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# Saving Private Pareto

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# Saving Private Pareto

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**Abstract**

We include initial holdings in the jungle economy of Piccione and Rubinstein (Economic Journal, 2007) and relax the assumptions on consumption sets and preferences. We show that initial holdings are irrelevant for lexicographic welfare maximization. Equilibria other than such maximizers can be jungle equilibria due to myopia. We show that farsightedness restores the equivalence between jungle equilibria and lexicographic welfare maximization. However, we also derive farsighted equilibria in which stronger agents withhold goods from weaker agents. Then, gift giving by stronger agents is needed to restore Pareto efficiency. Our results add to understanding coercion and the crucial assumptions underlying jungle economies.

*JEL* Classification: D51, D61, P52

**Keywords:** power, coercion, jungle economy, withholding

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# 1 Introduction

Piccione and Rubinstein (2007), PR07 hereafter, propose a stylized model to study jungle economies. Coercion governs the bilateral exchange of resources in the jungle. Coercion is driven by the agents' preferences over bounded consumption sets and by an exogenous ranking of agents according to their strength. Weaker agents concede to stronger agents without engaging in costly conflict. The jungle economy mirrors the standard model of an exchange economy. The distribution of power in the jungle is the counterpart of the distribution of initial endowments in the market.

In a jungle equilibrium, a stronger agent no longer wants to take goods from a weaker agent nor from a pile of common goods, that no other agent holds. PR07 specify certain conditions on consumption sets and preferences under which a unique and Pareto efficient jungle equilibrium exists. This jungle equilibrium coincides with the unique lexicographic welfare maximum in which all of the economy's resources are initially common goods and stronger agents take from the pile of common goods before weaker agents can.

It is tempting to conclude from PR07's intriguing analysis that exactly the particular strength relation assumed in their paper constitutes the main driving force behind the final distribution of resources obtained in the jungle. However, this conclusion is somewhat premature. The goal of our paper is to provide a more nuanced view on the interaction of strength, preferences and holdings behind the jungle equilibrium. Intentionally, we do not deviate from PR07's strength relation throughout the paper.

In our analysis, we first replace the pile of common goods by initial holdings and we modify lexicographic welfare maximization to include such holdings. Agents still take in the lexicographic order induced by the strength relation, but now stronger agents take bilaterally from a sequence of weaker agents, rather than once from the pile of common goods. Under weak assumptions on consumption sets and preferences[1], we show that initial holdings are

[1] These assumptions are compact and comprehensive consumption sets and complete, transitive and continuous preferences.

irrelevant for lexicographic welfare maximization. Furthermore, each such maximum is a jungle equilibrium and such maximizers exist. The economic intuition is that, since agents take in the lexicographic order induced by power, stronger agents are always able to obtain a lexicographic welfare-maximizing bundle through a sequence of bilateral takings. The initial distribution of resources among agents in jungle economies is relevant only to determine how much to take from whom.

Next, we investigate the role of preferences in conjunction with the possibility of jungle equilibria that differ from lexicographic welfare maxima. We provide an example of a jungle economy with two agents, three goods and Leontief preferences. In this example, initial holdings exist in which each agent is unable to gain by single bilateral takings.[2] There are at least three intriguing issues about this example.

First, Leontief preferences are monotone and convex and hence weaken the strongly monotone and strictly convex preferences underlying the jungle economy of PR07. We provide novel conditions on preferences that exclude our example and guarantee that all jungle equilibria coincide with lexicographic welfare maximizing allocations.

Second, the jungle equilibrium concept fails to recognize that the stronger agent can gain by a sequence of bilateral takings and, therefore, the jungle equilibrium includes some myopia that does not matter in the jungle economy of PR07. We show that if we include farsightedness into the equilibrium concept, then this restores the equivalence between jungle equilibria and lexicographic welfare maximization under the previously mentioned weak assumptions on consumption sets and preferences.

Third, the strongest agent can sustain his welfare level while holding excess goods. Pareto improvements can be achieved in case the strongest agent would give away these excess goods for consumption by weaker agents. Of course, the strongest agent has no incentives to do so and may withhold these excess goods from weaker agents, who cannot take it. In

[2] In the simplest case, the stronger agent holds everything of the first good, the weaker agent holds everything of the second good while everything of the third good is available as a common good. Then, the stronger agent has no use of the weaker agent's holding of the second good if he is unable to simultaneously obtain the common (third) good.

order to study the phenomenon of withholding further, we distinguish between holdings and consumption. We provide a condition under which withholding will not occur in farsighted equilibria, which includes the jungle economy in PR07.

In another example, we derive a continuum of farsighted equilibria in which the strongest agent holds goods in excess of his satiation point. This withholding of goods is Pareto inefficient and only gift giving, i.e. giving away nonnegative amounts of goods, by stronger agents can remedy this inefficiency. In a last example, we show that even such gift giving may sometimes be insufficient to restore efficiency. Here, trade is needed.

These latter results exactly exemplify why we believe that our analysis is important and adds to a better understanding of the crucial assumptions underlying jungle economies. Pareto efficiency in the jungle is not a result of coercion alone. On the contrary, depending on the kind of preferences present in the jungle, gift giving and trade, acts almost diametric to coercion, are needed to keep the jungle efficient.

We proceed as follows. Section 2 presents a formal account of a jungle economy with initial holdings. Section 3 investigates lexicographic welfare maximization and provides novel conditions under which all jungle equilibria are solutions to such maximization. The farsighted jungle equilibrium is investigated in Section 4. The subtle role of withholding goods and giving goods is discussed in Section 5. Section 6 concludes.

# 2 The Jungle Economy

We consider a finite set of agents $N = \{1, \ldots, n\}$ of size $n \geq 2$ and a finite number of $m \geq 1$ goods that are present in positive quantities. Agent $i \in N$ has a consumption set that is denoted $C^i \subseteq \mathbb{R}^m_+$. This set is nonempty, compact and strictly comprehensive, i.e., for all $z^i \in C^i$ and $\hat{z}^i \in \mathbb{R}^m_+$ such that $\hat{z}^i \leqq z^i$ it holds that $\hat{z}^i \in \text{Int } C^i$, the interior of $C^i$.[3] The preference relation of agent $i$ on $C^i$ is denoted $\succeq^i$ and it is complete, transitive and continuous.

[3] Vector inequalities: $a \leqq b$, $a \leq b$ and $a < b$. Furthermore, $\subseteq$ denotes a subset and $\subset$ a strict subset.

An allocation $z = (z^1, \ldots, z^n, z^{n+1})$ assigns a bundle $z^i \in C^i$ to each agent $i \in N$, which we refer to as agent $i$'s holdings, and $z^{n+1} \in \mathbb{R}^m_+$ indicates the bundle of common goods that is held by none of them. For convenience, we treat the bundle of common goods as agent $n+1$ and we define $N_+ = N \cup \{n+1\}$ and $C^{n+1} = \mathbb{R}^m_+$. The economy's total endowment equals $\bar{\omega} \in \mathbb{R}^m_+$. An allocation $z$ is feasible when $\sum_{i \in N_+} z^i = \bar{\omega}$ and $z^i \in C^i$ for all $i \in N$.[4] The initial holdings are defined as the feasible allocation $\omega = (\omega^1, \ldots, \omega^n, \omega^{n+1})$. In some results, we fix $\bar{\omega}$ and refer to the feasible allocation $\omega$ as the allocation $\omega$ of $\bar{\omega}$.

Coercion governs the bilateral exchange of goods and it is driven by the agents' preferences and by strength. As in the jungle economy of PR07, the strength structure is extreme. The strongest agent of any pair of agents has the power to take anything that the weaker agent possesses, while the weaker agent cannot take anything from the stronger agent.[5] We may renumber the agents such that agent 1 is stronger than agent 2, who is stronger than agent 3, etc. In what follows, we often think of agent $i$ as the strongest agent in the arbitrary pair of agents $i$ and $j$ meaning $j > i$. Formally, such pair of agents $i$ and $j$ is denoted $(i, j)$ and belongs to the relative-strength relation $S \equiv \{(i, j) \in N_+ \times N_+ | j > i\}$.

