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How Risk Sharing may enhance Efficiency in English Auctions

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How Risk Sharing May Enhance Efficiency in English Auctions

By Audrey Hu, Theo Offerman, and Liang Zou

We investigate the possibility of enhancing efficiency by awarding premiums to a set of highest bidders in an English auction—in a setting that extends Maskin and Riley (1984, *Econometrica* **52**: 1473-1518) in three aspects: (i) the seller can be risk averse, (ii) the bidders can have heterogeneous risk preferences, and (iii) the auction can have a binding reserve price. Our analysis reveals that the premium has an intricate joint effect on risk sharing and expected revenue, which in general benefits risk averse bidders. When the seller is more risk averse than the pivotal bidder – a condition often verifiable by deduction prior to the auction – the premium also benefits the seller and therefore leads to a Pareto improvement of the English auction. We discuss how this finding is related to the seller's degree of risk aversion, the reserve price, the riskiness of the object for sale, the degree of heterogeneity in risk preferences among the bidders, and the number of the potential bidders.

Keywords: Risk sharing, Pareto efficiency, Premium auction, English auction, Reserve price, Ensuing risk, Heterogeneous risk preferences.

JEL classification: D44

1 Introduction

Most of the auctions literature assumes either the seller, or bidders, or both to be risk neutral.[1] In this paper we consider a more general situation in which both the seller and bidders in an auction can be risk averse. This situation naturally arises when the seller and bidders are consumers, business persons or small firms with limited capital. For instance, the seller of a unique painting may be as risk averse as the bidders competing for it. When both the seller and bidders are risk averse, an important question arises as to whether, given an *ex post* efficient auction mechanism, the involved parties can benefit from a risk-sharing scheme that enhances *ex ante* efficiency of the mechanism without jeopardizing ex post efficiency.[2,3] Put differently, can we make *all players*, i.e., the seller and all types of the prospective bidders, better off by modifying the payment rule of a mechanism while maintaining its allocation rule? The main contribution of our paper is to provide a mechanism that practically achieves this goal.

We consider a single-object auction environment and take the English auction (EA) as our benchmark auction model. The EA is one of the most widely practiced, and most extensively studied, auction format. Its open ascending-bid procedure ensures simplicity, transparency, optimal use of information,[4] and under various conditions

[1]Exceptions that allow both the seller and buyers to be risk averse can be found in Hu, Matthews and Zou (2010) and Hu (2011), where the focus is on the optimal reserve prices.

[2]We use the term "ex ante" to mean the pre-auction stage when the auction rule may be subject to changes (by the seller, the auction designer, or as a result of bargaining). This may include the "interim" stage when each potential bidder has received his private information but does not know the others' information, as well as the stage when no bidder has received any private information (see, e.g., Holmstrom and Myerson, 1983; Crawford, 1985).

[3]Ex ante risk sharing should be distinguished from a separate problem of sharing *ensuing* risk between the seller and the winning bidder through joint ownership of the auctioned asset (e.g., the security design problem studied in DeMarzo, Kremer and Skrzypacz, 2005).

[4]See, e.g., Milgrom and Weber (1982), McAfee and McMillan (1987), McMillan (1994), Klemperer (2002), Ausubel (2004), and Perry and Reny (2005).

leads to ex post efficient outcomes.[5] Therefore, when the EA is Pareto *inefficient* ex ante, it is of practical as well as theoretical importance to find out how the situation can be improved.

We generalize the EA to a class of English premium auctions (EPA), which proceeds just like an EA except that the highest two bidders receive a "profit share", or "premium", from the seller that is equal to a fraction α of the difference between the second and the third highest bids. This class of EPA includes EA as a special case when $\alpha = 0$ and it maintains the EA's simple and "detail-free" (Wilson, 1987) feature to the seller. In practice, premium auctions are regularly used in Europe to sell houses, land, and machinery among others. Various premium auctions have been analyzed in several previous studies.[6] In particular, the existence of equilibrium for a related EPA model has been established in our previous work (Hu, Offerman and Zou, 2011), which allows us to focus on the welfare analysis of risk sharing in the present study.[7]

The auction environment we consider extends the classical setting of Maskin and Riley (1984) in three aspects: (i) the seller can be risk averse, (ii) the bidders can exhibit heterogeneous risk preferences,[8] and (iii) the seller can impose a binding

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[5] See, e.g., Maskin (1992), Wilson (1998), Krishna (2002), Dubra Echenique and Manelli (2009), Birulin and Izmalkov (2010), and Hu, Matthews and Zou (2013).

[6] For instance, see Goeree and Offerman (2004), Milgrom (2004), Hu, Offerman and Onderstal (2011), and Hu, Offerman and Zou (2011). Van Bochove, Boerner and Quint (2012) provide an interesting historical account of the "Anglo-Dutch premium auctions" used in the secondary market for financial securities in the eighteenth-century.

[7] In Hu, Offerman and Zou (2011) we considered the EPA in a symmetric interdependent-values setting of Milgrom and Weber (1982) with homogeneous bidders, finding that when bidders are risk averse, revenue maximization is unlikely to be a good reason for the seller to offer any premiums in an EA.

[8] Apart from being commonly recognized as a stylized fact (e.g., Arrow, 1971), heterogeneity in risk preferences has been confirmed in many experimental studies following Cox, Smith and Walker (1982, 1988). For instance, Harrison, List and Towe (2007, p.437) reported that they "observe considerable individual heterogeneity in risk attitudes, such that one should not readily assume homogeneous risk preferences for the population."

reserve price.[9] We also incorporate the possibility of ensuing risks in our model, e.g., where the auctioned object is, in essence, a risky asset. All these extensions are not necessary for our results, but the generality is certainly worthy of the endeavor as it provides more scope for potential applications. We do not seek optimal solutions that maximize the seller's expected utility, which typically involve ex post inefficiency (e.g., Myerson, 1981; Riley and Samuelson, 1981). As shown in Matthews (1983) and Maskin and Riley (1984), even if the seller is risk neutral and bidders have the same homogeneous utility function, the problem of finding an optimal auction mechanism is highly complicated. The existence of such an optimal mechanism requires strong assumptions on the utility and distribution functions as well as detailed knowledge of the seller about these functions. We therefore focus in this study on the detail-free and ex post efficient EPA with the objective of obtaining sharp and applicable results, treating the reserve price and the premium rule as exogenously given.

The key result of our study is that in plausible situations, the EPA provides a Pareto improvement on the EA. This result applies to situations in which it is reasonable to expect that the seller is more risk averse than the buyers. For instance, when a household sells an antique in an auction to wealthy collectors, an author his manuscript to publishers, an inventor his patent to venture capitalists, or a small firm its assets to large corporations, the prospective buyers are typically wealthier and, by a simple argument of decreasing absolute risk aversion (DARA), more risk tolerant than the seller. In some situations, the seller can even check without a cost whether potential buyers are more risk tolerant. For instance, consider a setting in which the auctioned item is a risky asset whose value distribution is commonly known. Then as long as the seller imposes a reserve price that makes him indifferent between selling and not selling, the prospective buyer, by the very fact that he is willing to pay more than the reserve price, can be inferred to be more risk tolerant than the seller.

[9] As we do not require that the seller imposes a reserve price strictly higher than his reservation value, there is no commitment problem or loss of efficiency that may arise in the event of no sale since the seller will then be the one who values the object the most among all players.

In Section 2, we present the general model and describe how the EPA works. We introduce an effective measure to tackle heterogeneous risk preferences, defined by a function(al) Q in (2). We show in Lemma 1 that, in the paradigm of expected utility theory, the quantity of Q has the same role of the Arrow-Pratt measure of absolute risk aversion for ordering risk preferences. But this measure of Q involves weaker assumptions and a broader scope of applications than the Arrow-Pratt measure.

In Section 3, we analyze the equilibrium properties of the EA and EPA, showing that a binding reserve price causes the equilibrium bids in the EPA to exhibit a "jump" at the reserve price (Theorem 1). An important consequence of this jump bidding, a phenomenon similar to the one observed in Jehiel and Moldovanu (2000) for their second-price auction equilibrium with externalities, is that at the interim stage the same reserve price will induce the same subset of active bidders in either the EA or the EPA. This property greatly simplifies our comparative welfare analysis.

Section 4 contains the main findings from this study. We first show in Theorem 2 that for any number (> 2) of active bidders, the difference in the seller's expected payoff between the EA and the EPA can be characterized by a functional of the utility function of the seller and that of the *pivotal* bidder—the one who determines the selling price in the EA. This result reveals an important role of the pivotal bidder for assessing the relative advantage of risk sharing in the EPA. It suggests that the seller will benefit from the premium tactics whenever he is *more risk averse than the pivotal bidder*. Although the utility function of the pivotal bidder may not be known, this condition can often be assured to hold prior to the auction by deduction.

From the seller's viewpoint, we consequently obtain the following predictions that are direct implications of Theorem 2. Given any premium rule $\alpha \in (0, 1/2)$ and reserve price $p_0 \geq 0$, the seller is better off in the EPA compared to the EA in either of the following scenarios.

Scenario 1: All active bidders are more risk tolerant than the seller (Proposition 1).

Scenario 2: The seller's preference belongs to that of the population of the bidders, and p_0 is no less than the seller's reservation value (Proposition 2 and Corollary 1).

Scenario 3: There is a sufficiently large number of potential bidders (Proposition 3).

Without any further restriction on the players' risk preferences, Scenario 1 is a significant generalization of the auction environment studied in Waehrer et al. (1998) and Eso and Futo (1999) in which the seller is risk averse and bidders are risk neutral. Eso and Futo (1999) obtained an interesting result that among all incentive compatible mechanisms, there is one that is deterministic to the seller and is therefore, by a simple argument of revenue equivalence under risk neutrality, ex ante efficient. As Eso and Futo noted, however, their mechanism may involve large "gambles" among the bidders and fail to hold as soon as some bidders are risk averse. For such a mechanism the seller also needs to have accurate knowledge of how bidders' types are distributed.

In Scenario 2, the situation is akin to a business-to-business transaction in which the seller imposes a reserve price to prevent losses from the sale. The seller is then better off using the EPA rather than the EA because of the deduced fact that the prospective buyer will be more risk tolerant. In Scenario 3, there may or may not be a binding reserve price. The prediction derives from the fact that as the bidder number increases, the probability increases toward 1 that the pivotal bidder is more risk tolerant than the seller. We obtain two more corollaries of Theorem 2 that when bidders are risk neutral, the EPA revenue is less risky than that of the EA in term of second-order stochastic dominance (Corollary 2); and that there exists an optimal $\alpha^* \in (0, 1/2]$ that maximizes the seller's expected utility among the class of EPAs considered (Corollary 3).

From the bidders' viewpoint, we show that under plausible conditions the risk averse bidders derive higher expected utilities in the EPA rather than the EA. The intuition lies in the twofold benefits that the premium offers to risk averse bidders: it reduces the average payment (Hu, Offerman and Zou, 2011) and it reduces the riskiness of the payment. This result extends Matthews' (1987) finding that in independent private values settings, the DARA bidders prefer the second-price auction, or its strategically equivalent format EA, to the first-price auction. Our result shows that as long as bidders have nonincreasing absolute risk aversion, they further prefer the EPA to the second-price auction.[10] We establish this result first assuming that bidders have the same utility function that exhibits constant absolute risk aversion (Theorem 3). The case with heterogeneous bidders turns out to be surprisingly complicated and requires additional, although plausible, assumptions and it is proved in Theorem 4.

