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Mixed Density based Copula Likelihood

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Abstract

We consider a new copula method for mixed marginals of discrete and continuous random variables. Unlike the Bayesian methods in the literature, we use maximum likelihood estimation based on closed-form copula functions. We show with a simulation that our methodology performs similar to the method of Hoff (2007) for mixed data, but is considerably simpler to estimate. We extend to a time series setting, where the parameters are allowed to vary over time. In an empirical application using data from the 2013 Household Finance Survey, we show how the copula dependence between income (continuous) and discrete household characteristics varies across groups who were affected differently by the recent economic crisis.

JEL Classification: C18, C22, C58.

Keywords: Copula, discrete data, time series.

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1 Introduction

Recently copula models have gained popularity in various fields. They offer a simple and neat environment to specify joint behaviour among random variables. The decomposition of the joint distribution through a copula function implies that the marginal characteristics of the random variables can be specified separately from the dependence structure. There is a vast array different copula functions, each of which can capture different, possibly complex dependence structures (asymmetric, tail dependence, etc.), see Joe (1997) for a detailed review of copula functions, or Nelsen (2007) for their theoretical properties. Copula models are used in wide variety of applications, such as Embrechts et al. (2002) Value-at-Risk analysis in finance to Munkin and Trivedi (1999) health utilization data application. Cherubini et al. (2004) present various financial applications using copula models, and provide a thorough survey of the copula literature. Trivedi and Zimmer (2006) similarly cover various aspects of discrete copulas and review their use in microeconometrics. literature.

This paper proposes a new technique to allow the dependence structure between discrete and continuous random variables to be specified through copula functions. The novelty of our approach is that the methodology does not require computationally intensive Bayesian routines, unlike some other proposals in the literature. As a second contribution, we show how to extend the model to a time series setting with time-varying marginals and possibly time-varying dependence structure as well. It is the first time in literature that a joint analysis among random variables of different types is conducted through simple likelihood evaluation. Due to the mixture of different marginals, the copula likelihood function does not take the standard density form nor the form of a probability mass function, but rather a mix of both. We refer to this mixed expression as the mixed copula density.

With a few exceptions, the copula literature either deals with having all random variables in a multivariate system belonging to continuous distributions or all to discrete distributions. According to Sklar (1959)'s theorem, a copula is uniquely identified if the marginal distributions are all continuous. For discrete marginals the copula is not unique and can only be defined on the range of the discrete marginal distributions. Despite the non-uniqueness, empirical researchers typically still look for parsimonious copula functions that describe the data at

hand, even though such copulas may also be defined outside the range of the discrete marginal distributions.

Recently there has been an interest in allowing a mixture of discrete and continuous random variables in a multivariate analysis. Pitt et al. (2006) propose a multivariate gaussian copula with mixed type marginals. The data in their case is generated through a continuous copula function, but for the discrete margins the appropriate coordinates from the copula draw are projected onto the range of the marginal distributions. During the estimation, these coordinates are considered latent and are integrated out. Smith and Khaled (2012) extend the previous methodology to the case of an arbitrary copula function. Both papers use Markov Chain Monte Carlo (MCMC) techniques. Hoff (2007) proposes a semi-parametric multivariate gaussian copula, where the marginal distributions are left unspecified. The only information he uses from the observed data to estimate the multivariate copula consists of the order statistics. The method, however, suffers from large posterior correlation, and hence requires intensive computations to obtain an independent posterior sample for inference. Our mixed copula likelihood function, by contrast, separates the impact of each type of random variable (discrete/continuous) in a simple and elegant way. The method is easy to implement and similar to the case of only discrete or only continuous marginals, and can be readily estimated through maximum likelihood.

We extend our method to the time series context. In time series, it is often important to allow parameters to vary over time. Therefore, Patton (2006) proposes a conditional copula approach where the copula parameters vary over time using average pairwise differences between the past marginal probabilities integral transforms. Rodriguez (2007) also considers time-varying copula parameters, but uses a Markov switching model. Our dynamic mixed density copula model also allows the parameters (both of the marginal and the copula) to change over time. Using the generalized autoregressive score approach of Creal et al. (2013). The approach is observation driven in the classification of Cox (1981) and results in a closed form expression for the likelihood function. The approach has been applied successfully in for example Creal et al. (2013), Lucas et al. (2014), Harvey and Luati (2014). Blasques et al. (2014) establish a number of information theoretical optimality properties for models with generalized autoregressive score dynamics.

We present an extensive simulation to show how our mixed density approach can consistently

and efficiently estimate the copula parameters. We benchmark our results against the approach of Hoff (2007) and the most commonly employed Pearson correlation measure. Over various sample sizes, we show how our method produces similar result to Hoff’s method and yet does not require the computational burden of a Bayesian approach. The standard pearson correlations, by contrast, produce large biases for the correlation estimates, especially for low count discrete data. Such biases persist through various sample sizes.

To illustrate our technique empirically, we employ a real data example based on the U.S. household finance survey data. We observe a number of key variables related to household wealth and income, such as the number of vehicles a household has (count), whether the house is owned (binary), if the household had any form of credit request turned down (binary), and the income (continuous). The multivariate analysis will be conducted over two different groups: households who ended up in bankruptcy, and stable households with no bankruptcy. The dependence structure varies across these groups.

The remainder of the paper is set up as follows: In Section 2 we introduce the mixed density copula and the time-varying extension. In Section 3 we present a detailed simulation study, where we compare our method to number of relevant benchmarks. In Section 4, we present an empirical application to household finance data. Section 5 concludes.

2 Copula Setup

Let $Y = (Y_1, \dots, Y_p)$ be a p -dimensional random variable. Following Sklar’s (1959) theorem, the distribution function H of Y can be decomposed into the marginal distributions F_i of Y_i for $i = 1, \dots, p$, and a Copula C . The copula fully captures the dependence between Y_i and Y_j for $i \neq j$. Formally, we have

$$H(y_1, \dots, y_p; \beta_1, \dots, \beta_p, \theta) = C(F_1(y_1; \beta_1), \dots, F_p(y_p; \beta_p); \theta),$$

where β_i is a parameter vector describing the i th marginal distribution, and θ is a parameter vector describing the shape of the copula. The copula satisfies $C : [0, 1]^p \mapsto [0, 1]$ and can be seen as a distribution function on a p -dimensional hypercube. Define the probability integral

transforms $U_i = F_i(Y_i; \beta_i)$ for $i = 1, \dots, p$, then

$$C(u_1, \dots, u_p; \theta) = P(U_1 \leq u_1, \dots, U_p \leq u_p).$$

As mentioned in the introduction, there is a large number of copula families available in the literature; see for example Nelsen (2007) for a detailed overview.

The domain of the copula function is $\text{Ran } F_1 \times \dots \times \text{Ran } F_p$, where $\text{Ran } F_i$ denotes the range of the marginal distribution functions F_i . If all the elements of Y are continuous with a well-behaved density function, the domain of C is the entire p -dimensional hypercube. In that case, the copula density can be defined as

$$c(u_1, \dots, u_p; \theta) = \frac{\partial^p C(u_1, \dots, u_p; \theta)}{\partial u_1 \dots \partial u_p}.$$

If, on the other hand, all elements of Y are discretely valued as integer counts, the copula has no density but rather has a probability mass function, which is given by

$$c(u_1, \dots, u_p; \theta) = \sum_{s_1=0,1} \dots \sum_{s_p=0,1} (-1)^{s_1+\dots+s_p} C(F_1(y_1 - s_1; \beta_1), \dots, F_1(y_p - s_p; \beta_p); \theta). \quad (2.1)$$

Evaluating the copula mass function requires 2^p evaluations of the copula function. For discrete marginals, the copula is not unique. This implies there can be several copula functions that describe exactly the same dependence structure. For more details about the discrete copula characteristics, we refer to Genest and Nešlehová (2007).