Thus far, we have defined an economy driven by coercion as a tuple $\langle \{C^i, \succeq^i\}_{i \in N}, \omega, S \rangle$. This tuple extends the jungle economy of PR07 by introducing the initial holdings $\omega$. Furthermore, each $C^i$ is a convex set and preferences in the jungle economy are strictly convex and strongly monotone.[6]

We express feasible consumption as a feasible set of bundles in the stronger agent's consumption set that this agent is able to reach from his own and the weaker agent's current holdings. This set depends upon the identity of the pair of agents and upon their current holdings. Formally, we define the correspondence $\Phi : S \times C^i \times C^j \to C^i$ as the set of feasible

[4] We include $z^i \in C^i$ into the definition of feasible allocations because technically speaking agent $i$'s preferences on $\mathbb{R}^m_+ \backslash C^i$ are undefined.

[5] The interpretation is that the weaker agent concedes to the stronger agent and does not initiate a costly conflict knowing it will be lost for sure.

[6] In fact, strictly convex and monotone preferences imply strongly monotone preferences (we omit a proof). However, for the sake of consistency with PR07, we keep refering to the preferences in the jungle as strictly convex and strongly monotone.

consumption

$$\Phi\left((i,j),z^i,z^j\right)=\left\{\hat{z}^i\in C^i|\hat{z}^i\leqq z^i+z^j\right\}. \tag{1}$$

Note that $\Phi\left((i,j),z^i,z^j\right)$ is increasing in $z^j$. Since the weaker agent $j$ cannot take from the stronger agent $i$, we have $z^i\in\Phi\left((i,j),z^i,z^j\right)$. By definition, $\Phi\left((i,j),z^i,z^j\right)$ is a nonempty, compact and comprehensive subset of $C^i$.

As a benchmark and point of departure, we adopt the jungle equilibrium of PR07 in which stability against bilateral taking by stronger agents is the key idea.

**Definition 1** *A jungle equilibrium is a feasible allocation $z$ such that there does not exist a pair of agents $(i,j)\in S$ and holdings $\hat{z}^i\in\Phi((i,j),z^i,z^j)$ for which $\hat{z}^i\succ^i z^i$.*

# 3 Lexicographic welfare maximization

In the jungle economy of PR07, the unique equilibrium is obtained by lexicographic welfare maximization in which a stronger agent's welfare is maximized over his consumption set before a weaker agent's welfare is maximized. Moreover, the lexicographic welfare maximization is performed as if all goods are initially common goods. We show that, for lexicographic welfare maximization, initial holdings in jungle economies are indeed irrelevant. However, other equilibria may exist and we provide sufficient conditions under which all jungle equilibria coincide with the lexicographic welfare maximum. These issues are analyzed in two separate subsections.

## 3.1 Initial holdings

In this subsection, we define lexicographic welfare maximization for arbitrary initial holdings, and then establish that initial holdings are irrelevant in such maximizer.

The lexicographic welfare maximization in PR07 is performed under $\omega=(0,\ldots,0,\bar{\omega})$, i.e., initially all goods are common goods. Since we impose almost no assumptions on the agents' consumption sets and their preferences, uniqueness of the lexicographic welfare maximization is no longer guaranteed. Therefore, we denote a lexicographic welfare maximum without initial holdings as the allocation $z^* = (z^{*1}, \ldots, z^{*n+1})$ and the set of all such maximizers as $Z^*$. For $j \in N$ and holdings $z^{*1}, \ldots, z^{*j-1}$, the $j$-th level of the lexicographic welfare maximization is defined as the bundle $z^{*j}$ that is a best element of net takings for agent $j$ according to $\succeq^j$ on $C^j$ among feasible bundles. Formally, for $j \in N$ and holdings $z^{*1}, \ldots, z^{*j-1}$, for any best element $z^{*j} \in \Phi\left((j, n+1), 0, \bar{\omega} - \sum_{i=1}^{j-1} z^{*i}\right)$ it holds that

$$z^{*j} \succeq^j z^j \text{ for all } z^j \in \Phi\left((j, n+1), 0, \bar{\omega} - \sum_{i=1}^{j-1} z^{*i}\right). \tag{2}$$

By definition, $z^{*n+1} = \bar{\omega} - \sum_{i=1}^{n} z^{*i}$.

When initial holdings are arbitrarily distributed across agents, stronger agents have to encounter several or all weaker agents in order to collect their preferred bundle through bilateral takings. This implies some sequence of such bilateral encounters for each agent. Instead of a dynamic approach, we propose a static approach that captures the essence of a sequence of such bilateral encounters and that foregoes specifying the order in which a stronger agent visits agents that are weaker.[7] As before, agents take from strongest to weakest, which still suggests some dynamics. The sequence of bilateral takings by a stronger agent from weaker agents is expressed as bilateral net takings from the weaker agents' remaining holdings after all stronger agents took before. The sequence of bilateral net takings of each agent is modeled as a static optimization program.

Formally, we define $y^{i,j} \in \mathbb{R}^m$, $(i,j) \in S$, as the bilateral net takings by agent $i$ from agent $j$.[8] All bilateral net takings by agent $i$ are denoted by the tuple $y^i = (y^{i,i+1}, \ldots, y^{i,n+1})$. For agent $j \in N$, after bilateral net takings $\bar{y}^1, \ldots, \bar{y}^{j-1}$ by agents that are stronger than agent $j$, agent $j$ and the weaker agent $k > j$ still hold

$$\omega^j - \sum_{i=1}^{j-1} \bar{y}^{i,j}, \quad \text{respectively, } \omega^k - \sum_{i=1}^{j-1} \bar{y}^{i,k}.$$

The $j$-th level of the lexicographic welfare maximization is defined as the tuple $y^j$ of bilateral

[7] Our static approach foregoes issues as whether the order should be exogenous or endogenous.

[8] These net takings may be negative for some goods to allow for substitution, which is necessary especially along the frontier of $C^i$.

net takings by agent $j$ that is a best element according to $\succeq_j$ on $C^j$ subject to[9]

$$\begin{aligned} z^j &\equiv \omega^j - \sum_{i=1}^{j-1} \bar{y}^{i,j} + \sum_{k=j+1}^{n+1} y^{j,k} \in C^j \\ &\text{and} \\ y^{j,k} &\leqq \omega^k - \sum_{i=1}^{j-1} \bar{y}^{i,k} \text{ for all } k \in \{j+1, \ldots, n\}. \end{aligned} \tag{3}$$

We denote a lexicographic welfare maximum with initial holdings as the tuple $(\bar{y}^1, \ldots, \bar{y}^{n+1})$ and the set of such maximizers as $\bar{Y}$. Typically, the set of best elements will be large because, for each best element $\bar{y}^j$, all feasible $y^{j,k}$ and $y^{j,k'}$ such that $y^{j,k} + y^{j,k'} = \bar{y}^{j,k} + \bar{y}^{j,k'}$ can be used to construct another best element. For that reason, we express best elements in terms of the associated consumption bundles. So, for $\bar{y} \in \bar{Y}$ we denote the associated allocation $\bar{z} = (\bar{z}^1, \ldots, \bar{z}^{n+1})$ and the set of such maximizers as $\bar{Z}$.

The following result holds for all consumption sets and preference relations within our domain of preferences. Its proof and all other proofs are deferred to the Appendix.

**Theorem 1** *Each lexicographic welfare maximizing allocation $\bar{z} \in \bar{Z}$ is a jungle equilibrium and initial holdings are irrelevant for the lexicographic welfare maximization, i.e., for fixed $\bar{\omega}$ and the allocation $\omega$ of $\bar{\omega}$ it holds that $\bar{Z} = Z^*$. Moreover, the set $\bar{Z}$ is nonempty.*

Theorem 1 confirms the implicit presumption that initial holdings do not matter for lexicographic welfare maximization because any arbitrary sequence of bilateral takings by stronger agents from weaker agents must result in a jungle equilibrium. In PR07, any allocation $z^* \in Z^*$ is immune against taking by stronger agents. Because of the equivalence between $\bar{Z}$ and $Z^*$, each agent's bundle of holdings in $\bar{z} \in \bar{Z}$ is also immune to taking in the economy with initial holdings and, hence, it is an equilibrium. In calculating the lexicographic welfare maximization, the initial holdings $\omega$ of $\bar{\omega}$ do not matter. The stronger

[9] Equation (3) implies that

$$\omega^j - \sum_{i=1}^{j-1} \bar{y}^{i,j} + y^{j,k} \in \Phi((j,k), \omega^j - \sum_{i=1}^{j-1} \bar{y}^{i,j}, \omega^k - \sum_{i=1}^{j-1} \bar{y}^{i,k}) \text{ for all } (j,k) \in S.$$

agent can sequentially take from weaker agents as if he simply takes from a pile of common goods as in PR07.