In Section 4.2.2 we provide a numerical example that illustrates the main results of the paper. Section 5 concludes the paper with remarks on future research. The proofs of the lemmas and propositions are relegated to the Appendix.

2 Model and Preliminaries

We consider selling an indivisible object to N (> 2) potential bidders via an English premium auction (EPA). The seller announces a reserve price for the object, p_0, and observes n ($\leq N$) active bidders. If $n \leq 2$, then the auction will be conducted as a standard (button-) English auction (EA), in which case for $n = 0$, the auction results in no sale, for $n = 1$, the only active bidder wins the object and pays the reserve price p_0, and for $n = 2$, the winner purchases the object for the price at which the other bidder quits.

For $n > 2$, the EPA will be conducted in two stages. In the first stage, a clock price rises from p_0. At each price level, bidders decide to stay in the auction or to exit. An exit decision is irrevocable. The first stage ends when only two bidders, or *finalists*, remain active. The price level X, or *bottom price*, at which the third-to-last bidder quits will serve as a new reserve price onwards in the second stage, in which the

[10] Eso and White (2004) extends Matthews (1987) in another direction, showing that under a given first-price, second-price, or English auction environment the symmetric DARA bidders prefer that the object for sale entails higher ensuing risk. It can be shown that this observation extends to our heterogeneous model under the EA. An interesting, yet unverified, conjecture is that heterogeneous bidders would prefer higher ensuing risk in the EPA as well.

price rises from X until one of the finalists quits. The remaining one wins the object and pays the price p at which the other finalist quits. Both finalists also receive a *premium* from the seller equal to $\alpha(p-X)$, where $\alpha \in (0, 1/2]$ is publicly known prior to the auction.[11] Any ties are resolved randomly: in the second stage, if both finalists withdraw at the same price p, then both will receive a premium equal to $\alpha(p-X)$ and one of them will be randomly chosen to receive the object and pay the price p; in the first stage, if two or more bidders simultaneously withdraw at price X, with only one (or no) bidder left, then the auction ends like an EA with the (randomly chosen) highest bidder paying price X for the object and no one receiving any premium.

Each potential bidder i has a private type $t_i \in [0, H] \subset \mathbb{R}$ that affects his preference for the object. Ex ante, the types t_i are independently distributed according to the same distribution function F. The density function $f = F'$ is strictly positive and continuously differentiable on $(0, H]$.

The preference of a typical bidder with type t is represented by

$$\begin{cases} w(x,t) & \text{if he wins the object and receives } x \\ u(x,t) & \text{if he loses and receives } x \end{cases} \tag{1}$$

We interpret function $u(\cdot, t)$ as type-t bidder's status-quo utility for income. The bidder with type t who drops out in the first stage will have utility $u(0, t)$.

For ease of exposition, we refer to the special case where $u(x, t)$ is independent of t in (1) as the *homogeneous-utility* model (e.g., Maskin and Riley, 1984),[12] and the more general case as the *heterogeneous-utility* model in which $u(\cdot, t)$ and $u(\cdot, t')$ can be two different utility functions given any $t \neq t'$.

The functions $u(x, t)$ and $w(x, t)$ are assumed to satisfy the following mild con-

[11] Hu, Offerman and Zou (2011) considered a more general premium rule that is an increasing, but not necessarily linear, function of $p - X$. In this paper, we restrict attention to linear premium rules for tractability, while noting that such simple rules are predominant in premium auction practices (e.g., Goeree and Offerman, 2004).

[12] The term "homogeneous utility" refers only to the fact that all *losing* bidders have the same utility for income. The *winning* bidders' utility functions can still vary with their private types.

ditions.[13]

A1. u and w are thrice differentiable.

For all $t \in [0, H]$,

A2. $w(-\infty, t) < u(0, t) < w(0, t)$.

A3. $u_1 > 0$, $w_1 > 0$, and $w_2 > 0$.

A4. $u(x, t) - u(z, t)$ is log-concave in x on (z, ∞), $\forall z$.

Condition A2 implies that all types of bidders would be better off receiving the object for free ($w(0, t) > u(0, t)$), but no bidder is willing to pay too high a price for the object ($w(-\infty, t) < u(0, t)$). A3 is the usual assumption that utilities increase in income. A4 is commonly invoked to guarantee the existence of equilibria in first-price sealed-bid auctions (e.g., Holt, 1980; Athey, 2001). It holds for risk averse bidders in general, and to some extent for risk preferring bidders as well.

The next two conditions involve the properties of the ratio[14]

$$Q(x, y, t) \equiv \frac{u(x, t) - w(x - y, t)}{u_1(x, t)} \tag{2}$$

A5. For all x, y, $Q(x, y, t)$ is decreasing in t.

A6. For all y, t, $Q(x, y, t)$ is nonincreasing in x.

The economic interpretations of A5 and A6 will become more transparent by considering some special cases of our model. We first present a lemma that will be frequently used later on for interpretations of the main results of this paper. The lemma can be seen as a corollary of Pratt (1964, Theorem 1), which helps connect the expression in (2) to the Arrow-Pratt measure of absolute risk aversion.[15]

[13] Subscripts denote the argument with respect to which a partial derivative is taken.

[14] The notational dependence of Q on the functions u and w is suppressed.

[15] Similar results are presented in Hu, Matthews and Zou (2013) in a more general setting with asymmetric interdependent-values and heterogeneous bidders.

Lemma 1 *Let $u, \hat{u} : \mathbb{R} \to \mathbb{R}$ be two increasing and twice continuously differentiable utility functions. Then the following conditions are equivalent, in either the strong form (indicated in brackets), or the weak form (with the bracketed material omitted):*

(i) $-u''/u' \geq -\hat{u}''/\hat{u}'$ [and $>$ for at least one x in every interval].

(ii) For all x and y such that $y \neq 0$,

$$\frac{u(x) - u(x-y)}{u'(x)} \geq [>] \frac{\hat{u}(x) - \hat{u}(x-y)}{\hat{u}'(x)}. \tag{3}$$

(iii) For all x and y, and for all nondegenerate random variables $\tilde{v}$ such that $E\tilde{v}$ exists,

$$\frac{u(x) - Eu(\tilde{v} + x - y)}{u'(x)} \geq [>] \frac{\hat{u}(x) - E\hat{u}(\tilde{v} + x - y)}{\hat{u}'(x)}. \tag{4}$$

For the homogeneous-utility model, the following four special cases have been considered in Maskin and Riley (1984) where U is an increasing von Neumann-Morgenstern utility function.

Case 1 $w(x, t) = U(t + x)$ and $u(x, t) \equiv U(x)$.

Case 2 $w(x, t) = U(t + \psi(x))$ and $u(x, t) \equiv U(\psi(x))$, where $\psi' > 0$, $\psi'' \leq 0$, and $\psi(0) = 0$.

Case 3 $w(x, t) = \int U(v + x) dK(v|t)$ and $u(x, t) \equiv U(x)$, where $K(v|t) > K(v|\hat{t})$ for all $t < \hat{t}$.

Case 4 $w(x, t) = (1 + t)U(t + x)$ and $u(x, t) \equiv U(x)$, where $U \geq 0$.

Case 1 is the standard private-values model. Case 2 allows a bidder to assign certain quality to the auctioned object, which may not have an equivalent monetary value. Case 3 allows the object to entail *ensuing risks*, where the true value v remains risky at the time when the auction concludes. In this case the conditional distribution of v for a higher type exhibits first-order stochastic dominance over that for a lower type. Case 4 provides an example in which winning the object gives the bidder a greater ability to derive pleasure, crudely translated into a higher marginal utility as well as utility for income.[16]

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[16] See Maskin and Riley (1984) for more detailed discussions of these cases.

It is easily seen that for Cases 1-4, conditions A1-A4 hold under proper assumptions on functions U and K.[17] The following lemma relates U to A5 and A6.

Lemma 2 *For Cases 1-4, $U' > 0$ implies A5. U exhibiting nonincreasing absolute risk aversion implies A6 for Cases 1-3. If in addition U is nonnegative and is log-concave, then A6 holds for Case 4.*

For the heterogeneous-utility model, it is clear that each of the Cases 1-4 can be generalized straightforwardly by replacing $U(x)$ with $U(x,t)$, so that $u(x,t) = U(x,t)$ (or $u(x,t) = U(\psi(x),t)$ for Case 2').

Case 1' $w(x,t) = U(v(t) + x, t)$ where v is twice continuously differentiable with $v, v' > 0$.

Case 2' $w(x,t) = U(v(t) + \psi(x), t)$ where $\psi' > 0$, $\psi'' \leq 0$, and $\psi(0) = 0$.

Case 3' $w(x,t) = \int U(v + x, t)dK(v|t)$ with $K(v|t) \geq K(v|\hat{t})$ for all $t < \hat{t}$.

Case 4' $w(x,t) = (1+t)U(v(t) + x, t)$ where $U \geq 0$.

Cases 1'-4' generalize Cases 1-4 also in some other details. For instance, Case 3' allows the distribution K to be independent of t so that all bidders have the same probability distribution over v. This can be a situation in which all available information has been "priced" into the object for sale but because the bidders have different risk attitudes they may still have different expected (utility) payoffs upon winning. More generally, Case 3' allows the bidders' types to affect their risk preferences as well as their expectations about the object's uncertain value. For instance, t may be correlated to a bidder's wealth, a higher wealth level giving the bidder more favorable conditions for using or deriving values from the object.

For Cases 1'-4', conditions A1-A4 also easily hold with proper assumptions on $U(x,t)$ and $K(v|t)$. The next lemma gives an interpretation of A5-A6 in terms of $U(x,t)$.

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[17] For instance, for Case 2 u is log-concave as long as U is log-concave in the sense of A4, since ψ is a (weakly) concave function.

Lemma 3 *Suppose A1-A4 hold. Further assume that $U(\cdot, t)$ exhibits nonincreasing absolute risk aversion and that $U(\cdot, t)$ is more risk averse than $U(\cdot, \hat{t})$ whenever $t < \hat{t}$. Then A5-A6 hold for Cases 1'-4'.*

Indeed, in all these cases the conditions A5-A6 can be replaced by the joint condition that $-u_{11}(x, t)/u_1(x, t)$ is nonincreasing in x and t. An important special case is where $u(x, t)$ exhibits CARA in x for all t, or that A6 holds with $Q_1 = 0$. If A6 holds with $Q_1 < 0$, then by Lemma 1 it corresponds to the cases in which $u(x, t)$ exhibits DARA in x for all t. However, since these cases are just special examples of our model and the function $w(x, t)$ can be given other forms or interpretations (e.g., non-expected utility preferences), we maintain A5-A6 in this paper for generality.

3 Equilibrium

In both the EA and the EPA we assume that the seller chooses the same reserve price p_0. We begin with the EA equilibrium, which serves as a benchmark for analyzing the EPA and welfare effects of risk sharing in Section 4.

3.1 English auction

In the EA, it is routine to check that there exists a unique symmetric equilibrium in our setting. In this equilibrium, it is a (weakly) dominant strategy for a type-t bidder to stay in the auction until the price reaches $\eta(t)$ such that

$$w(-\eta(t), t) = u(0, t) \tag{5}$$

By A1-A3 the bid function η is well defined on $[0, H]$, and it is increasing by A5.

