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Comparable Characterizations of Four Solutions for Permission Tree Games

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Abstract

In the field of cooperative games there is an extensive literature that studies situations of restricted cooperation. Myerson (1979) introduced communication graph games, where players can only cooperate if they are connected in an undirected graph representing the communication possibilities. The Myerson value of such a game is obtained by taking the Shapley value of the corresponding restricted game. For the special case that the graph is cycle-free and connected, Demange (2004) introduced for each player the corresponding hierarchical outcome, being the marginal contribution vector for a particular permutation of the player set induced by the graph. Gilles, Owen and van den Brink (1992) introduced games with a (hierarchical) permission structure modeled by a directed graph on the set of players. In the conjunctive (disjunctive) approach, a coalition is said to be feasible, if for every player in the coalition also all (at least one of) its predecessors (if any) belong(s) to the coalition. The conjunctive (disjunctive) permission value is obtained by taking the Shapley value of the associated conjunctive (disjunctive) restricted game. The two approaches coincide when the permission structure is given by a rooted tree.

In this paper we consider games with a hierarchical permission structure given by a rooted tree and modify the Myerson value to a value for such games. We also consider for these games the hierarchical outcome with respect to the root of the tree (top player), along with a new solution that assigns all payoff to the unique top player in the hierarchy. Then comparable characterizations are given of these three solutions and the (conjunctive) permission value.

Keywords: Cooperative TU-game, rooted tree, Myerson value, hierarchical outcome, permission value, axiomatization.

1 Introduction

A *cooperative game with transferable utility*, or simply a TU-game, consists of a finite set of players and a characteristic function that assigns a *worth* to any subset (coalition) of players. A (single-valued) *solution* is a function that assigns to every game a vector of individual payoffs to the players. One of the most applied efficient solutions for cooperative TU-games is the *Shapley value* (Shapley, 1953).

A classical TU-game describes a situation in which any coalition of players can cooperate and earn its worth. Besides this, in the literature various models involving restrictions on coalition formation have been developed. The well-known model of *communication graph games*, or shortly *graph games*, introduced by Myerson (1977), studies situations where it is no longer assumed that any coalition of players can cooperate. Coalitions may fall apart into disjunct subsets of players that can not cooperate together because they are unable to communicate. Myerson models restrictions on communication by an undirected graph in which the links represent the communication relations between the players. A coalition is unrestricted with respect to cooperation, or *feasible*, if it is connected in this communication graph. The *Myerson value* for communication graph games is the solution that assigns to every graph game the Shapley value of its corresponding restricted TU-game, defined as the game in which the worth of any coalition is given by the sum of the worths of its components (i.e. maximally connected subsets) within the graph. Myerson (1977) characterized his value by component efficiency and fairness. For the special situation that the communication graph is cycle-free and connected, Demange (2004) introduced the notion of *hierarchical outcome* as a solution for communication graph games. For every player there is a corresponding hierarchical outcome. This hierarchical outcome is defined by using the (directed) rooted tree obtained by taking the selected player as root player and orienting all edges in the communication graph 'away' from this root player. Then the corresponding hierarchical outcome assigns to any player its marginal contribution in the restricted game to the coalition of all its subordinates in the rooted tree. Demange (2004) illustrates that for a superadditive game any hierarchical outcome is in the Core of the restricted game. Therefore, she argues that 'hierarchy yields stability'.

Another type of restricted cooperation is the model of *games with a permission structure*, introduced by Gilles, Owen and van den Brink (1992). This model studies situations where the players are part of a hierarchical organization that is modeled by a directed graph. Cooperation is restricted in the sense that a player needs permission to cooperate from some of the players that are higher in the hierarchy (if any). Two approaches with respect to games with a permission structure are defined. In the *conjunctive approach* developed in Gilles, Owen and van den Brink (1992), a coalition is unrestricted with respect to cooperation, or *feasible*, if and only if for every player in the coalition it holds that all its predecessors in the directed graph also belong to the coalition. In the *disjunctive approach* developed in Gilles and Owen (1994), a coalition is feasible if and only if for every player in the coalition that has at least one predecessor in the digraph, at least one of his predecessors also belongs to the coalition. The corresponding conjunctively (disjunctively) restricted game is the TU-game with the worth of a coalition given by the worth of its maximal unrestricted subcoalition according to the conjunctive (disjunctive) approach. The conjunctive permission value was axiomatized in van den Brink and Gilles (1996), while the disjunctive permission value was axiomatized in van den Brink (1997). Note that for directed graphs in which every player has at most one predecessor, the conjunctive and the disjunctive approach coincide.

In this paper we consider games with a permission structure that is given by a rooted tree. For this subclass of games with a permission structure, called *permission tree games*, van den Brink, Herings, van der Laan and Talman (2013) introduce the *Average Tree permission value* and give comparable axiomatizations for this value and the conjunctive permission value. In this paper we modify the Myerson value and the hierarchical outcomes to define solutions for permission tree games. We modify the Myerson value by assigning to every permission tree game the Myerson value of the underlying undirected graph game. The hierarchical outcome of a permission tree game is simply the hierarchical outcome of the underlying communication graph game corresponding to the root (i.e. top player) of the permission tree. Besides these two solutions we define a third new solution which assigns to every permission tree game the hierarchical outcome of the corresponding permission restricted game. We compare these three solutions with each other and with the conjunctive permission value by providing a collection of axioms. It appears that any of these four solutions is characterized by a subset of these axioms and that for any two of these solutions the characterizations differ in at most two axioms.

A typology of the four solutions can be made based on a number of criteria. The cooperation restrictions underlying the Myerson value and the hierarchical outcome give the hierarchy some flavor of communication according to these solutions. Players located at upper levels of the hierarchy serve to connect players at lower levels, but players who are not connecting lower level players do not need to give permission to these players. Coalitions can be feasible even if not all predecessors are present. On the other hand, according to the permission value and the top value the hierarchy has the flavor of a permission structure. Coalitions are only feasible if for all players their predecessors are also present. Considering the distribution of payoff, the Myerson value and the permission value have in common that they treat players that are in some sense equally important in generating worth under their respective cooperation restrictions equally. The hierarchical outcome and top value have in common that they assign payoffs fully to their respective top players (being the root player as global top for the top value, and the local top player who is the highest in connecting a coalition for the hierarchical outcome).

The paper is organized as follows. Section 2 consists of preliminaries on cooperative TU games, graphs and digraphs, games on communication graphs and games with a permission structure. Section 3 discusses the four solutions for permission tree games. Section 4 gives comparable characterizations for these four solutions. In Section 5 the typology of the four solutions is discussed.

2 Preliminaries

2.1 Cooperative TU-games

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair (N, v), where $N \subseteq \mathbb{N}$ is a finite set of players and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* on N satisfying $v(\emptyset) = 0$. For any *coalition* $S \subseteq N$, $v(S)$ is the *worth* of coalition S, meaning that the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate. For a game (N, v) and $S \subseteq N$, the game (S, v_S) denotes the subgame of (N, v) on S, given by $v_S(T) = v(T)$, $T \subseteq S$. When needed for clarity of notation, we write $(v)_S$ instead of v_S. Further, for $i \in N$, we denote $N \setminus \{i\}$ by N_{-i} and the subgame $(N_{-i}, v_{N_{-i}})$ by (N_{-i}, v_{-i}). A TU-game (N, v) is superadditive if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$. A game (N, v) is said to be a monotone game if $v(S) \leq v(T)$ when $S \subseteq T \subseteq N$. We denote the class of all TU-games by $\mathcal{G}$, and the class of all monotone TU-games by $\mathcal{G}_M$.