### 3.2 Preferences

In the previous subsection, we showed that initial holdings do not matter. However, attention was restricted to lexicographic welfare maximizing allocations. In this subsection, we provide weaker sufficient conditions on preferences under which all equilibria coincide with such allocations.

First, consider the following example in which the strongly monotone and strictly convex preferences in PR07 are relaxed to monotone and convex preferences.

**Example 1** *Consider an economy with two agents and three goods. The economy's total resources are* $\bar{\omega} = (1, 1, 1)$. *The agents' consumption sets are identical and given by* $C^1 = C^2 = \left\{x \in \mathbb{R}^2_+ | x \leq \bar{\omega}\right\}$ *for simplicity. For* $i = 1, 2$, *agent* $i$*'s best element of* $\succeq^i$ *on* $C^i$ *maximizes the Leontief preferences* $\min\left\{z^i_1, z^i_2, z^i_3\right\}$. *For all* $\omega$ *of* $\bar{\omega}$, *the unique lexicographic welfare maximum* $\bar{z}$ *is given by* $\bar{z}^1 = \bar{\omega}$ *and* $\bar{z}^2 = \bar{z}^3 = 0$. *However, for* $\alpha \in [0, 1]$,

$$\omega^1 = (1, \alpha, \alpha)\,,\, \omega^2 = (0, 1 - \alpha, 0) \text{ and } \omega^3 = (0, 0, 1 - \alpha)$$

*form an equilibrium according to Definition 1 because neither agent* 1 *or agent* 2 *can gain from bilateral takings. For* $\alpha < 1$, *it differs from the lexicographic welfare maximum and it is Pareto inefficient because* $z^1 = (\alpha, \alpha, \alpha)$, $z^2 = (1 - \alpha, 1 - \alpha, 1 - \alpha)$ *and* $z^3 = (0, 0, 0)$ *is welfare improving. Besides permutations, no other equilibria exist.*

This example assumes Leontief preferences and initial holdings in which each agent is unable to improve himself by bilateral taking. In the simplest equilibrium, which we would identify as $\omega^1 = (1, 0, 0)$, $\omega^2 = (0, 1, 0)$ and $\omega^3 = (0, 0, 1)$, agent 1 has no use of agent 2's holding of good 2 if he is unable to simultaneously get access to the common good 3. Other initial holdings can be an equilibrium as well.

There are intriguing issues about this example. First, what conditions on preferences exclude these additional equilibria? We deal with this question in this subsection. Second, the jungle equilibrium fails to recognize that stronger agents might evaluate the aggregate of sequences of bilateral takings instead of single bilateral takings. The latter can be interpreted as myopia. Farsightedness is an issue that we postpone until Section 4. Finally, for initial holdings $\omega^1 = (1, \alpha, \alpha)$, the amount $1 - \alpha$ of good 1 does not contribute additional welfare to agent 1 and this agent is indifferent between keeping it or disposing it. This observation also reveals a glimpse of the issue of withholding goods, which is treated in Section 5.

Single bilateral takings are not enough to upset the initial holdings in Example 1 because weaker agents do not hold all goods and the Leontief preferences do not improve from taking a single good. Loosely speaking, the larger-than set at any $z^1$ is not a subset of the better-than set of Leontief preferences at this $z^1$. Formally, for $i \in N$, we define the larger-than set $L^i(z^i) = \{\hat{z}^i \in C^i | \hat{z}^i \geq z^i\}$ and the better-than set $B^i(z^i) = \{\hat{z}^i \in C^i | \hat{z}^i \succ^i z^i\}$. Strongly monotone preferences are equivalent to $L^i(z^i) \subset B^i(z^i)$, as is the case in the jungle economy. Example 1 shows that the strictly convex and strongly monotone preferences in the jungle economy cannot be weakened to convex and monotone preferences because for Leontief preferences all boundaries of the larger-than set $L^i(z^i)$ belong to the indifference set. In fact, a single boundary of the larger-than set $L^i(z^i)$ that belongs to the indifference set is enough to exclude $L^i(z^i) \subset B^i(z^i)$. To put it differently, for strongly monotone preferences the property $L^i(z^i) \subset B^i(z^i)$ implies that welfare improvements just outside the boundaries of $L^i(z^i)$ must exist in case $z^i \in \text{Int}\, C^i$, i.e., $L^i(z^i) \neq \emptyset$. That is, for any $z^i \in \text{Int}\, C^i$ such that $z_k > 0$ for some good $k$, agent $i$ can always improve his welfare by substituting an arbitrarily small amount of this good for some extra of any other good $l$. This insight is important, because agent $i$ with such holdings $z^i$ can always improve his welfare by a single encounter with any agent $j$ who holds a positive amount of good $l$. However, for $z^i$ at the frontier of $C^i$ it holds that $L^i(z^i) = \emptyset$ and then it depends on the curvature of the consumption set $C^i$ at $z^i$ and the curvature of the better-than set $B^i(z^i)$ whether a similar substitution of good

$k$ for good $l$ exist, i.e., whether a single encounter suffices.

The following result derives sufficient conditions under which all jungle equilibria coincide with the lexicographic welfare maximum.

**Proposition 1** *If $C^i$, $i \in N$, is a convex set and preferences $\succeq^i$ on $C^i$ are convex and strongly monotone, for all $z^i \in C^i$, then the set of jungle equilibria is $\bar{Z}$.*

The conditions on preferences are slightly weaker than the condition of strictly convex and strongly monotone preferences in PR07. The discussion above suggests that these conditions cannot be weakened further. Moreover, in the proof (as in PR07) it is established that for any other allocation $z$ there exists an agent $i \in N$ who can improve his welfare in a single encounter by substituting some of his holdings of one particular good by taking a small amount of some other good from a weaker agent $j \in N_+$. Proposition 1 illuminates how absolute strength and conditions on preferences are closely intertwined behind the surface of the jungle equilibrium concept.

# 4 Farsightedness

Recall that the jungle equilibrium exhibits myopia in that it requires stability against single bilateral takings. The myopia underlying the jungle equilibrium concept is the major reason why jungle equilibria other than lexicographic welfare maximizers can be equilibria. In this section, we include farsightedness into the equilibrium concept and show that all farsighted equilibria maximize lexicographic welfare.

Formally, under farsightedness in allocation $z$, agent $i$ should not only consider bilateral takings from a single weaker agent $j$, as in the jungle economy, but rather consider sequences of bilateral takings from (some or) all weaker agents. We model this dynamic sequence in a static manner, similar as before. As in Section 3.1, agent $i$'s bilateral net takings from all weaker agents is denoted by the tuple $y^i = (y^{i,i+1}, \ldots, y^{i,n+1})$. Given allocation $z$, agent $i$'s bilateral net takings $y^i$ are feasible if

$$z^i + \sum\nolimits_{j=i+1}^{n+1} y^{i,j} \in C^i \quad \text{and } y^{i,j} \leqq z^j \text{ for all } j \in \{i+1, \ldots, n+1\}. \tag{4}$$

Note that (4) implies that $z^i + y^{i,j} \in \Phi((i,j), z^i, z^j)$ for all $(i,j) \in S$. This additional notation allows us to define the farsighted version of the jungle equilibrium.

**Definition 2** *A farsighted jungle equilibrium is a feasible allocation $z$ such that there does not exist an agent $i \in N$ and feasible bilateral net takings $y^i$ for which $z^i + \sum_{j=i+1}^{n+1} y^{i,j} \succ^i z^i$.*

We show the following equivalence.

**Theorem 2** *Each lexicographic welfare maximizing $\bar{z} \in \bar{Z}$ is a farsighted equilibrium $\bar{z}$ and vice versa. Moreover, $\bar{Z}$ is nonempty.*

Recall that all consumption sets are non-empty, compact and strictly comprehensive and all preferences are complete, transitive and continuous. Therefore, the conditions for this result are rather weak.