If the reserve price $p_0 < \eta(0)$, it has no effect and all bidders will participate in the EA. If $p_0 > \eta(H)$ then no bidder will be interested in bidding. From now on we assume that $p_0 \in [\eta(0), \eta(H)]$. Then, there exists a *screening level* $t_0 \in [0, H]$ defined by $\eta(t_0) = p_0$. A bidder will abstain from bidding in the EA if and only if his type is lower than t_0.

Given any vector of types $(t_1, ..., t_N)$, we let $t_{(1)}, t_{(2)}$ and $t_{(3)}$ denote the highest, second highest, and third highest types from among $(t_1, ..., t_N)$. We call the bidder of type $t_{(2)}$ the *pivotal* bidder, who determines the selling price in the EA equilibrium.

An important property of the EA equilibrium is that the object for sale will be allocated to the one who has the highest willingness to pay for the object and therefore the EA is *ex post efficient*. It is important to note that as long as the absolute risk aversion $-u_{11}(x,t)/u_1(x,t)$ is nonincreasing in x and t, the winning bidder in the EA should be also (weakly) more risk tolerant than all other bidders. This observation is further strengthened when the object for sale entails ensuing risk.

3.2 English premium auction

According to the EPA rule, if the number of participating bidders $n \leq 2$, then the auction reduces to the EA and the preceding analysis of equilibrium strategy $\eta(t)$ holds for this special case.

Now suppose $n > 2$. As in the EA, we focus on symmetric equilibria in which bidders adopt the same bidding strategies. By backward induction, suppose that the first stage ends with bottom price $X \geq p_0$ and the two finalists adopt strategy $b(\cdot, X)$: $[r, H] \to [X, \infty)$ with updated lower bound r of the opponent's type distribution. We say that b is a *second-stage equilibrium* if conditional on X, adopting $b(\cdot, X)$ maximizes each finalist's expected utility given that the other finalist adopts the same strategy b.

At the start of the second stage, a finalist with type t who bids as though his type is s $(\geq r)$ derives a conditional expected utility equal to

$$\begin{aligned} U(t,s|r,X) &= \frac{1}{1-F(r)} \int_r^s w\left(\alpha(b(y,X)-X) - b(y,X), t\right) dF(y) \\ &\quad + \frac{1-F(s)}{1-F(r)} u\left(\alpha(b(s,X)-X), t\right) \qquad (6) \end{aligned}$$

Equilibrium requires $U_2(t,t|r,X) = 0$, and it can be readily verified that (see Hu, Offerman and Zou, 2011, Theorem 1), by A1-A6, there exists a second-stage equilibrium that is the unique solution of the differential equation in (7) under the boundary condition in (8):

$$b_1(t,X) = \frac{1}{\alpha}\frac{u\left(\alpha(b(t,X)-X),t\right) - w\left(\alpha(b(t,X)-X)-b,t\right)}{u_1\left(\alpha(b(t,X)-X),t\right)}\frac{f(t)}{1-F(t)} \tag{7}$$

$$b(H,X) = B(X) \tag{8}$$

such that $b_1 > 0$; where $B(X)$ is the solution B that solves[18]

$$w(\alpha(B-X)-B,H) = u(\alpha(B-X),H). \tag{9}$$

The following lemma shows how b is affected by the bottom price.

Lemma 4 *Assume A1-A6. Then, on its effective domain,*[19] *$b(t,X)$ is continuously differentiable such that (i) $b_2 = 0$ if $Q_1 = 0$, and (ii) $b_2 < 0$ if $Q_1 < 0$.*

The second-stage strategy $b(t,X)$ now induces a first-stage strategy β, which is given (implicitly) by

$$\beta(t) = b(t,\beta(t)) \tag{10}$$

By Lemma 4 and the implicit function theorem, $\beta(t)$ is well defined and is continuously differentiable, satisfying

$$\beta'(t) = \frac{b_1(t,\beta(t))}{1-b_2(t,\beta(t))} > 0 \tag{11}$$

Because β is increasing, in equilibrium we have $X = \beta(t_{(3)})$ and that both finalists have types in $[t_{(3)},H]$.

Summarizing, the EPA strategy, denoted b^*, can be fully described as follows.[20] For all N potential bidders,

[18] Because the left side in (9) increases in B and the right side decreases in B, by A1-A3 the solution $B = B(X)$ is uniquely defined and differentiable in X.

[19] By the EPA rules, the effective domain of $b(t,X)$ is

$$\Omega = \{(t,X) \in [0,H] \times [p_0, B(H)] : X \leq b(t,X) \leq B(X)\}.$$

[20] The dependence of b^* on p_0 and (η, β, b) is suppressed to ease notation.

(I) their pre-auction strategy is $\begin{cases} b^*(t) \geq p_0 & \text{if} \quad t \geq t_0 \\ b^*(t) < p_0 & \text{if} \quad t < t_0 \end{cases}$

This means that bidders with types lower than t_0 choose to abstain from bidding and the rest choose to participate. Once the auction begins, the active number n becomes common knowledge. Thus for the n active bidders,

(II) their first-stage strategy is $\begin{cases} b^*(t) = \beta(t) & \text{if} \quad n > 2 \\ b^*(t) = \eta(t) & \text{if} \quad n \leq 2 \end{cases}$

The case with $n \leq 2$ is straightforward. For $n > 2$, the first stage will end with a bottom price X and for the two finalists,

(III) their second-stage strategy is $b^*(t) = b(t, X)$.

We say that b^* is an *EPA equilibrium* if (i) $b(\cdot, X)$ is a second-stage equilibrium conditional on any bottom price X; (ii) in the first stage with $n > 2$ active bidders, conditional on any updated information it is optimal for each bidder to adopt strategy β providing the other bidders adopt β; and with $n = 2$ active bidders, it is a (weakly) dominant strategy for each bidder to adopt strategy η; and (iii) prior to the auction, a type-t bidder chooses to stay at price p_0 if and only if his expected payoff from the subsequent auction game is no less than $u(0, t)$.

Our next theorem establishes that b^* is indeed an EPA equilibrium.

Theorem 1 *Suppose A1-A6 hold. Then the strategy b^* constitutes an EPA equilibrium.*

Proof. The proof that given $n \geq 3$ active bidders, (β, b) is an EPA equilibrium follows similar (lengthy) arguments as in Hu, Offerman and Zou (2011, Theorem 1); hence is omitted. Because the effect of a binding reserve price has not been considered previously, to complete the proof we consider the pre-auction stage here assuming that active bidders will follow strategy β in the first stage of the EPA.

First, suppose a potential bidder has type $t < t_0$ so that $\eta(t) < p_0$. Prior to the auction, he is unsure about the number n of bidders who will choose to participate at p_0. We show that it is optimal for the bidder to abstain from bidding. If he stays at p_0, he faces three possible scenarios. (i) No other bidder stays at p_0. In this case

$n = 1$ and by (5) the bidder purchases the object at a loss. (ii) Only one other bidder stays at p_0. In this case $n = 2$ and the EA policy implies that the bidder will have no expected profit to be made. He has to quit immediately or else face a potential loss should he become the winner. (iii) There are $n \geq 3$ bidders staying at p_0 (including this one with type $t < t_0$). Then, given that the other bidders will adopt strategy β in the first stage, and given that these bidders have followed the pre-auction strategy so that their types are no less than t_0, the bidder with $t < t_0$ will have no chance to become a finalist unless he deviates from strategy β. This is suboptimal, however, as it is optimal to follow strategy β in the first-stage.

Consider next a bidder with type $t \geq t_0$. It is clear that in all the above possible scenarios (i)-(iii), he will have an expected payoff higher than (if $t > t_0$) or equal to (if $t = t_0$) his status-quo utility $u(0, t)$. Therefore, it is optimal for the bidder to participate in the auction.

We conclude that b^* is an EPA equilibrium. ■

Intuitively, with $n > 2$ active bidders the premium induces all types to bid higher in the EPA than in the EA, i.e., $\beta(t) > \eta(t)$ for all $t \geq t_0$.[21] By the pre-auction strategy b^*, this implies that observing $n > 2$ leads to a "jump" in bids at t_0 in the EPA, resulting in no bid in the price interval $(p_0, \beta(t_0))$. As shown in the proof of Theorem 1, this jump bidding is caused by the uncertainty at the pre-auction stage about the number n of active participants under reserve price p_0.[22]

[21] This follows from (7) that for $X = \beta(t)$, $b_1(t, \beta(t)) > 0$ is equivalent to $u(0, t) > w(-\beta(t), t)$. Comparing this with (5) gives $\beta(t) > \eta(t)$ for all $t \in [t_0, H)$.

[22] Jehiel and Moldovanu (2000) derive a similar jump-bidding property in their second-price auction equilibrium with negative externalities. In their model with two bidders and a binding reserve price, a subset of types with private values lower than the reserve price face the uncertainly whether the opponent will bid higher or lower than the reserve price (similar to our $n > 2$ or $n \leq 2$ scenarios). By deduction, under both scenarios these low-value types will not stand to gain and therefore will bid zero. As the reserve price does not affect the equilibrium bids by other types, which are higher than their true values due to the externality, the jump bidding occurs at the level of the reserve price.

Milgrom and Weber (1982) also observe a similar "jump" property in their analysis of second-price

The key implication of the jump bidding in the EPA is that when the seller chooses the same reserve price p_0 as in the EA, it will induce the same screening level t_0 so that the subsets of active types who are willing to participate in the EA and the EPA are the same. This implication is essential for our main results in Section 4.

4 Main Results

We now turn to investigating the welfare implications of the premiums, from the seller's perspective first, and then from that of the bidders.

4.1 Seller's perspective

Suppose the seller's utility function, V, is twice differentiable and that the seller has a certainty equivalent value for the object equal to $v_0 \leq p_0$.[23]

Let $f_{(2)}^N$ denote the density function of the second-highest type $t_{(2)}$, with the associated cumulative distribution $F_{(2)}^N$. The seller's expected utility in the EA can then be written as

$$\begin{aligned} & V_N(p_0|\text{EA}) \\ = \; & V(v_0)F(t_0)^N + V(p_0)NF(t_0)^{N-1}(1-F(t_0)) + \int_{t_0}^{H} V(\eta(y))\, dF_{(2)}^N(y) \quad (12) \end{aligned}$$

where the first term in (12) comes from event $t_{(1)} \leq t_0$, the second term from event $t_{(2)} \leq t_0 < t_{(1)}$, and the last term from event $t_0 < t_{(2)}$.

Now let $f_{(2)(3)}^N$ denote the joint density of the second- and the third-highest types.

equilibrium with interdependent values and a binding reserve price. For related analyses see also, e.g., Jehiel and Moldovanu (1996) and Caillaud and Jehiel (1998).

[23] In our setting, if the seller were able to choose the reserve price optimally for the EA, then $p_0 > v_0$ (e.g., Hu, Matthews and Zou, 2010).

