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# Behavioral Learning Equilibria

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## Abstract

We propose behavioral learning equilibria as a plausible explanation of coordination of individual expectations and aggregate phenomena such as excess volatility in stock prices and high persistence in inflation. Boundedly rational agents use a simple univariate linear forecasting rule and correctly forecast the unconditional sample mean and first-order sample autocorrelation. In the long run, agents learn the best univariate linear forecasting rule, without fully recognizing the structure of the economy. The simplicity of behavioral learning equilibria makes coordination of individual expectations on such an aggregate outcome more likely. In a first application, an asset pricing model with AR(1) dividends, a unique behavioral learning equilibrium exists characterized by high persistence and excess volatility, and it is stable under learning. In a second application, the New Keynesian Phillips curve, multiple equilibria co-exist, learning exhibits path dependence and inflation may switch between low and high persistence regimes.

*Keywords:* Bounded rationality; Stochastic consistent expectations equilibrium; Adaptive learning; Excess volatility; Inflation persistence

*JEL classification:* E30; C62; D83; D84

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# 1 Introduction

Expectations feedback plays a crucial role in economics and finance. Since the introduction by Muth (1961), and its application in macroeconomics by Lucas (1972), the Rational Expectations Hypothesis (REH) has become the predominant paradigm. A Rational Expectations Equilibrium (REE) is in fact a fixed point of an expectations feedback system. Typically it is assumed that rational agents perfectly know the correctly specified market equilibrium equations as well as their parameter values.

Despite its popularity, the REH has been criticized for its highly demanding and unrealistic information requirements. Adaptive learning models have been proposed as an alternative to rational expectations; see, e.g., Evans and Honkapohja (2001, 2011) and Bullard (2006) for extensive surveys and references. In contrast to rational expectations, adaptive learning models assume that agents do not have perfect knowledge about market equilibrium equations, but agents are assumed to have some belief, the perceived law of motion, about the actual law of motion; the relevant parameters are not known, but are estimated by recursive techniques based on past observations. The implied actual law of motion under adaptive learning is thus a time-varying self referential system, depending on the perceived law of motion. In this framework, a rational expectations equilibrium is simply a situation in which the implied law of motion exactly coincides with the perceived law of motion, and adaptive learning may converge to such a rational expectations equilibrium. Convergence of adaptive learning to a rational expectations equilibrium can occur when the perceived law of motion is correctly specified.

In general, however, a perceived law of motion will be misspecified. White (1994) argues that an economic model or a probability model is only a more or less crude approximation to whatever might be the “true” relationships among observed data and consequently it is necessary to view economic and/or probability models as misspecified to some greater or lesser degree. Sargent (1991) develops a notion of equilibrium as a fixed point of an operator that maps the perceived law of motion (a vector ARMA process) into a statistically optimal estimator of the actual law of motion. This may be viewed as an early example of a Restricted Perceptions Equilibrium (RPE), as defined by Evans and Honkapohja (2001), formalizing the idea that agents have misspecified beliefs, but within the context of their forecasting model they are unable to detect the misspecification. Branch (2006) gives an excellent survey and argues that RPE is a natural alternative to rational expectation equilibrium, because it is to some extent consistent with Muth’s

original hypothesis of REE while allowing for bounded rationality by restricting the class of perceived laws of motion. Fuster et al. (2010) and Fuster et al. (2011, 2012) propose a concept of natural expectations, where agents use simple, misspecified models, e.g., linear autoregressive models, for their perceived law of motion. They argue that economists and non-economists –statisticians, professional forecasters, and firms– regularly make some simplifications of models. Simple models are easier to understand, easier to explain, and easier to employ. Simplicity also reduces the risks of overfitting, which is the reasoning that underlies many formal model selection criteria. However, they do not pin down the parameters of the forecasting model through consistency requirements as in a restricted perceptions equilibrium nor do they allow the agents to learn an optimal misspecified model through empirical observations.

The main contribution of our paper is to develop a general *behavioral learning equilibrium* (BLE) concept. The general idea of a BLE is that agents try to learn a simple, but misspecified forecasting rule *with an intuitive, behavioral interpretation*. Here we model a BLE as a first order Stochastic Consistent Expectations Equilibrium (SCEE), which may be viewed as the simplest RPE. A BLE seems more likely as a description of aggregate behavior, because a large population of individual agents may coordinate their expectations more easily to learn a simple, parsimonious behavioral equilibrium. Suppose the actual law of motion (ALM) of the economy is a high dimensional linear stochastic system. But agents do not fully recognize the high dimensional structure and they use a simple univariate AR(1) rule to forecast the economy. In a first order SCEE the mean and the first-order autocorrelation of realized prices in the economy coincide with the corresponding mean and first-order autocorrelation of agents' AR(1) perceived law of motion (PLM). Moreover, a simple adaptive learning scheme –Sample Autocorrelation Learning (SAC-learning)– with an intuitive behavioral interpretation, enforces convergence to the (stable) BLE. In this paper we formalize the concept of BLE in the simplest class of models one can think of: a one-dimensional linear stochastic model driven by an exogenous linear stochastic AR(1) process. Agents do not recognize, however, that the economy is driven by an exogenous AR(1) process  $y_t$  and forecast the state of the economy  $x_t$  using a simple univariate AR(1) rule. Within this simple class of models we are able to fully characterize the BLE and their stability under learning. While this class of models is strikingly simple, it already yields rich dynamical behavior, including high persistence, excess volatility and multiple equilibria.

The simple class of models that we study, contains two standard and empirically rele-

vant applications. In the first - an asset pricing model driven by an exogenous stochastic dividend process - the BLE is unique and SAC-learning always converges to the BLE. The BLE is characterized by highly persistent prices (close to unit root) and excess volatility with asset price volatility more than doubled compared to REE. In the second application - a New Keynesian Philips curve (NKPC) - driven by an exogenous AR(1) process for the output gap and an independent and identically distributed (i.i.d.) stochastic shock to inflation - multiple stable BLE may co-exist. In particular, for empirically plausible parameter values a BLE with highly persistent inflation exists, matching the stylized facts of US-inflation data. Coordination on a behavioral learning equilibrium may thus explain excess volatility in stock prices and high persistence in inflation (cf. Milani, 2007).

## Related literature

Our behavioral equilibrium concept is closely related to the Consistent Expectations Equilibrium (CEE) introduced by Hommes and Sorger (1998), where agents believe that prices follow a linear AR(1) stochastic process, whereas the implied actual law of motion is a deterministic chaotic nonlinear process. Along a CEE, price realizations have the same sample mean and sample autocorrelation coefficients as the AR(1) perceived law of motion. A CEE is another early example of a RPE and may be seen as an “approximate rational expectations equilibrium”, in which the misspecified perceived law of motion is the *best linear approximation* within the class of perceived laws of motion of the actual (unknown) nonlinear law of motion. Hommes and Rosser (2001) investigate CEE in an optimal fishery management model and use numerical simulations to study adaptive learning of CEE in the presence of dynamic noise. The adaptive learning scheme used here is SAC-learning, where the parameters of the AR(1) forecasting rule are updated based on the observed sample average and first-order sample autocorrelation. Sögner and Mitlöhner (2002) apply the CEE concept to a standard asset pricing model with independent and identically distributed (i.i.d.) dividends and show that the unique CEE coincides with the REE. As we will see in the current paper, introducing autocorrelations in the stochastic dividend process will lead to a learning equilibrium *different* from REE. Tuinstra (2003) analyzes first-order consistent expectations equilibria numerically in a deterministic overlapping generations (OLG) model. Hommes et al. (2004) generalize the notion of CEE to *nonlinear stochastic* dynamic economic models, introducing the concept of stochastic consistent expectations equilibrium (SCEE). In a SCEE, agents’ perceptions

about endogenous variables are consistent with the actual realizations of these variables in the sense that the unconditional mean and autocorrelations of the unknown nonlinear stochastic process, which describes the actual behavior of the economy, coincide with the unconditional mean and autocorrelations of the AR(1) process agents believe in. In a SCEE agents use the optimal (univariate) linear forecasting rule in an unknown nonlinear stochastic economy. Although an SCEE is *not* a REE, because the linear forecast does not coincide with the true conditional expectation, along a SCEE forecasting errors are unbiased and uncorrelated. Hommes et al. (2004) apply this concept to an OLG model and study the stability of SCEE under sample autocorrelation learning (SAC-learning) by numerical simulations.

Showing theoretically existence of SCEE has proven to be technically difficult, while convergence of SAC-learning has been studied only by numerical simulations. The principle technical difficulty here is to calculate autocorrelation coefficients, prove existence of fixed points and analyze the relationship between SCEE and sample autocorrelation learning in a highly nonlinear system. Branch and McGough (2005) obtain existence results on *first-order* SCEE theoretically and analyze the stability of SCEE under real-time learning numerically in a stochastic non-linear self-referential model where expectations are based on an AR(1) process. Lansing (2009) considers a special class of SCEE in the same reduced-form New Keynesian Philips curve that we will use, where the value of the Kalman gain parameter in agents' forecast rule is pinned down using the observed autocorrelation of inflation changes. Lansing (2010) studies a Lucas-type asset pricing model where agents believe stock prices follow a geometric random walk without drift and he pins down the forecasting model to match the first order autocorrelation of the model and the data. Lansing (2010), however, does not contain theoretical existence results or multiple equilibria, while learning is based on numerical simulations. Similarly, Adam and Marcet (2009, 2010) consider a general risk-neutral asset pricing model where agents hold subjective priors about the price process, but they do not pin down parameters in agents' perceived law of motion through learning. Bullard et al. (2008, 2010) add judgment into agents' forecasts and use the concept of SCEE to provide a related interesting concept of exuberance equilibria. They study the resulting dynamics in the New Keynesian model and a standard asset pricing model, respectively, where the driving variables are white noises (no autocorrelations), and find high persistence and excess volatility.

The current paper studies first order SCEE in a simple, but general class of models, one-dimensional linear stochastic models driven by an exogenous linear AR(1) process.

Our paper makes two methodological contributions. First, we prove existence of first order SCEE and fully characterize the (multiple) equilibria in this simple class of models. Second, we present the first proof that the first order SCEE is stable under SAC-learning and provide simple and intuitive stability conditions. Although the class of models we study is simple, it contains two important standard applications: an asset pricing model and the New Keynesian Philips curve. For both applications learning of a behavioral learning equilibria generates empirically relevant stylized facts such as excess volatility, high persistence and multiple equilibria.

A number of other related papers studied the effects of learning from different perspectives. Timmermann (1993, 1996) shows that learning helps to explain excess volatility and predictability of stock returns in a similar present value asset pricing model. In Timmermann (1993, 1996), however, the perceived law of motion is correctly specified and the parameters are estimated by adaptive learning, so that in the long run learning converges to REE. Bullard and Duffy (2001) introduce adaptive learning into a general-equilibrium life-cycle economy with capital accumulation and show that in contrast to perfect-foresight dynamics, the system under least-squares learning possesses equilibria that are characterized by persistent excess volatility in returns to capital. Sargent et al. (2009) find that occasional shocks can trigger, via the learning dynamics, sudden departures from a rational expectations equilibrium. Huang et al. (2009) show that the self-confirming equilibrium under adaptive expectations is the same as the steady state rational expectations equilibrium for all admissible parameter values, but that the dynamics around the steady state are substantially different between the two equilibria. Guidolin and Timmermann (2007) characterize equilibrium asset prices under adaptive, rational and Bayesian learning schemes in a model where dividends evolve on a binomial lattice and find that learning introduces serial correlation and volatility clustering in stock returns. Branch and Evans (2010) find existence of multiple restricted perceptions equilibria and that the model under real-time learning is capable of matching key aspects of the data regarding regime-switching stock price returns and volatilities. Branch and Evans (2011) illustrate that agents are likely to temporarily believe that the price process is a random walk without drift. An important difference between our analysis and the literature above, though, is that here excess volatility and persistence are not just the result of transitory learning dynamics and do not vanish after the economy converges to the equilibrium. Another conceptual difference is our *behavioral interpretation* of the first order SCEE as what is perhaps the *simplest example* of a RPE. A behavioral learning equilibrium together with



an intuitive SAC-learning scheme may explain coordination of individual expectations on (almost) self-fulfilling equilibria.