# 5 Farsightedness, withholding and giving

In principle, an agent is not forced to consume all her holdings and may voluntarily dispose or waste some of the resources available to her. This property is called *free disposal.* Agents in the jungle economy of PR07 may freely dispose goods and this is captured by the assumption that the consumption set is comprehensive. Given strongly monotone preferences in the jungle economy of PR07, all agents consume their holdings in equilibrium. The Leontief preferences in Example 1, however, illustrate that the distinction between holdings and consumption is more subtle and can matter. Holdings that are not consumed are withheld from other agents in the economy. In the next subsection, we investigate withholding of goods in farsighted equilibria.

## 5.1 Farsightedness and withholding

First, we motivate our analysis with another example. It is similar in spirit to Example 1 but differs in that a continuum of farsighted jungle equilibria are Pareto inefficient due to withholding, a phenomenon that is not present in Example 1.

**Example 2** *Consider an economy with two agents and two goods. The economy's total resources are* $\bar{\omega} = (2, 1)$. *The agents' consumption sets are identical and given by* $C^1 = C^2 = \left\{x \in \mathbb{R}^2_+ | x \leq \bar{\omega}\right\}$ *for simplicity. Agent* 1*'s best element of* $\succeq^1$ *on* $C^1$ *maximizes the Leontief preferences* $\min\left\{z^1_1, z^1_2\right\}$ *and agent* 2*'s best element of* $\succeq^2$ *on* $C^2$ *maximizes* $\sqrt{z^2_1} + \sqrt{z^2_2}$. *For all allocations* $\omega$ *of* $\bar{\omega}$, *the set of lexicographic welfare maximizers is given by*

$$\bar{Z} = \left\{\left(\bar{z}^1, \bar{z}^2, \bar{z}^3\right) \in C^1 \times C^2 \times C^3 | \bar{z}^1 = (1 + \varepsilon, 1), \bar{z}^2 = (1 - \varepsilon, 0), \varepsilon \in [0, 1]\right\}.$$

*This set coincides with the set of jungle equilibria as well as the set of farsighted equilibria because neither agent* 1 *or agent* 2 *can gain from a sequence of bilateral takings. For* $\varepsilon > 0$, *the lexicographic welfare maximum is Pareto inefficient because* $z^1 = (1, 1)$, $z^2 = (1, 0)$ *and* $z^3 = (0, 0)$ *is welfare improving. So, only the lexicographic welfare maximum corresponding to* $\varepsilon = 0$ *is Pareto efficient. The economic issue is that agent* 1 *has no incentives to keep or give away his excess holdings* $\varepsilon > 0$ *of good* 1 *and, by keeping this* $\varepsilon$ *of good* 1, *this agent withholds it from agent* 2, *for whom it increases his welfare. So, withholding can be said to occur whenever* $\varepsilon > 0$.

Example 2 shows that withholding of goods is an issue that hampers Pareto efficiency. In this example, preferences are defined for holdings but one can raise the question whether someone with Leontief preferences consumes all holdings or simply consumes only the corner point while disposing the rest.[10] From here on, we distinguish between an agent's holdings and his consumption. This distinction also requires that we redefine the agents' preference relations. Whenever an agent compares two bundles of holdings, say $z^i$ and $\hat{z}^i$, she actually

[10] Such ambiguity between consumptions and holdings cannot occur for satiated preferences.

compares best elements attainable from $z^i$ with best elements attainable from $\hat{z}^i$ and prefers the holdings that allow the most favorable of such best elements.

Formally, given agent $i$'s holdings $z^i \in C^i$, we denote agent $i$'s consumption under free disposal as $x^i \in \mathbb{R}^m_+$ and define feasible consumption as $x^i \leqq z^i$ and $x^i \in C^i$. Then, by comprehensiveness of the consumption set, $x^i \in C^i$ whenever $x^i \leqq z^i$. The presence of free disposal implies that agent $i$'s preference relation over holdings, i.e., $\succeq^i$, must be distinguished from agent $i$'s preference relation over consumption bundles, denoted $\succeq^i_x$. Both preference relations have to be logically consistent. We take preference relations $\succeq^i_x$, $i \in N$, as the primitive and assume it is complete, transitive and continuous. These assumptions on $\succeq^i_x$ and feasibility on the set $\{x^i \in C^i | x^i \leqq z^i\}$ guarantee a nonempty and compact set of best elements, denoted as $\beta^i(z^i)$, on the set of feasible consumptions. By definition, $x^i, \hat{x}^i \in \beta^i(z^i)$ implies $x^i \sim^i \hat{x}^i$. The set $\beta^i(z^i)$ defines $\succeq^i$ as follows: Agent $i$ prefers holdings $z^i$ to $\hat{z}^i$, if she prefers the best elements attainable from $z^i$ to the best elements attainable to $\hat{z}^i$. That is, $z^i \succeq^i \hat{z}^i$ if and only if for every $x^i \in \beta^i(z^i)$ and $\hat{x}^i \in \beta^i(\hat{z}^i)$ it holds that $x^i \succeq^i_x \hat{x}^i$. The redefined preference relation $\succeq^i$ on $C^i$ is complete, transitive and continuous as well.

The distinction between consumption and holdings requires to modify the lexicographic welfare maximization based upon (3). As before, agents take from strongest to weakest. The sequence of bilateral takings by a stronger agent from weaker agents is expressed as bilateral net takings from the weaker agents' remaining holdings after stronger agents took before. What differs is maximization over the preferences over consumption bundles and an additional constraint. For $j \in N$ and after bilateral net takings $\bar{y}^1, \ldots, \bar{y}^{j-1}$ by agents that are stronger than agent $j$, the $j$-th level of the modified lexicographic welfare maximization is defined as the consumption bundle $x^j$, holdings $z^j$ and the tuple $y^j$ of bilateral net takings by agent $j$ that are a best element according to $\succeq^j_x$ on $C^j$ subject to

$$\begin{aligned} x^j &\leqq z^j, \\ z^j &= \omega^j - \sum_{i=1}^{j-1} \bar{y}^{i,j} + \sum_{k=j+1}^{n+1} y^{j,k} \in C^j \text{ and} \qquad (5) \\ y^{j,k} &\leqq \omega^k - \sum_{i=1}^{j-1} \bar{y}^{i,k} \text{ for all } k \in \{j+1, \ldots, n+1\}. \end{aligned}$$

We denote a modified lexicographic welfare maximum by an upper bar. As before, the best element $\bar{y}^j$ will not be uniquely determined and we suppress it from our notation. Also, for each best element $\bar{x}^j$, all feasible $z^j \geqq \bar{x}^j$ can be used to construct another best element, but in order to study excess holdings we maintain $\bar{z}^j$ in our notation. So, we write $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n)$ for the allocation of consumption bundles, $\bar{z} = (\bar{z}^1, \ldots, \bar{z}^{n+1})$ for the allocation of holdings and the set of such maximizers as $\bar{X} \times \bar{Z}$. The associated excess holdings are denoted as the tuple $\bar{e} = (\bar{z}^1 - \bar{x}^1, \ldots, \bar{z}^n - \bar{x}^n)$, where each vector $\bar{e}^j$ is nonnegative. The set of all excess holdings is $\bar{E}$.

The farsighted equilibrium of Definition 2 can be easily modified to include consumptions and holdings. Given holdings $z^i$ and consumption $x^i \leqq z^i$, agent $i$ should not prefer to change his holdings by a sequence of feasible bilateral net takings $y^i$ to obtain holdings $z^i + \sum_{j=i+1}^{n+1} y^{i,j} \in C^i$ from which to consume $\hat{x}^i \leqq z^i + \sum_{j=i+1}^{n+1} y^{i,j}$. In short, in a farsighted jungle equilibrium agent $i$ weakly prefers $x^i$ to any feasible $\hat{x}^i$. These considerations lead to the following modified version of the farsighted equilibrium.

**Definition 3** *A modified farsighted equilibrium is a feasible pair $(x, z)$ such that there does not exist an agent $i \in N$, a tuple of feasible bilateral net takings $y^i$ such that $z^i + \sum_{j=i+1}^{n+1} y^{i,j} \in C^i$ and a consumption bundle $\hat{x}^i \leqq z^i + \sum_{j=i+1}^{n+1} y^{i,j}$ for which $\hat{x}^i \succ_x^i x^i$.*

The following result holds for all consumption sets and preference relations within our domain of preferences. It is stated without proof.