The seller's expected utility in the EPA is then given by

$$\begin{aligned} V_N(\alpha, p_0|\text{EPA}) &= V(v_0)F(t_0)^N + V(p_0)NF(t_0)^{N-1}(1-F(t_0)) \\ &\quad + \int_0^{t_0}\int_{t_0}^{H} V(\eta(y))\, f_{(2)(3)}^N(y,z)dydz \qquad (13) \\ &\quad + \int_{t_0}^{H}\int_{z}^{H} V(R(y,\beta(z)))\, f_{(2)(3)}^N(y,z)dydz \qquad (14) \end{aligned}$$

where the term in (13) comes from event $t_{(3)} < t_0 < t_{(2)}$, the term in (14) from event $t_0 \leq t_{(3)}$, and

$$R(y,\beta(z)) \equiv b(y,\beta(z)) - 2\alpha\left(b(y,\beta(z)) - \beta(z)\right) \qquad (15)$$

is the seller's revenue conditional on $t_{(2)} = y$ and $t_{(3)} = z \geq t_0$.

The next theorem provides a key result concerning the lower bound for the difference between the seller's expected payoffs in the EPA and the EA. This bound is "tight" in that it is reached under condition (i) in the theorem.

Theorem 2 *Assume A1-A6, and that $V'' \leq 0$. Then for all $\alpha \in (0, 1/2]$,*

$$\begin{aligned} & V_N(\alpha, p_0|EPA) - V_N(p_0|EA) \\ \geq & \int_{t_0}^{H} \Phi(t)\left(1 - \left(\frac{F(t_0)}{F(t)}\right)^{N-2}\right) V'(\beta(t))\, dF_{(2)}^N(t) \qquad (16) \\ \textit{where } \Phi(t) = & \frac{V(\beta(t)) - V(\eta(t))}{V'(\beta(t))} - \frac{u(0,t) - w(-\beta(t),t)}{u_1(0,t)} \qquad (17) \end{aligned}$$

The inequality in (16) is (i) an equality if $Q_1 = 0$ and either $V'' = 0$ or $\alpha = 0.5$, and (ii) is a strict inequality if $Q_1 < 0$, or if $V'' < 0$ and $\alpha \in (0, 1/2)$.

Proof. The difference between the seller's expected payoffs in the EPA and the EA is uniquely determined by their difference in the event $t_0 \leq t_{(3)}$. Therefore

$$\begin{aligned} & V_N(\alpha, p_0|\text{EPA}) - V_N(p_0|\text{EA}) \\ = & \int_{t_0}^{H}\int_{z}^{H} \left[V(R(y,\beta(z))) - V(\eta(y))\right] f_{(2)(3)}^N(y,z)dydz \end{aligned}$$

Substituting $f_{(2)(3)}^N(y,z) = N(N-1)(N-2)F(z)^{N-3}(1-F(y))f(z)f(y)$ gives

$$\begin{aligned} & V_N(\alpha, p_0|\text{EPA}) - V_N(p_0|\text{EA}) \\ = & N(N-1)\int_{t_0}^{H}\left(\int_{z}^{H}\left[V(R(y,\beta(z))) - V(\eta(y))\right](1-F(y))dF(y)\right) dF(z)^{N-2} \end{aligned}$$

Integrating by parts, and noting that $R(z,\beta(z))=\beta(z)$, we obtain

$$\begin{aligned}
& V_N(\alpha,p_0|\text{EPA})-V_N(p_0|\text{EA}) \\
= & -N(N-1) \\
& \times\int_{t_0}^{H}\left[F(z)^{N-2}-F(t_0)^{N-2}\right]\frac{\partial}{\partial z}\int_{z}^{H}\left[V\left(R(y,\beta(z))\right)-V\left(\eta(y)\right)\right](1-F(y))dF(y)dz \\
= & N(N-1)\int_{t_0}^{H}\left[F(z)^{N-2}-F(t_0)^{N-2}\right]\left[V\left(\beta(z)\right)-V\left(\eta(z)\right)\right](1-F(z))dF(z) \\
& -N(N-1)\int_{t_0}^{H}\left[F(z)^{N-2}-F(t_0)^{N-2}\right]\int_{z}^{H}\frac{\partial}{\partial z}V\left(R(y,\beta(z))\right)(1-F(y))dF(y)dz \qquad (18)
\end{aligned}$$

The partial derivative

$$\frac{\partial}{\partial z}V\left(R(y,\beta(z))\right)=V'\left(R(y,\beta(z))\right)R_2(y,\beta(z))\beta'(z)$$

By Lemma 4, we have $b_2\leq 0$. So,

$$R_2 = 2\alpha+(1-2\alpha)b_2\leq 2\alpha \qquad (19)$$

$$\beta'(z) = \frac{b_1(z,\beta(z))}{1-b_2(z,\beta(z))}\leq b_1(z,\beta(z)) \qquad (20)$$

Since $R_1=(1-2\alpha)b_1\geq 0$, $R(y,\beta(z))\geq R(z,\beta(z))=\beta(z)$. So $V''\leq 0$ implies $V'\left(R(y,\beta(z))\right)\leq V'\left(\beta(z)\right)$ for all $y\geq z$. Consequently,

$$\frac{\partial}{\partial z}V\left(R(y,\beta(z))\right)\leq V'\left(\beta(z)\right)2\alpha b_1(z,\beta(z)) \qquad (21)$$

It follows that

$$\begin{aligned}
& \int_{z}^{H}\frac{\partial}{\partial z}V\left(R(y,\beta(z))\right)(1-F(y))dF(y) \\
\leq & V'\left(\beta(z)\right)2\alpha b_1(z,\beta(z))\left(\int_{z}^{H}(1-F(y))dF(y)\right) \\
= & V'\left(\beta(z)\right)\alpha b_1(z,\beta(z))(1-F(z))^2 \\
= & V'\left(\beta(z)\right)\frac{u\left(0,z\right)-w\left(-\beta(z),z\right)}{u_1\left(0,z\right)}(1-F(z))f(z)
\end{aligned}$$

where we used (7) to obtain the last equation. Substituting this inequality into (18), rearranging terms, and changing the notation of variable z to t, we obtain (16)-(17).

If $Q_1=0$, then by Lemma 4(i) $b_2=0$. This implies that both inequalities in (19)-(20) hold as an equality. In this case either $V''=0$ or $\alpha=0.5$ implies $V'\left(R(y,\beta(z))\right)=$

$V'(\beta(z))$. So (21) holds as an equality. The same deduction will then yield (16) as an equation.

If $V'' < 0$ and $\alpha \in (0, 0.5)$, then $V'(R(y, \beta(z))) < V'(\beta(z))$ by the fact that $R(y, \beta(z))$ is an increasing function of y. So the inequality in (21) holds strictly. This is also true with $Q_1 < 0$, which implies, by Lemma 4(ii), $b_2 < 0$ and therefore both inequalities in (19)-(20) hold strictly. The subsequent deduction will then lead to a strict inequality in (16). ■

By inspecting (16), we find that the relative performance of the EPA from the seller's perspective depends only on the distribution of the second-highest type $F_{(2)}^N$, where $(u(\cdot, t), w(\cdot, t))$ in (17) stands for the preference functions of the pivotal bidder in the EA (i.e., $t = t_{(2)}$). Therefore, a sufficient condition for the EPA to outperform the EA is that the function $\Phi(t)$ in (17) is positive for all $t \in (t_0, H]$. In light of a result in Hu, Offerman and Zou (2011) that the premium lowers expected revenue when bidders are risk averse (see also Lemma 5 in the next subsection), Theorem 2 suggests a strong risk sharing effect of the premium: even though the expected revenue is lower, the seller may strictly prefer the EPA for the reduction of revenue risk.

It is instructive to use Case 1' (and thus Case 1) as an example and see how the sign of $\Phi(t)$ can be determined. For Case 1', $w(x, t) = u(v(t) + x, t)$. So by (5), $w(-\eta(t), t) = u(0, t)$ implies $\eta(t) = v(t)$. Now assume that

$$-\frac{V''(x)}{V'(x)} \geq -\frac{u_{11}(y, t_0)}{u_1(y, t_0)}, \quad \forall x, y \in \mathbb{R} \tag{22}$$

Then, by Lemma 1

$$\frac{V(\beta(t)) - V(\eta(t))}{V'(\beta(t))} \geq \frac{u(x + \beta(t), t_0) - u(x + \eta(t), t_0)}{u_1(x + \beta(t), t_0)}, \qquad \forall x$$

In particular, for $x = -\beta(t)$ we have

$$\begin{aligned} & \frac{V(\beta(t)) - V(\eta(t))}{V'(\beta(t))} \geq \frac{u(0, t_0) - u(\eta(t) - \beta(t), t_0)}{u_1(0, t_0)} \\ > \; & \frac{u(0, t) - u(\eta(t) - \beta(t), t)}{u_1(0, t)}, \ \forall t > t_0, \text{ by A5 and Lemma 1} \\ = \; & \frac{u(0, t) - w(-\beta(t), t)}{u_1(0, t)} \text{ by } \eta(t) = v(t) \end{aligned} \tag{23}$$

This shows $\Phi(t) > 0$ and therefore $V_N(\alpha, p_0|\text{EPA}) > V_N(p_0|\text{EA})$.

The condition (22) means that regardless of the respective income levels, the seller is more risk averse than the type-t_0 bidder. This condition removes the "wealth effect" that may cause ambiguity in comparing relative risk aversion between individuals at different wealth levels. Indeed, in general, the bidders' degrees of risk aversion as being modelled depend on how we "normalize" their status quo wealth. This has been assumed to be zero in our model by convention. Such a normalization is innocuous if the bidders exhibit CARA, but in the case of DARA it makes the bidders "appear" to be more risk averse than they actually are – given the supposition that each bidder has sufficient funds to purchase the object for sale. To avoid such ambiguities, we therefore invoke the assumption $Q_1 = 0$ for the following propositions. This assumption is akin to the CARA assumption used elsewhere in auction theory (e.g., Milgrom and Weber, 1982; Matthews, 1983) as well as other fields of studies.[24]

Our first proposition generalizes the preceding observation for Case 1' to Cases 2'- 4'.

Proposition 1 *For Cases 1'- 4' (and therefore Cases 1-4), assume A1-A5, $Q_1 = 0$, and (22). Then a risk averse seller with reserve price p_0 has a higher expected utility in the EPA, given any premium rule $\alpha \in (0, 1/2)$, than in the EA.*

The next proposition highlights a "businessman's case" in which the seller's preference belongs to the same population of the bidders.

Proposition 2 *Suppose A1-A5 hold and $Q_1 = 0$. Suppose the seller's preference is the same as a bidder with type $t_0 \in [0, H)$, and he chooses reserve price p_0 according to $u(p_0, t_0) = w(0, t_0)$ (the seller's status-quo utility if there is no sale). Suppose $u_{11}(\cdot, t_0) \leq 0$. Then $V_N(\alpha, p_0|EPA) > V_N(p_0|EA)$ for all $\alpha \in (0, 1/2)$.*

[24] We agree with Milgrom (2004, p. 93-94) that using CARA is an analytical technique, and it by no means prejudges the importance of wealth effects. Of course, alternatively, the conclusions of these propositions can be arrived at by simply assuming that the bidders exhibit DARA and are sufficiently wealthier than the seller. For instance, assume $u(x, t) = U(c(t) + x, t)$ with $c' > 0$ and $c(t) >> 0$. But such "manipulations" will not add any new insight.

Not confined to Cases 1'- 4', this proposition illustrates why in some circumstances the expected sales above the reserve price logically imply that the seller is better off by employing the EPA rather than the EA.

The next proposition concerns the effect of the number of bidders N prior to the auction.

