals which satisfy condition (3.3). Let* R *and* Δ *be defined as in (3.4). Let*

$$s_0 := R \quad \textit{and} \quad s_i = R + \sum_{j=1}^{i} (1 - p_j), \quad i = 1, \ldots, n.$$

Let U *be a uniform*$(0,1)$ *random variable, and define*

$$U_i = U + (1 - s_i) 1_{\{s_{i-1} < U < s_i\}} - (1 - p_i) 1_{\{U > s_i\}}, \quad i = 1, \ldots, n. \tag{3.6}$$

Then $(F_1^{-1}(U_1), \ldots, F_n^{-1}(U_n))$ *is a* $\boldsymbol{p}$*-ME random vector in* $\mathcal{R}(F_1, \ldots, F_n)$*.*

Proof. It is straightforward to check that each U_i defined in (3.6) is uniformly distributed on $(0,1)$ and satisfies (3.5). □

The following result characterizes MET random vectors through their distribution functions.

Proposition 3.7. *Let* $\boldsymbol{X} = (X_1, \ldots, X_n)$ *be a random vector in* $\mathcal{R}(F_1, \ldots, F_n)$*. Suppose that* $0 < F_i(0) \le p_i \le 1$ *for all* i*. The random vector* $\boldsymbol{X}$ *is* $\boldsymbol{p}$*-ME if and only if its joint distribution function satisfies*

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \sum_{i=1}^{n} (F_i(x_i) - p_i)_+ + F_{\boldsymbol{X}}(\boldsymbol{x} \wedge \boldsymbol{a}), \quad \textit{for any } \boldsymbol{x} \ge \boldsymbol{0}, \tag{3.7}$$

where $\boldsymbol{a} := (F_1^{-1}(p_1), \ldots, F_n^{-1}(p_n))$*, and* $\boldsymbol{x} \wedge \boldsymbol{a} := (\min(x_1, a_1), \ldots, \min(x_n, a_n))$*.*

Proof. Suppose that $\boldsymbol{X}$ is $\boldsymbol{p}$-ME. Let $B, A_1, \ldots, A_n$ be subsets of $\mathbb{R}^n$ defined as in (3.1). For any $\boldsymbol{x} \ge \boldsymbol{0}$,

$$\begin{aligned} F_{\boldsymbol{X}}(\boldsymbol{x}) &= \sum_{i=1}^{n} \Pr(\{\boldsymbol{X} \le \boldsymbol{x}\} \cap \{\boldsymbol{X} \in A_i\}) + \Pr(\{\boldsymbol{X} \le \boldsymbol{x}\} \cap \{\boldsymbol{X} \in B\}) \\ &= \sum_{i=1}^{n} \Pr(a_i < X_i \le x_i) + \Pr(\{\boldsymbol{X} \le \boldsymbol{x}\} \cap \{\boldsymbol{X} \le \boldsymbol{a}\}) \\ &= \sum_{i=1}^{n} (F_i(x_i) - p_i)_+ + F_{\boldsymbol{X}}(\boldsymbol{x} \wedge \boldsymbol{a}), \end{aligned}$$

where the last equality follows from our convention that $p_i = F_i(a_i) = F_i(F_i^{-1}(p_i))$.

Conversely, suppose that $\boldsymbol{X}$ is a random vector whose joint distribution function satisfies (3.7) for some $(p_1, \ldots, p_n) \in (0,1)^n$. By symmetry, to show that $\boldsymbol{X}$ is $\boldsymbol{p}$-ME, it suffices to show that for any $x_1 > a_1$

$$\Pr(X_1 > x_1) = \Pr(X_1 > x_1, X_2 = 0).$$

Since X_1 and X_1 are positive random variables, it follows from (3.7) that

$$\begin{aligned}
\Pr(X_1 \le x_1, X_2 = 0) &= F_{\boldsymbol{X}}(x_1, 0, \infty, \ldots, \infty) \\
&= (F_1(x_1) - p_1) + F_{\boldsymbol{X}}(a_1, 0, a_3 \ldots, a_n)
\end{aligned}$$

because $x_1 > a_1$ implies that $F_1(x_1) \ge F_1(a_1) = F_1(F_1^{-1}(p_1)) \ge p_1$. Therefore,