**Theorem 3** *Each modified lexicographic welfare maximizing pair $(\bar{x}, \bar{z}) \in \bar{X} \times \bar{Z}$ is a modified farsighted equilibrium $(\bar{x}, \bar{z})$ and vice versa. Moreover, $\bar{X} \times \bar{Z}$ is nonempty.*

Obviously, each modified farsighted equilibrium induces nonnegative excess holdings and there exists a modified farsighted equilibrium with no excess holdings, i.e., $0 \in \bar{E}$.

One important issue is under what sufficient conditions none of the players will withhold goods from other agents.

**Proposition 2** *If all preferences $\succeq^i$ on $C^i$, $i \in N$, are strongly monotone, then $x^i = z^i$ in any modified farsighted equilibrium.*

Strongly monotone preferences exclude withholding or excess holdings, i.e., $\bar{E} = \{0\}$. This condition holds in the jungle economy of PR07. Example 2 shows that it is impossible to relax the conditions of this result to monotone preferences.

## 5.2 Farsightedness and giving

In general, withholding of goods is Pareto inefficient. It seems that only gift giving by stronger agents can remedy this inefficiency. For the moment, we interpret gift giving in the strict sense of giving away nonnegative amounts of goods. Of course, there are no incentives for such givings in the jungle economy. Nevertheless, we study what might happen if stronger agents allow weaker agents to consume their excess holdings.

Although gift giving can be Pareto improving, the following example shows that it may be insufficient to remedy the inefficiency of modified farsighted equilibria.[11]

**Example 3** *Consider an economy with two agents and two goods. The economy's total resources are $\bar{\omega} = (1, 1)$. Agent 1's consumption set is given by $C^1 = \left\{x \in \mathbb{R}^2_+ | x_1 + x_2 \leq 1\right\}$ and, for simplicity, agent 2's consumption set is given by $C^2 = \left\{x \in \mathbb{R}^2_+ | x \leq \bar{\omega}\right\}$. Agent 1's best element of $\succeq^1_x$ on $C^1$ maximizes the preferences $x^1_1 + x^1_2$ and agent 2's best element of $\succeq^2_x$ on $C^2$ maximizes $\sqrt{x^2_1} + \sqrt{x^2_2}$. Due to strongly monotone preferences, we obtain $\succeq^i$ on $C^i$ by substituting $z$ for $x$ in $\succeq^i_x$. For all allocations $\omega$ of $\bar{\omega}$, the set of lexicographic welfare maximizers is given by*

$$\bar{Z} = \left\{ \left(\bar{z}^1, \bar{z}^2, \bar{z}^3\right) \in C^1 \times C^2 \times C^3 | \bar{z}^1 = (\varepsilon, 1 - \varepsilon), \varepsilon \in [0, 1] \right\}.$$

*This set coincides with the set of jungle equilibria as well as the set of farsighted equilibria because neither agent 1 or agent 2 can gain from a sequence of bilateral takings. Obviously,*

[11] Note that, in this example, the frontier of the stronger agent's consumption set coincides with this agent's maximally attainable indifference curve in his consumption set. Therefore, this example is nongeneric, but has to be taken into account.

*no withholding occurs. For $\varepsilon \neq \frac{1}{2}$, however, the lexicographic welfare maximum is Pareto inefficient because $z^1 = (\frac{1}{2}, \frac{1}{2})$, $z^2 = (\frac{1}{2}, \frac{1}{2})$ and $z^3 = (0, 0)$ is welfare improving. So, only the lexicographic welfare maximum corresponding to $\varepsilon = \frac{1}{2}$ is Pareto efficient. It can only be reached from allocations with $\varepsilon \neq \frac{1}{2}$ by voluntary trade in which one unit of good 1 is exchanged for one unit of good 2. Again, the economic issue is that agent 1 has no incentives to trade.*

From here on, for ease of discussion, we interpret gift giving more general to include Pareto improving trade. The issue is that stronger agents have no incentives to give or trade. Including givings into the economy requires to modify the lexicographic welfare maximization based upon (5) one step further. As before, the order of bilateral net takings is from strongest to weakest agents and agents take from the weaker agents' remaining holdings after stronger agents took before. In the presence of givings, for any $k > j > i$ we assume that agent $j$ can take from agent $i$'s givings to agent $k$ and this reflects a situation in which agent $i$ does not protect his givings to weaker agents. Then, it is natural to assume that agents receive givings in the same order from strongest to weakest agents, which avoids the notational burden of accounting for taking from received givings.

Formally, we define $g^{i,j} \in \mathbb{R}^m$, $(i, j) \in S$, as the bilateral givings from agent $i$ to agent $j$. All bilateral givings received by agent $j$ are denoted by the tuple $g^j = (g^{1,j}, \ldots, g^{j-1,j})$, where $g^1 = \emptyset$. In the presence of givings, holdings $z^i$ already include bilateral givings received from stronger agents. Before agent $j$ receives bilateral givings from agent $i$, all agents between $i+1$ and $j-1$ received bilateral givings $\bar{g}^{i,i+1}, \ldots, \bar{g}^{i,j-1}$ from agent $i$. So, agent $i$ holds

$$\hat{\omega}^i \equiv z^i - x^i - \sum_{j'=i+1}^{j-1} \bar{g}^{i,j'} \in C^i$$

before agent $j$ receives bilateral givings $g^{i,j}$ from agent $i$. Agent $i$ will only allow bilateral givings $g^{i,j}$ that preserve his welfare and this imposes $\hat{\omega}^i - g^{i,j} \succeq^i \hat{\omega}^i$, where $\hat{\omega}^i - g^{i,j} \in C^i$ and $z^j + g^{i,j} \in C^j$.[12] This adds $j-1$ feasibility constraints to agent $j$'s welfare maximization and

[12] This includes the subset of true bilateral givings $g^{i,j} \in \mathbb{R}^m_+$ from agent $i$ to agent $j$ given by $0 \leqq g^{i,j} \leqq \hat{\omega}^i$, which is nonempty.

this increases this agent's holdings by the sum of all gifts received, i.e., $\sum_{i=1}^{j-1} g^{i,j}$. For $j \in N$ and after bilateral net takings $\bar{y}^1, \ldots, \bar{y}^{j-1}$ and bilateral givings $\bar{g}^1, \ldots, \bar{g}^{j-1}$ by agents that are stronger than agent $j$, the $j$-th level of the modified lexicographic welfare maximization is defined as the consumption bundle $x^j$, holdings $z^j$, the tuple $y^j$ of bilateral net takings by agent $j$ and the tuple $g^j$ of bilateral givings received by agent $j$ that are a best element according to $\succeq_x^j$ on $C^j$ subject to

$$\begin{aligned} x^j &\leqq z^j, \\ z^j &= \omega^j - \sum_{i=1}^{j-1} \bar{y}^{i,j} + \sum_{k=j+1}^{n+1} y^{j,k} + \sum_{i=1}^{j-1} g^{i,j} \in C^j, \\ y^{j,k} &\leqq \omega^k - \sum_{i=1}^{j-1} \bar{y}^{i,k} \text{ for all } k \in \{j+1, \ldots, n+1\}, \\ \hat{\omega}^i - g^{i,j} &\succeq {}^i\hat{\omega}^i \text{ for all } \hat{\omega}^i - g^{i,j} \in C^i \text{ and } i \in \{1, \ldots, j-1\}. \end{aligned} \tag{6}$$

Similar as for best elements $\bar{y}^j$ and $\bar{z}^j$, best elements $\bar{g}^j$ will not be uniquely determined either and we also suppress it from our notation. So, we write $(\bar{x}, \bar{z}) \in \bar{X} \times \bar{Z}$ with associated excess holdings $\bar{e} \in \bar{E}$.