Proposition 3 *Under the circumstances of either Proposition 1 or Proposition 2, except that the seller does not impose a reserve price (i.e., reserve price equal to zero). Then, for all* $\alpha \in (0, 1/2)$*, there exists a number* $N_\alpha > 2$ *such that* $V_N(\alpha, 0|EPA) > V_N(0|EA)$ *for all* $N > N_\alpha$*.*

It is easily seen from the proof of this proposition that the result holds not just for a reserve price equal to zero. The same prediction holds for any arbitrary reserve price, with a higher reserve price likely to be associated with a lower threshold number N_α of the bidders.

An immediate corollary concerning the expected revenue of Propositions 1-3 is as follows.

Corollary 1 *Suppose the bidder population includes a risk neutral type, say,* $t_0 \in [0, H)$*. Then, under A1-A6, for arbitrary reserve price* p*, (i)* $p \geq \eta(t_0)$ *implies that the expected revenue in the EPA is greater than that in the EA; and (ii)* $p < \eta(t_0)$ *implies that for all* $\alpha \in (0, 1/2)$*, there exists an* $N_\alpha > 2$ *such that the expected revenue in the EPA is greater than that in the EA for all* $N > N_\alpha$*.*

Proof. Because the corollary concerns expected revenues, it is consistent with a risk neutral seller in our model. Assume that both the seller and the type-t_0 bidder are risk neutral. Then, applying the results of Propositions 1 or 2 for Part (i), and of Proposition 3 for Part (ii) lead to the conclusions. ■

We conclude this subsection with two more corollaries that are of interest on their own.

Corollary 2 *Suppose the bidders are risk neutral. Then the EA revenue is a mean-preserving spread of that of the EPA for all* $\alpha \in (0, 1/2)$.

Proof. By the revenue equivalence theorem, under bidder risk neutrality the expected revenue is the same in the EA and in the EPA. For risk neutral bidders A1-A5 and $Q_1 = 0$ hold trivially. Hence, by Theorem 2, the EPA revenue is preferred by all types of risk averse sellers. Hence the conclusion (e.g., Rothschild and Stiglitz, 1970). ■

This corollary significantly generalizes a result of Goeree and Offerman (2004), who showed that for uniformly distributed types the EPA revenue has a lower variance than that of the EA.

All preceding results do not assume any knowledge of the seller (except knowledge of his own preference). Now, if we assume that the seller knows the utility functional forms of u and w, as well as the distribution function of the bidder types F, then we have the next corollary.

Corollary 3 *Under the assumptions of either Proposition 1 or Proposition 2, there exists an optimal* $\alpha^* > 0$ *that maximizes* $V_N(\alpha, p_0|EPA)$ *on* $[0, 1/2]$.

Proof. Obvious, given Propositions 1-2 and the fact that $V_N(\alpha, p_0|\text{EPA})$ is continuous in α on the closed interval $[0, 1/2]$. ■

4.2 Bidders' perspective

We now turn to bidders' preferences for the auction forms. Among n (> 2) active bidders in the EPA under the reserve price p_0, the first $n - 2$ bidders who drop out are the same in either the EA or the EPA. As these losing bidders end up with their status quo utility under either auction policy, it suffices to focus on the second stage EPA with any bottom price $X = \beta(r)$ given. To ease notation, we fix r and denote

$$\varphi(t) = \alpha(b(t, X) - X) \text{ and } h(t) = b(t, X) - \varphi(t)$$

Hence, $\varphi(t)$ is the premium and $h(t)$ is the *effective payment* by the winner if the EPA concludes at price $b(t, X)$.

In the EA, when only two bidders remain, the expected utility of a type-t bidder equals

$$U(t|\text{EA}) = \frac{1}{1-F(r)} \int_r^t w(-\eta(y), t) dF(y) + \frac{1-F(t)}{1-F(r)} u(0, t) \tag{24}$$

The same bidder in the EPA has an expected utility equal to

$$U(t|\text{EPA}) = \frac{1}{1-F(r)} \int_r^t w(-h(y), t) dF(y) + \frac{1-F(t)}{1-F(r)} u(\varphi(t), t) \tag{25}$$

Therefore, in order to compare the bidders' preferences over the two auction forms it suffices to consider the sign of

$$\Delta(s,t) \equiv \int_r^s \left(w\left(-h(y), t\right)\right) - w(-\eta(y), t)) \, dF(y) + \left(1 - F(s)\right) \left(u(\varphi(s), t) - u(0, t)\right) \tag{26}$$

and show that $\Delta(t,t) > 0$ for all $t \in (r, H]$.

4.2.1 Homogeneous utility

We first show a clear-cut result for the homogeneous-utility model.

Theorem 3 *For the homogeneous-utility model, assume A1-A4, $w_2 > 0$, and*[25]

$$-\frac{w_{12}(x,t)}{w_1(x,t)} \equiv \lambda(t), \quad \forall x \in [-\eta(t), 0)$$

Then, $\lambda > 0$ implies $U(t|EPA) > U(t|EA)$ for all $t \in (r, H]$.

Proof. For the homogeneous-utility model, $u_2(x,t) \equiv 0$ so that by the envelope theorem,

$$\frac{d}{dt}\Delta(t,t) = \Delta_2(t,t) = \int_r^s \left(w_2\left(-h(y), t\right)\right) - w_2(-\eta(y), t)) \, dF(y)$$

By the assumption $-w_{12}(x,t) = \lambda(t) w_1(x,t)$, integrating over x gives

$$w_2\left(x, t\right)) - w_2(y, t) = -\lambda(t) \left(w\left(x, t\right)\right) - w(y, t)\right), \quad \forall x, y \in [-\eta(t), 0)$$

[25] For Cases 1-4, this condition is implied by U exhibiting CARA.

Therefore

$$\Delta_2(t,t) = -\lambda(t) \int_r^t \left(w\left(-h(y),t\right)\right) - w(-\eta(y),t)\right) dF(y) \tag{27}$$

If $\Delta(t,t) \leq 0$ for some $t > r$, then $\int_r^t \left(w\left(-h(y),t\right)\right) - w(-\eta(y),t)\right) dF(y) < 0$ as $u(\varphi(t),t) > u(0,t)$. But then (27) implies $\Delta_2(t,t) > 0$ for $\lambda(t) > 0$. We know $\Delta(r,r) = 0$. Thus $\lambda(t) > 0$ implies $\Delta(t,t) > 0$, or $U(t|\text{EPA}) > U(t|\text{EA})$, for all $t \in (r,H]$. ■

Observe that, in the proof of this theorem, no assumption is made about risk aversion of the status quo utility u (apart from A4 that u is log-concave). The conclusion of the theorem depends only on risk aversion of bidders' utility w upon winning.

A straightforward implication of this theorem is that risk sharing in the EPA makes all risk averse bidders better off – at least when income effects are negligible. We provide a numerical example below to visualize the premium effects on the seller's, bidders', and total surplus of expected payoffs.

4.2.2 Example

Consider Case 1. Suppose $n = 3$ and t is uniformly distributed on $[0,1]$, and that the seller does not impose a reserve price. Suppose that bidders' utility U exhibits CARA:

$$U(x) = \frac{1 - \exp(-\lambda x)}{\lambda}, \qquad \lambda \in \mathbb{R}.$$

In the EA, Case 1 implies the equilibrium condition $w(-\eta(t),t) = U(t - \eta(t)) = 0$ so that $\eta(t) = t$. In the EPA, the differential equation (7) with boundary condition (8) has an explicit solution

$$b_\alpha(t) = -\frac{1}{\lambda} \ln \left(\frac{1}{\alpha} \int_t^1 \frac{e^{-\lambda y}}{1-y} \left(\frac{1-y}{1-t}\right)^{\frac{1}{\alpha}} dy \right) \tag{28}$$

where the bid function $b_\alpha(t)$ is independent of the bottom price X. By (10), this implies $\beta(t) = b_\alpha(t)$ so that all bidders will adopt the same strategy b_α in both the first and second stages. Now suppose the seller also has a CARA utility function

$$V(x) = \frac{1 - \exp(-\gamma x)}{\gamma}, \qquad \gamma > 0$$

The density function $f^n_{(2)(3)}(y,z)$ now equals $6(1-y)$, so the seller's expected utility equals[26]

$$V(\alpha) = 6\int_0^1\int_z^1 \frac{1-\exp(-\gamma\,(b_\alpha(y)+2\alpha\,(b_\alpha(y)-b_\alpha(z))))}{\gamma}(1-y)dydz$$

From any bidder's viewpoint, the density of the highest and second highest types from among the other bidders is $f^{n-1}_{(1)(2)}(y,z)$ $(=2)$. Thus, given the bidder's type t, his expected utility equals

$$\begin{aligned} U(t|\alpha) &= 2\int_0^t\int_z^t \frac{1-\exp(-\lambda(t-b_\alpha(y)+\alpha(b_\alpha(y)-b_\alpha(z)))}{\lambda}dydz \\ &\quad +2\int_0^t\int_t^1 \frac{1-\exp(-\lambda(\alpha(b_\alpha(t)-b_\alpha(z)))}{\lambda}dydz \end{aligned}$$

Ex ante, the expected utility of an active bidder equals

$$U(\alpha) = \int_0^1 U(t|\alpha)dt$$

Table 1 shows numerical results for the case with $\lambda = 1$ and $\gamma = 2$ under different premium rules of α. The column with $\alpha = 0$ corresponds to the EA. As can be seen, the seller obtains maximum expected utility at about $\alpha = 0.3$, and bidders prefer $\alpha = 0.5$. The total surplus is maximized at $\alpha = 0.5$.

Table 1: predictions of specific example

Premium rule α	0	0.1	0.2	0.3	0.4	0.5
Seller expected utility $V(\alpha)$	0.297	0.303	0.305	0.306*	0.305	0.304
Bidder expected utility $U(\alpha)$	0.059	0.063	0.066	0.068	0.070	0.073*
Total surplus $V(\alpha)+3U(\alpha)$	0.473	0.490	0.502	0.510	0.516	0.523*

4.2.3 Heterogeneous utility

We now extend the result of Theorem 3 to the heterogeneous-utility model. For concreteness, we will interpret t as a parameter that is positively associated with a bidder's wealth. A natural implication of this interpretation is that risk aversion is associated

[26] Without ambiguity, we use the same notation V (and U) to denote expected utility at each stage.