$$\begin{aligned}
&\Pr(X_1 > x_1, X_2 = 0) \\
&\quad = \Pr(X_2 = 0) - F_{\boldsymbol{X}}(x_1, 0, \infty, \ldots, \infty) \\
&\quad = F_{\boldsymbol{X}}(\infty, 0, \infty, \ldots, \infty) - F_{\boldsymbol{X}}(x_1, 0, \infty, \ldots, \infty) \\
&\quad = [(1 - p_1) + F_{\boldsymbol{X}}(a_1, 0, a_3 \ldots, a_n)] - [(F_1(x_1) - p_1) + F_{\boldsymbol{X}}(a_1, 0, a_3 \ldots, a_n)] \\
&\quad = 1 - F_1(x_1),
\end{aligned}$$

as desired. □

We can apply Proposition 3.7 to derive the joint distribution function of a mutually exclusive random vector. A direct derivation can be found in Dhaene and Denuit (1999).

Corollary 3.8. *A random vector* $\boldsymbol{X} = (X_1, \ldots, X_n) \in \mathcal{R}(F_1, \ldots, F_n)$ *is mutually exclusive if and only if its joint distribution function satisfies*

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \sum_{i=1}^{n} F_i(x_i) - n + 1, \quad \textit{for any } \boldsymbol{x} \ge \mathbf{0}. \tag{3.8}$$

Proof. Suppose that $\boldsymbol{X}$ is mutually exclusive. By the remark after the proof of Proposition 3.5, we have $\sum_i F_i(0) \ge n - 1$, and we can take $p_i(0) := F_i(0)$ in equation (3.7). Then $a_i = F_i^{-1}(p_i) = 0$ and equation (3.7) becomes

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = \sum_{i=1}^{n} (F_i(x_i) - F_i(0)) + F_{\boldsymbol{X}}(\mathbf{0}), \quad \text{for any } \boldsymbol{x} \ge \mathbf{0}.$$

Since $\boldsymbol{X}$ is mutually exclusive, we have

$$\begin{aligned}
F_{\boldsymbol{X}}(\mathbf{0}) &= \Pr(X_1 = \cdots = X_n = 0) \\
&= 1 - \Pr(X_i \neq 0 \text{ for some } i) \\
&= 1 - \sum_{i=1}^{n} \Pr(X_i \neq 0) = 1 - \sum_{i=1}^{n} \big(1 - F_i(0)\big) = \sum_{i=1}^{n} F_i(0) - n + 1.
\end{aligned}$$

The result follows. □

4 Tail behavior of MET random vectors

We begin with the following decomposition result which characterizes the upper tail of the distribution function of a sum of MET random variables. The second assertion of the following proposition can be found in Dhaene and Denuit (1999).

Proposition 4.1. *Let* $\boldsymbol{X}^* = (X_1^*, \dots, X_n^*) \in \mathcal{R}(F_1, \dots, F_n)$ *be a* $\boldsymbol{p}$*-ME random vector, and* $S^* := X_1 + \cdots + X_n$*. Then*

$$\Pr(S^* > t) = \sum_{i=1}^{n} \Pr(X_i^* > t) \quad \textit{for any } t \geq \sum_{i=1}^{n} F_i^{-1}(p_i).$$

In particular, if $\boldsymbol{X}^*$ *is mutually exclusive, then*

$$\Pr(S^* > t) = \sum_{i=1}^{n} \Pr(X_i^* > t) \quad \textit{for any } t \geq 0.$$

Proof. Suppose that $\boldsymbol{X}^*$ is $\boldsymbol{p}$-ME in $\mathcal{R}(F_1, \dots, F_n)$, and define

$$\phi := F_1^{-1}(p_1) + \cdots + F_n^{-1}(p_n),$$

where F_i is the distribution function of X_i^*. For any $t > \phi$, using the notation introduced in (3.1), we get

$$\Pr(S^* > t) = \Pr(S^* > t, \boldsymbol{X}^* \in B) + \sum_{i=1}^{n} \Pr(S^* > t, \boldsymbol{X}^* \in A_i) = 0 + \sum_{i=1}^{n} \Pr(X_i^* > t),$$

as desired. When $\boldsymbol{X}^*$ is mutually exclusive, it suffices to notice that $\phi = 0$ when $p_i = F_i(0)$ for all i. □

Using Proposition 4.1, we can easily obtain the following decomposition result for the stop-loss premium of a sum of MET random variables, which is a generalization of equation (2) in Dhaene and Denuit (1999).