The paper is organized as follows. Section 2 introduces the main concepts, the first-order SCEE, sample autocorrelation learning and their interpretation as a behavioral learning equilibrium. Section 3 focusses on existence of first-order SCEE and stability under SAC-learning within a simple linear class of one-dimensional models driven by an exogenous AR(1) process. Section 4 discusses two applications, an asset pricing model and a New Keynesian Philips curve, illustrating the empirical relevance of BLE in explaining excess volatility, inflation persistence and regime switching. Finally, Section 5 concludes.

## 2 Main concepts

This section introduces the main concepts. Suppose that the law of motion of an economic system is given by the stochastic difference equation

$$x_t = f(x_{t+1}^e, y_t, u_t), \quad (2.1)$$

where  $x_t$  is the state of the system (e.g. an asset price or inflation) at date  $t$  and  $x_{t+1}^e$  is the expected value of  $x$  at date  $t + 1$ . This denotation highlights that expectations may not be rational. Here  $f$  is a continuous function,  $\{u_t\}$  is an i.i.d. noise process with mean zero and finite absolute moments<sup>1</sup>, where the variance is denoted by  $\sigma_u^2$ , and  $y_t$  is a driving variable (e.g. dividends or the output gap), assumed to follow an exogenous stochastic AR(1) process

$$y_t = a + \rho y_{t-1} + \varepsilon_t, \quad 0 \leq \rho < 1, \quad (2.2)$$

where  $\{\varepsilon_t\}$  is another i.i.d. noise process with mean zero and finite absolute moments, with variance  $\sigma_\varepsilon^2$ , and uncorrelated with  $\{u_t\}$ . The mean of the stationary process  $y_t$  is  $\bar{y} = \frac{a}{1-\rho}$ , the variance is  $\sigma_y^2 = \frac{\sigma_\varepsilon^2}{1-\rho^2}$  and the  $k$ th-order autocorrelation coefficient of  $y_t$  is  $\rho^k$ , see for example, Hamilton (1994).

Agents are boundedly rational and do not know the exact form of the actual law of motion (2.1). We assume that, in order to forecast  $x_{t+1}$ , agents only use past observations  $x_{t-1}, x_{t-2}, \dots$ , etc. Hence, agents do not recognize that  $x_t$  is driven by an exogenous stochastic process  $y_t$ . Instead agents believe that the economic variable  $x_t$  follows a simple

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<sup>1</sup>The condition on finite absolute moments is required to obtain convergence results under SAC-learning.

linear stochastic process. More specifically, agents' perceived law of motion (PLM) is an AR(1) process<sup>2</sup>, as in Hommes et al. (2004) and Branch and McGough (2005), i.e.

$$x_t = \alpha + \beta(x_{t-1} - \alpha) + \delta_t, \quad (2.3)$$

where  $\alpha$  and  $\beta$  are real numbers with  $\beta \in (-1, 1)$  and  $\{\delta_t\}$  is a white noise process;  $\alpha$  is the unconditional mean of  $x_t$  and  $\beta$  is the first-order autocorrelation coefficient. Given the perceived law of motion (2.3), the 2-period ahead forecasting rule for  $x_{t+1}$  that minimizes the mean-squared forecasting error is

$$x_{t+1}^e = \alpha + \beta^2(x_{t-1} - \alpha). \quad (2.4)$$

Combining the expectations (2.4) and the law of motion of the economy (2.1), we obtain the implied actual law of motion (ALM)

$$x_t = f(\alpha + \beta^2(x_{t-1} - \alpha), y_t, u_t), \quad (2.5)$$

with  $y_t$  an AR(1) process as in (2.2).

### Stochastic Consistent Expectations Equilibrium (SCEE)

We are now ready to recall the definition of SCEE. Following Hommes et al. (2004)<sup>3</sup>, the concept of first-order SCEE is defined as follows.

**Definition 2.1** *A triple  $(\mu, \alpha, \beta)$ , where  $\mu$  is a probability measure and  $\alpha$  and  $\beta$  are real numbers with  $\beta \in (-1, 1)$ , is called a first-order stochastic consistent expectations equilibrium (SCEE) if the following three conditions are satisfied:*

- S1 The probability measure  $\mu$  is a nondegenerate invariant measure for the stochastic difference equation (2.5);*
- S2 The stationary stochastic process defined by (2.5) with the invariant measure  $\mu$  has unconditional mean  $\alpha$ , that is,  $E_\mu(x) = \int x d\mu(x) = \alpha$ ;*
- S3 The stationary stochastic process defined by (2.5) with the invariant measure  $\mu$  has unconditional first-order autocorrelation coefficient  $\beta$ .*

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<sup>2</sup>In this paper we focus on a univariate stochastic process (2.1) for the law of motion of the economy and an AR(1) PLM (2.3) and forecasting rule (2.4). More generally one may consider an N-dimensional state vector  $X_t$  and a higher-order linear AR(p) or a VAR forecasting model.

<sup>3</sup>In Hommes et al. (2004), the actual law of motion is  $x_t = f(x_{t+1}^e, u_t)$ , without the driving variable  $y_t$ . However, the definitions of SCEE and SAC-learning can still be applied here.

That is to say, a first-order SCEE is characterized by the fact that both the unconditional mean and the unconditional first-order autocorrelation coefficient generated by the actual (unknown) stochastic process (2.5) coincide with the corresponding statistics for the perceived linear AR(1) process (2.3). This means that in a first-order SCEE agents correctly perceive the mean and the first-order autocorrelation (i.e., the persistence) of economic variables although they do not correctly specify their model of the economy.

Our SCEE concept may be viewed as the simplest example of a RPE, where agents predict an unknown stochastic law of motion by a first-order linear approximation. It should be stressed that the SCEE has an intuitive behavioral interpretation, and therefore we refer to a first-order SCEE as a *behavioral learning equilibrium* (BLE). In a SCEE agents use a linear forecasting rule with two parameters, the mean  $\alpha$  and the first-order autocorrelation  $\beta$ . Both can be detected from past observations by inferring the average price (or inflation) level and the (first-order) persistence of the time series. For example,  $\beta = 0.5$  means that, on average, prices mean-revert toward their long-run mean by 50%. These observations could be made by “guestimating” the mean and the persistence from an observed time series of aggregate variables. It is interesting to note that in learning-to-forecast laboratory experiments with human subjects, for many individuals forecasting behavior is well described by simple rules, such as a simple AR(1) rule, see for example, Hommes et al. (2005), Adam (2007), Heemeijer et al. (2009) and Hommes (2011).

Finally, we note that in a first-order SCEE, the orthogonality condition imposed by Restricted Perceptions Equilibrium (RPE)<sup>4</sup>

$$Ex_{t-1}[x_t - \alpha - \beta(x_{t-1} - \alpha)] = E(x_{t-1} - \alpha)[x_t - \alpha - \beta(x_{t-1} - \alpha)] = 0$$

is satisfied. The orthogonality condition shows that agents can not detect the correlation between their forecasting errors and perceived model, see Branch (2006). The first-order SCEE is a RPE where agents have their model incorrect; but within the context of their forecasting model agents are unable to detect their misspecification.

### Sample autocorrelation learning

In the above definition of first-order SCEE, agents’ beliefs are described by the linear forecasting rule (2.4) with fixed parameters  $\alpha$  and  $\beta$ . However, the parameters  $\alpha$  and  $\beta$  are usually unknown. In the adaptive learning literature, it is common to assume

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<sup>4</sup>Readers are referred to Evans and Honkapohja (2001) and Branch (2006) for further discussion on the orthogonality condition and RPE.

that agents behave like econometricians using time series observations to estimate the parameters as additional observations become available. Following Hommes and Sorger (1998), we assume that agents use sample autocorrelation learning (SAC-learning) to learn the parameters  $\alpha$  and  $\beta$ . That is, for any finite set of observations  $\{x_0, x_1, \dots, x_t\}$ , the sample average is given by

$$\alpha_t = \frac{1}{t+1} \sum_{i=0}^t x_i, \quad (2.6)$$

and the first-order sample autocorrelation coefficient is given by

$$\beta_t = \frac{\sum_{i=0}^{t-1} (x_i - \alpha_t)(x_{i+1} - \alpha_t)}{\sum_{i=0}^t (x_i - \alpha_t)^2}. \quad (2.7)$$

Hence  $\alpha_t$  and  $\beta_t$  are updated over time as new information arrives. It is easy to check that, independently of the choice of the initial values  $(x_0, \alpha_0, \beta_0)$ , it always holds that  $\beta_1 = -\frac{1}{2}$ , and that the first-order sample autocorrelation  $\beta_t \in [-1, 1]$  for all  $t \geq 1$ .<sup>5</sup>

Adaptive learning is sometimes referred to as statistical learning, because agents act as statisticians or econometricians and use a statistical procedure, such as OLS, to estimate and update parameters over time. SAC-learning may be viewed as another statistical learning procedure. We would like to stress, however, that SAC-learning has a simple behavioral interpretation that agents simply infer the sample average and persistence (i.e. first-order autocorrelation) from time series observations. We focus on the sample average for  $\alpha_t$  in (2.6) and sample first-order autocorrelation for  $\beta_t$  in (2.7) over the entire time-horizon, but one could also restrict the learning to the last  $T$  observations with  $T$  relatively small (e.g.,  $T = 100$  or even smaller). In fact, it is relatively easy for agents to “guestimate” the mean and first-order autocorrelation directly based on an observed time series.

Define

$$R_t = \frac{1}{t+1} \sum_{i=0}^t (x_i - \alpha_t)^2,$$

then the SAC-learning is equivalent to the following recursive dynamical system (see

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<sup>5</sup>The definition of the first-order sample autocorrelation coefficient in (2.7) is only slightly different from least-squares learning, where in fact  $\beta_t = (\sum_{i=0}^{t-1} (x_i - \bar{x}_t^-)(x_{i+1} - \bar{x}_t^+)) / (\sum_{i=0}^{t-1} (x_i - \bar{x}_t^-)^2)$ , with  $\bar{x}_t^- = \frac{1}{t} \sum_{i=0}^{t-1} x_i$ ,  $\bar{x}_t^+ = \frac{1}{t} \sum_{i=1}^t x_i$ . However, the sample autocorrelation coefficient in (2.7) always satisfies  $|\beta_t| \leq 1$ , while the OLS estimate does not. This property is a natural “projection facility” for the SAC-learning process, which is the terminology used in Evans and Honkapohja (2001) to bound parameters in ordinary least-squares learning to avoid explosive dynamics.

Appendix A).

$$\begin{cases} \alpha_t = \alpha_{t-1} + \frac{1}{t+1}(x_t - \alpha_{t-1}), \\ \beta_t = \beta_{t-1} + \frac{1}{t+1}R_t^{-1} \left[ (x_t - \alpha_{t-1}) \left( x_{t-1} + \frac{x_0}{t+1} - \frac{t^2 + 3t + 1}{(t+1)^2} \alpha_{t-1} - \frac{1}{(t+1)^2} x_t \right) \right. \\ \quad \left. - \frac{t}{t+1} \beta_{t-1} (x_t - \alpha_{t-1})^2 \right], \\ R_t = R_{t-1} + \frac{1}{t+1} \left[ \frac{t}{t+1} (x_t - \alpha_{t-1})^2 - R_{t-1} \right]. \end{cases} \quad (2.8)$$

The actual law of motion under SAC-learning is therefore given by

$$x_t = f(\alpha_{t-1} + \beta_{t-1}^2(x_{t-1} - \alpha_{t-1}), y_t, u_t), \quad (2.9)$$

with  $\alpha_t, \beta_t$  as in (2.8) and  $y_t$  as in (2.2).