2.1 Mixed Copula Density

Our main interest in this paper lies in the case we have a mix of discrete and continuously valued responses. Using this mixed data set-up, we propose a way to specify a copula function for the dependence structure and subsequently generalize it to a time series setting. If data are a mix of discrete and continuous variables, the copula does not have full support on the p -dimensional hypercube. On the other hand, the domain is more than a finite set of points, as would be the case if all the data were discretely valued.

Let $p \geq 2$, and let k denote the number of discrete random variables, with $0 < k < p$. The number of continuous random variables therefore equals $p - k$. Without loss of generality, let

F_1, \dots, F_k be the discrete marginals. Given the discrete nature of part of the observations, the copula C is still not unique. The range in this case is given by

$$(\text{Ran } F_1 \times \dots \times \text{Ran } F_k) \times [0, 1]^{p-k}.$$

As a result, we neither have a proper copula density nor a copula probability mass function. Rather, we have a mix of both, which we call the mixed copula density from now on. There will be continuous segments of the copula along the dimension of the continuous elements $F_i(Y_i)$ for $i = k + 1, \dots, p$. These continuous segments are clearly separated in dimensions $1, \dots, k$ due to the discrete nature of the first k coordinates.

This is best understood by a simple bivariate simulation example. Consider a setting where Y_1 follows a Poisson distribution, whereas Y_2 is normally distributed. We link Y_1 and Y_2 through a Clayton copula structure. The result is shown in Figure 1. We use a value $\theta = 6$ for the Clayton copula parameter, which corresponds to a Kendall's τ rank correlation of approximately 0.75. There are 9 different vertical segments in the figure, corresponding to the 9 distinct realizations for Y_1 in the simulations. The Clayton copula particularly results in strong dependence in the lower tail. This is clear from the fact that the spread of the probability integral transforms U_2 of the continuously valued normal random variable Y_2 are much tighter for small values of U_1 than for large values.

To define our mixed copula density, we first differentiate the joint copula distribution with respect to the continuous coordinates. We obtain the conditional copula function of the first k coordinates given the value of the last $p - k$ coordinates,

$$H_{1, \dots, k | k+1, \dots, p}(y_1, \dots, y_k | y_{k+1}, \dots, y_p) = \frac{\partial^{p-k} C(F_1(y_1), \dots, F_p(y_p))}{\partial F_{k+1}(y_{k+1}) \cdots \partial F_p(y_p)} \cdot \prod_{j=k+1}^p f_j(y_j). \quad (2.2)$$

The availability of closed-form expressions for the derivatives in (2.2) for most copula families makes helps to speed up the subsequent computations needed for our model. Using the conditional copula in (2.2), we compute the probability mass function for the discrete parts of the copula using equation (2.1). We assume that the discrete data takes integer values and obtain

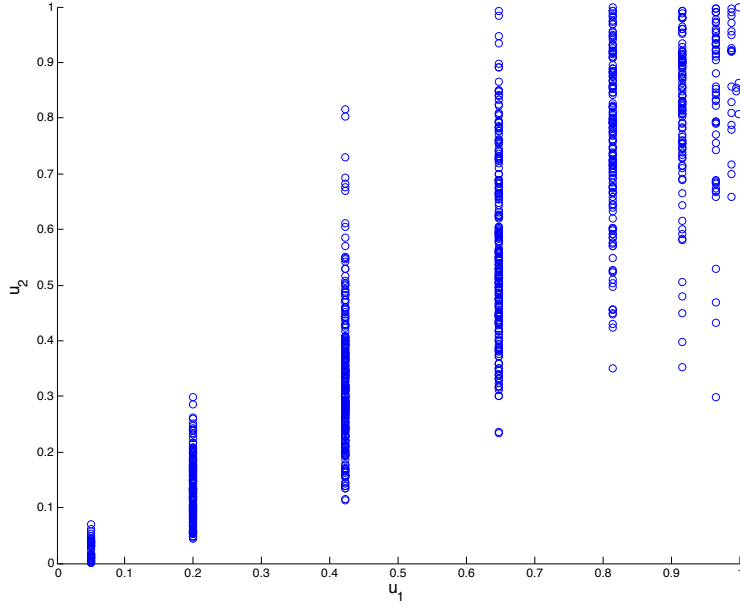


Figure 1: Clayton copula scatter

the mixed copula density

$$h(y_1, \dots, y_p; \beta_1, \dots, \beta_p, \theta) = \prod_{j=k+1}^p f_j(y_j; \beta_j) \cdot \sum_{s_1=0,1} \dots \sum_{s_k=0,1} (-1)^{s_1+\dots+s_k} H_{1,\dots,k|k+1,\dots,p}(y_1 - s_1, \dots, y_k - s_k | y_{k+1}, \dots, y_p).$$

Computing the mixed copula density is invariant to whether one first takes derivatives and then computes the discrete changes, or the other way around. Unlike the case for a completely discrete copula, the number of times the (conditional) copula distribution needs to be evaluated is reduced to $2^k + 1$. We present two mixed density copula examples in Appendix A, namely for a Gaussian and a Clayton copula. Estimation of the copula parameter θ and the marginal parameters β_i , $i = 1, \dots, p$, now proceeds as is standard in the literature, either through Inference for Margins (IFM) or using 1-step estimation by maximum likelihood.

2.2 Dynamic Mixed Copula Density

So far, our model was formulated as a static model. In many settings of interest, however, the parameters of the marginal distribution, the copula, or both, may vary over time. For

example, economic and financial time series often exhibit different characteristics over time, such as changing means or variances, changing correlations or dependence structures, or changing (default) probabilities.

We follow the generalized autoregressive score approach of Creal et al. (2013) and Harvey (2013) to allow the parameters to vary over time. Score driven models possess information theoretic optimality properties as argued by Blasques et al. (2014). Intuitively, they increase the local fit of the model by decreasing the Kullback-Leibler divergence based on the most recent observation that has become available, and the current estimates of the time varying parameters. Score driven models are observation driven in the terminology of Cox (1981). As such, estimating the static parameters that describe the dynamics in such models is straightforward using maximum likelihood, as a closed-form expression is available for the likelihood function. The expression for the likelihood builds directly on the standard prediction error decomposition, which can be also be obtained in a framework with both discrete and continuous random variables.

Using copulas in a dynamic context requires a few changes to our previous set-up. Patton (2006) showed that the copula set-up may also be used in a dynamic setting if one specifies the copula as a time-varying conditional copula, and the marginals as conditional marginals. The conditioning sets for the marginals has to coincide with that of the copula for the framework to apply. Patton's copula dynamics are driven by past average distances between the pairwise probability integral transforms $|u_{i,t} - u_{j,t}|$ at time t , for $i \neq j$. Rodriguez (2007) specifies similar copula dynamics, whereas Creal et al. (2011) and Lucas et al. (2014) specify score based copula dynamics for purely continuously valued random variables.

To introduce the parameter dynamics for the dynamic mixed copula density model, define the parameter vector $f_t = (\beta_{1,t}, \dots, \beta_{p,t}, \theta_t)$. We allow both the copula and marginal parameters to vary over time and write the mixed copula density as $h(y_t; f_t)$. The score based updating scheme of Creal et al. (2013) for δ_t is given by

$$f_{t+1} = \omega + A s_t + B f_t, \tag{2.3}$$

$$s_t = S_t \cdot \nabla_t = S_t \cdot \frac{\partial \log h(y_t | f_t)}{\partial f_t}, \tag{2.4}$$

where ω is a const parameter vector, A and B are fixed parameter matrices of appropriate

dimensions, and S_t is a positive semi-definite scaling matrix that may depend on f_t and on past data. Further dynamics can be specified in (2.3) by including more lags of s_t or f_t on the right-hand side of (2.3); see Creal et al. (2013). Also, further parsimony can be imposed by restricting the parameter matrices A and B to be diagonal or even scalar.