Corollary 4.2. *Let* $\boldsymbol{X}^* = (X_1^*, \dots, X_n^*) \in \mathcal{R}(F_1, \dots, F_n)$ *be a* $\boldsymbol{p}$*-ME random vector, and* $S^* := X_1 + \cdots + X_n$*. Then*

$$\mathbb{E}[(S^* - d)_+] = \sum_{i=1}^{n} \mathbb{E}[(X_i^* - d)_+] \quad \textit{for any } d \geq \sum_{i=1}^{n} F_i^{-1}(p_i).$$

Recall from Cheung and Vanduffel (2013) that given two random variables X and Y, X is said to precede Y in the *tail convex order*, denoted as $X \preceq_{\text{tcx}} Y$, if there exists a real number k, called the tail index, such that $\Pr(Y > k) > 0$ and $\mathbb{E}[(X-d)_+] \leq \mathbb{E}[(Y-d)_+]$ for all $d \geq k$.

The next result shows that among all the risks with given marginals F_i and such that (3.3) is fulfilled, the $\boldsymbol{p}$-ME risks lead to the safest portfolio, in the sense that this kind of dependence leads to the smallest stop-loss premiums for deductibles high enough. This generalizes Theorem 10 of Dhaene and Denuit (1999).

Corollary 4.3. *Let $\boldsymbol{X}^*$ be a $\boldsymbol{p}$-ME random vector in $\mathcal{R}(F_1, \ldots, F_n)$. Then for any random vector $\boldsymbol{X} \in \mathcal{R}(F_1, \ldots, F_n)$,*

$$\sum_{i=1}^{n} X_i^* \preceq_{tcx} \sum_{i=1}^{n} X_i.$$

Proof. It suffices to note that for any $d \geq \phi := F_1^{-1}(p_1) + \cdots + F_n^{-1}(p_n)$,

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i^* - d\right)_+\right] = \sum_{i=1}^{n} \mathbb{E}[(X_i^* - d)_+] = \sum_{i=1}^{n} \mathbb{E}[(X_i - d)_+] \leq \mathbb{E}\left[\left(\sum_{i=1}^{n} X_i - d\right)_+\right],$$

where the first equality follows from Corollary 4.2. The corresponding tail index can be taken as ϕ. □

5 MET and TVaR lower bounds

We are now ready to prove the main result of this paper, which states that the lower bound (2.1) derived in Cheung and Lo (2013b) is attained under MET, provided the probability level ε is large enough. In other words, mutual exclusivity in the tail gives rise to the smallest value of Tail-VaR of a sum of risks within a given Fréchet space, provided that the probability level of the Tail-VaR is close to one. The conditions required are that all risks have a point mass at 0, and that at least one of the risks is unbounded above. Both conditions are mild and natural from a modelling perspective when one is considering insurance risks.

Theorem 5.1. *Consider $\mathcal{R}(F_1, \ldots, F_n)$ with*

(i) $F_i(0) > 0$ *for all* i*, and*

(ii) $F_i^{-1}(1) = +\infty$ *for at least one* i*.*

Then, there exist a MET random vector $(X_1^*, \ldots, X_n^*)$ *in* $\mathcal{R}(F_1, \ldots, F_n)$ *and some probability level* $\varepsilon^* \in (0,1)$ *such that*

$$TVaR_\varepsilon\left(\sum_{i=1}^n X_i^*\right) = \min_{\boldsymbol{X}\in\mathcal{R}(F_1,\ldots,F_n)} TVaR_\varepsilon\left(\sum_{i=1}^n X_i\right) = \max_{\sum \varepsilon_i = 1-\varepsilon} \sum_{i=1}^n \frac{\varepsilon_i}{1-\varepsilon} TVaR_{1-\varepsilon_i}(X_i^*)$$

for any $\varepsilon > \varepsilon^*$. *If we further assume that each* F_i *is continuous on* $(F_i^{-1}(p_i), F_i^{-1}(1))$, *then the maximum on the right most expression is attained at*

$$(\varepsilon_1^*, \ldots, \varepsilon_n^*) = \left(\overline{F}_1(F_{S^*}^{-1}(\varepsilon)), \ldots, \overline{F}_n(F_{S^*}^{-1}(\varepsilon))\right), \tag{5.1}$$

where $S^* := X_1^* + \cdots + X_n^*$.