In Hommes and Sorger (1998), the map  $f$  in (2.9) is a nonlinear deterministic function depending only on  $\alpha_{t-1} + \beta_{t-1}^2(x_{t-1} - \alpha_{t-1})$ , without the driving variable  $y_t$  and the noise  $u_t$ . Hommes et al. (2004) extend the CEE framework to SCEE, with  $f$  a nonlinear stochastic process (without exogenous driving variable  $y_t$ ), but existence and stability under learning are hard to obtain in a nonlinear framework. In this paper to make the model analytically tractable, the map  $f$  is assumed to be a linear function, depending on the forecast  $\alpha_{t-1} + \beta_{t-1}^2(x_{t-1} - \alpha_{t-1})$ , the noise  $u_t$ , and also on an exogenous AR(1) process  $y_t$ .

### 3 Main results in a simple linear framework

Assume that the true law of motion of the economy is a one-dimensional linear stochastic process  $x_t$ , driven by an exogenous AR(1) process  $y_t$ . More precisely, the actual law of motion of the economy is given by

$$x_t = f(x_{t+1}^e, y_t, u_t) = b_0 + b_1 x_{t+1}^e + b_2 y_t + u_t, \quad (3.1)$$

$$y_t = a + \rho y_{t-1} + \varepsilon_t, \quad (3.2)$$

where  $0 < \rho < 1$ , as before, and  $b_1$  is in the interval  $(-1, 1)$ .<sup>6</sup> Before turning to SCEE, consider rational expectations first.

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<sup>6</sup>This assumption is made to ensure stationarity; for  $|b_1| > 1$  the dynamics under learning easily becomes explosive.

### 3.1 Rational expectations equilibrium

Under the assumption that agents are rational, a straightforward computation (see Appendix B) shows that the rational expectations equilibrium  $x_t^*$  satisfies

$$x_t^* = \frac{b_0}{1-b_1} + \frac{ab_1b_2}{(1-b_1\rho)(1-b_1)} + \frac{b_2}{1-b_1\rho}y_t + u_t. \quad (3.3)$$

Thus based on the expression of the rational expectations equilibrium  $x_t^*$  in (3.3), its unconditional mean and variance are, respectively,

$$\bar{x}^* := E(x_t^*) = \frac{b_0(1-\rho) + ab_2}{(1-b_1)(1-\rho)}, \quad (3.4)$$

$$Var(x_t^*) = E(x_t^* - \bar{x}^*)^2 = \frac{b_2^2\sigma_\varepsilon^2}{(1-b_1\rho)^2(1-\rho^2)} + \sigma_u^2. \quad (3.5)$$

Furthermore, the first-order autocovariance and autocorrelation of rational expectations equilibrium  $x_t^*$  are, respectively,

$$E(x_t^* - \bar{x}^*)(x_{t-1}^* - \bar{x}^*) = \frac{b_2^2\rho\sigma_\varepsilon^2}{(1-b_1\rho)^2(1-\rho^2)},$$

$$Corr(x_t^*, x_{t-1}^*) = \frac{\rho b_2^2}{b_2^2 + (1-b_1\rho)^2(1-\rho^2)\frac{\sigma_u^2}{\sigma_\varepsilon^2}}.$$

Note that in the special case  $\sigma_u = 0$ , the above expression reduces to  $Corr(x_t^*, x_{t-1}^*) = \rho$ , that is, when there is no exogenous noise  $u_t$  in (3.1), the persistence of the REE coincides exactly with the persistence of the exogenous driving force  $y_t$ .

### 3.2 Existence of first-order SCEE

Now assume that agents do not recognize that the economy is driven by an exogenous AR(1) process, but use a simple univariate linear rule to forecast the state of the economy. Given that agents' perceived law of motion is an AR(1) process (2.3), the actual law of motion becomes

$$x_t = b_0 + b_1[\alpha + \beta^2(x_{t-1} - \alpha)] + b_2y_t + u_t, \quad (3.6)$$

where  $y_t$  is given in (3.2). The mean of  $x_t$  in (3.6), denoted by  $\bar{x}$ , is computed as

$$\bar{x} = \frac{b_0 + b_1\alpha(1-\beta^2) + b_2a/(1-\rho)}{1-b_1\beta^2}. \quad (3.7)$$

Imposing the first consistency requirement of a SCEE on the mean, i.e.  $\bar{x} = \alpha$ , and solving for  $\alpha$  yields

$$\alpha^* = \frac{b_0(1-\rho) + ab_2}{(1-b_1)(1-\rho)}. \quad (3.8)$$

Comparing with (3.4), we conclude that in a SCEE the unconditional mean  $\alpha^*$  coincides with the REE mean. That is to say, in a SCEE the state of the economy  $x_t$  fluctuates on average around its fundamental value  $x^*$ .

Next consider the second consistency requirement of a SCEE on the first-order autocorrelation coefficient  $\beta$  of the PLM. A straightforward computation (see Appendix C) shows that the first-order autocorrelation coefficient  $\text{Corr}(x_t, x_{t-1})$  of the ALM (3.6) satisfies

$$\text{Corr}(x_t, x_{t-1}) = b_1\beta^2 + \frac{b_2^2\rho(1 - b_1^2\beta^4)}{b_2^2(b_1\beta^2\rho + 1) + (1 - \rho^2)(1 - b_1\beta^2\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}} =: F(\beta). \quad (3.9)$$

The second consistency requirement of first-order autocorrelation coefficient  $\beta$  yields

$$F(\beta) = \beta. \quad (3.10)$$

The actual law of motion (3.1-3.2) depends on seven parameters  $b_0, b_1, b_2, a, \rho, \sigma_\varepsilon^2$  and  $\sigma_u^2$ . The constants  $b_0$  and  $a$  only affect the level of fluctuations through the mean  $\alpha^*$  in (3.8), but not the persistence, i.e. they do not affect  $F(\beta)$  in (3.9). Moreover, only the ratio  $\sigma_u^2/\sigma_\varepsilon^2$  of noise terms matters for the persistence  $F(\beta)$  in (3.9). Hence, the existence of first-order SCEE ( $\alpha^*, \beta^*$ ) depends on four parameters  $b_1, b_2, \rho$  and  $\sigma_u^2/\sigma_\varepsilon^2$

Define  $G(\beta) := F(\beta) - \beta$ . Since  $0 < \rho < 1$  and  $|b_1| < 1$ ,

$$G(0) = \frac{b_2^2\rho}{b_2^2 + (1 - \rho^2)\frac{\sigma_u^2}{\sigma_\varepsilon^2}} > 0$$

and

$$\begin{aligned} G(1) &= \frac{b_2^2(b_1 + \rho) + b_1(1 - \rho^2)(1 - b_1\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}}{b_2^2(b_1\rho + 1) + (1 - \rho^2)(1 - b_1\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}} - 1 \\ &= \frac{-b_2^2(1 - b_1)(1 - \rho) - (1 - b_1)(1 - \rho^2)(1 - b_1\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}}{b_2^2(b_1\rho + 1) + (1 - \rho^2)(1 - b_1\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}} \\ &< 0. \end{aligned}$$

Therefore, there exists at least one  $\beta^* \in (0, 1)$ , such that  $G(\beta^*) = 0$ , i.e.  $F(\beta^*) = \beta^*$ . That is,

**Proposition 1** *In the case that  $0 < \rho < 1$  and  $|b_1| < 1$ , there exists at least one nonzero first-order stochastic consistent expectations equilibrium (SCEE) ( $\alpha^*, \beta^*$ ) for the economic system (3.6) with  $\alpha^* = \frac{b_0(1-\rho)+ab_2}{(1-b_1)(1-\rho)} = \bar{x}^*$  and  $0 < \beta^* < 1$ .*

It is useful to discuss the special case without dependence on an exogenous AR(1) driving variable  $y_t$ , that is,  $b_2 = 0$  (no driving variable),  $\rho = 0$  (no autocorrelation in the driving variable), or  $\sigma_\varepsilon^2 = 0$  (no stochasticity in the driving variable). In all these cases, (3.9) reduces to  $F(\beta) = b_1\beta^2$ . Hence, without a driving exogenous AR(1) process, the unique first-order SCEE  $\beta^* = 0$  and coincides with the REE.

Since the solutions  $\beta^*$  of the consistency requirement (3.10) depend continuously on the parameters, we conclude that for  $b_2 \approx 0$  (a weak driving variable),  $\rho \approx 0$  (almost no autocorrelation in the driving variable), or  $\sigma_\varepsilon^2 \approx 0$  (weak stochasticity in the driving variable) the system has a unique SCEE  $\beta^* \approx 0$ . Hence, when the exogenous AR(1) driving force is weak, there is a unique low persistence SCEE.

On the other hand, consider the other extreme case with strong dependence on the AR(1) driving variable  $y_t$ , i.e.  $|b_2| \rightarrow \infty$  (strong dependence on the AR(1) driving variable) or  $\sigma_u^2 = 0$  (no exogenous shock  $u_t$ , but only an AR(1) driving variable  $y_t$ ). In both cases, (3.9) reduces to

$$F(\beta) = \frac{b_1\beta^2 + \rho}{b_1\beta^2\rho + 1}. \quad (3.11)$$

In this case we have a unique SCEE (see Appendix D). Furthermore, in the case of positive expectations feedback, i.e.,  $b_1 > 0$ , because  $F(0) = \rho$  and  $F'(\beta) = \frac{2b_1\beta(1-\rho^2)}{(\rho b_1\beta^2 + 1)^2} > 0$  for  $\beta \in (0, 1)$ , we have  $F(\beta) > \rho$ . Consequently

$$\beta^* > \rho.$$

In the special case where also  $b_1 = 0$ ,  $F(\beta) \equiv \rho$  and hence  $\beta^* = \rho$ . Based on the above analysis, we have the following proposition.

**Proposition 2** *Under the conditions in Proposition 1, if  $b_2 \rightarrow \infty$  or  $\sigma_u^2 \rightarrow 0$ , then the nonzero first-order stochastic consistent expectations equilibrium (SCEE)  $(\alpha^*, \beta^*)$  is unique. Furthermore in the case  $0 \leq b_1 < 1$ , the unique SCEE satisfies  $\beta^* \geq \rho$ .*

The fact that  $\beta^* \geq \rho$  means that along the first-order SCEE the persistence of the economy is larger than under REE. Hence, the fact that agents do not recognize that the economy is driven by a relatively strong exogenous AR(1) process leads to *excess volatility*.

To summarize, when the dependence on the AR(1) driving variable is weak, a unique low persistence SCEE exists. If, on the other hand, the dependence is strong, a unique high persistence, excess volatility SCEE exists. It turns out that for intermediate values of the parameter  $b_2$ , multiple SCEE may coexist. The next proposition states, however, that at most three different SCEE coexist (the proof is given in appendix E).



**Proposition 3** *For the economic system (3.1-3.2) with  $0 < \rho < 1$  and  $|b_1| < 1$ , at most three first-order stochastic consistent expectations equilibria (SCEE)  $(\alpha^*, \beta^*)$  coexist.*

In our applications in Section 4 we will see that the asset pricing model has a unique excess volatility SCEE, while the New Keynesian Philips curve can have multiple SCEE.

### 3.3 Stability under SAC-learning

In this subsection we study the stability of SCEE under SAC-learning. The ALM of the economy under SAC-learning is given by

$$\begin{cases} x_t = b_0 + b_1[\alpha_{t-1} + \beta_{t-1}^2(x_{t-1} - \alpha_{t-1})] + b_2y_t + u_t, \\ y_t = a + \rho y_{t-1} + \varepsilon_t. \end{cases} \quad (3.12)$$

with  $\alpha_t, \beta_t$  updated based upon realized sample average and sample autocorrelation as in (2.8). Appendix F shows that the E-stability principle applies and that the stability under SAC-learning is determined by the associated ordinary differential (ODE) equation<sup>7</sup>

$$\begin{cases} \frac{d\alpha}{d\tau} = \bar{x}(\alpha, \beta) - \alpha = \frac{b_0 + \alpha(b_1 - 1) + b_2a/(1 - \rho)}{1 - b_1\beta^2}, \\ \frac{d\beta}{d\tau} = F(\beta) - \beta = \frac{b_2^2(b_1\beta^2 + \rho) + b_1\beta^2(1 - \rho^2)(1 - b_1\beta^2\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}}{b_2^2(b_1\beta^2\rho + 1) + (1 - \rho^2)(1 - b_1\beta^2\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}} - \beta, \end{cases} \quad (3.13)$$

where  $\bar{x}(\alpha, \beta)$  is the implied mean given by (3.7) and  $F(\beta)$  the implied first-order autocorrelation given by (3.9).