For continuous responses, updating the time varying parameter by the scaled score s_t of the predictive likelihood has information theoretic optimality properties; see Blasques et al. (2014). Though only one part of our measurements consists of continuous responses, whereas the other part consists of discrete responses, a similar argument can be made in the present context.

For most copula models the score ∇_t is easy to compute. In Appendix B we present the score based derivations for the Gaussian and Clayton copula with normal and Poisson based marginals. For the scaling S_t , Creal et al. (2013) provide a number of choices. In this paper, we opt for unit scaling ($S_t = I$). This avoids the need for computationally time-consuming numerical integration procedures to compute the information matrices with respect to the copula parameters for every time t .

2.3 Estimation

Estimation of our score driven time-varying parameter model is straightforward. We can either estimate the whole model in 1-step where the parameters of the marginals and the copula are estimated jointly. Alternatively, we can use a 2-step approach, where the marginal parameters are estimated first, and the copula parameters are estimated in a second step using the probability integral transforms from the first step. This latter method is also referred to as the Inference for Margins method (IFM).

For the constant copula case, we can proceed with IFM. This is known to produce consistent and asymptotically efficient results. First, we estimate the parameters of suitable marginal distributions as

$$\hat{\beta}_j^{IFM} = \arg \max_{\beta_j} \sum_{t=1}^n \log f_j(y_{j,t}; \beta_j). \quad (2.5)$$

All $\hat{\beta}_i^{IFM}$ for $i = 1, \dots, p$, are the ML estimators for the marginal parameters. Conditional on the estimates from the first step, the copula parameter vector θ can then be estimated by

maximum likelihood as

$$\hat{\theta} = \arg \max_{\theta} \sum_{t=1}^n \log \left[\sum_{s_1=0,1} \dots \sum_{s_p=0,1} (-1)^{s_1+\dots+s_p} C(F_1(y_{1,t} - s_1; \hat{\beta}_1^{IFM}), \dots, F_p(y_{t,p} - s_p; \hat{\beta}_p^{IFM})) \right]. \quad (2.6)$$

Alternatively, the estimation steps in (2.5) and (2.6) can be integrated into a 1-step full maximum likelihood (ML) procedure.

For $p > 2$, IFM makes the estimation problem simpler and computationally less intensive than full ML. In particular, IFM helps to simplify the estimation problem in high dimensions p . Joe (1997) discusses that both the 1-step ML and 2-step IFM are asymptotically the same for a multivariate Gaussian copula with normal margins. This equivalence does not hold in general for all copulas. The IFM estimator for both the marginals and the copula parameters is consistent and asymptotically normal under appropriate regularity conditions. The computation of the covariance matrix, however, can be quite cumbersome. Joe (1997) advises the Jackknife and other such related techniques. This, combined with the fact that the IFM approach is difficult to implement for dynamic marginal distributions leads us to use the 1-step approach for estimation purposes.

Given that we adopt a 1-step estimation methodology, estimating the static parameters in our time-varying mixed copula density model is straightforward. The appropriate estimators are defined as

$$(\hat{\omega}, \hat{A}, \hat{B}) = \arg \max_{\omega, A, B} \sum_{t=1}^n \log h(y_t; f_t), \quad (2.7)$$

where f_t follows the recursion in equation (2.3). We estimate the covariance matrix of the parameters using the numerical Hessian computed at the optimum.

3 Simulation

3.1 Constant Gaussian Copula

To investigate the adequacy of our new dynamic mixed copula density framework, we proceed in two steps. First, we investigate the adequacy of the approach in the static setting.

Second, we consider the effect of dynamic parameters. We use a similar simulation set-up as in Azam (2014).

We opt for two benchmark methodologies. First, we use the standard Pearson correlations between the observed y_i series. Second, we consider the recent Bayesian method of Hoff (2007) as an alternative to our approach. Hoff proposes a multivariate Gaussian copula to estimate the correlation among random variables of mixed type (binary, ordered and continuous variables) and uses a sampling approach in a Bayesian set-up to estimate the parameters. Simulations in Azam (2014) reveal that Hoff’s method outperforms a range of simpler alternatives in terms of bias and mean squared error (MSE) for the copula parameters.

In our simulations we illustrate the performance of the different methods using a Gaussian copula for a vector of three mixed measurements,

$$z \sim N(0, \Sigma), \quad \Sigma = \begin{pmatrix} 1.0 & 0.8 & 0.4 \\ 0.8 & 1.0 & 0.6 \\ 0.4 & 0.6 & 1.0 \end{pmatrix}, \quad (3.1)$$

$$y_1 = F_1^{-1}\{\Phi(z_1)|1.5\} \sim \text{Exponential}(y_1|\lambda_1),$$

$$y_2 = F_2^{-1}\{\Phi(z_2)|6\} \sim \text{Poisson}(y_2|\lambda_2),$$

$$y_3 = F_3^{-1}\{\Phi(z_3)|0.6\} \sim \text{Bernoulli}(y_2|p),$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function (cdf). We thus obtain three different types of random variables: an exponential (continuous), a poisson (count), and a bernoulli (binary) random variable. The method of Hoff (2007) is semiparametric in that it treats all three marginals F_1 , F_2 , and F_3 to be completely unknown. Through the order statistics of the data $y_t = (y_{1,t}, y_{2,t}, y_{3,t})$, the unknown $z_t = (z_{1,t}, z_{2,t}, z_{3,t})$ are sampled. Subsequently, the Gaussian copula parameters are sampled conditional upon z_t . See Hoff (2007) for further details.

To have comparable results across the two methodologies, we assume empirically distributed marginals for our mixed copula density approach and do not try to unsettle any of the ties observed in the ranks of the discrete data. The mixed Gaussian copula density is given in Appendix A for the bivariate case, and the trivariate case needed in equation (3.1) is a straightforward extension.

Table 1: Bias results

The table contains the average bias of the estimates of the Gaussian copula parameters for the trivariate system in equation (3.1) for different sample sizes n . Averages are computed over 250 Monte Carlo replications.

	$n = 25$	$n = 50$	$n = 100$	$n = 250$	$n = 500$
Mixed Copula Density Approach					
$\Sigma_{[1,2]}$	-0.0010	-0.0017	0.0029	-0.0010	0.0001
$\Sigma_{[1,3]}$	0.0072	-0.0002	-0.0018	0.0064	-0.0007
$\Sigma_{[2,3]}$	-0.0015	0.0080	-0.0086	0.0066	0.0002
Hoff's Method					
$\Sigma_{[1,2]}$	-0.0747	-0.0466	-0.0268	-0.0168	-0.0010
$\Sigma_{[1,3]}$	-0.0270	-0.0166	-0.0116	0.0010	-0.0033
$\Sigma_{[2,3]}$	-0.0826	-0.0363	-0.0348	-0.0105	-0.0110
Pearson Correlations					
$\Sigma_{[1,2]}$	-0.0585	-0.0600	-0.0558	-0.0597	-0.0595
$\Sigma_{[1,3]}$	-0.1244	-0.1243	-0.1235	-0.1201	-0.1254
$\Sigma_{[2,3]}$	-0.1381	-0.1311	-0.1416	-0.1303	-0.1363

Table 1 presents the average bias for all three different ways of computing the Gaussian copula parameters. The Pearson correlations at the bottom of the table obviously display the largest biases. The biases do not decrease as the sample size grows. The bias in the Pearson correlations is due to the fact that the observed data y_t are very coarse measurements of the unobserved z_t s. The measurement errors bias the Pearson correlation estimates towards zero. As can be expected, the biases are largest for the dependence parameter between the Poisson counts and the Bernoulli random variable ($\Sigma_{[2,3]}$), followed by the dependence between the continuous random variable y_1 and the Bernoulli random variable y_3 ($\Sigma_{[1,3]}$). The bias for $\Sigma_{[1,2]}$ is roughly half this size. Unsettling the ties in the data using an additional randomization step does not change these results; see Azam (2014) for further details.