Proof. We first choose a vector $(p_1, \ldots, p_n) \in (0,1)^n$ that satisfies $\sum_i p_i \geq n-1$ and (3.3). As explained in Property 3.2, we may further assume that $p_i = F_i(F_i^{-1}(p_i)$ for all i. By Proposition 3.5, there exists a $\boldsymbol{p}$-ME random vector $\boldsymbol{X}^* = (X_1^*, \ldots, X_n^*)$ with marginals $F_1, \ldots, F_n$. Define

$$S^* := X_1^* + \cdots + X_n^* \quad \text{and} \quad \phi := F_1^{-1}(p_1) + \cdots + F_n^{-1}(p_n).$$

For any $t > \phi$, it follows from Proposition 4.1 that

$$\Pr(S^* > t) = \sum_{i=1}^n \Pr(X_i^* > t) > 0. \tag{5.2}$$

where the last inequality follows from assumption (ii).

Next, we define new distribution functions $G_1, \ldots, G_n$ by

$$G_i(t) := \begin{cases} F_i(t), & t \geq F_i^{-1}(p_i), \\ p_i, & 0 \leq t < F_i^{-1}(p_i), \\ 0, & t < 0. \end{cases}$$

By our choice of p_i, $\sum_{i=1}^n G_i(0) \geq n-1$, so condition (2.3) is fulfilled. Therefore there exists a mutually exclusive random vector $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ in $\mathcal{R}(G_1, \ldots, G_n)$. From our construction, it is clear that for any i,

$$\mathrm{TVaR}_\alpha(Y_i) = \mathrm{TVaR}_\alpha(X_i^*) \text{ for any } \alpha \geq p_i. \tag{5.3}$$

The mutual exclusivity of $\boldsymbol{Y}$ implies that

$$\Pr(Y_1 + \cdots + Y_n > t) = \sum_{i=1}^n \Pr(Y_i > t) \text{ for all } t \geq 0, \tag{5.4}$$

and the TVaR of the sum $Y_1 + \cdots + Y_n$ attains the lower bound described in (2.1):

$$\mathrm{TVaR}_\varepsilon(Y_1 + \cdots + Y_n) = \max_{\sum \varepsilon_i = 1-\varepsilon} \sum_{i=1}^n \frac{\varepsilon_i}{1-\varepsilon} \mathrm{TVaR}_{1-\varepsilon_i}(Y_i), \quad \varepsilon \in (0,1). \tag{5.5}$$

From (5.4) and the fact that G_i and F_i have the same tail from $F_i^{-1}(p_i)$ onward, the sums $Y_1 + \cdots + Y_n$ and S^* are related by

$$0 < \Pr(S^* > t) = \sum_{i=1}^n \Pr(X_i^* > t) = \sum_{i=1}^n \Pr(Y_i > t) = \Pr(Y_1 + \cdots + Y_n > t), \quad t > \phi,$$

which in turn implies that

$$\mathrm{TVaR}_\alpha(Y_1 + \cdots + Y_n) = \mathrm{TVaR}_\alpha(S^*) \text{ for any } \alpha \geq F_{S^*}(\phi).$$

Together with (5.3) and (5.5), we obtain

$$\mathrm{TVaR}_\varepsilon(S^*) = \max_{\sum \varepsilon_i = 1-\varepsilon} \sum_{i=1}^n \frac{\varepsilon_i}{1-\varepsilon} \mathrm{TVaR}_{1-\varepsilon_i}(Y_i) = \max_{\sum \varepsilon_i = 1-\varepsilon} \sum_{i=1}^n \frac{\varepsilon_i}{1-\varepsilon} \mathrm{TVaR}_{1-\varepsilon_i}(X_i^*)$$

whenever $\varepsilon > \varepsilon^* := \max(F_{S^*}(\phi), p_1, \ldots, p_n) < 1$.