A first-order SCEE  $(\alpha^*, \beta^*)$  corresponds to a fixed point of the ODE (3.13). Moreover, a SCEE  $(\alpha^*, \beta^*)$  is locally stable under SAC-learning, if it is a stable fixed point of the ODE (3.13).

A straightforward computation shows that the eigenvalues of the Jacobian  $JG(\alpha^*, \beta^*)$  of (3.13) are given by  $(b_1 - 1)/(1 - b_1(\beta^*)^2)$  (the coefficient of  $\alpha$  in the first ODE) and  $F'(\beta^*) - 1$  (since the second ODE is independent of  $\alpha$ ). Since, by assumption,  $b_1 < 1$  the first eigenvalue is always  $< 0$ . Hence, the local stability of a first-order SCEE  $(\alpha^*, \beta^*)$  under SAC-learning only depends on the slope  $F'(\beta^*)$ :

**Proposition 4** *A first-order SCEE  $(\alpha^*, \beta^*)$  is locally stable under SAC-learning if*

$$F'(\beta^*) < 1,$$

where  $F$  is the implied first-order autocorrelation in (3.9).

<sup>7</sup>See Evans and Honkapohja (2001) for discussion and a mathematical treatment of E-stability.

**Proof.** See Appendix F.

Recall from Subsection 3.2 that  $F(0) > 0$  and  $F(1) < 1$ , so that at least one first-order SCEE exists. If the SCEE is unique, then by continuity of  $F$  it must be that at the unique intersection point  $F'(\beta^*) < 1$  and, according to proposition 4, the unique SCE is (locally) stable under SAC-learning.<sup>8</sup> Numerical simulations suggest that a unique SCEE is even globally stable under SAC-learning. In the case of multiple first-order SCEE, the graph of the map  $F$  has multiple fixed points. Since  $F(0) > 0$  and  $F(1) < 1$ , typically  $F$  will then have three fixed points, two locally stable first-order SCEE separated by an unstable SCEE. Indeed in the application of the New Keynesian Philips curve in Subsection 4.2 we will meet exactly this situation.

## 4 Two applications

In this section we discuss two applications: an asset pricing model driven by AR(1) dividends and a New Keynesian Philips curve driven by an exogenous AR(1) process for the output gap. In both applications we study existence of first-order SCEE and stability under SAC-learning with parameters taken from the empirical literature.

### 4.1 An asset pricing model with AR(1) dividends

A simple example of the general framework (3.1-3.2) is given by the standard present value asset pricing model with stochastic dividends; see for example Campbell et al. (1997) and Brock and Hommes (1998). Here we consider AR(1) dividends instead of i.i.d. dividends.

#### 4.1.1 The model

Assume that agents can invest in a risk free asset or in a risky asset. The risk-free asset is perfectly elastically supplied at a gross return  $R > 1$ .  $p_t$  denotes the price (ex dividend) of the risky asset and  $y_t$  denotes the (random) dividend process. Let  $\tilde{E}_t, \tilde{V}_t$  denote the subjective beliefs of a representative agent about the conditional expectation and conditional variance of excess return  $p_{t+1} + y_{t+1} - Ry_t$ . The representative agent is a

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<sup>8</sup>The only exception is a hairline case where the graph of  $F$  is tangent to the diagonal at its unique fixed point  $\beta^*$  and  $F'(\beta^*) = 1$ . In such a hairline case, the SCEE may also be locally stable under SAC-learning, but stability does not follow directly from the E-stability principle.

myopic mean-variance maximizer of tomorrow's wealth. Optimal demand  $z_t$  for the risky asset by the representative agent is then given by

$$z_t = \frac{\tilde{E}_t(p_{t+1} + y_{t+1} - Rp_t)}{\tilde{a}\tilde{V}_t(p_{t+1} + y_{t+1} - Rp_t)} = \frac{\tilde{E}_t(p_{t+1} + y_{t+1} - Rp_t)}{\tilde{a}\sigma^2},$$

where  $\tilde{a} > 0$  denotes the risk aversion coefficient and the belief about the conditional variance of the excess return is assumed to be constant over time<sup>9</sup>, i.e.  $\tilde{V}_t(p_{t+1} + y_{t+1} - Rp_t) \equiv \sigma^2$ .

Equilibrium of demand and supply implies

$$\frac{\tilde{E}_t(p_{t+1} + y_{t+1} - Rp_t)}{\tilde{a}\sigma^2} = z_s,$$

where  $z_s$  denotes the supply of outside shares in the market, assumed to be constant over time. Without loss of generality<sup>10</sup>, we assume zero supply of outside shares, i.e.  $z_s = 0$ . The market clearing price in the standard asset pricing model is then given by

$$p_t = \frac{1}{R} [p_{t+1}^e + y_{t+1}^e], \quad (4.1)$$

where  $p_{t+1}^e$  is the conditional expectation of next period price  $p_{t+1}$  and  $y_{t+1}^e$  is the conditional expectation of next period dividend  $y_{t+1}$ .

Dividend  $\{y_t\}$  is assumed to follow an AR(1) process (2.2). Suppose that the risky asset (share) is traded, after payment of real dividends  $y_t$ , at a competitively determined price  $p_t$ , so that  $y_t$  is known by agents, and<sup>11</sup>

$$y_{t+1}^e = a + \rho y_t. \quad (4.2)$$

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<sup>9</sup>This assumption is consistent with the assumption that agents believe that prices follow an AR(1) process and dividends follow a stochastic AR(1) process with finite variance. Of course, as discussed in Branch and Evans (2010, 2011), agents might also not know  $\tilde{V}_t$  and need to learn it. However in this paper we focus on the theoretical analysis of SCEE and its stability under SAC-learning in a relatively simple framework. We leave the case of learning of the variance for future work.

<sup>10</sup>In the case  $z_s > 0$ , the difference with the analysis below only lies in the mean of the SCEE  $\alpha^* = \frac{\bar{y} - \tilde{a}\sigma^2 z_s}{R-1}$ . The analysis on autocorrelations and variances remains the same.

<sup>11</sup>Notice that agents are assumed to know the exogenous dividend process and forecast it correctly. An exogenous dividend process is easier to forecast than endogenously determined equilibrium prices. In a rational homogeneous world, agents believe that prices are completely determined by dividends and use the dividend process to compute rational equilibrium prices. Our agents however are boundedly rational, believing that prices are not completely determined by dividends, but that "other factors" may affect prices in an economy whose structure is not fully understood. As a first order approximation of these "other factors", our boundedly rational agents use a simple AR(1) rule to forecast endogenous prices.

The market clearing price in the standard asset pricing model with AR(1) dividends is then given by

$$p_t = \frac{1}{R} [p_{t+1}^e + a + \rho y_t], \quad (4.3)$$

Compared to the general framework (3.1), here we have  $b_0 = \frac{a}{R}$ ,  $b_1 = \frac{1}{R}$ ,  $b_2 = \frac{\rho}{R}$  and  $\sigma_u = 0$ .

#### 4.1.2 Theoretical results

Following the general results on SCEE in Section 3, the rational expectations equilibrium  $p_t^*$  becomes

$$p_t^* = \frac{aR}{(R-1)(R-\rho)} + \frac{\rho}{R-\rho} y_t. \quad (4.4)$$

In particular, if  $\{y_t\}$  is i.i.d., i.e.,  $a = \bar{y}$  and  $\rho = 0$ , then  $p_t^* \equiv \frac{a}{R-1} = \frac{\bar{y}}{R-1}$  is constant. The corresponding unconditional mean and the unconditional variance of the rational expectation price  $p_t^*$  are given by, respectively,

$$\bar{p}^* := E(p_t^*) = \frac{a}{(R-1)(1-\rho)} = \frac{\bar{y}}{R-1}, \quad (4.5)$$

$$Var(p_t^*) = E(p_t^* - \bar{p}^*)^2 = \frac{\rho^2 \sigma_\varepsilon^2}{(R-\rho)^2(1-\rho^2)}. \quad (4.6)$$

Furthermore, the first-order autocovariance and autocorrelation coefficient of the rational expectation price  $p_t^*$  are given by, respectively,

$$\begin{aligned} E(p_t^* - \bar{p}^*)(p_{t-1}^* - \bar{p}^*) &= \frac{\rho^3 \sigma_\varepsilon^2}{(R-\rho)^2(1-\rho^2)}, \\ Corr(p_t^*, p_{t-1}^*) &= \rho. \end{aligned} \quad (4.7)$$

Under the assumption that agents are boundedly rational and believe that the price  $p_t$  follows a univariate AR(1) process, the implied actual law of motion for prices is

$$\begin{cases} p_t = \frac{1}{R} [\alpha + \beta^2(p_{t-1} - \alpha) + a + \rho y_t], \\ y_t = a + \rho y_{t-1} + \varepsilon_t. \end{cases} \quad (4.8)$$

Since  $0 \leq \frac{\beta^2}{R} < 1$  and  $0 \leq \rho < 1$ , the price process (4.8) is stationary and ergodic. It is easy to see that the mean price is

$$\bar{p} = \frac{\alpha(1-\beta^2) + \bar{y}}{R-\beta^2}. \quad (4.9)$$

Imposing the first consistency requirement of a SCEE on the mean, i.e.  $\bar{p} = \alpha$ , yields

$$\alpha = \frac{\bar{y}}{R-1} =: \alpha^*. \quad (4.10)$$

The corresponding first-order autocorrelation coefficient  $F(\beta)$  of the ALM satisfies

$$F(\beta) = \frac{\beta^2 + R\rho}{\rho\beta^2 + R}. \quad (4.11)$$

Using the results from propositions 2 and 4 (and the discussion thereafter) we have the following property for the asset pricing model.

**Corollary 1** *For the asset pricing model (4.8), the first-order SCEE  $(\alpha^*, \beta^*)$  is unique,  $\alpha^* = \frac{\bar{y}}{R-1} = \bar{p}^*$  and  $\beta^* > \rho$  (excess volatility), and it is stable under SAC-learning.*

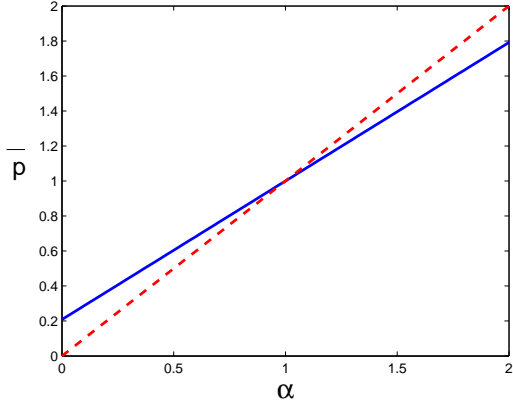
### 4.1.3 Numerical analysis

Now we illustrate the above results by numerical simulations for empirically plausible parameter values. For example, consider  $R = 1.05, \rho = 0.9, a = 0.005, \varepsilon_t \sim i.i.d. U(-0.01, 0.01)$  (i.e. uniform distribution on  $[-0.01, 0.01]$ )<sup>12</sup>. Figures 1a and 1b illustrate the existence of a unique first-order SCEE, where  $(\alpha^*, \beta^*) = (1, 0.997)$ , stable under SAC-learning. The time series of fundamental prices and market prices along the first-order SCEE, i.e., with  $(\alpha, \beta) = (\alpha^*, \beta^*)$ , are shown in Figure 1c, illustrating that the market price fluctuates around the fundamental price but has much more persistence and exhibits excess volatility. Recall from Corollary 1, that in a SCEE the mean of the market prices is equal to that of the fundamental prices and the first-order autocorrelation coefficient  $\beta^*$  of the market prices is greater than that of the fundamental prices  $\rho$ , implying that the market prices have higher persistence. The autocorrelation functions of the market prices and the fundamental prices are shown in Figure 1d. The autocorrelation coefficients of the market prices along a SCEE are much higher than those of the fundamental prices and hence the market prices have much higher persistence.