A *payoff vector* of an n-player TU-game (N, v) is an n-dimensional vector $x \in \mathbb{R}^N$ giving a payoff $x_i \in \mathbb{R}$ to any player $i \in N$. A (single-valued) *solution* for TU-games is a function f that assigns to every game $(N, v) \in \mathcal{G}$ a payoff vector $f(N, v) \in \mathbb{R}^N$, $N \subseteq \mathbb{N}$. One of the most famous solutions for TU-games is the *Shapley value* (Shapley (1953)) given by

$$Sh_i(N, v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(|N| - |S|)!(|S| - 1)!}{|N|!}(v(S) - v(S \setminus \{i\})) \text{ for all } i \in N.$$

We define the null game $(N, null)$ as the game given by $null(S) = 0$ for every $S \subseteq N$. Further, for each $R \subseteq N$, the *unanimity* game (N, u^R) is given by $u^R(S) = 1$ if $R \subseteq S$, and $u^R(S) = 0$ otherwise. It is well-known that every characteristic function of a game $(N, v) \in \mathcal{G}$ can be written uniquely as a linear combination of the characteristic function of the unanimity games, i.e., $v = \sum_{R \subseteq N,\ R \neq \emptyset} \Delta_v(R) u^R$, where the coefficients

$\Delta_v(R)$ are the *Harsanyi dividends*, see Harsanyi (1959). Notice that, by definition, $v(S) = \sum_{\emptyset \neq T \subseteq S} \Delta_v(T)$, $T \subseteq S$, $S \subseteq N$.

In a TU-game any subset $S \subseteq N$ is assumed to be able to form a coalition and earn the worth $v(S)$. However, in most economic and political organizations not every set of participants can form a feasible coalition. Therefore, in cooperative game theory models have been developed in which there are restrictions on coalition formation using graphs and digraphs to represent the structure of these organizations.

2.2 Graphs and Digraphs

An undirected graph is a pair (N, L) where N is a set of nodes and $L \subseteq \{\{i,j\} | i,j \in N, i \neq j\}$ is a set of unordered pairs of distinct elements of N. In this paper the nodes represent the players in a game. We therefore refer to them as players. The elements of L are called (undirected) links or *edges*. We denote the set of all graphs by $\mathcal{L}$. For $S \subseteq N$, the graph $(S, L(S))$ with $L(S) = \{\{i,j\} \in L | i,j \in S\}$ is called the subgraph of (N, L) on S. Note that $L(N) = L$. Given a graph (N, L), a sequence of k different players $(i_1, ..., i_k)$ is a *path* from i_1 to i_k if $\{i_l, i_{l+1}\} \in L$ for $l = 1, ..., k-1$. Two players $i, j \in N$ are called *connected* in (N, L) if $i = j$ or there is a path from i to j. A coalition $S \subseteq N$ is said to be connected if every two players in S are connected in $(S, L(S))$. A graph (N, L) is said to be connected if N is connected. A coalition $K \subseteq N$ is a *component* of (N, L) if and only if (i) K is connected, and (ii) $K \cup \{i\}$ is not connected for every $i \in N \backslash K$. The set of components of $(S, L(S))$ is denoted by $C_L(S)$.[1] Note that every player in $S \subseteq N$ that is not linked with any other player in S is a (singleton) component in $(S, L(S))$. A *cycle* is a path $(i_1, \ldots, i_k)$ with $k \geq 3$, where $\{i_1, i_k\} \in L$. A graph (N, L) is *cycle-free*, if it does not contain any cycle. A graph that is both cycle-free and connected is called a *tree*. The collection of trees is denoted by $\mathcal{L}_T$. Note that in a tree any two players are connected by exactly one path.

A directed graph or digraph is a pair (N, D) where N is a set of nodes and $D \subseteq \{(i,j) | i,j \in N, i \neq j\}$ is a set of ordered pairs of distinct elements of N. As with undirected graphs, the nodes represent the players in a game (N, v), and we refer to them as players. The elements of D are called directed links or *arcs*. We denote the set of all digraphs by $\mathcal{D}$. For $S \subseteq N$, the digraph $(S, D(S))$ with $D(S) = \{(i,j) \in D | i,j \in S\}$ is called the subgraph of D on S. For $i \in N$ the nodes in $F_D(i) := \{j \in N \mid (i,j) \in D\}$ are called the *followers* of i, and the nodes in $P_D(i) := \{j \in N \mid (j,i) \in D\}$ are called the *predecessors* of i. Given $(N, D) \in \mathcal{D}$, a sequence of k different players $(i_1, ..., i_k)$ is a (directed) *path* if $(i_l, i_{l+1}) \in D$ for $l = 1, ..., k-1$. The *transitive closure* of $(N, D) \in \mathcal{D}$ is the digraph $(N, tr(D))$ where

[1] Note that $C_L(S)$ is a partition of S.

for any pair $i, j \in N, i \neq j$, it holds that $(i,j) \in tr(D)$ if and only if there is a directed path from i to j. By $\widehat{F}_D(i) = F_{tr(D)}(i)$ we denote the set of followers of i in the transitive closure of (N, D), and refer to these players as the *subordinates* of i. We refer to the players in $\widehat{P}_D(i) = \{j \in N \mid i \in \widehat{F}_D(j)\}$ as the *superiors* of i. A digraph (N, D) is transitive if $D = tr(D)$. For a set of players $S \subseteq N$ we denote by $F_D(S) = \bigcup_{i \in S} F_D(i)$ the set of all followers of the players in S and by $P_D(S) = \bigcup_{i \in S} P_D(i)$, the set of all predecessors of the players in S. Also, for $S \subseteq N$, we denote $\widehat{F}_D(S) = \cup_{i \in S} \widehat{F}_D(i)$ and $\widehat{P}_D(S) = \cup_{i \in S} \widehat{P}_D(i)$. For a digraph (N, D) we define the associated undirected graph (N, L_D) by $\{i, j\} \in L_D$ if and only if $\{(i,j),(j,i)\} \cap D \neq \emptyset$. In a digraph (N, D) player $i \in N$ is called *top player* when it has no predecessors ($P_D(i) = \emptyset$). A digraph (N, D) is a rooted tree with root i when (i) player i is the unique top player, (ii) $|P_D(j)| = 1$ for all $j \neq i$ and (iii) (N, L_D) is cycle-free and connected. In the sequel we denote by $\mathcal{D}_t$ the class of directed rooted trees and an element of $\mathcal{D}_t$ by (N, T). We denote the unique top player in a rooted tree (N, T) by $Top(N, T)$.

2.3 Communication graph games

A well-known game theoretic model of restrictions on coalition formation is that of games on communication graphs of Myerson (1977). A *communication graph game* or *graph game* is a triple (N, v, L) with $N \subset \mathbb{N}$ a finite set of players, (N, v) a TU-game and (N, L) an undirected communication graph. In a communication graph game (N, v, L) a coalition of players can cooperate if and only if they are able to communicate with each other, i.e. the players in a coalition S can cooperate if and only if S is connected. Myerson (1977) introduced the *restricted game* of a communication graph game (N, v, L) as the TU-game (N, v_L) in which every connected coalition S can earn its worth $v(S)$. Whenever S is not connected it can earn the sum of the worths of its components in $(S, L(S))$. The restricted game (N, v^L) corresponding to communication graph game (N, v, L) is therefore given by

$$v^L(S) = \sum_{T \in C_L(S)} v(T) \text{ for all } S \subseteq N. \tag{2.1}$$

The *Myerson value* μ on the class of communication graph games is defined by $\mu(N, v, L) = Sh(N, v^L)$, i.e., the Myerson value is the solution that assigns to every communication graph game the Shapley value of the restricted game.