Now we prove the last assertion. Suppose that each F_i is continuous on $(F_i^{-1}(p_i), F_i^{-1}(1))$. Since the solution set of the maximum is characterized by (2.2), we need to verify that the point $(\varepsilon_1^*, \ldots, \varepsilon_n^*)$ defined in (5.1) satisfies

$$\sum_{i=1}^n \varepsilon_i^* = 1 - \varepsilon \quad \text{and} \quad \bigcap_{i=1}^n \left[F_i^{-1}(1-\varepsilon_i^*), F_i^{-1+}(1-\varepsilon_i^*)\right] \neq \emptyset.$$

To verify the first equation, we observe that $\varepsilon > \varepsilon^* \geq F_{S^*}(\phi)$ implies that $\phi < F_{S^*}^{-1}(\varepsilon)$. As $\boldsymbol{X}^*$ is $\boldsymbol{p}$-ME, Proposition 4.1 yields that

$$\sum_{i=1}^n \varepsilon_i^* = \sum_{i=1}^n \overline{F}_1(F_{S^*}^{-1}(\varepsilon)) = \overline{F}_{S^*}(F_{S^*}^{-1}(\varepsilon)).$$

By Proposition 4.1 again and our hypothesis that each F_i is continuous on $(F_i^{-1}(p_i), F_i^{-1}(1))$, F_{S^*} is continuous on at $F_{S^*}^{-1}(\varepsilon)$. Hence we obtain

$$\sum_{i=1}^n \varepsilon_i^* = \overline{F}_{S^*}(F_{S^*}^{-1}(\varepsilon)) = 1 - \varepsilon.$$

Finally, it follows from the continuity of F_i that

$$F_{S^*}^{-1}(\varepsilon) \in \bigcap_{i=1}^n \left[F_i^{-1}(1-\varepsilon_i^*), F_i^{-1+}(1-\varepsilon_i^*)\right],$$

which completes the proof. □

Theorem 5.1 means that under the two stated hypotheses on the marginal distributions, the TVaR lower bound in (2.1) is reachable in $\mathcal{R}(F_1, \ldots, F_n)$. With slightly more effort, one can dispense with the assumption that $F_i(0) > 0$ for all i and show that the lower bound in (2.1) is *asymptotically tight*. The only required condition is that at least one of the risks is unbounded above.

Theorem 5.2. *Suppose that $F_i^{-1}(1) = \infty$ for at least one i. Then*

$$0 \leq \lim_{\varepsilon \to 1} \left\{ \inf_{Y_i \sim F_i} TVaR_\varepsilon \left(\sum_{i=1}^n Y_i \right) - \max_{\sum \varepsilon_i = 1-\varepsilon} \sum_{i=1}^n \frac{\varepsilon_i}{1-\varepsilon} TVaR_{1-\varepsilon_i}(F_i) \right\} = 0.$$

Proof. We need to show that for any $\delta > 0$, there exists some $\varepsilon^* \in (0,1)$ such that for any $\varepsilon^* < \varepsilon < 1$ there exists a random vector $(Y_1^*, \ldots, Y_n^*) \in \mathcal{R}(F_1, \ldots, F_n)$ such that

$$\mathrm{TVaR}_\varepsilon \left(\sum_{i=1}^n Y_i^* \right) - \max_{\sum \varepsilon_i = 1-\varepsilon} \sum_{i=1}^n \frac{\varepsilon_i}{1-\varepsilon} \mathrm{TVaR}_{1-\varepsilon_i}(F_i) \leq \delta. \tag{5.6}$$