The biases in the mixed copula density approach appear smallest, and tend to go to zero as

Table 2: Monte Carlo Means and Standard Deviation

The table contains the Monte Carlo averages of the Gaussian copula parameters for the trivariate system in equation (3.1) for different sample sizes n , as well as the Monte Carlo standard deviations of the estimates (in parentheses). Averages are computed over 250 Monte Carlo replications.

	True	$n = 25$	$n = 50$	$n = 100$	$n = 250$	$n = 500$
Mixed Copula Density						
$\Sigma_{[1,2]}$	0.8	0.7991 (0.0855)	0.7983 (0.0855)	0.8029 (0.0399)	0.7990 (0.0217)	0.8001 (0.0167)
$\Sigma_{[1,3]}$	0.4	0.4072 (0.2682)	0.4072 (0.2682)	0.3982 (0.1170)	0.4046 (0.0699)	0.3993 (0.0510)
$\Sigma_{[2,3]}$	0.6	0.5985 (0.2105)	0.5985 (0.2105)	0.5914 (0.0911)	0.6066 (0.0580)	0.6003 (0.0420)
Hoff's Method						
$\Sigma_{[1,2]}$	0.8	0.7253 (0.0958)	0.7534 (0.0630)	0.7732 (0.0408)	0.7832 (0.0217)	0.7910 (0.0165)
$\Sigma_{[1,3]}$	0.4	0.3730 (0.2093)	0.3834 (0.1503)	0.3884 (0.1069)	0.4009 (0.0678)	0.3967 (0.0499)
$\Sigma_{[2,3]}$	0.6	0.5174 (0.1783)	0.5637 (0.1346)	0.5652 (0.0861)	0.5895 (0.0560)	0.5890 (0.0410)

the sample size increases. The biases in Hoff's method appear to be slightly larger than those in the mixed copula density approach, but are also generally small and tending to zero as the sample size increases.

Table 2 shows the actual estimates of the correlation parameters rather than the biases, alongside with the Monte Carlo standard deviations of the estimates. Accounting for the Monte Carlo variation, we see that the methods behave quite similarly, even though they are based on entirely different paradigms. In particular, the mixed copula density approach does not require an iterative sampling scheme for estimation, but directly proceeds by maximizing the explicit expression for the likelihood function as given in equation (2.7). In this sense, it is

computationally more straightforward. The similarity of the performance of the two approaches is also evidenced in Appendix C, where we present kernel density plots for the estimates of both methods.

3.2 Time-varying Mixed Copula Density

For our time-varying mixed copula density approach, we use a simulation set-up with a bivariate Clayton copula. In this way, we can also study the performance of our method if there is asymmetric tail dependence in the data. For simplicity, we only consider the case of time variation in the copula parameter θ_t , while keeping the parameters of the marginal distributions fixed ($\beta_{i,t} \equiv \beta_i$ for $i = 1, \dots, p$). Experiments with both sets of parameters time-varying yielded similar results.

First, we generate a bivariate time series $(u_{1,t}, u_{2,t})$ from a Clayton copula $C(u_{1,t}, u_{2,t}; \alpha_t)$ with time-varying dependence parameter α_t ¹,

$$C(u_{1,t}, u_{2,t}; \alpha_t) = (u_{1,t}^{-\alpha_t} + u_{2,t}^{-\alpha_t} - 1)^{-1/\alpha_t},$$

where

$$\alpha_t = 2 \sin(4\pi t/n) + 3,$$

for $t = 1, \dots, n$. The sinusoid pattern results in prolonged periods of high and low dependence in the data. We then transform $u_{1,t}$ into a Poisson count $y_{1,t}$ with intensity parameter 3 as $y_{1,t} = F_1^{-1}(u_t|3)$, where $F_1(\cdot | 3)$ denotes the Poisson cdf with intensity parameter equal to 3. Similarly, we transform $u_{2,t}$ into a normal random variable by setting $y_{2,t} = \Phi^{-1}(u_{2,t})$, where Φ is the standard normal cdf.

The statistical model uses the generalized autoregressive score scheme in equation (2.3) as explained in Section 2 to filter out the time-varying dependence parameter α_t . To enforce that the dependence parameter only takes positive values, we formulate the autoregressive score scheme in terms of $f_t = \log \alpha_t$, such that α_t is always positive, irrespective of the value of f_t . We use unit scaling ($S_t = 1$) and samples of size $n = 200, 500, 1000, 2000$.

¹The copula parameter vector θ equals to α , for Clayton copula

Given the time-variation of the copula, there is no obvious benchmark method to compare to. A comparison would be further complicated if we would also allow for time-varying marginal distributions. In that case, also the semiparametric first step in Hoff’s methodology would break down, as it obtains the empirical estimates of the marginal distribution functions by pooling all the data. Pooling the data is no longer appropriate if the marginal distributions are time-varying.

We report the performance of the mixed copula density approach in terms of its ability to capture the time-variation in the copula dependence parameter α_t over repeated replications. Figure 2 presents the results. The true copula parameter for each sample size is plotted as a reference and ranges between 0 and 6, corresponding with a Kendall’s τ value between 0 and 0.75. For every sample size n and every time t , we plot the median estimate of θ_t over repeated Monte Carlo samples, as well as the 5th and 95th percentile. As the sample size increases, the median estimate of the time-varying dependence parameter gets closer and closer to its true value. It clearly shows that the score based dynamic specification from Section 2 tracks the unobserved dynamics in the dependence parameter quite well.

For small samples like $n = 200$ and $n = 500$, it is harder to accurately identify large dependence parameters α_t , much unlike lower values of the dependence parameter. This stems from the fact that large values of α_t correspond to independence. The method is thus more suited to pick up episodes of strong dependence from the data, i.e., lower values of α_t , as is to be expected. This is in line with common findings for static archimedean copulas.

We also note that for lower values of α_t , the score based parameter estimates take some time to adjust to the lower (as well as the higher) values of α_t . This is due to the observation driven nature of the score based approach; see the classification of Cox (1981). The parameter estimates of $\alpha_t = \exp(f_t)$ as described in equation (2.3) react to past observations. This implies that some observations are required to signal that the parameter α_t has moved from its previous values. The speed at which new information is incorporated is dictated by the parameters A and B in (2.3), which are estimated from the data using maximum likelihood.