For the class of cycle-fee and connected communication graph games, Demange (2004) introduced the notion of *hierarchical outcomes*. These outcomes are defined by associating for every $i \in N$ and $(N, L) \in \mathcal{L}$ a cycle-free and connected graph, the rooted tree (N, T^i_L), where $i \in N$ is the unique root player in (N, T^i_L) and any undirected link $\{k, j\}$ on the unique path from i to $j \neq i$ is converted into a directed link orienting it 'away from' i, so $T_L^i = \{(h,j) \in N \times N \mid \{h,j\} \in L$ and h is on the path from i to $j\}$. The hierarchical outcome with respect to i is the solution h^i defined by

$$h_j^i(N,v,L) = v(\widehat{F}_{T_L^i}(j) \cup \{j\}) - \sum_{h \in F_{T_L^i}(j)} v(\widehat{F}_{T_L^i}(h) \cup \{h\}) \text{ for all } j \in N.$$

The hierarchical outcome with respect to i assigns to any player $j \in N$ its marginal contribution in the graph restricted game (N, v^L) to the coalition of its subordinates in the rooted tree (N, T_L^i).

2.4 Games on a permission structure

A model that studies restrictions in coalition formation arising from hierarchies is that of *games with a permission structure*. In those games it is assumed that players who participate in a cooperative TU-game are part of a hierarchical organization in which there are players that need permission or approval from certain other players before they are allowed to cooperate.

A *game with a permission structure* is a triple (N, v, D) with $N \subset \mathbb{N}$ a finite set of players, $(N, v) \in \mathcal{G}$ a TU-game and $(N, D) \in \mathcal{D}$ a digraph on N. In the *conjunctive approach* to permission structures as developed in Gilles, Owen and van den Brink (1992) and van den Brink and Gilles (1996) a coalition is feasible if and only if for any player in the coalition all its predecessors are also in the coalition. We define

$$\Phi_D^c = \{S \subseteq N \,|\, P_D(i) \subseteq S \text{ for all } i \in S\}$$

as the set of *conjunctively feasible coalitions* in (N, D).

For any $S \subseteq N$, let $\sigma_D^c(S) = \bigcup_{\{T \in \Phi_D^c | T \subseteq S\}} T$ be the largest conjunctively feasible subset[2] of S in the collection Φ_D^c. Then, the induced *conjunctive permission restricted* game of (N, v, D) is the game $(N, r_{N,v,D}^c) \in \mathcal{G}$ given by

$$r_{N,v,D}^c(S) = v(\sigma_D^c(S)) \text{ for all } S \subseteq N. \tag{2.2}$$

The *conjunctive permission value* φ^c on the class of games with a permission structure is defined by

$$\varphi^c(N, v, D) = Sh(N, r_{N,v,D}^c) \text{ for all } (N, v) \in \mathcal{G},\ (N, D) \in \mathcal{D},$$

i.e., φ^c assigns to every game with a permission structure the Shapley value of the conjunctive permission restricted game.

A different approach to acyclic permission structures is given by the disjunctive approach developed in Gilles and Owen (1994) and van den Brink (1997). In this approach

[2] Every coalition having a unique largest feasible subset follows from the fact that Φ_D^c is union closed.

it is assumed that every non-top player needs permission from at least one of its predecessors. Consequently, a coalition is feasible if and only if for every non-top player in the coalition at least one of its predecessors is also in the coalition. In case the digraph is a rooted tree (N, T) the conjunctive and disjunctive approach coincide and we refer to the conjunctive permission restricted game as the permission restricted game and to the conjunctive permission value as the *permission value*, denoted by φ instead of φ^c. We denote the collection of all games with a permission structure (N, v, T) as $\mathcal{G}_\mathcal{T}$ and refer to these games as *permission tree games*.

3 Three new solutions for permission tree games

A single-valued solution f on $\mathcal{G}_\mathcal{T}$ assigns a unique payoff vector $f(N, v, T) \in \mathbb{R}^N$ to every $(N, v, T) \in \mathcal{G}_\mathcal{T}$. The permission value φ as defined in the previous section assigns to any (N, v, T) the Shapley value of the permission restricted game. In this section we define three other solutions for permission tree games.

Consider a directed rooted tree $(N, T) \in \mathcal{D}_t$. Then $(N, L_T) \in \mathcal{L}$ is the cycle-free and connected graph given by $L_T = \{\{i, j\} : (i, j) \in T \text{ or } (j, i) \in T\}$. As first solution, we define the permission Myerson value for permission tree games as the solution obtained by taking the Myerson value of the underlying communication graph game. We denote this solution by μ^p.

Definition 3.1 *The permission Myerson value for permission tree games is the solution* μ^p *given by*

$$\mu^p(N, v, T) = \mu(N, v, L_T),\ (N, v, T) \in \mathcal{G}_\mathcal{T}.$$

Second we define the hierarchical outcome for permission tree games as the solution that for every $(N, v, T) \in \mathcal{G}_\mathcal{T}$ assigns the hierarchical outcome with respect to the unique top player $Top(N, T)$ in the graph game (N, v, L_T). We denote this solution by η.[3]

Definition 3.2 *The hierarchical outcome for permission tree games is the solution* η *given by*

$$\eta(N, v, T) = h^{Top(N,T)}(N, v, L_T),\ (N, v, T) \in \mathcal{G}_\mathcal{T}.$$

Third, we introduce the top value as the solution that for every $(N, v, T) \in \mathcal{G}_\mathcal{T}$ assigns the worth $v(N)$ of the grand coalition N fully to the (unique) top player $Top(N, T)$, while the other players get a payoff of zero. We denote this solution by τ.

[3]When there is no confusion we will refer to this solution as well as the payoff vector it assigns to a permission tree game as hierarchical outcome.

Definition 3.3 *The top value for permission tree games is the solution* τ *given by*

$$\tau_i(N,v,T) = \begin{cases} v(N) & \text{if } i = Top(N,T) \\ 0 & \text{if } i \neq Top(N,T) \end{cases}, \quad (N,v,T) \in \mathcal{G}_{\mathcal{T}}.$$

Notice that for the permission restricted game $(N, r^c_{N,v,T})$ we have that $r^c_{N,v,T}(S) = 0$ for every $S \subseteq N \backslash \{Top(N,T)\}$. From this it follows that $\tau(N,v,T) = h^{Top(N,T)}(N, r^c_{N,v,T}, L_T)$, i.e. the top value assigns to each (N,v,T) the hierarchical outcome with respect to $Top(N,T)$ of the permission tree game $(N, r^c_{N,v,T}, L_T)$.

In the remaining of this section we give the four solutions in terms of the Harsanyi dividends of the underlying game (N,v). Given a tree $(N,T) \in \mathcal{D}_t$, the *connected hull* of a coalition $R \subseteq N$, denoted by $\gamma_T(R)$, is defined as the smallest connected coalition in (N, L_T) containing R. Also the *permission hull* of R, denoted by $\alpha_T(R)$, is defined as $\alpha_T(R) = R \cup \widehat{P}_T(R)$, i.e., the permission hull is the smallest coalition in Φ^c_T (so the smallest coalition with full permission in (N,T)) containing R.[4] Then we have the following results.

Proposition 3.4 *The permission Myerson value* μ^p *can be written in terms of dividends as*

$$\mu^p_i(N,v,T) = \sum_{\{R \subseteq N : i \in \gamma_T(R)\}} \frac{\Delta_v(R)}{|\gamma_T(R)|}, \;\; (N,v,T) \in \mathcal{G}_{\mathcal{T}}, \;\; i \in N.$$

So, the permission Myerson value allocates the dividend of a coalition R equally over the players in the connected hull of R. This result follows from Owen (1986).

Proposition 3.5 *The hierarchical outcome* η *can be written in terms of dividends as*

$$\eta_i(N,v,T) = \sum_{\{R \subseteq N : i = Top(\gamma_T(R), T(\gamma_T(R)))\}} \Delta_v(R), \;\; (N,v,T) \in \mathcal{G}_{\mathcal{T}}, \;\; i \in N.$$

So, the hierarchical outcome assigns the dividend of a coalition R exclusively to the top player in the subtree on the connected $\gamma_T(R)$ of R. This follows straightforward from rewriting the expression of the hierarchical outcome in terms of dividends.