The modified farsighted equilibrium of Definition 3 can be easily adapted to include bilateral givings. Given holdings $z^i$ and consumption $x^i \leqq z^i$, agent $i$ should not prefer to change his holdings to $\hat{z}^i$ and consumption $\hat{x}^i \leqq \hat{z}^i$ by either a sequence of feasible bilateral net takings $y^i$ or (beg for and) obtain a sequence of bilateral givings from stronger agents (or both) to obtain holdings $\hat{z}^i = z^i + \sum_{j=i+1}^{n+1} y^{i,j} + \sum_{i=1}^{j-1} g^{i,j} \in C^i$. In short, agent $i$ weakly prefers $x^i$ to such $\hat{x}^i$. These considerations lead to the following modified version of the farsighted equilibrium.

**Definition 4** *A modified farsighted equilibrium with gift giving is a feasible pair $(x, z)$ such that there does not exist an agent $i \in N$, a tuple of feasible bilateral net takings $y^i$ and a tuple of feasible givings $g^i$ such that $\hat{z}^i = z^i + \sum_{j=i+1}^{n+1} y^{i,j} + \sum_{i=1}^{j-1} g^{i,j} \in C^i$ and a consumption bundle $\hat{x}^i \leqq \hat{z}^i$ for which $\hat{x}^i \succ_x^i x^i$.*

The following result holds for all consumption sets and preference relations within our domain of preferences. It establishes the equivalence between lexicographic welfare maximizing pairs associated with (6) and modified farsighted equilibria. More importantly, gift givings (including trade) are a feasible way for economies with coercion to restore Pareto efficiency. This means that any inefficient withholding of goods needs to be ruled out. The result is stated without proof.

**Theorem 4** *Each lexicographic welfare maximizing pair* $(\bar{x}, \bar{z}) \in \bar{X} \times \bar{Z}$ *associated with (6) is a modified farsighted equilibrium* $(\bar{x}, \bar{z})$ *with gift giving and vice versa. Moreover,* $\bar{X} \times \bar{Z}$ *is nonempty and every* $(\bar{x}, \bar{z})$ *is Pareto efficient.*

# 6 Conclusion

This paper provides an extensive analysis of lexicographic welfare maximization. Our work can be interpreted as a sensitivity analysis on the crucial assumptions underlying jungle economies, as first described by PR07. Throughout the paper, we maintain the assumption that stronger agents have coercive power over weaker agents. We discuss two extensions of PR07. First, we allow initial holdings of the agents, while PR07 assumes that all goods are taken from a common pool. We find that initial holdings are irrelevant for lexicographic welfare maximization. Second, we allow agents in the jungle to be endowed with Leontief preferences. In this case, as our Example 1 shows, stronger agents may not be able to gain by any bilateral takings. Hence, jungle equilibria are no longer lexicographic welfare maximizers. This result is due to myopia in the equilibrium concept, where only bilateral takings are considered. We suggest the concept of a farsighted jungle equilibrium as an alternative. In the latter, we require that no player can gain through a sequence of takings. Farsighted jungle equilibrium restores lexicographic welfare maximization. However, Example 2 shows that a farsighted jungle equilibrium need not be Pareto efficient. Stronger agents can withhold from weaker agents goods that they do not wish to consume. Only voluntary giving can restore Pareto efficiency in this case. In Example 3 we show that, in some nongeneric cases, even trade is necessary to restore Pareto efficiency. Therefore, we introduce the notion of farsighted equilibrium with giving (including trade) that allows a weaker agent to consume a stronger agent's excess holdings. Note that the stronger agent does not have an incentive to object. Hence, we interpret giving as the default option for the disposal of excess goods. Initial holdings again do not matter in a farsighted jungle equilibrium with giving. Agents can take in the order of strength and pass on any excess goods to the weaker agents.

Our notion of farsighted jungle equilibrium with giving implements private property rights according to the strength relation between agents. However, acquisition is limited, such that no resources are wasted. This microeconomic idea of an efficient jungle has its philosophical counterpart in John Locke's (1690, section 31) spoilage proviso. In his famous "Second Treatise of Government" Locke argues that legitimate property rights are incompatible with wasting resources. Locke's second proviso that one can only privately acquire goods from the common pool as long as "there is enough, and as good left in common for others" (section 27) is, however, violated in the jungle.

# 7 Appendix: Mathematical Proofs

**Proof of Theorem 1**

We first show that $\bar{Z} = Z^*$ independent of the allocation $\omega$ of $\bar{\omega}$, which we do by proving three claims. In the first claim, we will construct net takings $(y^{*1}, \ldots, y^{*n+1})$ from $z^*$ that are feasible according to (3). In the second claim, we show the reverse direction. The third claim establishes the equivalence between $\bar{Z}$ and $Z^*$.

**Claim A:** Fix $\bar{\omega}$ and the allocation $\omega$ of $\bar{\omega}$. For any $z^* \in Z^*$, there exist net takings $(y^{*1}, \ldots, y^{*n+1})$ consistent with $z^*$ such that if allocation $z$ is feasible to (2) then $z$ is also feasible to (3) for $\bar{z}$ substituted by $z^*$.

For any $z^* \in Z^*$, (2) holds. We provide an algorithm to construct net takings $(y^{*1}, \ldots, y^{*n+1})$ from $z^*$. In step $j \in N$ of this algorithm, the auxiliary variable $x_c^j(k_j)$, $k_j \in \{k \in N_+ | k > j\}$, indicates how much of good $c = 1, \ldots, m$ is left unallocated at the start of iteration $k_j$. Since we let $k_j$ descend,[13] we initialize $x_c^j(n+1) = z_c^{*j}$. For ascending $j \in N$, descending $k_j \in$

[13] Descending $k_j$ fits a motivating story in which agent $j$ first takes from the common goods and then from

$\{k \in N_+ | k > j\}$ and ascending $c = 1, \ldots, m$, we take $y_c^{*j,k_j} = \max\left\{0, \omega_c^{k_j} - \sum_{i=1}^{j-1} y_c^{*i,k_j} - x_c^j(k_j)\right\}$ and $x_c^{*j}(k_j - 1) = x_c^{*j}(k_j) - y_c^{*j,k_j}$. Without going into further details, this algorithm constructs net takings $(y^{*1}, \ldots, y^{*n+1})$ that are feasible to (3) for $\bar{z}$ substituted by $z^*$.

**Claim B:** Fix $\bar{\omega}$ and the allocation $\omega$ of $\bar{\omega}$. For any lexicographic welfare maximizing $(\bar{y}^1, \ldots, \bar{y}^{n+1})$ (i.e., $\bar{z} \in \bar{Z}$), if allocation $z$ is feasible to (3) then $z$ is also feasible to (2) for $z^*$ substituted by $\bar{z}$.

From substitution in (3) we obtain

$$\begin{aligned} z^j &\leq \sum_{k=j}^{n+1} \omega^k - \sum_{i=1}^{j-1} \sum_{k=j}^{n+1} \bar{y}^{i,k} \\ &= \sum_{k=j}^{n+1} \omega^k - \sum_{k=j}^{n+1} \bar{y}^{j-1,k} - \sum_{i=1}^{j-2} \sum_{k=j}^{n+1} \bar{y}^{i,k} \\ &= \sum_{k=j}^{n+1} \omega^k - \left(\bar{z}^{j-1} - \omega^{j-1} + \sum_{i=1}^{j-2} \bar{y}^{i,j-1}\right) - \sum_{i=1}^{j-2} \sum_{k=j}^{n+1} \bar{y}^{i,k} \\ &= \sum_{k=j-1}^{n+1} \omega^k - \bar{z}^{j-1} - \sum_{i=1}^{j-2} \sum_{k=j-1}^{n+1} \bar{y}^{i,k}. \end{aligned}$$

By repeating the arguments of the last three lines, we eventually obtain $z^j \leq \bar{\omega} - \sum_{i=1}^{j-1} \bar{z}^i$. Combined with $z^j \in C^j$, we obtain that $z^j$ satisfies (2) for $z^*$ substituted by $\bar{z}$.