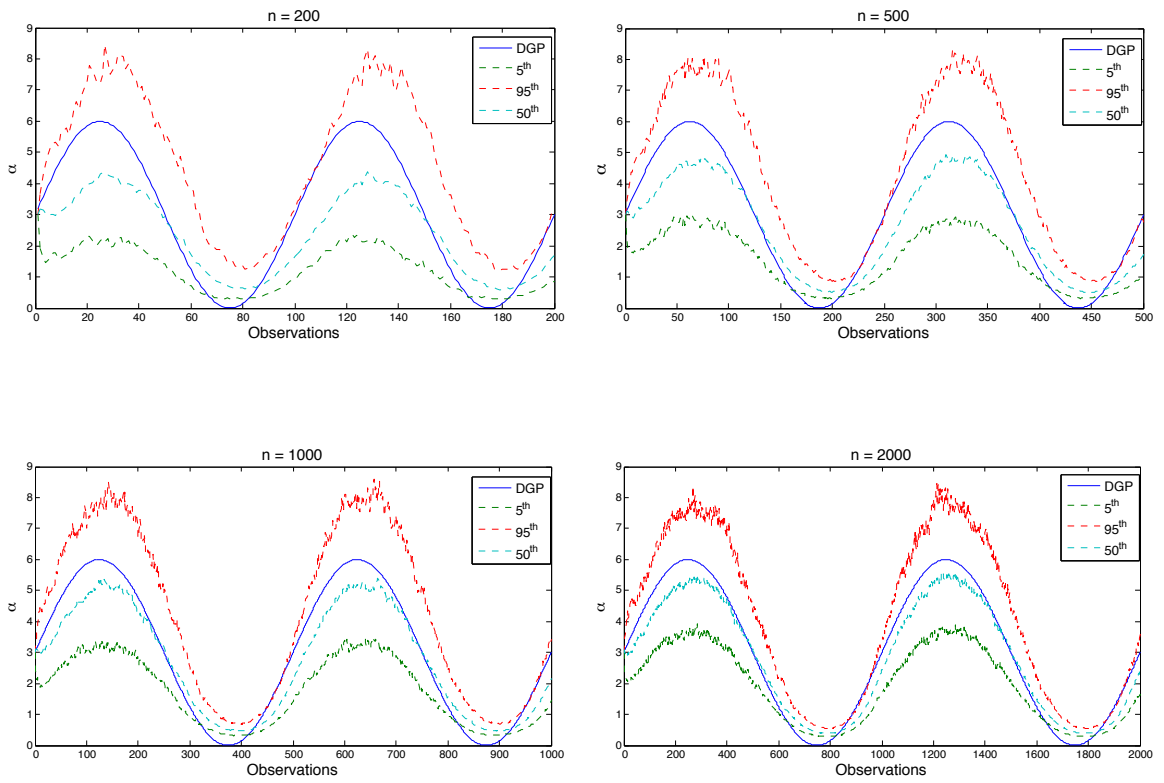


Figure 2: Quantile plots from simulation

Table 3: Summary Statistics

Variable	Financially Stable				Bankrupted since 2008			
	Mean	Std. Dev.	Min	Max	Mean	Std. Dev.	Min	Max
Ln(Income)	11.350	1.193	8.0	15.1	10.874	0.650	8.2	12.9
Credit turned down (since 2008)	0.160	0.367	0	1	0.421	0.494	0	1
No. Vehicles Owned	1.970	1.269	0	10	1.850	1.193	0	10
Household Owns Home	0.665	0.472	0	1	0.507	0.500	0	1

4 Empirical illustration

For our static mixed density copula application, we consider data from the Survey on Consumer Finances sponsored by the Federal Reserve Board and collected by the NORC at the University of Chicago. We use the summary variables (SDA) as used in the Federal Reserve Bulletin, which contain information about household demographics, assets, debts, methods of payment, shopping patterns, income, net worth and expenditure. We restrict our analysis to the cross-section interviewed during the most recent survey of 2013. We exclude all cases where the household head is fully enrolled as a student, unemployed or retired.² This process leaves us with a sample of somewhat more than 20,000 households. We split the sample into two groups, according to whether the households indicates that bankruptcy was declared over the last five years, or not. This results in 765 households that experienced bankruptcy and 19,000 that did not. We randomly sample 4% from the 19,000 households to end up with 779 financially stable households that did not experience bankruptcy. The roughly equal sample sizes between the two groups facilitates the comparison of the accuracy measures of the dependence parameters. Table 3 reports descriptive statistics on some of the main variables describing the average financial conditions of these two groups.

The variables chosen for our application span different aspects of household finance: net income, house ownership status, total number of vehicles owned and whether access to a new credit line was ever denied since 2008. We are interested in whether the correlations between

²We also exclude the top fifth percentile of the income distribution, as well as the households with a disabled family member, where we expect special financial arrangements to be applicable.

these variables are similar between the two groups of households. Different correlations between these groups may be particularly interesting, as the interview period relates to the time of the global financial crisis. Histograms of the four variables are provided in Appendix C. The house ownership and credit turned down variables are both Bernoulli distributed. We assume a Poisson distribution for the number of vehicles per household, and a Student's t distribution for the log of income. We choose a Gaussian specification for our mixed density copula and estimate the dependence structure using IFM.

Table 4 reports the estimation results. We find that home ownership is negatively dependent with the probability of being turned down for a credit line, but only significantly so for financially solid households. The collateral value of real estate properties is apparently eroded by earlier bankruptcy experiences of the same household. Home ownership is positively significantly dependent with income and with the number of vehicles owned. This holds for both groups. Financially stable households with high income have a lower probability of being denied a new credit line. Bankrupted households seem to be in a different situation. Their probability of being denied a new credit line is positively dependent with both the number of vehicles owned and the volume of their income. This might emerge from different spending habits: for example a high number of vehicles in a bankrupted household might indicate a high tendency to indebtedness. The different signs of the dependence coefficient between log income and being turned down a loan might also point to an endogeneity concern: not only might higher income lead to a lower probability of being turned down for a loan, it might also be the case that having been turned down for a loan, the household has worked to improve its income towards 2013 in order to not be turned down when applying for a loan again.

5 Conclusion

Conducting a multivariate analysis among variables of different types (discrete and continuous) can be quite challenging within a maximum likelihood framework. There are only a few possible methods typically based on a multivariate normal distribution. However, the assumption for all the marginals to be normally distributed is not always suitable. As an alternative, people often adopt a regression analysis, where a direction of causality is assumed

Table 4: Gaussian copula dependence parameter estimates

Estimates are obtained using binary marginals for home ownership and being turned down a loan over the past 5 years, a Poisson marginal for the number of vehicles owned, and a Student's t marginal for the logarithm of income. The estimates of the copula parameters are obtained using the static mixed copula density approach described in Section 2 and the inference for margins (IFM) estimation method.

	Financially Stable			Bankrupted since 2008		
	Home own.	Turned down	Vehic.	Home own.	Turned down	Vehic.
Turned down	-0.2242 (.0491)			-0.0538 (.051)		
Vehic.	0.5698 (.0370)	0.0138 (.0457)		0.3873 (.0473)	0.1849 (.0471)	
ln(Inc)	0.6363 (.0278)	-0.2061 (.0432)	0.4666 (.0254)	0.2873 (.036)	0.199 (.0387)	0.4167 (.0316)

ex-ante. Both approaches suffer from the inability to recover the full joint distribution from the conditional distributions and the dependence structure. In this paper, we proposed a flexible yet computationally simple copula based alternative. Unlike earlier proposals such as those of Pitt et al. (2006), Smith and Khaled (2012) or Hoff (2007), our method is entirely classical and uses straightforward maximum likelihood methods for estimation rather than a Bayesian sampling approach. The latter typically is computationally more expensive. Moreover, given the explicit formulation of the likelihood function in our framework, our method does not suffer from posterior correlations between the sampled dependence parameters, which typically requires large posterior samples for conducting reliable inference.

As a second contribution, we formulated a framework for a completely dynamic extension of the copula framework for mixed continuous and discrete data. The framework allowed for possibly time-varying marginal distributions as well as a time-varying dependence structure.

Using the score driven dynamics of Creal et al. (2013), we retained the closed form expression for the likelihood function, such that parameter estimation remained straightforward.

Using simulations, we showed that the new method was computationally fast and stable, and that the results were at par or even better than those of a fully Bayesian semiparametric set-up of Hoff (2007). Also, in a dynamic set-up the score based dynamic model was able to accurately capture the unobserved dependence dynamics from the observed data. In such a dynamic setting, no alternative framework exists as yet.

The good performance of the new method extended to empirical data. We presented a multivariate analysis for four variables of different types (discrete, count, continuous) and showed how correlation among different financial indicators varied for households who did and did not experience prior bankruptcy since the financial crisis.