Proposition 3.6 *The top value* τ *can be written in terms of dividends as*

$$\tau_i(N,v,T) = \begin{cases} \sum_{\substack{R \subseteq N \\ R \neq \emptyset}} \Delta_v(R) & \text{if } i = Top(N,T) \\ 0 & \text{if } i \neq Top(N,T) \end{cases}, \quad (N,v,T) \in \mathcal{G}_{\mathcal{T}}.$$

[4]In the literature, the set $\alpha_T(R)$ is also known as the authorizing set of R in (N,T).

The top value assigns the dividend of a coalition R exclusively to the top player in the subtree on the permission hull $\alpha_T(R)$ of R, which is always the top player $Top(N,T)$ of the tree itself. The proof follows from the fact that $\tau_i(N,v,T) = v(N)$ when $i = Top(N,T)$, $\tau_i(N,v,T) = 0$ otherwise and $v(N) = \sum\limits_{S\subseteq N, S\neq\emptyset} \Delta_v(S)$.

Proposition 3.7 *The permission value φ can be written in terms of dividends as:*

$$\varphi_i(N,v,T) = \sum_{\{R\subseteq N: i\in\alpha_T(R)\}} \frac{\Delta_v(R)}{|\alpha_T(R)|},\ (N,v,T)\in\mathcal{G}_\mathcal{T},\ i\in N.$$

From Gilles, Owen and van den Brink (1992), we obtain that the dividend of a coalition R in the conjunctive restricted game $r^c_{N,v,T}$ is given by $\sum\limits_{\{S\subseteq R: R=\alpha_T(S), S\neq\emptyset\}} \Delta_v(S)$. The expression of the permission value in terms of dividends now follows from the fact that the permission value of (N,v,T) is given by the Shapley value of $r^c_{N,v,T}$.

4 Characterization of the four solutions for permission tree games

In this section, we provide comparable axiomatizations of the four solutions discussed before (the three solutions introduced in the previous section and the permission value). We first state the axioms for a solution f on the class of permission tree games that will be used in characterizing the four solutions. The first two axioms are standard. Efficiency states that the payoffs assigned to players must sum to $v(N)$, the worth of the grand coalition.

Efficiency For every $(N,v,T)\in\mathcal{G}_\mathcal{T}$, it holds that $\sum_{i\in N} f_i(N,v,T) = v(N)$.

Additivity states that the solution applied to the sum of two permission tree games on the same permission tree (N,T) gives the same payoff vector as the sum of the two payoff vectors obtained when applying the solution to each of the two permission tree games.

Additivity For every $(N,v,T),(N,w,T)\in\mathcal{G}_\mathcal{T}$, it holds that $f(N,v+w,T) = f(N,v,T) + f(N,w,T)$.

Call a player $i\in N$ a *pending null player* in $(N,v,T)\in\mathcal{G}_\mathcal{T}$ if it is both a null player in game (N,v) (meaning that $v(S\cup\{i\}) = v(S)$ for all $S\subseteq N\setminus\{i\}$) and a pending player in the undirected graph (N,L_T) (meaning there exists only one player $j\in N$ such that $\{i,j\}\in L_T$). Notice that in tree (N,T), player i is a pending player if i has no followers.

Further, the top player is a pending player if and only it has only one follower. All other players have one predecessor and at least one follower and thus are not pending. Notice that when i is pending on rooted tree (N, T), then also the subgraph (N_{-i}, T_{-i}) is a rooted tree, where (N_{-i}, T_{-i}) denotes the subtree $(N_{-i}, T(N_{-i}))$. The pending null player out property states that pending null players can be removed from the game without affecting the payoff distribution of the other players.[5] Let (N_{-i}, v_{-i}, T_{-i}) denote the permission tree game given by the subgame (N_{-i}, v_{-i}) of (N, v) on the subtree (N_{-i}, T_{-i}) of (N, T).

Pending null player out property For every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$, when i is a pending null player in (N, v, T), it holds that $f_j(N, v, T) = f_j(N_{-i}, v_{-i}, T_{-i})$ for $j \in N_{-i}$.

The weak pending null player out property weakens the pending null player out property by only requiring the pending null player out property for those pending null players that are not the top player.

Weak pending null player out property For every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$, when $i \in N \setminus \{Top(N, T)\}$ is a pending null player in (N, v, T), it holds that $f_j(N, v, T) = f_j(N_{-i}, v_{-i}, T_{-i})$ for $j \in N_{-i}$.

A player $i \in N$ is called *necessary* in game (N, v) if $v(S) = 0$ for all $S \subseteq N \setminus \{i\}$. The necessary player property[6] states that in a permission tree game (N, v, T) a necessary player in (N, v) should get at least as much as each of the other players when (N, v) is monotone.

Necessary player property For every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ with (N, v) monotone, if $i \in N$ is a necessary player in (N, v) then $f_i(N, v, T) \geq f_j(N, v, T)$ for all $j \in N$.

The weak necessary player property weakens the necessary player property by stating that necessary players should only get at least as much as their subordinates in the permission structure, if the game is monotone.

Weak necessary player property For every $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ such that (N, v) is monotone, if $i \in N$ is a necessary player in (N, v) then $f_i(N, v, T) \geq f_j(N, v, T)$ for all $j \in \widehat{F}_T(i)$.

[5]For TU-games, Derks and Haller (1999) consider the null player out property meaning that removing a null player from a TU-game has no effect on the payoffs of the remaining players.

[6]In van den Brink and Gilles (1996) this axiom is used to axiomatizatize the conjunctive permission value.

A player $i \in N$ is said to veto player $j \in N$ in game (N, v) if $v(S \cup \{j\}) - v(S) = 0$ for any coalition $S \subseteq N_{-i}$. Let (N, v^i_j) be the game derived from (N, v) when player i vetoes player j, i.e.

$$v^i_j(S) = \begin{cases} v(S \setminus \{j\}) & \text{if } i \notin S \\ v(S) & \text{if } i \in S. \end{cases}$$

When player i vetoes player j in this way, then the dividend of a coalition R in game (N, v) such that $i \notin R$ and $j \in R$ is shifted to that of coalition $R \cup \{i\}$ in game (N, v^i_j). We therefore have the following expression:

$$v^i_j = v + \sum_{\{R \subseteq N : j \in R\}} \Delta_v(R)[u_{R\cup\{i\}} - u_R]. \tag{4.3}$$

Predecessor necessity[7] states that the payoff distribution does not change if for a ordered pair $(i, j) \in T$ the game (N, v) is replaced by the game (N, v^i_j), i.e., if predecessor i is going to veto follower j.

Predecessor necessity For every $(N, v, T) \in \mathcal{G}_\mathcal{T}$ and $i, j \in N$ such that $(i, j) \in T$, it holds that $f(N, v, T) = f(N, v^i_j, T)$.

In the following weaker version of predecessor necessity it is only required that the payoff distribution does not change when i is going to veto his follower j in the game (N, v^i_j) when in the game (N, v) the marginal contribution of player j to any coalition that is a subset of his subordinates is zero.

Weak predecessor necessity For every $(N, v, T) \in \mathcal{G}_\mathcal{T}$ such that $(i, j) \in T$ and $v(S \cup \{j\}) - v(S) = 0$ for all $S \subseteq \widehat{F}_T(j)$, it holds that $f(N, v, T) = f(N, v^i_j, T)$.

Finally, the one player property states that in a game where every player is necessary, only one of them can have a non-zero payoff.