To this end, define $g(x; a) = x 1_{\{x > a\}}$ on $\mathbb{R}_+$ for $a \geq 0$. Let $G_{i;a}$ be the distribution function of the random variable $g(X_i; a)$, where X_i is any random variable with distribution function F_i. Obviously, $g(x; 0) \equiv x$ for any $x \geq 0$, and

$$G_{i;a}(t) = F_i(t \vee a), \quad t \geq 0. \tag{5.7}$$

Now consider the case where both a and ε_i are close enough, but not equal, to zero. From (5.7),

$$\mathrm{TVaR}_{1-\varepsilon_i}(F_i) = \mathrm{TVaR}_{1-\varepsilon_i}(G_{i;a}); \tag{5.8}$$

moreover, $G_{i;a}(0) = F_i(a) > 0$ for all i and $G_{i;a}(t) \neq 1$ for all t for at least one i. Therefore, the marginals $G_{1;a}, \ldots, G_{n;a}$ satisfy the two conditions stated in Theorem 5.1, and hence there exists a MET random vector $\boldsymbol{X}^* = (X_1^*, \ldots, X_n^*) \in \mathcal{R}(G_{1;a}, \ldots, G_{n;a})$ such that the the TVaR of the sum $X_i^* + \cdots + X_n^*$ achieves its corresponding lower bound whenever the probability level is higher than some $\varepsilon^* \in (0,1)$:

$$\mathrm{TVaR}_\varepsilon \left(\sum_{i=1}^n X_i^* \right) = \max_{\sum \varepsilon_i = 1-\varepsilon} \sum_{i=1}^n \frac{\varepsilon_i}{1-\varepsilon} \mathrm{TVaR}_{1-\varepsilon_i}(G_{i;a}) \quad \text{for all } \varepsilon > \varepsilon^*. \tag{5.9}$$

Next, we denote by C the an n-copula associated with the random vector $\boldsymbol{X}^*$, which always exists by Sklar's theorem (see, for instance, Theorem 2.10.9 of Nelsen (2006)). Let

$(Y_1^*, \ldots, Y_n^*)$ be a random vector whose joint distribution function is given by $(x_1, \ldots, x_n) \mapsto C(F_1(x_1), \ldots, F_n(x_n))$. We claim that

$$(X_1^*, \ldots, X_n^*) \stackrel{d}{=} (g(Y_1^*; a), \ldots, g(Y_n^*; a)). \tag{5.10}$$

In fact, for any $t_i \geq 0$,

$$\begin{aligned} \Pr(g(Y_i^*; a) \leq t_i, i = 1, \ldots, n) &= \Pr(Y_i^* \leq t_i \vee a, i = 1, \ldots, n) \\ &= C(F_1(t_1 \vee a), \ldots, F_n(t_n \vee a)) \\ &= C(G_{1;a}(t_1), \ldots, G_{n;a}(t_n)) \\ &= \Pr(X_i^* \leq t_i, i = 1, \ldots, n), \end{aligned}$$

in which the second equality follows from the definition of $(Y_1^*, \ldots, Y_n^*)$, and the third equality follows from (5.7); if $t_i < 0$ for some i, then both probabilities are zero.

With all these constructions, we are ready to complete the proof. Whenever a and $1 - \varepsilon$ (and hence every ε_i) are close enough to zero, (5.8),(5.9), and (5.10) imply that the random vector $(Y_1^*, \ldots, Y_n^*)$ constructed in the previous paragraph satisfies

$$\mathrm{TVaR}_\varepsilon \left(\sum_{i=1}^n g(Y_i^*; a) \right) = \max_{\sum \varepsilon_i = 1 - \varepsilon} \sum_{i=1}^n \frac{\varepsilon_i}{1 - \varepsilon} \mathrm{TVaR}_{1 - \varepsilon_i}(F_i). \tag{5.11}$$

Since TVaR is continuous from from below (see, for instance, Lemma 4.21 and Theorem 4.52 of Föllmer and Schied (2011)):

$$\lim_{a \downarrow 0} \mathrm{TVaR}_\varepsilon \left(\sum_{i=1}^n g(Y_i^*; a) \right) = \mathrm{TVaR}_\varepsilon \left(\sum_{i=1}^n Y_i^* \right),$$

letting $a \downarrow 0$ in (5.11) yields our desired (5.6). □

Theorems 5.1 and 5.2 show that under the appropriate conditions, the TVaR lower bound (2.1) is (approximately) reachable. The results demonstrate the relevance of and the fundamental role played by tail mutual exclusivity, which describes the strongest negative dependence in the upper tail.

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