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A Mixed Density Copula

The derivations will be presented for the case of a bivariate copula, where one marginal is of continuous type and the other is of discrete type. Let H be the joint distribution for the random variables Y_1 and Y_2 with

their respective margins F_1 and F_2 , where F_1 is a discrete type margin and F_2 is continuous. The copula based decomposition can then be stated a

$$H(y_1, y_2; \beta_1, \beta_2, \theta) = C(F_1(y_1; \beta_1), F_2(y_2; \beta_2); \theta),$$

where β_1, β_2 and θ are the parameter vectors (scaler) for the marginals F_1 and F_2 and the copula C respectively. Taking the derivative in the direction of Y_2 of the above equation would yield

$$H_{1|2}(y_1|y_2; \beta_1, \beta_2, \theta) = C_{u|v}(F_1(y_1; \beta_1)|F_2(y_2; \beta_2); \theta)f_2(y_2; \beta_2)$$

The copula input arguments are obtained through the standard probability integral transformations, $u = F_1(y_1; \beta_1)$ and $v = F_2(y_2; \beta_2)$. $f_2(y_2; \beta_2)$ is the marginal density for the variable y_2 . We can now state the mixed density as

$$h(y_1, y_2; \beta_1, \beta_2, \theta) = f_2(y_2; \beta_2) [c(F_1(y_1; \beta_1)|F_2(y_2; \beta_2); \theta) - c(F_1(y_1 - 1; \beta_1)|F_2(y_2; \beta_2); \theta)].$$

The difference above can only be computed for $y_1 > 0$ of course.

A.1 Gaussian Copula

For Gaussian copula the conditional CDF is given as

$$C_{1|2}(u_1|u_2; \rho) = \Phi\left(\frac{\Phi^{-1}(u_1) - \rho\Phi^{-1}(u_2)}{\sqrt{1 - \rho^2}}\right),$$

and the Gaussian mixed density is

$$c(u_1, ud_1, u_2; \rho) = \Phi\left(\frac{\Phi^{-1}(u_1) - \rho\Phi^{-1}(u_2)}{\sqrt{1 - \rho^2}}\right) - \Phi\left(\frac{\Phi^{-1}(ud_1) - \rho\Phi^{-1}(u_2)}{\sqrt{1 - \rho^2}}\right).$$

In the above equation $u_1 = F_1(y_1; \beta_1)$ and $ud_1 = F_1(y_1 - 1; \beta_1)$. ρ is the copula parameter defined over the range $[-1, 1]$.

A.2 Clayton Copula

For Clayton copula the conditional CDF is given as

$$C_{1|2}(u_1|u_2; \alpha) = u_2^{-1-\alpha}(u_1^{-\alpha} + u_2^{-\alpha} - 1)^{-1-1/\alpha},$$

where $\alpha \in [1, \infty)$. The mixed density is then given as

$$c(u_1, ud_1, u_2; \alpha) = u_2^{-1-\alpha} [(u_1^{-\alpha} + u_2^{-\alpha} - 1)^{-1-1/\alpha} - (ud_1^{-\alpha} + u_2^{-\alpha} - 1)^{-1-1/\alpha}],$$

Clayton copula captures any possible dependence in the joint left tail of the variables.

B Time-varying Mixed Density

Here we present the formulations for having all parameters (marginals and copula) be Time-varying i.e. $\theta_t, \beta_{1,t}$ and $\beta_{2,t}$. All the parameters are specified through Observation-Driven model, specifically through the score of the joint density of the given parameter.

B.1 Time-varying Gaussian Copula

B.1.1 Copula paramater

For the Gaussian copula, there is only one parameter, the correlation which has to be allowed to vary with time (ρ_t). The parameter ρ_t can be transformed on to the real-line through

$$\rho_t = \frac{1 - e^{-\eta_t}}{1 + e^{-\eta_t}},$$

where $\eta_t \in [-\infty, \infty]$. We specify the dynamics of η_t to be given by Observation driven model based on the score of the joint likelihood. The Score for the Gaussian copula is given as

$$\begin{aligned} \nabla_t &= \frac{\partial \ln h(y_{1,t}, y_{2,t}; \rho_t, \beta_{1,t}, \beta_{2,t})}{\partial \rho_t} \frac{\partial \rho_t}{\partial \eta_t}, \\ \frac{\partial \rho_t}{\partial \eta_t} &= (1 - \rho_t^2)/2. \end{aligned}$$

Through the copula decomposition the joint density can be stated as

$$\begin{aligned} \nabla_t &= \frac{\partial \ln c(u_{1,t}, ud_{1,t}, u_{2,t}; \rho_t, \beta_{1,t}, \beta_{2,t})}{\partial \rho_t} \frac{\partial \rho_t}{\partial \eta_t}, \\ &= \frac{1 - \lambda^2}{2d_t} \left\{ \varphi(z_{1,t}) \left[\frac{-\Phi^{-1}(u_{2,t})}{\sqrt{1 - \rho_t^2}} + \frac{\rho_t z_{1,t}}{1 - \rho_t^2} \right] - \varphi(zd_{1,t}) \left[\frac{-\Phi^{-1}(u_{2,t})}{\sqrt{1 - \rho_t^2}} + \frac{\rho_t zd_{1,t}}{1 - \rho_t^2} \right] \right\}, \end{aligned}$$

where

$$\begin{aligned} d_t &= f_2(y_{2,t}; \beta_{2,t}) c(u_{1,t}, ud_{1,t}, u_{2,t}; \rho_t), \\ z_{1,t} &= \frac{\Phi^{-1}(u_{1,t}) - \rho_t \Phi^{-1}(u_{2,t})}{\sqrt{1 - \rho_t^2}}, \\ zd_{1,t} &= \frac{\Phi^{-1}(ud_{1,t}) - \rho_t \Phi^{-1}(u_{2,t})}{\sqrt{1 - \rho_t^2}}. \end{aligned}$$

B.1.2 Time-varying Poisson Marginal

Using a bivariate Gaussian copula, if one of the marginal distribution is Poisson, then the score with respect to the poisson parameter ($\beta_{1t} = \lambda_t$). Transforming λ_t on to the real-line through

$$\lambda_t = \exp(\eta_t).$$

The score of the joint density is given as

$$\begin{aligned} \nabla_t &= \frac{\partial \ln c(u_{1,t}, ud_{1,t}, u_{2,t}; \rho_t, \lambda_t, \beta_{2t})}{\partial \lambda_t} \frac{\partial \lambda_t}{\partial \eta_t}, \\ &= \frac{\lambda_t}{dt} \left\{ \varphi(z_{1,t}) \left[\frac{1}{\varphi(\Phi^{-1}(u_{1,t}))\sqrt{1-\rho_t^2}} \frac{\partial F_1(y_{1,t}; \lambda_t)}{\partial \lambda_t} \right] - \varphi(zd_{1,t}) \left[\frac{1}{\varphi(\Phi^{-1}(ud_{1,t}))\sqrt{1-\rho_t^2}} \frac{\partial F_1(y_{1,t}-1; \lambda_t)}{\partial \lambda_t} \right] \right\}, \end{aligned}$$

where

$$\frac{\partial F_1(y_{1,t}; \lambda_t)}{\partial \lambda_t} = -\frac{\lambda_t^{y_{1,t}} e^{-\lambda_t}}{y_{1,t}!}.$$