One player property For every $(N, v, T) \in \mathcal{G}_\mathcal{T}$ such that every $i \in N$ is a necessary player in (N, v), it holds that there is at most one player $j \in N$ such that $f_j(N, v, T) \neq 0$.

Before giving the characterizations we first state some propositions that will be used.

Proposition 4.1 *Let $\emptyset \neq R \subseteq N$ and $c \in \mathbb{R}$. Then*

[7] In van den Brink, Herings, van der Laan and Talman (2015) this axiom is used to axiomatizatize the permission value as well as the AT-permission value.

(i) *For any solution f on the class $\mathcal{G}_\mathcal{T}$ satisfying efficiency and the pending null player out property, it holds that $f_i(N, cu^R, T) = 0$ if $i \in N \setminus \gamma_T(R)$, and $f_i(N, cu^R, T) = f_i(\gamma_T(R), (cu^R)_{\gamma_T(R)}, T(\gamma_T(R)))$ for $i \in \gamma_T(R)$.*

(ii) *For any solution f satisfying efficiency and the weak pending null player out property, it holds that $f_i(N, cu^R, T) = 0$ if $i \in N \setminus \alpha_T(R)$, and $f_i(N, cu^R, T) = f_i(\alpha_T(R), (cu^R)_{\alpha_T(R)}, T(\alpha_T(R)))$ for $i \in \alpha_T(R)$.*

(iii) *For any solution f satisfying predecessor necessity, it holds that $f(N, cu^R, T) = f(N, cu^{\alpha_T(R)}, T)$.*

(iv) *For any solution f satisfying weak predecessor necessity, it holds that $f(N, cu^R, T) = f(N, cu^{\gamma_T(R)}, T)$.*

Proof. Consider any permission tree game $(N, v, T) \in \mathcal{G}_\mathcal{T}$. For any player $i \in N$ being a null player in (N, v) we have $v(N) = v(N_{-i}) = v_{N_{-i}}(N_{-i})$.

(i) Let $i \in N$ be a pending null player. Therefore i is a null player. By efficiency it holds that $\sum_{j \in N_{-i}} f_j(N_{-i}, v_{-i}, T_{-i}) = v_{-i}(N_{-i}) = v(N)$. By efficiency it also holds that $\sum_{j \in N} f_j(N, v, T) = v(N)$. By the pending null player out property it holds that $f_j(N, v, T) = f_j(N_{-i}, v_{-i}, T_{-i})$ for $j \in N_{-i}$. Therefore $\sum_{j \in N_{-i}} f_j(N, v, T) = v(N)$ and thus $f_i(N, v, T) = 0$. By repeated application of the pending null player out property and efficiency in this way it follows that $f_i(N, cu^R, T) = 0$ if $i \in N \setminus \gamma_T(R)$ and $f_i(N, cu^R, T) = f_i(\gamma_T(R), (cu^R)_{\gamma_T(R)}, T(\gamma_T(R)))$ for $i \in \gamma_T(R)$.

(ii) Similar to the proof of (i), but applying the weak pending null player out property only for pending players that do not have followers.

(iii) and (iv) The proofs follow straightforwardly from repeated application of predecessor necessity and weak predecessor necessity respectively. □

We first characterize the permission Myerson value on the class of permission tree games.

Theorem 4.2 *A solution on $\mathcal{G}_\mathcal{T}$ is equal to the permission Myerson value μ^p if and only if it satisfies efficiency, additivity, the pending null player out property, weak predecessor necessity and the necessary player property.*

Proof. It is straightforward to verify that the Myerson value satisfies efficiency, additivity, the pending null player out property and the necessary player property.

To show that the permission Myerson value satisfies weak predecessor necessity we argue as follows. By Proposition 3.4, $\mu^p_k(N,v,T) = \sum\limits_{\{R\subseteq N: k\in\gamma_T(R)\}} \frac{\Delta_v(R)}{|\gamma_T(R)|}$ for all $k \in N$. Consider those coalitions R such that $j \in R$ and $i \notin R$, $(i,j) \in T$. Denote by $\mathcal{V}$ the collection of these coalitions R such that $j \in R$ and $i \in \gamma_T(R)$. For $R \in \mathcal{V}$ it holds that $\gamma_T(R) = \gamma_T(R \cup \{i\})$. Denote by $\mathcal{W}$ the collection of those coalitions R such that $j \in R$ and $i \notin \gamma_T(R)$. By Proposition 3.4 and Equation (4.3), $\mu^p_k(N, v^i_j, T) = \sum_{\{R\subseteq N | k\in\gamma_T(R)\}} \frac{\Delta_{v^i_j}(R)}{|\gamma_T(R)|} = \sum\limits_{\{R\subseteq (N\setminus(\mathcal{V}\cup\mathcal{W}): k\in\gamma_T(R)\}} \frac{\Delta_v(R)}{|\gamma_T(R)|} + \sum\limits_{\{R\in\mathcal{V}: k\in\gamma_T(R\cup\{i\})\}} \frac{\Delta_v(R)}{|\gamma_T(R\cup\{i\})|} + \sum\limits_{\{R\in\mathcal{W}: k\in\gamma_T(R\cup\{i\})\}} \frac{\Delta_v(R)}{|\gamma_T(R\cup\{i\})|} = \sum\limits_{\{R\subseteq (N\setminus\mathcal{W}: k\in\gamma_T(R)\}} \frac{\Delta_v(R)}{|\gamma_T(R)|} + \sum\limits_{\{R\in\mathcal{W}: k\in\gamma_T(R\cup\{i\})\}} \frac{\Delta_v(R)}{|\gamma_T(R\cup\{i\})|}$ for all $k \in N$.
Weak predecessor necessity can be applied when $(i,j) \in T$ and $v(S) - v(S \setminus \{j\}) = 0$ for all $S \subseteq (\widehat{F}_T(j) \cup \{j\})$ with $j \in S$. In that case $\Delta_v(R) = 0$ for coalitions R such that $j \in R$ and $i \notin \gamma_T(R)$. These are exactly the coalitions in $\mathcal{W}$. We obtain $\mu^p(N,v,T) = \mu^p(N, v^i_j, T)$, showing that the Myerson value satisfies weak predecessor necessity.

To prove uniqueness, let f be a solution satisfying the axioms. For some $c > 0$, first we consider the permission tree game (N, cu^R, T) for a coalition R connected in the underlying undirected graph (N, L_T), so $R = \gamma_T(R)$. By Proposition 4.1.(i) and f satisfying efficiency and the pending null player out property, the players in $N \setminus \gamma_T(R) = N \setminus R$ obtain a payoff of 0. Therefore, by efficiency the players in R together obtain $cu^R(N) = c$. Since the players in R are all necessary players in (N, cu^R, T) and cu^R is monotone because $c > 0$, the necessary player property implies that $f_i(N, cu^R, T) = \frac{c}{|R|}$ for all $i \in R$. So, $f(N, cu^R, T)$ is uniquely determined for $c > 0$ and R connected in (N, L_T).

Now, for some $c > 0$, consider those coalitions R not connected in (N, L_T), so $R \neq \gamma_T(R)$. By Proposition 4.1.(iv) and f satisfying weak predecessor necessity, it holds that $f(N, cu^R, T) = f(N, cu^{\gamma_T(R)}, T)$. Since $\gamma_T(R)$ is a connected coalition in (N, L_T), $f(N, cu^{\gamma_T(R)}, T)$ has been uniquely determined above and therefore also $f(N, cu^R, T)$ is uniquely determined.

Consider the permission tree game $(N, null, T)$, with $(N, null)$ the null game. Then every $i \in N$ is a null player in $(N, null)$. By repeated application of the pending null player out property and efficiency it holds that $f_i(N, null, T) = 0$ for $i \in N$. Next, consider (N, cu^R, T) for some $c < 0$. Since $null = cu^R + (-cu^R)$, it follows from additivity of f that $f(N, cu^R, T) = f(N, null, T) - f(N, -cu^R, T) = -f(N, -cu^R, T)$ is uniquely determined above because $-c > 0$.