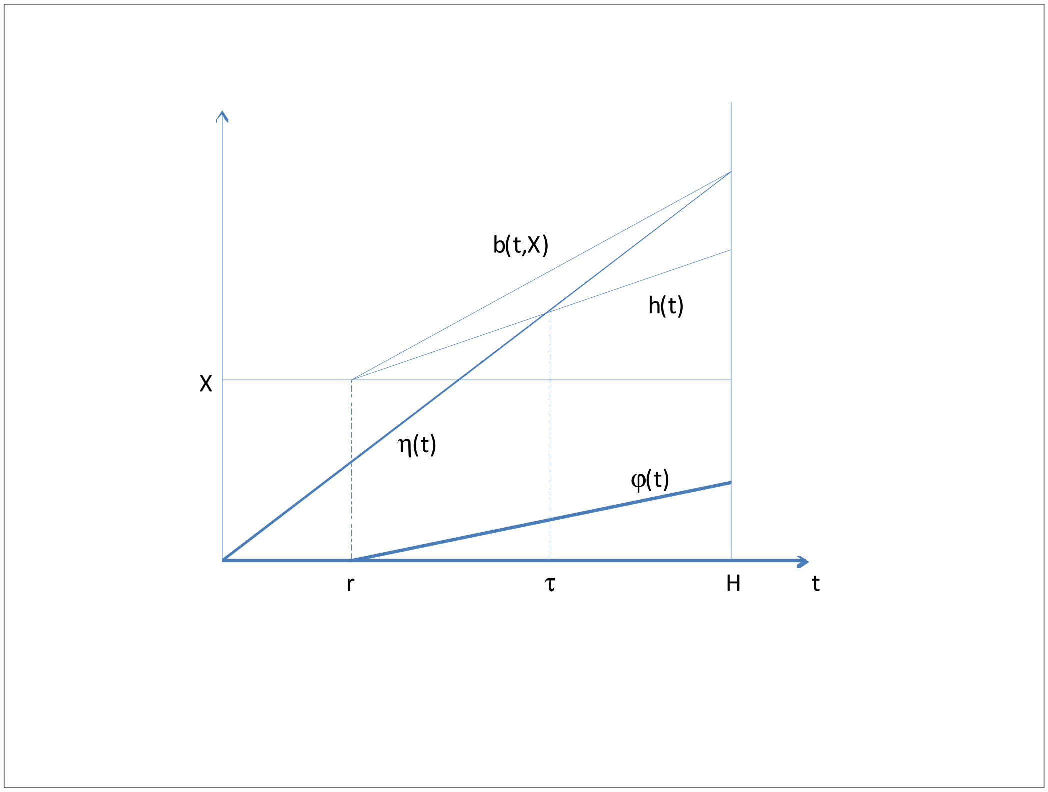

Figure 1: There is a threshold point τ at which the EA bid function $\eta(t)$ crosses the EPA effective payment function $h(t)$ from below.

with the property that $u_{12} \leq 0$, with $u_{12} = 0$ being a special case for homogeneous or risk neutral bidders.

For generality, we allow u and w to exhibit nonincreasing absolute risk aversion. As implied by the boundary condition (8) and (9), the effective payment function $h(t)$ satisfies $h(r) > \eta(r)$ and $h(H) < \eta(H)$. Consequently, there is a crossing point τ at which $\eta(t) - h(t)$ switches the sign from negative to positive. To simplify the analysis, in what follows we assume that $\eta(t) - h(t)$ has a *single crossing property* that for any $t \in [r, H)$, $\eta(t) \geq h(t)$ implies $\eta(\hat{t}) \geq h(\hat{t})$ for all $\hat{t} \in (t, H]$ (see Figure 1).[27]

[27] The crossing point τ can be more generally defined by $\tau = \inf\{t \in [r, H) : \eta(t) \geq h(t)\}$. Sufficient conditions for such a single crossing property to hold can be identified for specific cases, which we do not pursue here given the space limit.

To ease exposition, let x^e and x^p denote respectively the *effective income* of a finalist in the EA and EPA, such that $E_t U(\tilde{x}^e, t) = U(t|\text{EA})$ and $E_t U(\tilde{x}^p, t) = U(t|\text{EPA})$ as defined in (24) and (25).[28] In general, the effective income is a random variable that depends on a bidder's type t, the opponent's type y and possibly the ensuing risk. Because $U_1(x,t) > 0$, the effective income can be always well defined given any specification of $w(x,t)$. For example, for Cases 1' and 2', we define for any function ψ such that $\psi' > 0$ (with $\psi' \equiv 1$ for Case 1'),

$$\tilde{x}^e(y,t) = \begin{cases} v(t) + \psi(-\eta(y)) & \text{for } y \leq t \\ 0 & \text{for } y > t \end{cases}$$

$$\tilde{x}^p(y,t) = \begin{cases} v(t) + \psi(-h(y)) & \text{for } y \leq t \\ \psi(\varphi(t)) & \text{for } y > t \end{cases}$$

The following lemma shows that when bidders are risk averse, the premium increases the expected effective income for any type of a finalist under various circumstances. The lemma generalizes our previous work (Hu, Offerman and Zou, 2011) in that bidders now exhibit heterogeneous risk preferences, and that the auctioned object may carry ensuing risks.

Lemma 5 *For Cases 1'-3' suppose A1–A3, A5-A6 hold, and $U_{11}(\cdot,t) < 0$ for all $t \in [0,H]$. Suppose further for Case 3' that $Q_1 = 0$. Then the expected effective income by any type of the bidders is higher in the EPA than in the EA.*[29]

The analysis of the risk sharing effects on bidders' expected utilities is complicated by the fact that, although in the EA no bidder expects to end up with a loss, in the

[28] The expectation operator $E_t(\cdot)$ indicates that the expectation may be taken conditional on t, such as in Case 3 and Case 3'.

[29] The marginal utility in Case 4' exhibits a jump upon winning and consequently $U_{11}(x,t) < 0$ does not imply risk aversion "in the large." In the homogeneous-utility model of Maskin and Riley (1984), the absolute risk aversion in Case 4 is required to be greater than 2. In the more general Case 4', it can be shown that the result of this lemma holds if U is "sufficiently" risk averse. We drop this case for clarity of the conclusions.

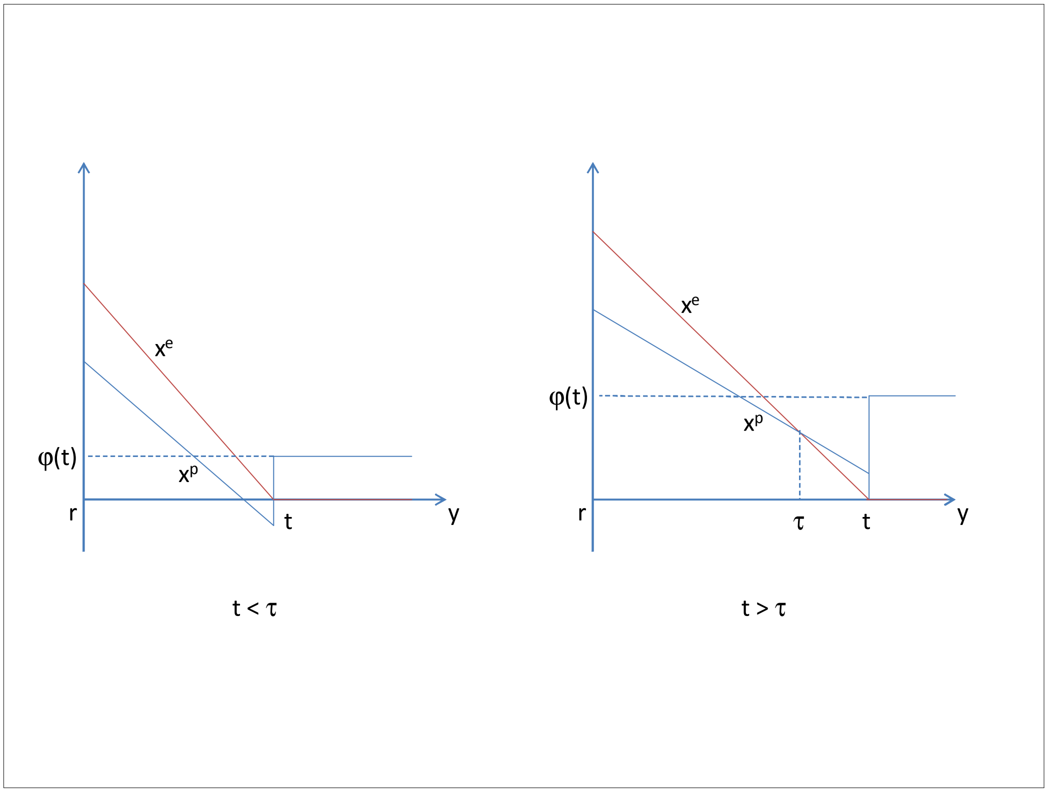

Figure 2: In situation $t < \tau$ the type-t finalist has the potential risk of losing money. This will happen when the opponent has a type $y < t$ that is sufficiently close to t. In situation $t > \tau$ the type-t finalist is ensured to earn a positive surplus.

EPA this can happen to a subset of low-type bidders when $t < \tau$ (see Figure 2). As it turns out, the case with $t < \tau$ requires some mild additional assumptions for an unambiguous determination of the premium effect. The situation with $t > \tau$ does not require these additional assumptions, and has a stronger implication that $\tilde{x}^p$ dominates $\tilde{x}^e$ by second-order stochastic dominance.

Our last theorem concerns the general result that risk averse bidders prefer the EPA to the EA under various circumstances.

Theorem 4 *For the heterogeneous-utility model, suppose A1-A6 hold and* u_{11}*,* $w_{11} < 0$*. Then* $U(t|EPA) > U(t|EA)$

(i) for all $t \in (r, \tau]$ if[30]

$$\frac{\partial}{\partial t}\frac{w_{11}(x,t)}{w_1(x,t)} \geq 0, \quad \forall x \in [-\eta(t), 0) \text{ and} \tag{29}$$

$$\frac{w_{12}(x,t)}{w_1(x,t)} < \frac{u_{12}(y,t)}{u_1(y,t)} \leq 0, \quad \forall x, y : -\eta(t) \leq x < 0 \leq y. \tag{30}$$

(ii) for all $t \in (\tau, H]$ if $\eta(t) - h(t)$ has the single crossing property on $[r, H]$ and $E_t(\tilde{x}^p) \geq E_t(\tilde{x}^e)$ (e.g., Lemma 5).

Proof. (i) It can be readily verified that condition (29) is equivalent to

$$\frac{\partial}{\partial x}\frac{w_{12}(x,t)}{w_1(x,t)} \geq 0, \tag{31}$$

and by Lemma 1, (29) implies

$$\frac{\partial}{\partial t}\frac{w(x,t) - w(y,t)}{w_1(x,t)} \leq 0, \quad \forall x, y \in [-\eta(t), 0)$$

Consequently, for all $x, y \in [-\eta(t), 0)$,

$$w_2\,(y,t)) - w_2(x,t) \geq (w\,(y,t)) - w(x,t))\,\frac{w_{12}(x,t)}{w_1(x,t)} \tag{32}$$

Consider now $\Delta(s,t)$ given in (26). We have

$$\begin{aligned}\Delta_2(s,t) &= \int_r^s (w_2\,(-h(y),t)) - w_2(-\eta(y),t))\,dF(y) \\ &\quad +(1 - F(s))\,(u_2(\varphi(s),t) - u_2(0,t))\end{aligned}$$

Substituting $-\eta(y)$ for x and $-h(y)$ for y in (32), we deduce

$$\begin{aligned}\Delta_2(s,t) &\geq \int_r^s (w\,(-h(y),t)) - w(-\eta(y),t))\,\frac{w_{12}(-\eta(y),t)}{w_1(-\eta(y),t)}dF(y) \\ &\quad +(1 - F(s))\,(u_2(\varphi(s),t) - u_2(0,t))\end{aligned}$$

[30] The condition in (29) says that the absolute risk aversion of $w(x,t)$ is nonincreasing in t. Condition (30) could be interpreted as requiring that the rate of change in the marginal utility of w is relatively lower than that of u as t increases. For the homogeneous-utility model, this is a natural consequence of risk aversion as $w_{12} < 0 \equiv u_{12}$. For the heterogeneous-utility model, (30) holds for Cases 1'-3' as long as $u_{11} < 0$ and $-u_{11}/u_1$ does not decrease "too fast" as income increases over the relevant domain. This nests CARA as a special case. For Case 4', the condition requires u to be "sufficiently" risk averse.