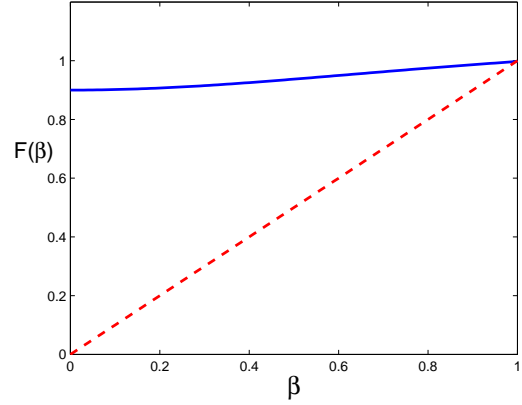
We now investigate how the excess volatility of market prices along a SCEE depends on the autoregressive coefficient of dividends  $\rho$ , which is also the first-order autocorrelation of fundamental prices. Consistent with Corollary 1, Figure 2a illustrates that the first-order autocorrelation  $\beta^*$  of market prices is significantly higher than that of fundamental prices, especially as  $\rho > 0.4$ . For  $\rho \geq 0.5$  we have  $\beta^* > 0.9$ , implying that asset prices are close

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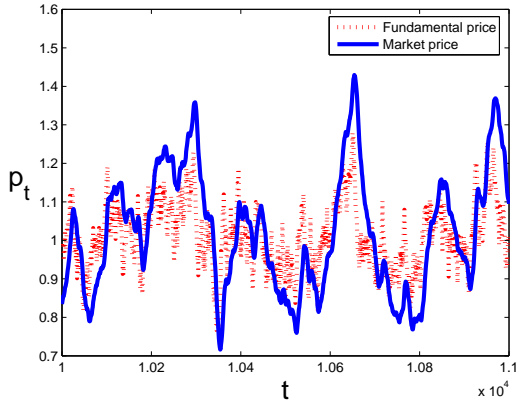
<sup>12</sup>As shown theoretically above, the numerical results are independent of selection of the parameter values within plausible ranges, sample paths, initial values and distribution of noise.



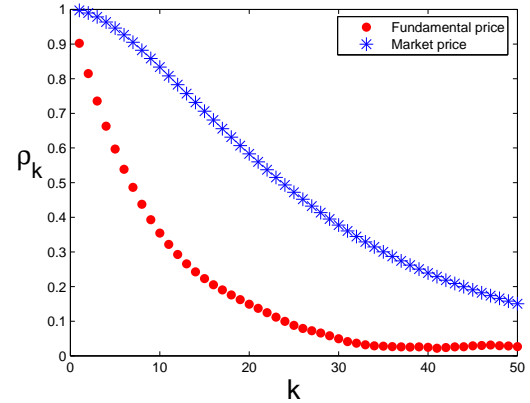
(a) SCEE for  $\alpha^*$



(b) SCEE for  $\beta^*$

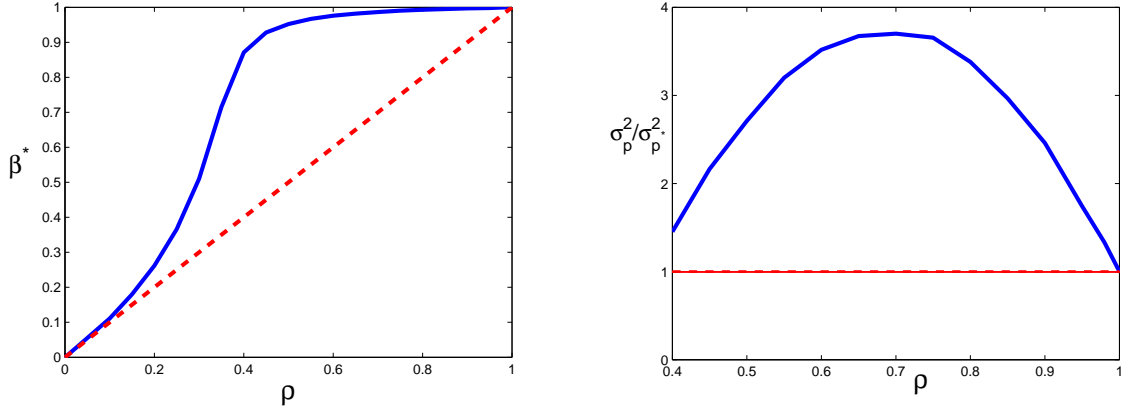


(c) Time series of prices in SCEE



(d) Autocorrelation function in SCEE

Figure 1: (a) SCEE  $\alpha^*(= 1)$  is the intersection point of the mean  $\bar{p} = \frac{\alpha(1-\beta^2)+\bar{y}}{R-\beta^2}$  (bold curve) with the perceived mean  $\alpha$  (dotted line); (b) SCEE  $\beta^*(= 0.997)$  is the intersection point of the first-order autocorrelation coefficient  $F(\beta) = \frac{\beta^2+R\rho}{\rho\beta^2+R}$  (bold curve) with the perceived first-order autocorrelation  $\beta$  (dotted line); (c) fundamental prices (dotted curve) and market prices (bold curve) in the SCEE; (d) autocorrelation functions of fundamental prices (lower dots) and market prices (higher stars) in the SCEE.



(a) first-order autocorrelation in SCEE

(b) Ratio unconditional variances in SCEE

Figure 2: (a) first-order SCEE  $\beta^*$  with respect to  $\rho$ ; (b) ratio of unconditional variances of market prices and fundamental prices with respect to  $\rho$ , where  $R = 1.05$ .

to a random walk and therefore quite unpredictable. In fact, based on empirical findings, e.g. Timmermann (1996) and Branch and Evans (2010), the autoregressive coefficient of dividends  $\rho$  is about 0.9, where the corresponding  $\beta^* \approx 0.997$ , very close to a random walk. In the case  $\rho > 0.4$ , the corresponding unconditional variance of market prices is larger than that of fundamental prices. As illustrated in Figure 2b, the ratio of the variance of market prices and the variance of fundamental prices is greater than 1 for  $0.4 < \rho < 1$ , with a peak around 3.5 for  $\rho = 0.7$ . For  $\rho = 0.9$ ,  $\frac{\sigma_p^2}{\sigma_{p^*}^2} \approx 2.5$ . Given the variance of fundamental prices (4.6) and the variance of market prices (C.8), with  $b_1 = \frac{1}{R}$ ,  $b_2 = \frac{\rho}{R}$  and  $\sigma_u = 0$ ,

$$\frac{\sigma_p^2}{\sigma_{p^*}^2} = \frac{(\beta^2 \rho + R)(R - \rho)^2}{(R^2 - \beta^4)(R - \rho \beta^2)} \Big|_{\beta = \beta^*(\rho)}.$$

Corollary 1 demonstrates  $\rho < \beta^*(\rho) < 1$  for  $0 < \rho < 1$ , and hence  $\beta^*(\rho)$  converges to 1 as  $\rho$  tends to 1. Thus as  $\rho$  tends to 1,  $\frac{\sigma_p^2}{\sigma_{p^*}^2}$  converges to 1, consistent with Figure 2b. So for plausible parameter values of  $\rho$ , the variance of market prices is significantly greater than that of fundamental prices, indicating that market prices exhibit excess volatility in the SCEE.

Figure 3 illustrates that the unique SCEE  $(\alpha^*, \beta^*)$  is stable under SAC-learning. Figure 3a shows that the mean of the market prices under SAC-learning,  $\alpha_t$ , tends to the mean  $\alpha^* = 1$  in the SCEE, while Figure 3b shows that the first-order autocorrelation coefficient of the market prices under SAC-learning,  $\beta_t$ , tends to the first-order autocorrelation coefficient  $\beta^* = 0.997$  in the SCEE. Figure 3c shows the asset price under SAC-learning,

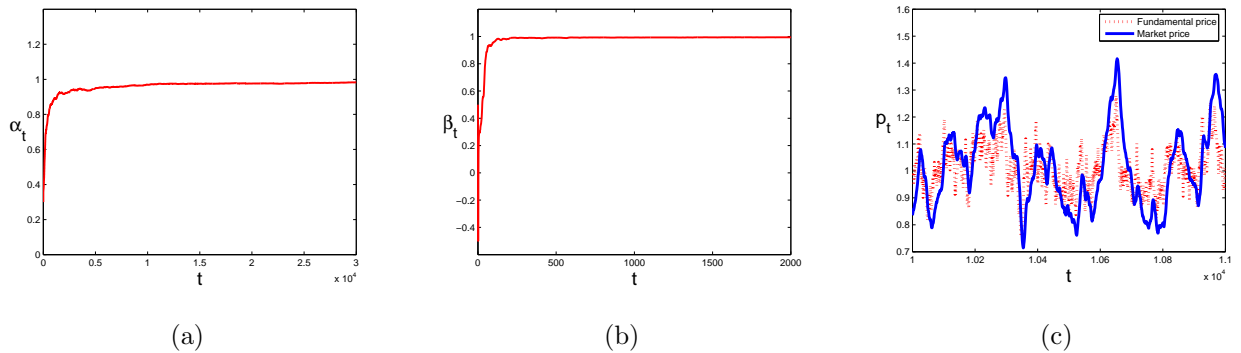


Figure 3: (a) Time series  $\alpha_t \rightarrow \alpha^*(1.0)$ ; (b) time series  $\beta_t \rightarrow \beta^*(0.997)$ ; (c) time series of market prices under SAC-learning and fundamental prices.

using the same sample path of noise, as the time series of the SCEE in Figure 1c. Since the times series are almost the same, SAC-learning converges to the SCEE rather quickly.

In summary, the first-order SCEE and SAC-learning offer an explanation of high persistence, excess volatility and bubbles and crashes in asset prices within a stationary time series framework.<sup>13</sup>

## 4.2 A New Keynesian Philips curve

Our second application of SCEE and SAC-learning is in macroeconomics. We use the New Keynesian Philips curve with inflation driven by an exogenous AR(1) process for the output gap (often measured by detrended real GDP) or the firm's real marginal cost (often measured by labor's share of income), as in Lansing (2009).

### 4.2.1 The model

We derive the model from microfoundations with monopolistic competition and staggered price setting. There is a continuum of firms indexed by  $i \in [0, 1]$ . Each firm produces a differentiated good, but they all use the same technology which uses labor as the only

<sup>13</sup>We also simulated SAC-learning with a constant gain parameter (not shown here) and, similar to Branch and Evans (2011), obtained persistent near unit root bubble and crash dynamics. When the autocorrelation in the driving process is low these unit root bubble and crash dynamics are transitory and recurrent after a series of shocks; for higher values of  $\rho \geq 0.4$  persistent near-unit root bubble and crash dynamics arise, because of the existence of a unique stable high persistence SCEE; cf. Figure 2a, where the SCEE  $\beta^*$  is plotted as a function of  $\rho$ .



factor of production. The demand curve for the good produced by firm  $i$  is given by

$$Y_t(i) = Y_t \left( \frac{P_t(i)}{P_t} \right)^{-\eta_t}, \quad (4.12)$$

where  $Y_t$  is the aggregator function defined as  $Y_t = [\int_0^1 Y_t(i)^{(\eta_t-1)/\eta_t} di]^{\eta_t/(\eta_t-1)}$ ,  $P_t$  is the aggregate price level defined as  $P_t = [\int_0^1 P_t(i)^{1-\eta_t} di]^{1/(1-\eta_t)}$ , and  $\eta_t$  is the elasticity of substitution among goods which varies over time according to some stationary stochastic process.