Finally, since for every $(N, v, T) \in \mathcal{G}_\mathcal{T}$ it holds that v can be written as $v = \sum_{\substack{R\subseteq N\\ R\neq\emptyset}} \Delta_v(R) u^R$, additivity uniquely determines $f(N, v, T) = \sum_{\substack{R\subseteq N\\ R\neq\emptyset}} f(N, \Delta_v(R) u_R, T)$ for any $(N, v, T) \in \mathcal{G}_\mathcal{T}$. □

Logical independence of the axioms used in this theorem, as well as for the following characterizations of the other three solutions is shown in the appendix.

We characterize the hierarchical outcome for permission tree games by replacing the necessary player property in Theorem 4.2 by the weak necessary player property and the one player property.

Theorem 4.3 *A solution on $\mathcal{G}_{\mathcal{T}}$ is equal to the hierarchical outcome η if and only if it satisfies efficiency, additivity, the pending null player out property, weak predecessor necessity, the weak necessary player property and the one player property.*

Proof. It is straightforward to verify that the hierarchical outcome satisfies efficiency, additivity, the pending null player out property, the weak necessary player property and the one player property. By arguments similar to the proof that the Myerson value satisfies weak predecessor necessity it follows that the hierarchical outcome satisfies weak predecessor necessity.

To prove uniqueness, let f be a solution satisfying the axioms. For some $c > 0$, again first we consider the permission tree game (N, cu^R, T) for a coalition R connected in the underlying undirected graph (N, L_T), so $R = \gamma_T(R)$. By Proposition 4.1.(i) and f satisfying efficiency and the pending null player out property, the players in $N \setminus \gamma_T(R) = N \setminus R$ obtain a payoff of 0 and $f_i(N, cu^R, T) = f_i(R, (cu^R)_R, T(R))$ for $i \in R$. Since in $(R, (cu^R)_R, T(R))$ all players are necessary we can apply the one player property to obtain that there is only one player $j \in R$ such that $f_j(R, (cu^R)_R, T(R)) \neq 0$. Let $r_0 = Top(R, R(T))$ be the top player in the subtree $(R, T(R))$. Since r_0 is a necessary player in $(R, (cu^R)_R)$, every $j \in R \setminus \{r_0\}$ is a subordinate of r_0 in $(R, T(R))$, and cu^R is monotone (because $c > 0$), the weak necessary player property implies that $f_{r_0}(R, (cu^R)_R, T(R)) \geq f_i(R, (cu^R)_R, T(R))$ for all $i \in R \setminus \{r_0\}$. So, the one player property, the weak necessary player property and efficiency imply that $f_{r_0}(R, (cu^R)_R, T(R)) = c$ and $f_i(R, (cu^R)_R, T(R)) = 0$ for all $i \in R \setminus \{r_0\}$. Therefore $f(N, cu^R, T)$ is uniquely determined.

The cases (i) $R \neq \gamma_T(R)$ and $c > 0$, (ii) $c < 0$ and (iii) $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ follow along similar lines to those in the proof of Theorem 4.2 for the permission Myerson value. □

Next the top value is characterized by replacing in Theorem 4.3 the pending null player out property by the weak pending null player out property and by replacing weak predecessor necessity by predecessor necessity.

Theorem 4.4 *A solution on $\mathcal{G}_{\mathcal{T}}$ is equal to the top value τ if and only if it satisfies efficiency, additivity, the weak pending null player out property, predecessor necessity, the weak necessary player property and the one player property.*

Proof. It is straightforward to verify that the top value satisfies efficiency, additivity, the weak pending null player out property, the weak necessary player property and the one player property. The top value satisfying predecessor necessity follows from $v^i_j(N) = v(N)$ and the fact that for any two games (N, v, T) and (N, v', T) such that $v(N) = v'(N)$ it holds that $\tau(N, v, T) = \tau(N, v', T)$.

To prove uniqueness, let f be a solution satisfying the axioms. For some $c > 0$, first we consider the permission tree game (N, cu^R, T) for a coalition R with R having full permission in T, so R equal to the permission hull $\alpha_T(R)$. By Proposition 4.1.(ii) and f satisfying efficiency and the weak pending null player out property, the players in $N \setminus \alpha_T(R) = N \setminus R$ obtain a payoff of 0 and $f_i(N, cu^R, T) = f_i(R, (cu^R)_R, T(R))$ for $i \in R$. Since all players in the game $(R, (cu^R)_R, T(R))$ are necessary we can apply the one player property to obtain that there is only one player $j \in R$ such that $f_j(R, (cu^R)_R, T(R)) \neq 0$. Let $i_0 = Top(N, T)$. Since $i_0 \in R$ (because $R = \alpha_T(R)$), i_0 is necessary in $(R, (cu^R)_R)$, every $j \in R \setminus \{i_0\}$ is a subordinate of r_0 in $(R, T(R))$, and cu^R is monotone because $(c > 0)$, the weak necessary player property implies that $f_{i_0}(R, (cu^R)_R, T(R)) \geq f_i(R, (cu^R)_R, T(R))$ for all $i \in R \setminus \{i_0\}$. So, the one player property, the weak necessary player property and efficiency imply that $f_{i_0}(R, (cu^R)_R, T(R)) = c$ and $f_i(R, (cu^R)_R, T(R)) = 0$ for all $i \in R \setminus \{i_0\}$. Therefore $f(N, cu^R, T)$ is uniquely determined.

Next consider those coalitions R that do not have full permission in (N, T), so $R \neq \alpha_T(R)$. By Proposition 4.1.(iv) and f satisfying predecessor necessity, it holds that $f(N, cu^R, T) = f(N, cu^{\alpha_T(R)}, T)$. Since $\alpha_T(R)$ has full permission in (N, T), $f(N, cu^{\alpha_T(R)}, T)$ has been uniquely determined above and therefore also $f(N, cu^R, T)$ is uniquely determined.

The cases $c < 0$ and $(N, v, T) \in \mathcal{G}_\mathcal{T}$ follow along similar lines to those in the proof of Theorem 4.2 for the permission Myerson value. □

Finally, the permission value is characterized by using only axioms that have been used in the three axiomatizations before.

Theorem 4.5 *A solution on $\mathcal{G}_\mathcal{T}$ is equal to the permission value φ if and only if it satisfies efficiency, additivity, the weak pending null player out property, predecessor necessity and the necessary player property.*

Proof. The permission value satisfying efficiency, additivity and the necessary player property follows from van den Brink and Gilles (1996), φ satisfying predecessor necessity follows from van den Brink, Herings, van der Laan and Talman (2015), and it is straightforward to show that φ satisfies the weak pending null player out property.

To prove uniqueness, let f be a solution satisfying the axioms. For some $c > 0$, first we consider the permission tree game (N, cu^R, T) for a coalition R with $R = \alpha_T(R)$.

By Proposition 4.1.(ii) and f satisfying efficiency and the weak pending null player out property, the players in $N \setminus \alpha_T(R) = N \setminus R$ obtain a payoff of 0. Therefore by efficiency the players in R obtain $cu^R(N) = c$. Since all players in R are necessary and cu^R is monotone, the necessary player property implies that $f_i(N, cu^R, T) = \frac{c}{|R|}$ for every $i \in R$, and thus $f(N, cu^R, T)$ is uniquely determined.

Now consider those coalitions R that do not have full permission in (N, T), so $R \neq \alpha_T(R)$. By Proposition 4.1.(iv) and f satisfying predecessor necessity it holds that $f(N, cu^R, T) = f(N, cu^{\alpha_T(R)}, T)$. Since $\alpha_T(R)$ has full permission in (N, T), $f(N, cu^{\alpha_T(R)}, T)$ has been uniquely determined above and therefore also $f(N, cu^R, T)$ is uniquely determined.