**Claim C:** Fix $\bar{\omega}$ and the allocation $\omega$ of $\bar{\omega}$. Then, $\bar{Z} = Z^*$.

We apply an induction argument to show equivalence of $\bar{Z}$ and $Z^*$. For $j \in N_+$, let $\bar{z}^i = z^{*i}$, $i < j$. Then, Claim A and B imply that the constraints underlying (2) and (3) are equivalent to $z^j \leq \bar{\omega} - \sum_{i=1}^{j-1} z^{*i} = \bar{\omega} - \sum_{i=1}^{j-1} \bar{z}^i$ and $z^j \in C^j$. For $j \in N$, the sets of best elements according to $\succeq_j$ on $C^j$ coincide. This allows us to select one of these best elements and assign it to both $z^{*j}$ and $\bar{z}^j$ to arrive at $\bar{z}^j = z^{*j}$. We then proceed to $j+1$ for which the induction hypothesis holds. For $j = n+1$, $z^{*n+1} = \bar{\omega} - \sum_{i=1}^{j-1} z^{*i} = \bar{\omega} - \sum_{i=1}^{j-1} \bar{z}^i = \bar{z}^{n+1}$ and the algorithm stops.

Claims A-C establish the equivalence between $\bar{Z}$ and $Z^*$ for fixed $\bar{\omega}$ and arbitrary allocation $\omega$ of $\bar{\omega}$. It is therefore without loss of generality to consider only the simpler $Z^*$ in the remaining steps.

Next, we show existence of a $z^* \in Z^*$. For any $j \in N$ and arbitrary consumption bundles

weaker to stronger agents.

$z^1, \ldots, z^{j-1} \in \mathbb{R}^m_+$ such that $z^1 + \ldots + z^{j-1} \leq \bar{\omega}$, our assumptions on $\{C^j, \succsim_j\}_{j \in N}$ guarantee that agent $j$ has a best element in the set $\Phi((j, n+1), 0, \bar{\omega} - z^1 - \ldots - z^{j-1})$. By definition, such best element belongs to $C^j$ and is bounded from above by $\bar{\omega} - z^1 - \ldots - z^{j-1}$. So, by induction, there exists a best element $z^{*1} \in \Phi((1, n+1), 0, \bar{\omega})$ for agent 1 such that $z^{*1} \leq \bar{\omega}$, there exists a best element $z^{*2} \in \Phi((2, n+1), 0, \bar{\omega} - z^{*1})$ for agent 2 such that $z^{*2} \leq \bar{\omega} - z^{*1}$, etc.

Finally, we show that $z^* \in Z^*$ is an equilibrium. For any pair $(j, k) \in S$, we have that

$$\Phi\left((j, n+1), 0, \bar{\omega} - \sum_{i=1}^{j-1} z^{*i}\right) = \Phi\left((j, n+1), z^{*j}, \bar{\omega} - z^{*j} - \sum_{i=1}^{j-1} z^{*i})\right) \supseteq$$
$$\Phi((j, n+1), z^{*j}, z^{*k}) = \Phi((j, k), z^{*j}, z^{*k}).$$

Furthermore, $z^{*j}$ is an element of all these sets and $z^{*j}$ is a best element of $\Phi((j, n+1), 0, \bar{\omega} - \sum_{i=1}^{j-1} z^{*i}))$. By the axiom of independence of irrelevant alternatives, $z^{*j}$ is also a best element of $\Phi((j, k), z^{*j}, z^{*k})$. Therefore, for any pair $(j, k) \in S$, there does not exist a bundle $\hat{z}^j \succ^j z^{*j}$ such that $\hat{z}^j \in \Phi((j, k), z^{*j}, z^{*k})$. Hence, allocation $z^*$ is an equilibrium. QED

**Proof of Proposition 1**

By Theorem 1, the set $\bar{Z} = Z^*$ is nonempty and independent of allocation $\omega$ of $\bar{\omega}$. Suppose some feasible initial holdings $\omega \notin \bar{Z}$ also form a farsighted equilibrium. For this $\omega$, we apply (3) in order to compute a lexicographic welfare maximum $\bar{z} \in \bar{Z}$ with bilateral net takings $\bar{y}^1, \ldots, \bar{y}^{n+1}$ under the following tie-breaking rule: Although $\omega \notin \bar{Z}$ implies that $\bar{z} \neq \omega$, it might be the case that for some $j \in N$ all bundles $\omega^1, \ldots, \omega^{j-1}$ are already best elements in the lexicographic welfare maximization. For the largest of such $j \in N$, we take $\bar{z}^i = \omega^i$ for $i < j$ and $\bar{y}^1 = 0$, …, $\bar{y}^{j-1} = 0$ in the computations of the best element $\bar{z}^j$. So, for $j \in N$ we encounter the first bundle $\bar{z}^j \neq \omega^j$ and tuple of net takings $\bar{y}^j = (\bar{y}^{j,j+1}, \ldots, \bar{y}^{j,n+1})$ with $\bar{y}^{j,k} \neq 0$ for at least one pair $(j, k) \in S$ such that $\bar{z}^j \succ^j \omega^j$.

There are several cases.

Case 1: There exists a $(j, k) \in S$ for which $\bar{y}^{j,k} \geq 0$. Then, $\omega^j + \bar{y}^{j,k} \in \Phi\left((j, k), \omega^j, \omega^k\right)$ and, by strong monotone preferences, $\omega^j + \bar{y}^{j,k} \succ^j \omega^j$, which contradicts $\omega \notin \bar{Z}$ is an equilibrium.

This case is impossible.

Case 2: There does not exist a $(j,k) \in S$ for which $\bar{y}^{j,k} \geq 0$. This can only occur when $\omega^j$ belongs to the frontier of $C^j$. So, $L^j(\omega^j) = \emptyset$ and $\omega^j + \sum_{k=j+1}^{n+1} \bar{y}^{j,k} \in B^j(\omega^j)$ with $\sum_{k=j+1}^{n+1} \bar{y}^{j,k} \neq 0$. The nontrivial case features more than one $\bar{y}^{j,k} \neq 0$. Because $\omega^j$ is suboptimal, there does not exist a hyperplane separating the convex sets $C^i$ and $B^j(\omega^j)$. Instead, there do exist sets of hyperplanes separating $\omega^j$ from $B^j(\omega^j)$ and, similarly, $\omega^j$ from $C^j$. Formally, denote as the set $A^j \in \mathbb{R}^n$ of vectors $a^j$ such that $a^j(z^j - \omega^j) > 0$ for all $z^j \in B^j(\omega^j)$ and the set $D^j \in \mathbb{R}^n$ of vectors $d^j$ such that $d^j(z^j - \omega^j) \leq 0$ for all $z^j \in C^j$. Then, $A^j \cap D^j = \emptyset$. By $\succeq^j$ strongly monotone and $C^j$ strictly comprehensive, $a^j > 0$ for all $a^j \in A^j$ and $d^j > 0$ for all $d^j \in D^j$. Furthermore, the sets $A^j$ and $D^j$ are each nonempty and compact. Note that these are singletons under the smoothness conditions in PR07. For the general case, choose $a^j \in A^j$ and $d^j \in D^j$ such that they produce the smallest cone containing $B^j(\omega^j)$. Such $a^j$ and $d^j$ exist. Then, from here on, the arguments in PR07 apply. We summarize their main argument. Because $\omega^j$ on the frontier of $C^j$ and $\omega^j + \sum_{j'=j+1}^{n+1} \bar{y}^{j,j'} \in B^j(\omega^j)$, there exist goods $k$ and $l$ such that $\omega^j_k + \sum_{j'=j+1}^{n+1} \bar{y}^{j,j'}_k > \omega^j_k$ and $\omega^j_l + \sum_{k=j+1}^{n+1} \bar{y}^{j,k}_l < \omega^j_l$ for some other good $l$. Then it also holds that, for any $\varepsilon > 0$, there exist an $\varepsilon_l \in (0,\varepsilon)$ and an $\varepsilon_k \in (0,\varepsilon)$ for which $\hat{z}^j \in C^j$ given by $\hat{z}^j_k = z^j_k + \varepsilon_k$, $\hat{z}^j_l = z^j_l - \varepsilon_l$ and $\hat{z}^j_{k'} = z^j_{k'}$ for $k' \neq l, k$ improves agent $j$'s welfare, i.e., $\hat{z}^j \in B^j(z^j)$. That is, agent $j$ is also able to improve his welfare in a single encounter with some weaker agent $j' \in N_+$ no matter how little the weaker agent possesses of good $l$.