### Aggregate price dynamics

Following Calvo (1983) we assume that in every period only a fraction  $(1 - \omega)$  of firms are able to reset their prices, while a fraction  $\omega$  keep their price unchanged. In such an environment the aggregate price dynamics are described by

$$P_t = (\omega P_{t-1}^{1-\eta_t} + (1 - \omega)(P_t^*)^{1-\eta_t})^{\frac{1}{1-\eta_t}}, \quad (4.13)$$

where  $P_t^*$  is the price set in period  $t$  by firms reoptimizing their price in that period. Notice that, as shown below, all firms will set the same price since they face an identical problem.

### Optimal price setting

We assume that firms have the same subjective beliefs, denoted by  $\tilde{E}_t$ , and that each firm hires labor from the same integrated economy-wide labor market. Therefore, all firms face the same optimization problem and they will set the same price when reoptimizing.

A firm reoptimizing in period  $t$  will choose the price  $P_t^*$  to maximize expected discounted profits, which are given by

$$\max_{P_t^*} \tilde{E}_t \sum_{s=0}^{\infty} \omega^s \Lambda_{t,t+s} \left( \frac{P_t^*}{P_{t+s}} - \Phi_{t+s} \right) \left( \frac{P_t^*}{P_{t+s}} \right)^{-\eta_{t+s}} Y_{t+s},$$

where  $\Lambda_{t,t+s}$  is the stochastic discount factor and  $\Phi_{t+s}$  are real marginal costs of production. The stochastic discount factor is defined as  $\Lambda_{t,t+s} = \delta^s (Y_{t+s}/Y_t)^{-\sigma}$ , where  $\delta$  is the time discount factor.

The first-order condition associated with the problem above is given by

$$\frac{P_t^*}{P_t} = \frac{\tilde{E}_t \sum_{s=0}^{\infty} \omega^s \Lambda_{t,t+s} \Phi_{t+s} \eta_{t+s} \left( \frac{P_{t+s}}{P_t} \right)^{\eta_{t+s}}}{\tilde{E}_t \sum_{s=0}^{\infty} \omega^s \Lambda_{t,t+s} (\eta_{t+s} - 1) \left( \frac{P_{t+s}}{P_t} \right)^{\eta_{t+s}-1}}. \quad (4.14)$$

Define  $Q_t^* = P_t^*/P_t$  and log-linearize Eq. (4.14) around a zero inflation steady state to get

$$\widehat{q}_t^* = \widetilde{E}_t \sum_{s=0}^{\infty} (\omega\delta)^s \left( (1 - \omega\delta)\widehat{\varphi}_{t+s} + \frac{1 - \omega\delta}{1 - \eta}\widehat{\eta}_{t+s} + \omega\delta\widehat{\pi}_{t+s+1} \right), \quad (4.15)$$

where “hatted” lower case letters denote log-deviations from steady state,  $\eta$  is the mean of the stochastic process  $\{\eta_t\}$ , and  $\widehat{\pi}_t$  is the inflation rate.

Log-linearizing Eq. (4.13) around the zero inflation steady state we get

$$\widehat{q}_t^* = \frac{\omega}{1 - \omega}\widehat{\pi}_t. \quad (4.16)$$

Combining Eqs. (4.15) and (4.16), and dropping hats for notational convenience, we get

$$\pi_t = \widetilde{E}_t \sum_{s=0}^{\infty} (\omega\delta)^s (\gamma\varphi_{t+s} + \lambda\eta_{t+s} + \delta(1 - \omega)\pi_{t+s+1}),$$

where  $\gamma = \frac{(1-\omega\delta)(1-\omega)}{\omega}$  and  $\lambda = \frac{\gamma}{1-\eta}$  are functions of the structural parameters, which can be rewritten as

$$\pi_t = \delta\widetilde{E}_t\pi_{t+1} + \gamma\varphi_t + u_t, \quad (4.17)$$

where  $u_t = \lambda\eta_t$ . Note that in deriving Eq. (4.17) we used the law of iterated expectations at the individual level. Even in the presence of bounded rationality, this is a reasonable and intuitive assumption which is standard in the learning literature; see, e.g., Evans and Honkapohja (2001) and Branch and McGough (2009). In order to rewrite (4.17) in terms of the output gap we can use the relationship  $\varphi_t = ky_t$ .

In the New Keynesian Philips curve (NKPC) with inflation driven by an exogenous AR(1) process  $y_t$  for the firm’s real marginal cost or the output gap, inflation and the output gap (real marginal cost) evolve according to

$$\begin{cases} \pi_t = \delta\pi_{t+1}^e + \gamma y_t + u_t, \\ y_t = a + \rho y_{t-1} + \varepsilon_t, \end{cases} \quad (4.18)$$

where  $\pi_t$  is the inflation at time  $t$ ,  $\pi_{t+1}^e$  is the subjective expected inflation at date  $t + 1$  and  $y_t$  is the output gap or real marginal cost,  $\delta \in [0, 1)$  is the representative agent’s subjective time discount factor,  $\gamma > 0$  is related to the degree of price stickiness in the economy and  $\rho \in [0, 1)$  describes the persistence of the AR(1) driving process.  $u_t$  and  $\varepsilon_t$  are i.i.d. stochastic disturbances with zero mean and finite absolute moments with variances  $\sigma_u^2$  and  $\sigma_\varepsilon^2$ , respectively. The most important difference with the asset pricing model in Subsection 4.1 is that (4.18) includes two stochastic disturbances, not only the

noise  $\varepsilon_t$  of the AR(1) driving variable, but also an additional noise term  $u_t$  in the New Keynesian Philips curve. We refer to  $u_t$  as a markup shock, which is often motivated by the presence of an uncertain, variable tax rate and to  $\varepsilon_t$  as a demand shock, that is uncorrelated with the markup shock. Compared with our general framework (3.1), the corresponding parameters are  $b_0 = 0$ ,  $b_1 = \delta$  and  $b_2 = \gamma$ .

#### 4.2.2 Theoretical results

Following the general results in Section 3, the rational expectations equilibrium

$$\pi_t^* = \frac{\gamma\delta a}{(1-\delta)(1-\delta\rho)} + \frac{\gamma}{1-\delta\rho}y_t + u_t. \quad (4.19)$$

The corresponding unconditional mean and the unconditional variance of the rational expectations equilibrium  $\pi_t^*$  are given by, respectively,

$$\bar{\pi}^* := E(\pi_t^*) = \frac{\gamma a}{(1-\delta)(1-\rho)}, \quad (4.20)$$

$$Var(\pi_t^*) = E(\pi_t^* - \bar{\pi}^*)^2 = \frac{\gamma^2 \sigma_\varepsilon^2}{(1-\delta\rho)^2(1-\rho^2)} + \sigma_u^2. \quad (4.21)$$

Furthermore, the first-order autocovariance and autocorrelation of rational expectations equilibrium  $\pi_t^*$  are, respectively,

$$E(\pi_t^* - \bar{\pi}^*)(\pi_{t-1}^* - \bar{\pi}^*) = \frac{\gamma^2 \rho \sigma_\varepsilon^2}{(1-\delta\rho)^2(1-\rho^2)},$$

$$Corr(\pi_t^*, \pi_{t-1}^*) = \frac{\rho \gamma^2}{\gamma^2 + (1-\delta\rho)^2(1-\rho^2) \frac{\sigma_u^2}{\sigma_\varepsilon^2}}.$$

Note that, the larger the noise level  $\sigma_u^2$  in the markup shock, the smaller the first-order autocorrelation in the fundamental rational equilibrium inflation.

Under the assumption that agents are boundedly rational and believe that inflation  $\pi_t$  follows a univariate AR(1) process, the implied actual law of motion becomes

$$\begin{cases} \pi_t = \delta[\alpha + \beta^2(\pi_{t-1} - \alpha)] + \gamma y_t + u_t, \\ y_t = a + \rho y_{t-1} + \varepsilon_t. \end{cases} \quad (4.22)$$

Since  $0 \leq \delta\beta^2 < 1$  and  $0 \leq \rho < 1$ , the inflation process (4.22) is stationary and ergodic. The implied sample mean is given by

$$\bar{\pi} = \frac{\delta\alpha(1-\beta^2) + \gamma a/(1-\rho)}{1-\delta\beta^2}. \quad (4.23)$$

Imposing the consistency requirement,  $\bar{\pi} = \alpha$ , yields the SCEE sample mean

$$\alpha^* = \frac{\gamma \bar{y}}{1 - \delta} = \frac{\gamma a}{(1 - \delta)(1 - \rho)}. \quad (4.24)$$

The corresponding first-order autocorrelation coefficient  $F(\beta)$  of the implied ALM is

$$F(\beta) = \delta \beta^2 + \frac{\gamma^2 \rho (1 - \delta^2 \beta^4)}{\gamma^2 (\delta \beta^2 \rho + 1) + (1 - \rho^2)(1 - \delta \beta^2 \rho) \cdot \frac{\sigma_u^2}{\sigma_\varepsilon^2}}. \quad (4.25)$$

Applying Proposition 1 in Section 3.2 we obtain

**Corollary 2** *In the case that  $0 < \rho < 1$  and  $0 \leq \delta < 1$ , there exists at least one nonzero first-order stochastic consistent expectations equilibrium (SCEE)  $(\alpha^*, \beta^*)$  for the New Keynesian Philips curve (4.22) with  $\alpha^* = \frac{\gamma a}{(1 - \delta)(1 - \rho)} = \bar{\pi}^*$ .*

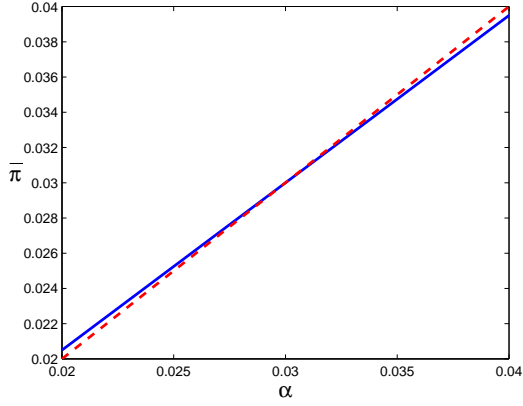
For the New Keynesian Philips curve (4.18), however, multiple SCEE may coexist.

### 4.2.3 Numerical analysis

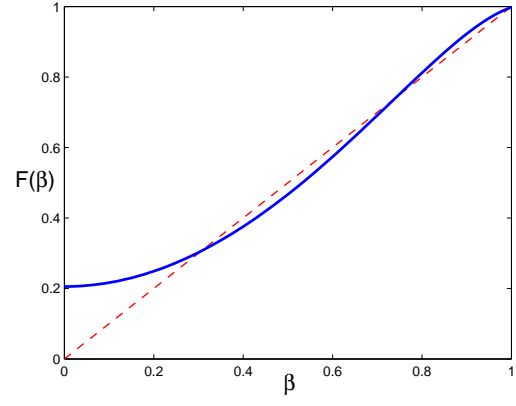
In this subsection we investigate how the multiplicity of SCEE and their stability under learning depends upon parameters. Based on empirical findings, e.g., in Lansing (2009), Gali et al. (2001) and Fuhrer (2006, 2009), we examine a range of empirically plausible parameter values.<sup>14</sup> First, fix the parameters  $\delta = 0.99$ ,  $\gamma = 0.075$ ,  $a = 0.0004$ ,  $\rho = 0.9$ ,  $\sigma_\varepsilon = 0.01$  [ $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ ], and  $\sigma_u = 0.003162$  [ $u_t \sim N(0, \sigma_u^2)$ ], so that  $\frac{\sigma_u^2}{\sigma_\varepsilon^2} = 0.1$ .