The cases $c < 0$ and $(N, v, T) \in \mathcal{G}_\mathcal{T}$ follow along similar lines to those in the proof of Theorem 4.2 for the permission Myerson value. □

5 Comparing the four solutions

Table 1:Axioms satisfied by the four solutions

Property	μ^p	η	τ	φ
Efficiency	++	++	++	++
Additivity	++	++	++	++
Weak pending null player out	+	+	++	++
Weak necessary player	+	+	+	++
Weak predecessor necessity	++	++	+	+
Pending null player out	++	++	-	-
Necessary player	++	-	-	++
Predecessor necessity	-	-	++	++
One player property	-	++	++	-

Table 1 gives an overview of the axioms used to characterize the four solutions for permission tree games (those marked with a ++). Moreover, it shows which other axioms are satisfied by the solutions (those marked with a +). In all four axiomatizations we use the first two axioms: efficiency and additivity. All four solutions also satisfy the next two axioms: the weak pending null player out property and weak predecessor necessity. Although not appearing explicitly in all axiomatizations, they do appear implicitly in all axiomatizations since the two axiomatizations that do not use the weak pending null player out property use the stronger pending null player out property, and the two axiomatizations that do not use weak predecessor necessity use the stronger predecessor necessity. The four solutions are distinguished by the last four axioms in the table. Each of the four solutions satisfies exactly two of the final four axioms in the table, and together with the previous axioms these give an axiomatization of the corresponding solution. Moreover, each of these four axioms appears in precisely two axiomatizations.

In these axiomatizations we used four of the six combinations containing two out of these last four axioms. This leaves two combinations that have not been considered. It turns out that these four combinations of two out of four axioms are actually the only possible ones. We show that a solution satisfying efficiency, additivity, the weak pending null player out property, the weak necessary player property and weak predecessor necessity and one of these two combinations of axioms leads to a contradiction and therefore cannot exist. In other words, a solution satisfying efficiency, additivity, the weak pending null player out property and weak predecessor necessity cannot also satisfy both the pending null player out property and predecessor necessity nor satisfy both the necessary player property and the one player property.

Proposition 5.1 *There does not exist a solution f satisfying efficiency, additivity, the pending null player out property, the weak necessary player property and predecessor necessity.*

Proof. Consider a permission tree game $(N, u^{\{i\}}, T)$, where $|N| > 1$ and $(N, u^{\{i\}})$ is the unanimity game of a player $i \neq Top(N, T)$. By Proposition 4.1.(i) and f satisfying efficiency and the pending null player out property, it holds that $f_i(N, u^{\{i\}}, T) = u^{\{i\}}(N) = 1$ and $f_j(N, u^{\{i\}}, T) = 0$ for $j \in N \setminus \{i\}$. By repeatedly applying predecessor necessity, we also obtain that $f(N, u^{\{i\}}, T) = f(N, u^{\alpha_T(i)}, T)$. According to the weak necessary player property it holds that $1 = f_i(N, u^{\{i\}}, T) = f_i(N, u^{\alpha_T(i)}, T) \leq f_j(N, u^{\alpha_T(i)}, T) = f_j(N, u^{\{i\}}, T) = 0$ for $j \in \widehat{P}_T(i)$. Since $\widehat{P}_T(i) \neq \emptyset$ we obtain a contradiction and f can not exist. □

Proposition 5.2 *There does not exist a solution f satisfying efficiency, additivity, the weak pending null player out property, the necessary player property and weak predecessor necessity and the one player property.*

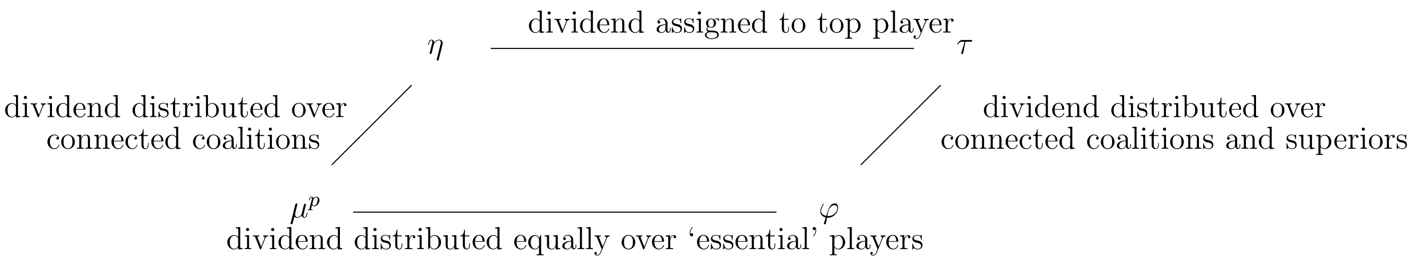
Proof. For $|N| > 1$, consider a permission tree game (N, u^N, T). According to the one player property only one player can have a payoff that is non-zero. Efficiency implies that the payoff to this player must be $u^N(N) = 1$. However the necessary player property implies that $f_i(N, u^N, T) = f_j(N, u^N, T)$ for any two players $i, j \in N$. Since in u^N all players in N are necessary and $|N| > 1$ we obtain a contradiction and f can not exist. □

In this way we have comparable axiomatizations of the four solutions for permission tree games. Next, we describe this comparison in more detail.

The pending null player out property is satisfied by the permission Myerson value and the hierarchical outcome. It implies that players who do not contribute anything in the game nor connect any contributing players, obtain a zero payoff. This follows from the communication feature of these two solutions. The permission and top value may still grant such players a nonzero payoff, by domination of contributing followers. This follows from the hierarchical feature of these two solutions. This is also shown by these solutions satisfying the predecessor necessity property, whereas the permission Myerson value and hierarchical outcome do not. Therefore, the permission Myerson value and the hierarchical outcome may be thought of as 'communication solutions', whereas the permission value and the top value might be considered 'hierarchy solutions'.

However this does not imply that 'bottom players' will always obtain a higher payoff from a 'communication value' than from a 'hierarchy value'. For example, considering the unanimity game of a connected coalition R with $|R| \geq 2$ containing bottom player j, player j gets zero payoff according to the hierarchical outcome, but earns a positive payoff according to the permission value. The reason is that the hierarchical outcome satisfies the one player property, assigning the full unanimity payoff to the 'local' top player in the connected coalition, i.e. the player in the coalition who is closest to the root, while the permission value assigns equal payoffs to player j and all of its superiors because it satisfies the necessary player property. In this sense there is another distinction between the four solutions. On the one hand the permission Myerson value and the permission value satisfy the necessary player property, equally distributing the dividend of a coalition among the players needed to make that coalition feasible. The hierarchical outcome and the top value on the other hand satisfy the one player property, assigning the payoff of a coalition to the unique top player among the players needed to make that coalition feasible. We summarize this in the following diagram.

Diagram 1: Solution classifications



The interpretation of these solutions with respect to the hierarchy can be looked at in the following way. The permission Myerson value and the hierarchical outcome take a more local approach to hierarchies; coalitions within the hierarchy have some sense of autonomy in that they can operate without needing their predecessors to sign off on everything they do. The permission value and the top value take a global approach to hierarchies; everything a coalition does, needs to be approved by players located at a higher position in the hierarchy. The permission Myerson value and the permission value interpret the players of any feasible coalition in the hierarchy to play an equally important role. They should therefore be rewarded equally. The hierarchical outcome and the top value interpret the (local or global) top player of any feasible coalition to be the one that is in control, therefore this player should obtain all of the rewards.