Because $w(-h(y),t)) \leq w(-\eta(y),t)$, by the (first) mean value theorem in integral calculus,

$$\begin{aligned} & \int_r^s (w(-h(y),t)) - w(-\eta(y),t)) \frac{w_{12}(-\eta(y),t)}{w_1(-\eta(y),t)} dF(y) \\ = \; & \frac{w_{12}(-\eta(\xi),t)}{w_1(-\eta(\xi),t)} \int_r^s (w(-h(y),t)) - w(-\eta(y),t)) \, dF(y), \;\; \xi \in [r,s] \end{aligned} \tag{33}$$

On the other hand, we have for $s > r$,

$$\begin{aligned} & u_2(\varphi(s),t) - u_2(0,t) \\ = \; & (u(\varphi(s),t) - u(0,t)) \frac{u_2(\varphi(s),t) - u_2(0,t)}{u(\varphi(s),t) - u(0,t)} \\ = \; & (u(\varphi(s),t) - u(0,t)) \frac{u_{12}(\zeta,t)}{u_1(\zeta,t)}, \quad \zeta \in [0,\varphi(s)], \end{aligned} \tag{34}$$

where the last equation derives from the generalized mean value theorem.

Because $\int_r^s (w(-h(y),t)) - w(-\eta(y),t)) \, dF(y) < 0$, we have by (30), (33), and (34) that

$$\begin{aligned} \Delta_2(s,t) \;\geq\; & \frac{w_{12}(-\eta(\xi),t)}{w_1(-\eta(\xi),t)} \int_r^s (w(-h(y),t)) - w(-\eta(y),t)) \, dF(y) \\ & + (u(\varphi(s),t) - u(0,t)) \frac{u_{12}(\zeta,t)}{u_1(\zeta,t)} \\ > \; & \frac{u_{12}(\zeta,t)}{u_1(\zeta,t)} \int_r^s (w(-h(y),t)) - w(-\eta(y),t)) \, dF(y) \\ & + (u(\varphi(s),t) - u(0,t)) \frac{u_{12}(\zeta,t)}{u_1(\zeta,t)} \\ = \; & \Delta(s,t) \frac{u_{12}(\zeta,t)}{u_1(\zeta,t)}, \quad \forall s \leq t \leq \tau. \end{aligned}$$

Because $u_{12} \leq 0$, it follows by the envelope theorem that $\Delta(t,t) \leq 0$ implies $\frac{d}{dt}\Delta(t,t) = \Delta_2(t,t) > 0$. We have $\Delta(r,r) = 0$. Thus $\Delta(t,t) > 0$ or $U(t|\text{EPA}) > U(t|\text{EA})$ for all $t \in (r,\tau]$.

(ii) We prove this part of the theorem by a simpler graphical argument. Define the induced cumulative probability distributions of $\tilde{x}^e$ and $\tilde{x}^p$ by

$$G^e(x|t) \equiv \Pr(\tilde{x}^e \leq x|t) \text{ and } G^p(x|t) \equiv \Pr(\tilde{x}^p \leq x|t) \tag{35}$$

It can be readily verified by inspecting Figure 2 that the relations between $G^e(x|t)$ and $G^p(x|t)$ can be depicted as in Figure 3. By assumption, we have

$$E_t(\tilde{x}^p) = \int x dG^p(x|t) \geq \int x dG^e(x|t) = E_t(\tilde{x}^e) \tag{36}$$

and for $t > \tau$, G^p crosses G^e from below exactly once. We therefore conclude that (e.g., Jewitt, 1987, Theorem 1) $\tilde{x}^p$ dominates $\tilde{x}^e$ in the sense of second-order stochastic dominance,[31] which implies $U(t|\text{EPA}) > U(t|\text{EA})$ for all $t \in (\tau, H]$. ■

Summarizing, this section has demonstrated a variety of circumstances in which the EPA Pareto dominates the EA at the interim, and therefore also at the ex ante stage of the auction game with incomplete information.

5 Summary and Conclusion

We have presented an analysis of risk sharing effects in English premium auctions (EPA) with risk averse seller and bidders.[32] Our study reveals that when both the seller and bidders are risk averse, the English auction is in general inefficient at the interim stage. By simple modifications of the payment rule of the English auction, the auction designer can often make the auction more attractive to both the sellers and buyers when they are risk averse. This finding has significant normative and positive implications. Because of the overwhelming evidence that the majority of individuals are risk averse and that people differ in their risk attitudes, the EPA format presented in this paper could be of interest to designers of auctions in practice. On the positive side, the result of our study provides a plausible risk-sharing motive that helps explain why premium auctions have stood the test of time and remain a class of regularly adopted auctions in Europe.

[31] Obviously, the same conclusion does not hold for $t < \tau$ as the left end of the support of G^p is lower than that of G^e.

[32] Hu, Matthews and Zou (2013) study a similar English auction model with ensuing risk and heterogeneous bidders, allowing for a more general setting. Their focus is on the existence of ex post efficient equilibria and the effects of ensuing risks.

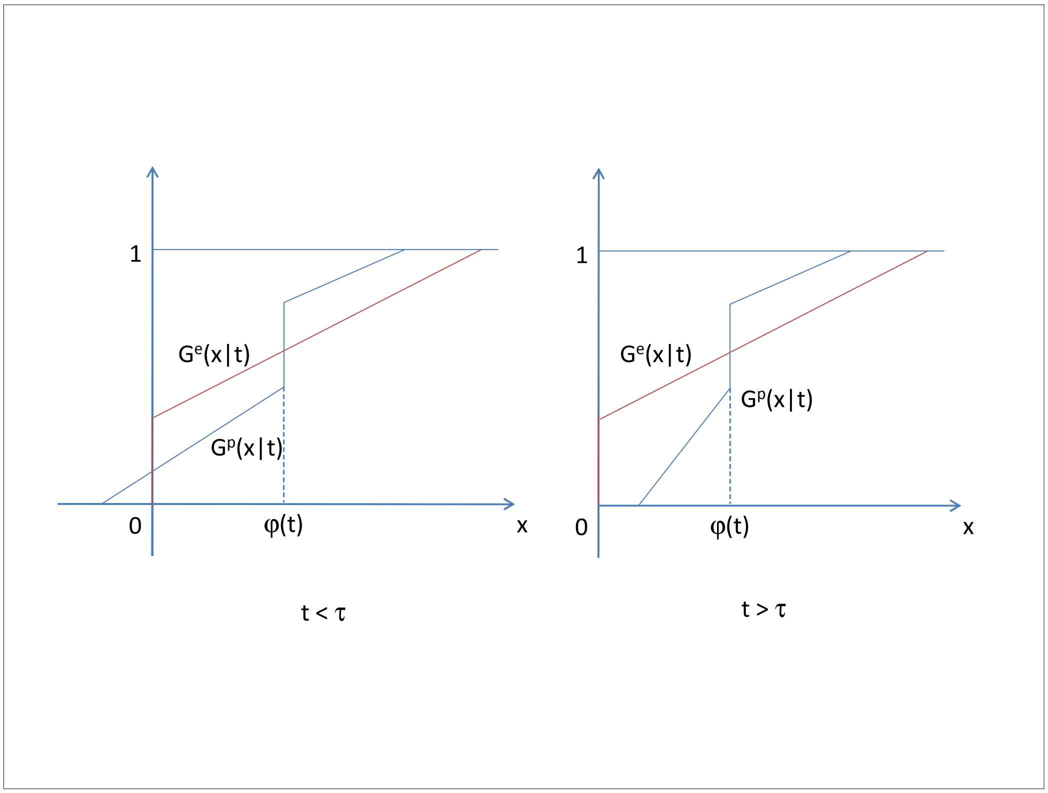

Figure 3: $G^e(x|t)$ and $G^p(x|t)$ are the induced cumulative distributions of a finalist's effective income under the EA and the EPA, respectively. When $t > \tau$, G^p crosses G^e only once from below and therefore dominates G^e in the sense of second-order stochastic dominance given that the expected effective income is higher under G^p rather than G^e.

An important related issue not considered in this paper is how rewarding premiums would affect the potential bidders' entry decisions (e.g., Levin and Smith, 1994; Smith and Levin, 1996; Bulow and Klemperer, 1996, 2009). We have assumed a fixed number of potential bidders and showed that risk averse bidders will unanimously prefer the EPA to the EA irrespective of the seller risk preference. Therefore, it is conceivable that when potential bidders make entry decisions based on their expected payoffs, and acquire information at some costs after entry, the EPA will be more conducive to entry than the English auction. From the seller's viewpoint, this could increase revenue by more than an optimally structured auction does with fewer bidders (e.g., Bulow and Klemperer, 1996). So, even if the seller is risk neutral, the use of EPA could make sense for attracting more bidders. With endogenous entry, risk sharing between the seller and bidders in an EPA could improve ex post allocation efficiency when more bidders are attracted to the trade, while an auction format that is more attractive for sellers could also encourage its actual usage.[33]

In light of the unambiguous benefit of risk sharing among the players in the English premium auctions, one might also wish to know whether, and to what extent, similar improvement in Pareto efficiency can be found on other ex post efficient auctions[34] when players are heterogeneous and risk averse. These will be interesting topics for future research.

[33]For instance, Engelbrecht-Wiggans and Nonnenmacher (1999) documented how implementing a "seller friendlier" auction design in early nineteenth-century New York attracted more imports to the city and supported its subsequent economic growth. See van Bochove, Boerner and Quint (2013) for a historic account about the use of premium tactics in Europe.

[34]For example, Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), Ausubel (2004), and Perry and Reny (2002, 2005). Along the lines of the VCG mechanisms (Vickrey, 1961; Clark, 1971; Groves, 1973), all these auction procedures were designed under the assumption that the players are risk neutral or have quasilinear utility functions.