Figure 4a illustrates the existence of a unique (stable) sample mean  $\alpha^*$ , where  $\alpha^* = 0.03$ . Figure 4b shows that  $F(\beta)$  has three fixed points  $\beta_1^* \approx 0.3066$ ,  $\beta_2^* \approx 0.7417$  and  $\beta_3^* \approx 0.9961$ . Hence, we have coexistence of three first-order SCEE. Figures 4c and 4d illustrate the time series of inflation along the coexisting SCEE. Inflation has low persistence along the SCEE  $(\alpha^*, \beta_1^*)$ , but very high persistence along the SCEE  $(\alpha^*, \beta_3^*)$ . The time series of inflation along the high persistence SCEE in Figure 4d has in fact similar persistence characteristics and amplitude of fluctuation as in empirical inflation data, e.g., in Tallman (2003). Furthermore, Figure 4d illustrates that inflation in the high persistence SCEE has much stronger persistence than REE inflation, where the first-order autocorrelation coefficient of REE inflation is 0.865, significantly less than  $\beta_3^* = 0.9961$ .

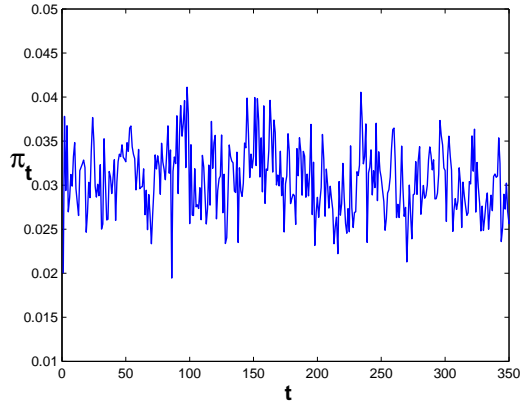
<sup>14</sup>As shown in Lansing (2009), based on regressions using either the output gap or labor's share of income over the period 1949:Q1 to 2004:Q4,  $\rho = 0.9, \sigma_\varepsilon = 0.01$ . Estimates of the NKPC parameters  $\delta, \gamma, \sigma_u$  are sensitive to the choice of the driving variable, the sample period, and the econometric model, etc., but our choices are within a plausible range. Furthermore, based on the above theoretical results, the constant  $a$  only affects the mean of inflation  $\bar{\pi}$ , and not its autocorrelation coefficient  $F(\beta)$ . Moreover,  $F(\beta)$  only depends on the ratio  $\sigma_u/\sigma_\varepsilon$ , but not on their absolute values.



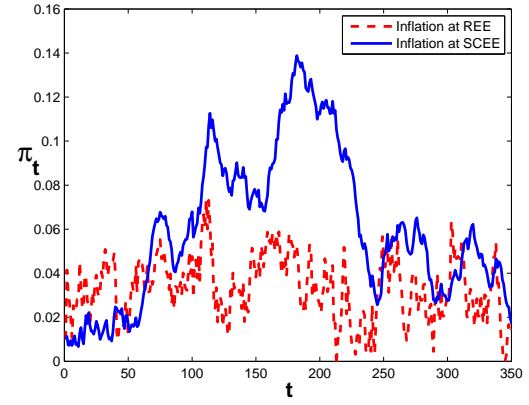
(a) SCEE for  $\alpha^*$



(b) SCEE for  $\beta^*$

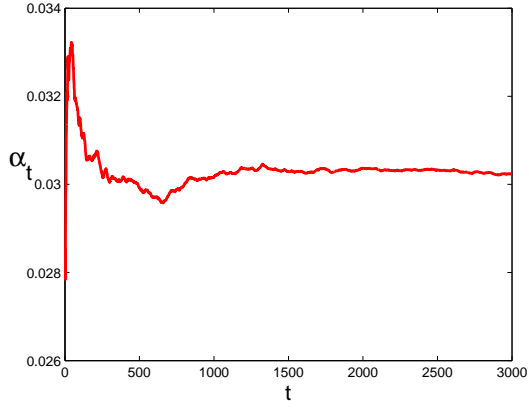


(c) time series at  $(\alpha^*, \beta^*) = (0.03, 0.3066)$

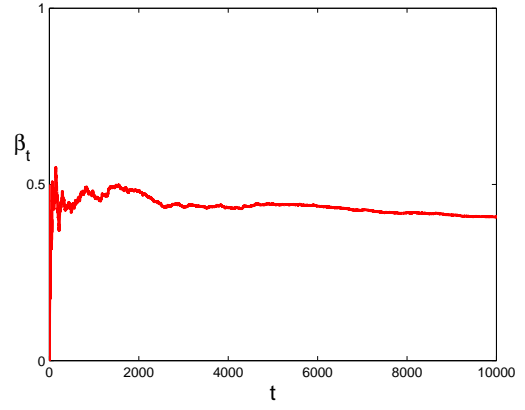


(d) time series at  $(\alpha^*, \beta^*) = (0.03, 0.9961)$

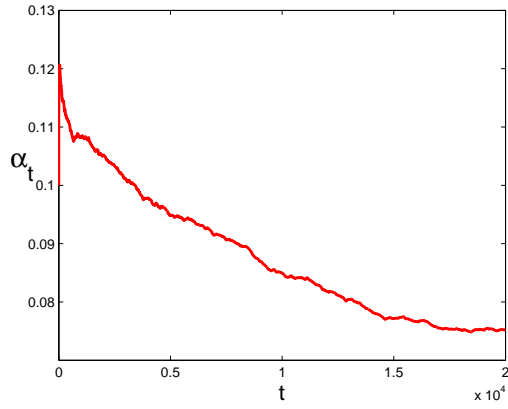
Figure 4: (a) The mean  $\alpha^*$  of the SCEE is the unique intersection point of mean inflation  $\bar{\pi}$  in (4.23) (bold curve) with the perceived mean  $\alpha$  (dotted line); (b) the first-order autocorrelation  $\beta^*$  of the SCEE correspond to the three intersection points of  $F(\beta)$  in (4.25) (bold curve) with the perceived first-order autocorrelation  $\beta$  (dotted line); (c) time series of inflation in low-persistence SCEE  $(\alpha^*, \beta_1^*) = (0.03, 0.3066)$ ; (d) times series of inflation in high-persistence SCEE  $(\alpha^*, \beta_3^*) = (0.03, 0.9961)$  (bold curve) and time series of REE inflation (dotted curve).



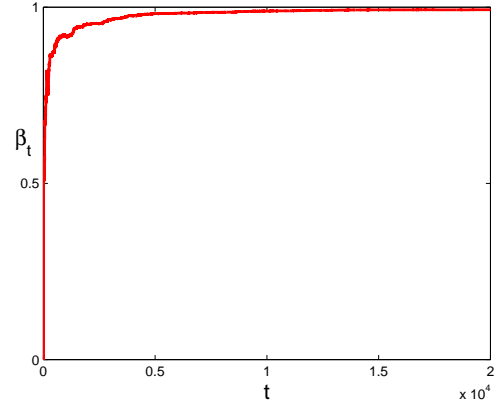
(a)  $\alpha_t \rightarrow 0.03$



(b)  $\beta_t \rightarrow 0.3066$



(c)  $\alpha_t \rightarrow 0.03$



(d)  $\beta_t \rightarrow 0.9961$

Figure 5: Time series of  $\alpha_t$  and  $\beta_t$  under SAC-learning for different initial values. (a-b) For  $(\pi_0, y_0) = (0.028, 0.01)$  SAC-learning converges to the low persistence SCEE  $(\alpha^*, \beta_1^*) = (0.03, 0.3066)$ ; (c-d) For  $(\pi_0, y_0) = (0.1, 0.15)$  SAC-learning converges to the high persistence SCEE  $(\alpha^*, \beta_1^*) = (0.03, 0.9961)$ .

If multiple SCEE coexist, the convergence under SAC-learning depends on the initial state of the system, as illustrated in Figure 5. Since  $0 < F'(\beta_j^*) < 1$ , for  $j = 1$  and  $j = 3$ , while  $F'(\beta_2^*) > 1$ , (see Figure 4b), Proposition 4 implies that the first-order SCEE  $(\alpha^*, \beta_1^*)$  and  $(\alpha^*, \beta_3^*)$  are (locally) stable under SAC-learning, while  $(\alpha^*, \beta_2^*)$  is unstable. For initial state  $(\pi_0, y_0) = (0.028, 0.01)$  (Figures 5a and 5b), the SAC-learning dynamics  $(\alpha_t, \beta_t)$  converges to the stable low-persistence SCEE  $(\alpha^*, \beta_1^*) = (0.03, 0.3066)$ . Figure 5b also illustrates that the convergence of the first-order autocorrelation coefficient  $\beta_t$  to the low-persistence first-order autocorrelation coefficient  $\beta_1^* = 0.3066$  is very slow. For a different initial state,  $(\pi_0, y_0) = (0.1, 0.15)$ , our numerical simulation shows that the sample mean  $\alpha_t$  still tends to  $\alpha^* = 0.03$ , but only slowly<sup>15</sup> (see Figure 5c), while  $\beta_t$  tends to the high persistence SCEE  $\beta_3^* \approx 0.9961$ <sup>16</sup> (see Figure 5d).

Numerous simulations (not shown) show that for initial values  $\pi_0$  of inflation higher than the mean  $\alpha^* = 0.03$ , the SAC-learning  $\beta_t$  generally enters the high-persistence region. In particular, a large shock to inflation may easily cause a jump of the SAC-learning process into the high-persistence region.<sup>17</sup> In the following we further indicate how high and low persistence SCEE depend on different parameters.

#### 4.2.4 Multiple equilibria and parameter dependence

Figure 6 illustrates how the number of SCEE depends on the parameter  $\gamma$ . For sufficiently small  $\gamma (< 0.05)$ , there exists only one, low persistence SCEE  $\beta^*$  (Figure 6a). This is similar to the case  $\gamma = 0$ , where correspondingly  $F(\beta) = \delta\beta^2$  and hence the unique SCEE  $\beta^* = 0$ . Moreover, since

$$\frac{\partial F}{\partial \gamma} = \frac{2\rho(1 - \delta^2\beta^4)(1 - \rho^2)(1 - \delta\beta^2\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}}{\gamma^3[(\delta\beta^2\rho + 1) + (1 - \rho^2)(1 - \delta\beta^2\rho)\frac{1}{\gamma^2}\frac{\sigma_u^2}{\sigma_\varepsilon^2}]^2} > 0,$$

the graph of  $F(\beta)$  in (4.25) shifts upward as  $\gamma$  increases. At  $\gamma \approx 0.05$ , a new SCEE,  $\beta^* \approx 0.975$ , is created in a tangent bifurcation, where  $F(\beta)$  is tangent to the diagonal (Figure 6b). Immediately thereafter, there exist three SCEE,  $\beta_1^*$ ,  $\beta_2^*$  and  $\beta_3^*$  (see Figures 6c and 6d). The low persistence SCEE  $\beta_1^*$  and the high persistence SCEE  $\beta_3^*$  are stable under

<sup>15</sup>The slow convergence is caused by the slope coefficient  $\frac{\delta - \delta\beta^2}{1 - \delta\beta^2}$  for  $\alpha$  in the expression  $\bar{\pi}$  in (4.23), which is very close to 1 for  $\delta = 0.99 \approx 1$ , as illustrated in Figure 4a.

<sup>16</sup>As shown in Figure 4b,  $F'(\beta_3^*)$  is close to 1 and, hence, the convergence of SAC-learning is very slow.

<sup>17</sup>We also simulated the NKPC under SAC-learning with a constant gain parameter (not shown here) and, similar to Branch and Evans (2010), obtained irregular regime switching between phases of very low persistence and phases of high persistence with near unit root behavior.

SAC-learning, since  $0 < F'(\beta_j^*) < 1$ ,  $j = 1$  and  $j = 3$ , and separated by an unstable SCEE  $\beta_2^*$ , with  $F'(\beta_2^*) > 1$ . As  $\gamma$  further increases, at  $\gamma \approx 0.084$ , another tangent bifurcation occurs, where the low and intermediate persistence SCEE  $\beta_1^*$  and  $\beta_2^*$  coincide (Figure 6e). For  $\gamma > 0.084$ , the low persistence SCEE disappears and a unique high persistence SCEE exists, which is stable under SAC-learning (Figure 6f).