This completes the proof. QED

**Proof of Theorem 2**

By Theorem 1, the set $\bar{Z}$ is nonempty and independent of allocation $\omega$ of $\bar{\omega}$. Suppose some feasible initial holdings $\omega \notin \bar{Z}$ also form a farsighted equilibrium. For this $\omega$, we apply the algorithm described in the proof of Proposition 1. As in the latter proof, for some $j \in N$ we encounter a bundle $\bar{z}^j \neq \omega^j$ and tuple of net takings $\bar{y}^j = (\bar{y}^{j,j+1}, \ldots, \bar{y}^{j,n+1})$ with $\bar{y}^{j,k} \neq 0$ for at least one pair $(j,k) \in S$ such that $\bar{z}^j \succ^j \omega^j$, which contradicts Definition 2. Hence, initial holdings $\omega \notin \bar{Z}$ cannot form a farsighted equilibrium, i.e., $\bar{Z} \subseteq Z^{*F}$. Finally, it can be verified that all initial holdings $\omega \in \bar{Z}$ satisfy Definition 2 and, therefore, the set of farsighted equilibria is $\bar{Z}$, i.e., $\bar{Z} = Z^{*F}$. QED

**Proof of Proposition 2**

Suppose not, then there exists an $i \in N$ such that $x^i \leq z^i$, which excludes equality. Consider the feasible $\hat{x}^i = z^i$. Then $\succeq^i$ is strongly monotone implies $\hat{x}^i \succ^i x^i$, which contradicts the optimality of $(x^i, z^i)$ in the $i$-th level of the modified lexicographic welfare maximization. QED

# 8 References

# References


[1] Piccione, M. and A. Rubinstein, 2007, Equilibrium in the jungle, *Economic Journal* 117(July), 883-896.

[2] Locke, John, 1690, *Two Treatises of Government*. Edited by Peter Laslett. 2nd ed. Cambridge, 1967: Cambridge University Press.

# 9 Proofs available upon request

**Some details of the proof of Theorem 1**

The following arguments prove Claim B. Condition (3) states

$$z^j = \omega^j - \sum_{i=1}^{j-1} \bar{y}^{i,j} + \sum_{k=j+1}^{n+1} y^{j,k}, \qquad y^{j,k} \leq \omega^k - \sum_{i=1}^{j-1} \bar{y}^{i,k},$$

Substitution of the weak inequality into the equality yields

$$\begin{aligned}
z^j &\leq \omega^j - \sum\nolimits_{i=1}^{j-1} \bar{y}^{i,j} + \sum\nolimits_{k=j+1}^{n+1} \left[\omega^k - \sum\nolimits_{i=1}^{j-1} \bar{y}^{i,k}\right] = \sum\nolimits_{k=j}^{n+1} \left[\omega^k - \sum\nolimits_{i=1}^{j-1} \bar{y}^{i,k}\right] \\
&= \sum\nolimits_{k=j}^{n+1} \omega^k - \sum\nolimits_{k=j}^{n+1} \sum\nolimits_{i=1}^{j-1} \bar{y}^{i,k} = \sum\nolimits_{k=j}^{n+1} \omega^k - \sum\nolimits_{i=1}^{j-1} \sum\nolimits_{k=j}^{n+1} \bar{y}^{i,k} \\
&= \sum\nolimits_{k=j}^{n+1} \omega^k - \sum\nolimits_{k=j}^{n+1} \bar{y}^{j-1,k} - \sum\nolimits_{i=1}^{j-2} \sum\nolimits_{k=j}^{n+1} \bar{y}^{i,k} \\
&= \sum\nolimits_{k=j}^{n+1} \omega^k - \left(\bar{z}^{j-1} - \omega^{j-1} + \sum\nolimits_{i=1}^{j-2} \bar{y}^{i,j-1}\right) - \sum\nolimits_{i=1}^{j-2} \sum\nolimits_{k=j}^{n+1} \bar{y}^{i,k} \\
&= \sum\nolimits_{k=j-1}^{n+1} \omega^k - \bar{z}^{j-1} - \sum\nolimits_{i=1}^{j-2} \sum\nolimits_{k=j-1}^{n+1} \bar{y}^{i,k} \\
&= \ldots \\
&= \sum\nolimits_{k=1}^{n+1} \omega^k - \sum\nolimits_{i=1}^{j-1} \bar{z}^i = \bar{\omega} - \sum\nolimits_{i=1}^{j-1} \bar{z}^i.
\end{aligned}$$

Combined with $z^j \in C^j$ we obtain $z^j \in \Phi\left((j, n+1), 0, \bar{\omega} - \sum\nolimits_{i=1}^{j-1} \bar{z}^i\right)$. QED

**Derivations of Example 2**

Consider an economy with two agents and three goods. The economy's total resources are $\bar{\omega} = (1, 1, 1)$. The agents' consumption sets are identical and given by $C^1 = C^2 = \left\{x \in \mathbb{R}^2_+ | x \leq \bar{\omega}\right\}$ for explanatory reasons. For $i = 1, 2$, agent $i$'s best element of $\succeq^i$ on $C^i$ maximizes the Leontief preferences $\min\left\{z_1^i, z_2^i, z_3^i\right\}$. For all $\omega$ of $\bar{\omega}$, the unique lexicographic welfare maximum $\bar{z}$ is given by $\bar{z}^1 = \bar{\omega}$ and $\bar{z}^2 = \bar{z}^3 = 0$.

In this example, we characterize which initial holdings $\omega$ of $\bar{\omega}$ can also be supported as an equilibrium. After renumbering the goods, $\omega_1^1 < \omega_2^1 \leq \omega_3^1$ cannot be supported as an equilibrium. To see this, whatever $\omega$ of $\bar{\omega}$ is, agent 1 is better off by taking $\varepsilon \in (0, \omega_2^1]$ of good 1 from either agent 2 or the common goods. Such $\varepsilon > 0$ is unavailable if agent 1 holds everything of good 1 and then the inequalities of this case contradict $\bar{\omega}$. So, consider

$\omega_1^1 = \omega_2^1 \le \omega_3^1$. If this can be supported as an equilibrium, then it is necessary that a single bilateral taking by agent 1 from either agent 2 or the common goods cannot improve agent 1's well being. This requires either $[\omega_2^2 = 0$ and $\omega_1^3 = 0]$, i.e., agent 2 does not own good 2 and good 1 is unavailable as a common good, or the other way around $[\omega_1^2 = 0$ and $\omega_2^3 = 0]$, or both. In what follows, it is without loss of generality to assume $\omega_1^2 = 0$ and $\omega_2^3 = 0$. Then, the initial holdings $\omega$ of $\bar{\omega}$ must also satisfy $\omega_1^3 = 1 - \omega_1^1 \ge 0$, $\omega_2^2 = 1 - \omega_2^1 \ge 0$ and $\omega_3^2 + \omega_3^3 = 1 - \omega_3^1 \ge 0$. In order to simplify notation, for $0 \le \alpha \le \beta \le 1$ and $0 \le \gamma \le 1 - \beta$, we have characterized the following necessary structure on $\omega$ of $\bar{\omega}$ :

$$\omega^1 = (\alpha, \alpha, \beta)\,, \qquad \omega^2 = (0, 1-\alpha, \gamma)\,, \qquad \omega^3 = (1-\alpha, 0, 1-\beta-\gamma)$$

in which agent 1 has utility $\alpha \ge 0$ and agent 2 utility 0. It is also necessary that agent 2 has no incentive to take from the common goods $\omega^3$. By such taking, the best agent 2 can achieve is $\min\{1-\alpha, 1-\alpha, 1-\beta\}$, and given $\beta \ge \alpha$, a utility level of 0 requires $\beta = 1$. Then also, $\gamma = 0$. To summarize, for $\alpha \in [0,1]$, if $\omega$ of $\bar{\omega}$ satisfies $\omega^1 = (\alpha, \alpha, 1)$, $\omega^2 = (0, 1-\alpha, 0)$ and $\omega^3 = (1-\alpha, 0, 0)$, then $\omega$ is an equilibrium according to Definition 1. For $\alpha < 1$, it differs from the lexicographic welfare maximum and it is Pareto inefficient because $\omega^1 = (\alpha, \alpha, \alpha)$, $\omega^2 = (1-\alpha, 1-\alpha, 1-\alpha)$ and $\omega^3 = (0,0,0)$ is welfare improving. QED