Appendix

Proofs of the lemmas

Proof of Lemma 1. (i)$\Rightarrow$(ii) follows from Pratt (1964, Eqs. (21) and (22) for $y<0$ and $y>0$, respectively). (ii)$\Rightarrow$(iii) holds by replacing y in (3) by $y-\tilde{v}$, and taking expectation over $\tilde{v}$. (iii)$\Rightarrow$(i) holds by noting that if the weak [strong] form of (i) does not hold, then the strong [weak] form of (i) holds on some interval with u and $\hat{u}$ interchanged. Thus (iii) cannot hold true for all x, y, and $\tilde{v}$ (Pratt, 1964; p. 129). ■

Proof of Lemma 2. Because $u(x,t)$ ($\equiv U(x)$) does not depend on t, $Q_3<0$ if and only if $w_2>0$. Hence, $U'>0$ is equivalent to A5 for Cases 1-3, and implies A5 for Case 4. Now suppose that U has nonincreasing absolute risk aversion. Then, by Lemma 1, for Cases 1-3 $Q(x,y,t)$ is nonincreasing in x and therefore A6 holds. To see that it is also true with Case 4, note that in this case

$$\begin{aligned} Q(x,y,t) &= \frac{U(x)-(1+t)U(t+x-y)}{U'(x)} \\ &= (1+t)\frac{U(x)-U(t+x-y)}{U'(x)}-t\frac{U(x)}{U'(x)} \end{aligned}$$

By Lemma 1 the first part is nonincreasing in x. By log-concavity, $\frac{U(x)}{U'(x)}$ is nondecreasing and therefore Q is nonincreasing in x. ■

Proof of Lemma 3. Fix any t, and assume that $U(x,t)$ is nondecreasing in absolute risk aversion as x increases. Then, as with Cases 1-4, A6 holds for Cases 1'-4'. Now if in addition $U(x,t)$ decreases in absolute risk aversion as t increases, then it can be shown that $Q(x,y,t)$ is a decreasing function of t and therefore condition A5 holds. We check this for Case 3', which is less obvious than the other cases. By Lemma 1, the assumption that $U(\cdot,t)$ is more risk averse than $U(\cdot,\hat{t})$ for $t<\hat{t}$ implies

$$\frac{U(x-(y-v),\hat{t})-U(x,\hat{t})}{U_1(x,\hat{t})}>\frac{U(x-(y-v),t)-U(x,t)}{U_1(x,t)}, \quad \forall x,y,v \tag{37}$$

Because $K(v|\hat{t})$ exhibits first-order stochastic dominance over $K(v|t)$, and because both sides of the above inequality increase in v, taking expectations maintains the inequality:

$$\frac{\int U(x-(y-v),\hat{t})dK(v|\hat{t})-U(x,\hat{t})}{U_1(x,\hat{t})}>\frac{\int U(x-(y-v),t)dK(v|t)-U(x,t)}{U_1(x,t)} \quad (38)$$

This shows $Q(x,y,t)>Q(x,y,\hat{t})$, verifying A5. ■

Proof of Lemma 4. The differential equation in (7) can be more succinctly written as

$$b_1(t,X)=\frac{1}{\alpha}Q(\alpha(b(t,X)-X),b(t,X),t)\frac{f(t)}{1-F(t)}$$

where Q is defined in (2). Because the right-hand side of (7) is continuously differentiable in b, t, and X, the solution $b(t,X)$ is continuously differentiable in t and X on its effective domain (e.g., Hale, 2009, Chapter 1, Theorem 3.3). Differentiating w.r.t. X gives

$$b_{12}(t,X)=-(1-b_2)Q_1+\frac{1}{\alpha}Q_2b_2\frac{f(t)}{1-F(t)} \quad (39)$$

where

$$Q_1=1-\frac{w_1}{u_1}-Q\frac{u_{11}}{u_1}\text{ and }Q_2=\frac{w_1}{u_1} \quad (40)$$

By (8), substituting $b(H,X)$ for B in (9) gives

$$u(\alpha(b(H,X)-X),H)=w(-b(H,X)+\alpha(b(H,X)-X),H),\quad \forall X$$

Differentiating w.r.t. X yields, at $t=H$,

$$u_1\times\alpha(b_2-1)=w_1\times(-b_2+\alpha(b_2-1)) \quad (41)$$

Part (i). Assume $Q_1=0$ and fix an arbitrary $X<B(H)$. Then from (40) we obtain $w_1=u_1$ whenever $Q=0$. Consequently, (41) implies $b_2(H,X)=0$ as $w_1>0$. Now by (39), $b_2(t,X)\geq 0$ implies $b_{12}(t,X)\geq 0$. Hence $b_2\leq 0$ for all $t\leq H$ such that $b(t,X)\geq X$ (see, e.g., Hu et al. 2011, Lemma 1). But this logic holds also for $-b_2$. Therefore, we must have $b_2(t,X)\equiv 0$ on the effective domain of b.

Part (ii). Assume $Q_1<0$. Then by (40), $Q=0$ implies $w_1>u_1$. Equation (41) now implies $w_1b_2=(w_1-u_1)\,\alpha(b_2-1)<(w_1-u_1)\,\alpha b_2$ and therefore $b_2(H,X)<0$.

We first show that $b_2(t,X)<1$ for all $t\in[r,H]$. This follows because by (39), $b_2(t,X)=1$ implies $b_{12}(t,X)>0$, which is impossible given $b_2(H,X)<0$.

Now by (39), $b_2 = 0$ implies $b_{12}(t, X) > 0$. This implies that $b_2(t, X) < 0$ for all $t \leq H$ such that $b(t, X) \geq X$ (see, e.g., Hu et al. 2011, Lemma 1). ■

Proof of Lemma 5. For Cases 1' and 2', it suffices to show that

$$A(t) \equiv \int_r^t (\psi(-h(y)) - \psi(-\eta(y)))\, dF(y) + (1 - F(t))\psi(\varphi(t)) > 0$$

Differentiating yields $A'(t) = (\psi(-h(t)) - \psi(-\eta(t)) - \psi(\varphi(t)))\, f(t) + (1 - F(t))\alpha b_1(t, X)$. Substituting (7), and noting that $v(t) + \psi(-\eta(t)) = 0$, we have

$$\begin{aligned} A'(t) &= \left(v(t) + \psi(-h(t)) - \psi(\varphi(t)) + \psi'(\varphi(t)) \frac{u\,(\varphi(t), t) - w\,(-h(t), t)}{u_1\,(\varphi(t), t)} \right) f(t) \\ &= \left(\frac{U(\psi(\varphi(t)), t) - U(v(t) + \psi(-h(t)), t)}{U_1(\psi(\varphi(t)), t)} - \frac{\psi(\varphi(t)) - (v(t) + \psi(-h(t)))}{1} \right) f(t) \\ &> 0 \end{aligned}$$

where the inequality holds because of $U_{11} < 0$ by Lemma 1.

For Case 3', the effective income also depends on the realization of the value of v. So

$$\begin{aligned} \tilde{x}^e(v, y, t) &= \begin{cases} v - \eta(y) & \text{for } y \leq t \\ 0 & \text{for } y > t \end{cases} \\ \tilde{x}^p(v, y, t) &= \begin{cases} v - h(y) & \text{for } y \leq t \\ \varphi(t) & \text{for } y > t \end{cases} \end{aligned}$$

If suffices to show for this case that

$$A(t) \equiv \int_r^t (\eta(y) - h(y))\, dF(y) + (1 - F(t))\varphi(t) > 0$$

Because $h = b - \varphi$, we have

$$\begin{aligned} A'(t) &= \left(\frac{u\,(\varphi, t) - w\,(-b + \varphi, t)}{u_1\,(\varphi, t)} - (b - \eta) \right) f(t) \\ &= [Q(\varphi, b, t) - (b - \eta)]\, f(t) \end{aligned}$$

Now $Q_1 = 0$ implies (by adding $b - \eta - \varphi$ to the first argument of Q)

$$\begin{aligned} A'(t) &= \left(\frac{u\,(b - \eta, t) - w\,(-\eta, t)}{u_1\,(b - \eta, t)} - (b - \eta) \right) f(t) \\ &= \left(\frac{u\,(b - \eta, t) - u\,(0, t)}{u_1\,(b - \eta, t)} - (b - \eta) \right) f(t) > 0 \end{aligned}$$

where the inequality comes from u being risk averse. Since $A(r) = 0$, we have $A(t) > 0$ for all $t \in (r, H]$. ■

Proofs of the propositions

Proof of Proposition 1. By Theorem 2, it suffices to show that for all the cases considered, the function $\Phi(t)$ defined in (17) is positive. The conclusion has been established for Cases 1 and 1' in (23). For Cases 2'- 4', $Q_1 = 0$ and $w(-\eta(t), t) = u(0, t)$ imply $w(-\beta(t), t) = u(\eta(t) - \beta(t), t)$. So by (23) $\Phi(t) > 0$ for all $t > t_0$ under (22). ■

Proof of Proposition 2. By $Q_1 = 0$, the seller's break even condition $u(p_0, t_0) = w(0, t_0)$ holds iff $u(0, t_0) = w(-p_0, t_0)$. This implies that, in equilibrium, a sale occurs at a price greater than p_0 iff the pivotal bidder has a type $t > t_0$. By the EA equilibrium $u\,(0, t) = w(-\eta(t), t)$, the assumption $Q_1 = 0$ also implies $u\,(\eta(t) - \beta(t), t) = w(-\beta(t), t)$. Hence,

$$
\begin{aligned}
\frac{V\,(\beta(t)) - V\,(\eta(t))}{V'\,(\beta(t))} &= \frac{u\,(\beta(t), t_0) - u\,(\eta(t), t_0)}{u_1\,(\beta(t), t_0)} \quad \text{by assumption} \\
&= \frac{u\,(0, t_0) - u\,(\eta(t) - \beta(t), t_0)}{u_1\,(0, t_0)} \quad \text{by } Q_1 = 0 \\
&> \frac{u\,(0, t) - u\,(\eta(t) - \beta(t), t)}{u_1\,(0, t)} \quad \text{for all } t > t_0, \text{ by A5} \\
&= \frac{u\,(0, t) - w\,(-\beta(t), t)}{u_1\,(0, t)}
\end{aligned}
$$

This shows that $\Phi(t) > 0$ so that by Theorem 2, $V_N(\alpha, p_0|\text{EPA}) > V_N(p_0|\text{EA})$. ■

Proof of Proposition 3. When the reserve price is zero, by A2 all bidders participate in the EA and hence in the EPA (Theorem 1). The inequality in (16) reduces to

$$
V_N(\alpha, 0|\text{EPA}) - V_N(0|\text{EA}) \geq \int_0^H \Phi(t) V'\,(\beta(t))\, dF_{(2)}^N(t) \tag{42}
$$

as can be seen by replacing 0 for t_0 in (16). We have

$$
\begin{aligned}
&\int_0^H \Phi(t) V'\,(\beta(t))\, dF_{(2)}^N(t) \\
= &\int_0^{t_0} \Phi(t) V'\,(\beta(t))\, dF_{(2)}^N(t) + \int_{t_0}^H \Phi(t) V'\,(\beta(t))\, dF_{(2)}^N(t)
\end{aligned}
$$

where the last term is positive because $\Phi(t) > 0$ (by assumption that the seller is more risk averse than all types $t > t_0$). This term comes from event $t_{(2)} \geq t_0$, and the probability of this event tends to 1 as N tends to infinity. Hence, because the term $\Phi(t)V'(\beta(t))$ is independent of N, for all $\alpha \in (0, 1/2)$ there exists an $N_\alpha > 2$ such that $\int_0^H \Phi(t)V'(\beta(t))\, dF_{(2)}^N(t) > 0$.

By A5, $\Phi(t)V'(\beta(t))$ has a single crossing property that if $\Phi(\hat{t})V'\left(\beta(\hat{t})\right) \geq 0$ for any $\hat{t}$ then $\Phi(t)V'(\beta(t)) > 0$ for all $t > \hat{t}$. Further notice that

$$\frac{f_{(2)}^{N+1}(t)}{f_{(2)}^{N}(t)} = \frac{(N+1)\, F(t)}{(N-1)}$$

is a positive and increasing function of t. Thus (e.g., by Persico, 2000, Lemma 1)

$$\int_0^H \Phi(t)V'(\beta(t))\, dF_{(2)}^N(t) \geq 0$$

implies

$$\int_0^H \Phi(t)V'(\beta(t))\, dF_{(2)}^{N+1}(t) = \int_0^H \Phi(t)V'(\beta(t))\, \frac{f_{(2)}^{N+1}(t)}{f_{(2)}^{N}(t)} dF_{(2)}^N(t) \geq 0$$

The conclusion of the proposition thus holds true. ■

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