B.1.3 Time-varying Normal Marginal

For having the continuous variable be normally distributed, we would need score with respect to two parameters, $\beta_{2t} = [\mu_t, \sigma_t]$. The score with respect to the mean μ_t is

$$\begin{aligned} \nabla_t &= \frac{\partial \ln c(u_{1,t}, ud_{1,t}, u_{2,t}; \rho_t, \mu_t, \sigma_t, \beta_{1,t})}{\partial \mu_t} + \frac{\partial \ln f_2(u_{2,t}; \mu_t, \sigma_t)}{\partial \mu_t}, \\ &= \frac{\partial F_2(y_{2,t}; \mu_t, \sigma_t)}{\partial \mu_t} \frac{1}{dt} \left\{ \varphi(z_{1,t}) \left[\frac{-\rho_t}{\varphi(\Phi^{-1}(u_{1,t}))\sqrt{1-\rho_t^2}} \right] - \varphi(zd_{1,t}) \left[\frac{-\rho_t}{\varphi(\Phi^{-1}(ud_{1,t}))\sqrt{1-\rho_t^2}} \right] \right\}, \end{aligned}$$

where

$$\frac{\partial F_2(y_{2,t}; \mu_t, \sigma_t)}{\partial \mu_t} = \frac{-e^{-(y_{2,t}+\mu_t)^2/2\sigma_t^2}}{\sqrt{2\pi}\sigma_t},$$

and with respect to σ_t we can transform σ on to the real-line through

$$\sigma_t = \exp(\eta_t).$$

$$\begin{aligned} \nabla_t &= \left[\frac{\partial \ln c(u_{1,t}, ud_{1,t}, u_{2,t}; \rho_t, \mu_t, \sigma_t, \beta_{1,t})}{\partial \sigma_t} + \frac{\partial \ln f_2(u_{2,t}; \mu_t, \sigma_t)}{\partial \sigma_t} \right] \frac{\partial \sigma_t}{\partial \eta_t}, \\ &= \frac{\partial F_2(y_{2,t}; \mu_t, \sigma_t)}{\partial \mu_t} \frac{\sigma_t}{dt} \left\{ \varphi(z_{1,t}) \left[\frac{-\rho_t}{\varphi(\Phi^{-1}(v_t))\sqrt{1-\rho_t^2}} \right] - \varphi(zd_{1,t}) \left[\frac{-\rho_t}{\varphi(\Phi^{-1}(u_{2,t}))\sqrt{1-\rho_t^2}} \right] \right\}, \end{aligned}$$

where

$$\frac{\partial F_2(y_{2,t}; \mu_t, \sigma_t)}{\partial \sigma_t} = \frac{-e^{-(y_{2,t}+\mu_t)^2/2\sigma_t^2}}{\sqrt{2\pi}\sigma_t^2} (y_{2,t} - \mu_t).$$

B.2 Time-varying Clayton Copula

B.2.1 Copula parameter

Similar to the Gaussian copula, Clayton copula only has one parameter which measures the dependence (α_t). This can be transformed on to the real-line through

$$\alpha_t = \ln(\eta_t).$$

The score of the Clayton copula is given as

$$\nabla_t = \frac{\partial \ln h(y_{1,t}, y_{2,t}; \alpha_t, \beta_{1,t}, \beta_{2,t})}{\partial \alpha_t} \frac{\partial \alpha_t}{\partial \eta_t},$$

Through the copula decomposition the joint density can be stated as

$$\begin{aligned} \nabla_t &= \frac{\partial \ln c(u_{1,t}, ud_{1,t}, u_{2,t}; \alpha_t, \beta_{1,t}, \beta_{2,t})}{\partial \alpha_t} \frac{\partial \alpha_t}{\partial \eta_t}, \\ &= \frac{-\alpha_t \ln(u_{2,t}) c(u_{2,t}, ud_{1,t}, u_{2,t}; \alpha_t)}{d_t} + \frac{\alpha_t u_{2,t}^{-1-\alpha_t}}{d_t} \left\{ r_{1,t}^{-1-1/\alpha_t} \left[\left(\frac{-\alpha_t - 1}{\alpha_t} \right) \left(\frac{-u_{1,t}^{-\alpha_t} \ln(u_{1,t}) - u_{2,t}^{-\alpha_t} \ln(u_{2,t})}{r_{1,t}} \right) + \right. \right. \\ &\quad \left. \left. \frac{\ln(r_{1,t})}{\alpha_t^2} \right] - r d_{1,t}^{-1-1/\alpha_t} \left[\left(\frac{-\alpha_t - 1}{\alpha_t} \right) \left(\frac{-ud_{2,t}^{-\alpha_t} \ln(ud_{1,t}) - u_{2,t}^{-\alpha_t} \ln(u_{2,t})}{rd_{1,t}} \right) + \frac{\ln(rd_{1,t})}{\alpha_t^2} \right] \right\}, \end{aligned}$$

where

$$d_t = f_2(y_{2,t}; \beta_{2t}) c(u_{1,t}, ud_{1,t}, u_{2,t}; \alpha_t),$$

$$r_{1,t} = u_{1,t}^{-\alpha_t} + u_{2,t}^{-\alpha_t} - 1,$$

$$rd_{1,t} = ud_{1,t}^{-\alpha_t} + u_{2,t}^{-\alpha_t} - 1.$$

B.2.2 Time-varying Poisson Marginal

Using a bivariate Clayton copula, if one of the marginal distribution is Poisson, then the score with respect to the poisson parameter ($\beta_{1t} = \lambda_t$). Transforming λ_t on to the real-line through

$$\lambda_t = \exp(\eta_t).$$

We can then write the score with respect to the mean λ_t as

$$\begin{aligned} \nabla_t &= \frac{\partial \ln c(u_{1,t}, ud_{1,t}, u_{2,t}; \alpha_t, \lambda_t, \beta_{2,t})}{\partial \lambda_t} \frac{\partial \lambda_t}{\partial \eta_t} \\ &= \frac{(\alpha_t^2 + \alpha_t)}{d_t} u_t^{-1-\alpha_t} \left[u_{1,t}^{-1-\alpha_t} r_{1,t}^{-2-1/\alpha_t} \frac{\partial F_1(y_{1,t}; \lambda_t)}{\partial \lambda_t} - ud_{1,t}^{-1-\alpha_t} rd_{1,t}^{-2-1/\alpha_t} \frac{\partial F_1(y_{1,t} - 1; \lambda_t)}{\partial \lambda_t} \right] \end{aligned}$$

where

$$\frac{\partial F_1(y_{1,t}; \lambda_t)}{\partial \lambda_t} = -\frac{\lambda_t^{y_{1,t}} e^{-\lambda_t}}{y_{1,t}!}.$$

B.2.3 Time-varying Normal Marginal

With Clayton copula and having one margin be normally distributed, the score with respect to the marginal parameters $\beta_{2,t} = [\mu_t, \sigma_t]$ will be

$$\begin{aligned}\nabla_t &= \frac{\partial \ln c(u_{1,t}, ud_{1,t}, u_{2,t}; \alpha_t, \mu_t, \sigma_t, \beta_{1,t})}{\partial \mu_t} + \frac{\partial \ln f_2(u_{2,t}; \mu_t, \sigma_t)}{\partial \mu_t}, \\ &= \frac{\partial F_2(y_{2,t}; \mu_t, \sigma_t)}{\partial \mu_t} \frac{u_{2,t}^{-2-\alpha_t}}{d_t} \left\{ (-1 - \alpha_t) u_{2,t}^{-\alpha_t} \left[-r_{1,t}^{-2-1/\alpha_t} + rd_{2,t}^{-2-1/\alpha_t} \right] - (1 + \alpha_t) \left[r_{1,t}^{\frac{-1-\alpha_t}{\alpha_t}} - rd_{1,t}^{\frac{-1-\alpha_t}{\alpha_t}} \right] \right\}\end{aligned}$$

where

$$\frac{\partial F_2(y_{2,t}; \mu_t, \sigma_t)}{\partial \mu_t} = \frac{-e^{-(y_{2,t} + \mu_t)^2 / 2\sigma_t^2}}{\sqrt{2\pi}\sigma_t},$$