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Appendix: Logical independence of axiomatizations in Section 4

Logical independence of the five axioms stated in Theorem 4.2 is shown by the following alternative solutions for permission tree games.

1. The solution $f_i(N,v,T) = 0$, $(N,v,T) \in \mathcal{G}_\mathcal{T}$, $i \in N$, satisfies additivity, the pending null player out property, weak predecessor necessity and the necessary player property. It does not satisfy efficiency.

2. For $(N,v,T) \in \mathcal{G}_\mathcal{T}$ let $Null(v)$ be the set of null players in v. Let f be the solution that for $(N,v,T) \in \mathcal{G}_\mathcal{T}$ divides the worth $v(N)$ of the grand coalition equally over all non-null players and the players that connect these players in the graph, and assigns a 0 payoff otherwise. So for $(N,v,T) \in \mathcal{G}_\mathcal{T}$ solution f is given by $f_i(N,v,T) = \frac{v(N)}{|\gamma_T(N \setminus Null(v))|}$ for $i \in \gamma_T(N \setminus Null(v))$ and $f_i(N,v,T) = 0$ for $i \in N \setminus \gamma_T(N \setminus Null(v))$. This solution satisfies efficiency, the pending null player out property, weak predecessor necessity and the necessary player property. It does not satisfy additivity.

3. The permission value satisfies efficiency, additivity, weak predecessor necessity and the necessary player property. It does not satisfy the pending null player out property.

4. The solution $f(N,v,T) = Sh(N,v)$, $(N,v,T) \in \mathcal{G}_\mathcal{T}$ satisfies efficiency, additivity, the pending null player out property and the necessary player property. It does not satisfy weak predecessor necessity.

5. The hierarchical outcome satisfies efficiency, additivity, the pending null player out property and weak predecessor necessity. It does not satisfy the necessary player property.

Logical independence of the six axioms stated in Theorem 4.3 is shown by the following alternative solutions for permission tree games.

1. The solution $f_i(N,v,T) = 0$, $(N,v,T) \in \mathcal{G}_\mathcal{T}$, $i \in N$, satisfies additivity, the pending null player out property, weak predecessor necessity, the weak necessary player property and the one player property. It does not satisfy efficiency.

2. Consider the class of games (N,v,T), for which there is a coalition $R \subseteq N$ such that $\Delta_v(S) = 0$ if $\gamma_T(S) \neq R$. The solution that on this class of games assigns the hierarchical outcome, and on games not in this class assigns the permission Myerson value satisfies efficiency, the pending null player out property, weak predecessor necessity, the weak necessary player property and the one player property. It does not satisfy additivity.

3. The top value satisfies efficiency, additivity, weak predecessor necessity, the weak necessary player property and the one player property. It does not satisfy the pending null player out property.

4. For a coalition S define $A(S) = (\gamma_T(S) \setminus (\widehat{F}_T(S) \cup S))$ to be those players in the connected hull $\gamma_T(S)$ that are not included in S nor are they subordinates of players in S. Now consider $f_i(N,v,T) = \sum_{\{S\subseteq N: i\in S, i=Top(\gamma_T(S),T(\gamma_T(S)))\}} \Delta_v(S) + \sum_{\{S\subseteq N: S\neq\emptyset, Top(\gamma_T(S),T(\gamma_T(S)))\notin S, i\in A(S)\}} \frac{\Delta_v(S)}{|A(S)|}$ $i \in N$, $(N,v,T) \in \mathcal{G}_{\mathcal{T}}$. This solution assigns the dividend of coalitions S, such that the top player of the connected hull $\gamma_T(S)$ is included in S uniquely to that top player, and equally distributes the dividend of coalitions S not containing this top player over those players in the connected hull $\gamma_T(S)$ that are not included in S nor are they followers of players in S. This solution satisfies efficiency, additivity, the pending null player out property, the weak necessary player property and the one player property. It does not satisfy weak predecessor necessity.

5. Consider the solution that for $(N,v,T) \in \mathcal{G}_{\mathcal{T}}$ assigns the dividend $\Delta_v(S)$ of a coalition S to the player $i \in \gamma_T(S) \setminus \{Top(\gamma_T(S), T(\gamma_T(S)))\}$, such that $i > j$ for $j \in (\gamma_T(S) \setminus \{Top(\gamma_T(S), T(\gamma_T(S)))\}) \setminus \{i\}$ for games $(N,v,T) \in \mathcal{G}_{\mathcal{T}}$. This solution satisfies efficiency, additivity, the pending null player out property, weak predecessor necessity and the one player property. It does not satisfy the weak necessary player property.

6. The Myerson value satisfies efficiency, additivity, the pending null player out property, weak predecessor necessity and the weak necessary player property. It does not satisfy the one player property.

Logical independence of the six axioms stated in Theorem 4.4 is shown by the following alternative solutions for permission tree games.

1. The solution $f_i(N,v,T) = 0$, $(N,v,T) \in \mathcal{G}_{\mathcal{T}}$, $i \in N$, satisfies additivity, the weak pending null player out property, predecessor necessity, the weak necessary player property and the one player property. It does not satisfy efficiency.

2. Let V be the class of games (N,v,T), for which there is a coalition $R \subseteq N$ such that $\Delta_v(S) = 0$ if $\alpha_T(S) \neq R$. The solution that on this class of games assigns the top value and on games not in this class assigns the permission value satisfies efficiency, the weak pending null player out property, predecessor necessity, the weak necessary player property and the one player property. It does not satisfy additivity.

3. Consider the solution that for $(N,v,T) \in \mathcal{G}_{\mathcal{T}}$ assigns the dividend $\Delta_v(S)$ of coalitions S such that $\alpha_T(S) = N$ to the $Top(N,T)$ and distributes the dividend $\Delta_v(S)$ equally over N otherwise. This solution satisfies efficiency, additivity, predecessor necessity, the weak necessary player property and the one player property. It does not satisfy the weak pending null player out property.

4. The hierarchical outcome satisfies efficiency, additivity, the weak pending null player out property, the weak necessary player property and the one player property. It does not satisfy predecessor necessity.

5. Consider the solution that for $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ assigns the dividend $\Delta_v(S)$ of a coalition S to the player $i \in \alpha_T(S)\backslash\{Top(N,T)\}$ such that $i > j$ for $j \in (\alpha_T(S)\backslash\{Top(N,T)\})\backslash \{i\})$ for games $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$. This solution satisfies efficiency, additivity, the weak pending null player out property, predecessor necessity and the one player property. It does not satisfy the weak necessary player property.

6. The permission value satisfies efficiency, additivity, the weak pending null player out property, predecessor necessity and the weak necessary player property. It does not satisfy the one player property.

Logical independence of the five axioms stated in Theorem 4.5 is shown by the following alternative solutions for permission tree games.

1. The solution $f_i(N, v, T) = 0$, $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$, $i \in N$, satisfies additivity, the weak pending null player out property, weak predecessor necessity, the weak necessary player property and the one player property. It does not satisfy efficiency.

2. The solution that for $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ equally divides the worth $v(N)$ of the grand coalition over all non-null players and their predecessors in the graph and assigns a 0 payoff otherwise satisfies efficiency, the weak pending null player out property, predecessor necessity and the necessary player property. It does not satisfy additivity.

3. The solution that for $(N, v, T) \in \mathcal{G}_{\mathcal{T}}$ equally divides the worth $v(N)$ of the grand coalition over the players in N satisfies efficiency, additivity, predecessor necessity and the necessary player property. It does not satisfy the weak pending null player out property.

4. The solution $f(N, v, T) = Sh(N, v)$ satisfies efficiency, additivity, the weak pending null player out property and the necessary player property. It does not satisfy predecessor necessity.

5. The top value satisfies efficiency, additivity, the weak pending null player out property and predecessor necessity. It does not satisfy the necessary player property.