and with respect to σ_t we can transform σ on to the real-line through

$$\sigma_t = \exp(\eta_t).$$

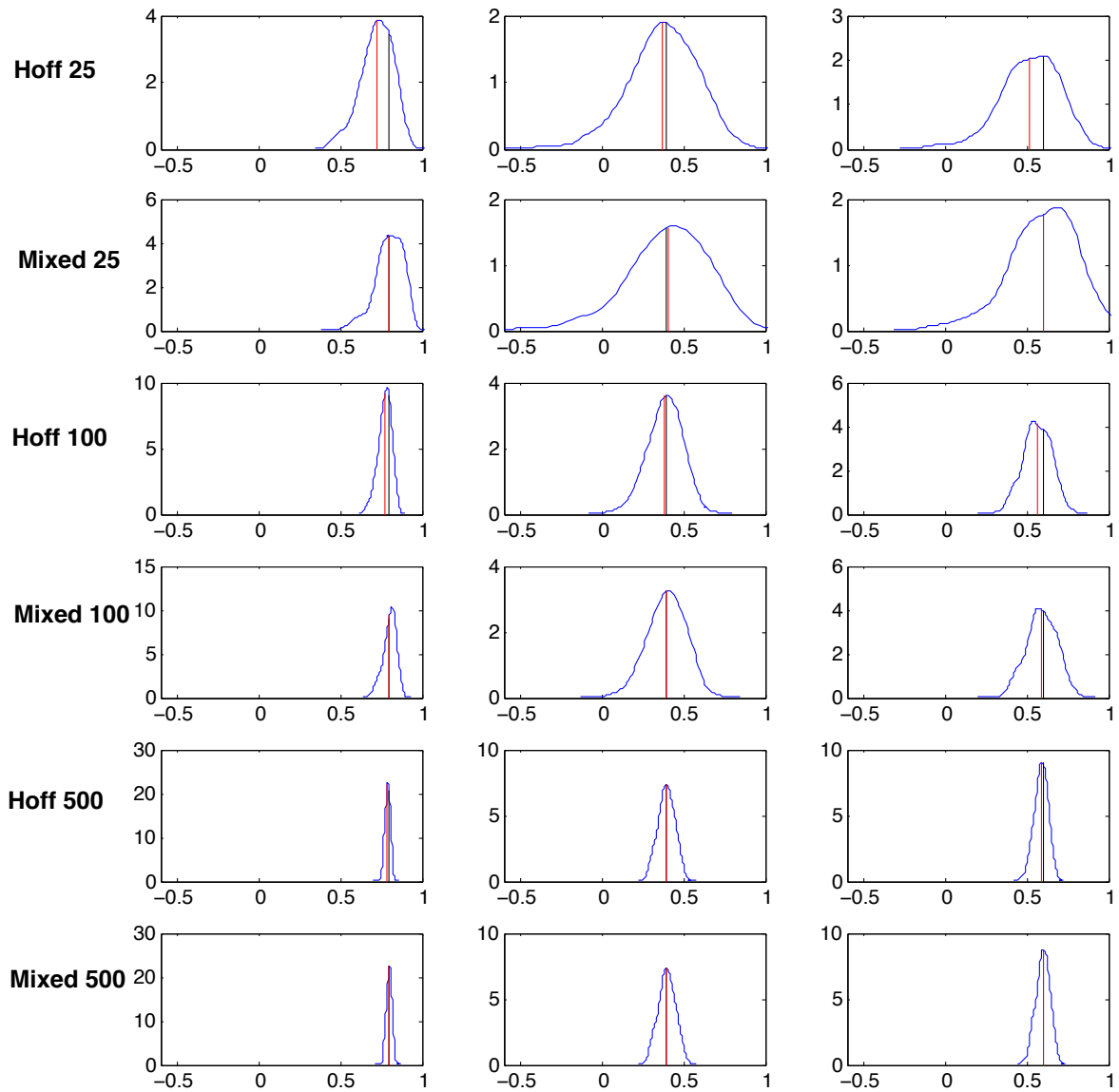
$$\begin{aligned}\nabla_t &= \left[\frac{\partial \ln c(u_{1,t}, ud_{1,t}, u_{2,t}; \alpha_t, \mu_t, \sigma_t, \beta_{1,t})}{\partial \sigma_t} + \frac{\partial \ln f_2(u_{2,t}; \mu_t, \sigma_t)}{\partial \sigma_t} \right] \frac{\partial \sigma_t}{\partial \eta_t}, \\ &= \frac{\partial F_2(y_{2,t}; \mu_t, \sigma_t)}{\partial \sigma_t} \frac{\sigma_t}{d_t} u_{2,t}^{-2-\alpha_t} \left\{ (-1 - \alpha_t) u_{2,t}^{-\alpha_t} \left[-r_{1,t}^{-2-1/\alpha_t} + rd_{1,t}^{-2-1/\alpha_t} \right] - (1 + \alpha_t) \left[r_{1,t}^{\frac{-1-\alpha_t}{\alpha_t}} - rd_{1,t}^{\frac{-1-\alpha_t}{\alpha_t}} \right] \right\}\end{aligned}$$

where

$$\frac{\partial F_2(y_{2,t}; \mu_t, \sigma_t)}{\partial \sigma_t} = \frac{-e^{-(y_{2,t} + \mu_t)^2 / 2\sigma_t^2}}{\sqrt{2\pi}\sigma_t^2} (y_{2,t} - \mu_t).$$

C Kernel Density Plots

Figure 3: Hoff & Mixed Density estimates



The Kernel density plots are presented for various samples; $n=25, 100$ & 500

Figure 4: Stable Household

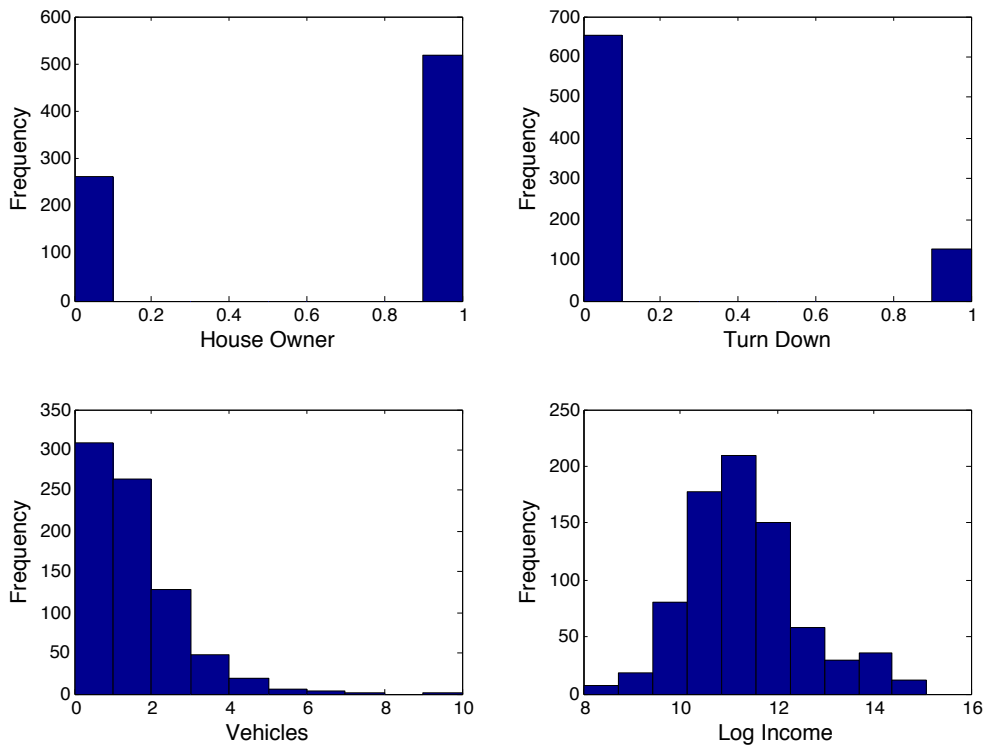


Figure 5: Bankrupt Household