The dependence of the number of SCEE and their persistence upon the parameter  $\gamma$  are quite intuitive. Recall that  $\gamma$  in (4.18) measures the relative strength of the driving variable, the output gap or marginal costs, to inflation.<sup>18</sup> When the driving force is relatively weak, a unique, stable low persistence SCEE prevails, with much weaker autocorrelation than in the driving variable. At the other extreme, when the driving force is sufficiently strong, a unique, stable high persistence SCEE prevails, with significantly stronger autocorrelation and higher persistence than in the driving variable. In the intermediate case, multiple SCEE coexist and the system exhibits path dependence, where, depending on initial conditions, inflation converges to a low or a high persistence SCEE.

In a similar way, the dependence of the SCEE upon the noise ratio  $\frac{\sigma_u^2}{\sigma_\varepsilon^2}$  can be analyzed.  $F(\beta)$  in (4.25) can be rewritten as

$$F(\beta) = \delta\beta^2 + \frac{\rho(1 - \delta^2\beta^4)}{(\delta\beta^2\rho + 1) + (1 - \rho^2)(1 - \delta\beta^2\rho) \cdot \frac{\sigma_u^2}{\sigma_\varepsilon^2} \frac{1}{\gamma^2}}.$$

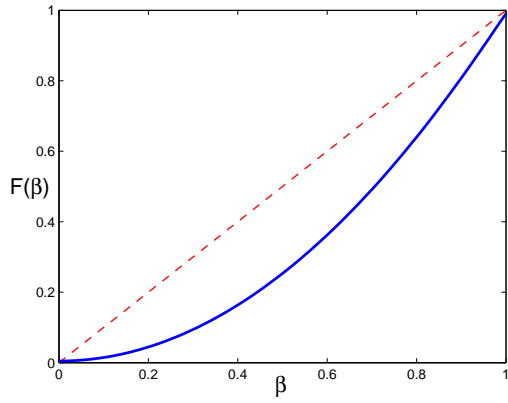
Consequently, the effect of the noise ratio  $\frac{\sigma_u^2}{\sigma_\varepsilon^2}$  is inversely related to the effect of  $\gamma$ . Hence, when the ratio  $\frac{\sigma_u^2}{\sigma_\varepsilon^2}$  is high, that is, when the markup shocks to inflation are high compared to the noise of the driving variable, a unique, stable low persistence SCEE prevails. If on the other hand, the markup shocks to inflation are low compared to the noise of the driving variable, a unique, stable high persistence SCEE prevails.

Furthermore, Figure 7 illustrates how the SCEE  $\beta^*$ , together with the first-order autocorrelation coefficient of REE inflation, depends upon the parameter  $\rho$ , measuring the persistence in the driving variable. For intermediate values of  $\rho$  ( $\in [0.84, 0.918]$ ), two stable SCEE  $\beta^*$  coexist separated by an unstable SCEE. In the high persistence SCEE,  $\beta^*$  is larger than the first-order autocorrelation coefficient of REE inflation, while in the low persistence SCEE  $\beta^*$  is smaller than the first-order autocorrelation coefficient of REE inflation. For small values of  $\rho$ ,  $\rho < 0.84$ , a unique, stable low persistence SCEE prevails, while for large values of  $\rho$ ,  $\rho > 0.918$ , a unique, stable high persistence SCEE prevails.

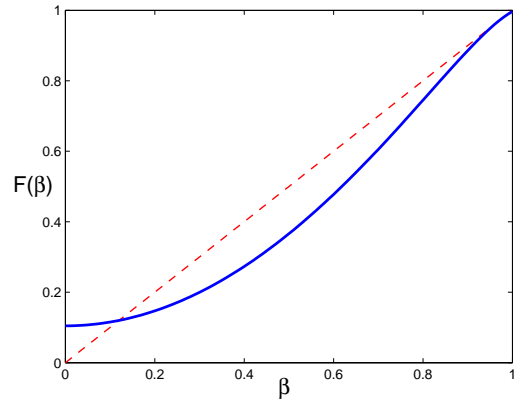
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<sup>18</sup>Note that  $\gamma$  corresponds to the parameter  $b_2$  in the general linear specification (3.1-3.2). See Proposition 2 and the discussion in Subsection 3.2 on how the low respectively high persistence SCEE depends on  $b_2$ .

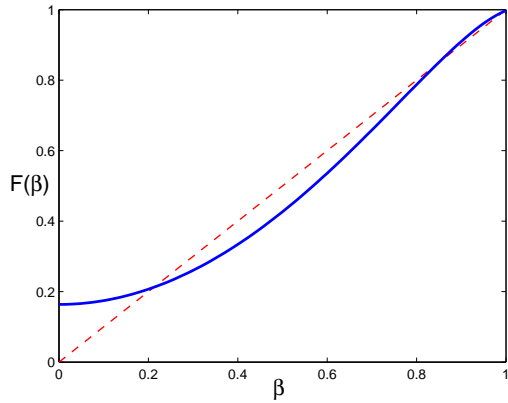




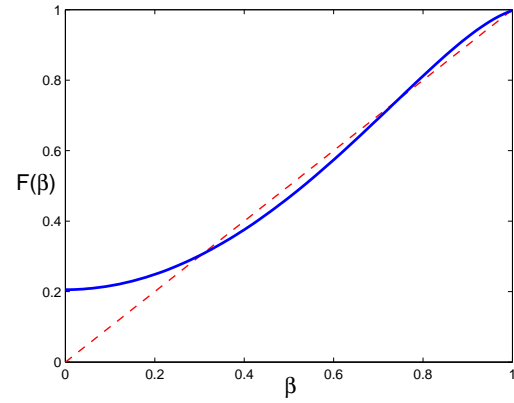
(a)  $\gamma = 0.01$



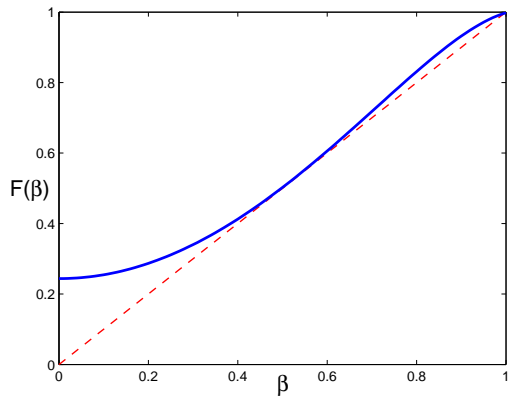
(b)  $\gamma = 0.05$



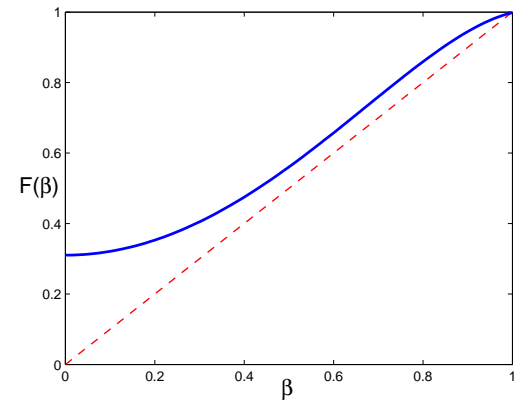
(c)  $\gamma = 0.065$



(d)  $\gamma = 0.075$



(e)  $\gamma = 0.084$



(f)  $\gamma = 0.1$

Figure 6: The figure illustrates how the (co-)existence of low and high persistence SCEE  $\beta^*$  depends upon the parameter  $\gamma$ , measuring the relative strength of inflation upon the driving variable, the output gap. (a)  $\gamma = 0.01$ ; (b)  $\gamma = 0.05$ ; (c)  $\gamma = 0.065$ ; (d)  $\gamma = 0.075$ ; (e)  $\gamma = 0.084$ , and (f)  $\gamma = 0.1$ . Other parameters:  $\frac{\sigma_u^2}{\sigma_\varepsilon^2} = 0.1$ ,  $\rho = 0.9$  and  $\delta = 0.99$ .

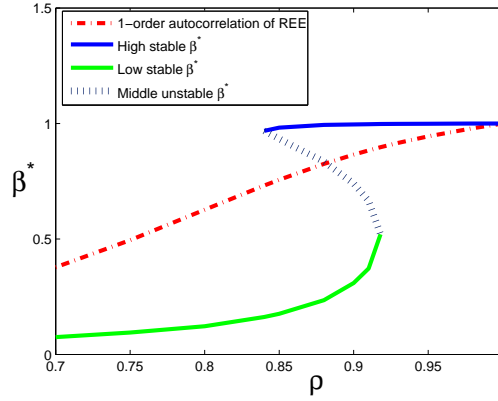


Figure 7: First-order autocorrelation coefficient of REE inflation (dotted real curve), stable SCEE  $\beta^*$  with respect to  $\rho$  (bold curves), unstable SCEE  $\beta^*$  (dotted curve), where  $\gamma = 0.075, \sigma_u = 0.003162, \sigma_\varepsilon = 0.01, \delta = 0.99$ .

Simulations show that, for plausible values of  $\rho$  around 0.9, for a large range of initial values of inflation, the SAC-learning converges to the stable, high persistence SCEE  $\beta^*$  with very strong persistence in inflation (see e.g. Figure 5d). This result is consistent with the empirical finding in Adam (2007) that the Restricted Reception Equilibrium (RPE) describes subjects' inflation expectations surprisingly well and provides a better explanation for the observed persistence of inflation than REE.

In summary, the dependence of the number of equilibria and whether their persistence is high or low are quite intuitive. This intuition essentially follows from the signs of the partial derivatives of the first-order autocorrelation coefficient  $F(\beta)$  of the implied ALM (4.25) satisfying (see Appendix G):

$$\frac{\partial F}{\partial \gamma} > 0 \quad \frac{\partial F}{\partial(\frac{\sigma_u^2}{\sigma_\varepsilon^2})} < 0 \quad \frac{\partial F}{\partial \rho} > 0 \quad \frac{\partial F}{\partial \delta} > 0. \quad (4.26)$$

Hence, as in Figure 6, the graph of  $F(\beta)$  shifts upwards when  $\gamma$  increases,  $\frac{\sigma_u^2}{\sigma_\varepsilon^2}$  decreases,  $\rho$  increases or  $\delta$  increases, and consequently, the equilibria shift from low persistence to high persistence equilibria. Depending on the shape of  $F(\beta)$  there are then two possibilities. When  $F$  is only weakly nonlinear, e.g., as in Figure 1b for the asset pricing model, the equilibrium is unique and only a shift from a low to a high persistence equilibrium arises. When the nonlinearity is stronger and  $F$  is S-shaped, e.g., as in Figure 6 for empirically relevant parameter values in the NKPC, both the persistence and the number of equilibria shift, and a transition from a unique stable low persistence SCEE, through coexisting stable low and high persistence equilibria, to a unique stable high persistence

equilibrium occurs. Such a transition from a unique low persistence SCEE, through coexisting low and high persistence SCEE, toward a unique high persistence SCEE occurs when the strength of the AR(1) driving force (the parameter  $\gamma$ ) increases, when the ratio of the model noise compared to the noise of the driving force (i.e.  $\frac{\sigma_u^2}{\sigma_\varepsilon^2}$ ) decreases, when the autocorrelation (i.e., the parameter  $\rho$ ) in the driving force increases, and when the strength of the expectations feedback (i.e., the parameter  $\delta$ ) increases.

## 5 Concluding remarks

In this paper we have introduced the concept of behavioral learning equilibrium, a very simple type of misspecification equilibrium together with an intuitive behavioral interpretation and learning process. Boundedly rational agents use a univariate linear forecasting rule and in equilibrium correctly forecast the unconditional sample mean and first-order sample autocorrelation. Hence, to a first order approximation the simple linear forecasting rule is consistent with observed market realizations. Sample autocorrelation learning simply means that agents are slowly updating the two coefficients –sample mean and first-order autocorrelation– of their linear rule. In the long run, agents thus learn the best univariate linear forecasting rule, without fully recognizing the structure of the economy.

We have applied our behavioral learning equilibrium concept to a standard asset pricing model with AR(1) dividends and a New Keynesian Philips curve driven by an AR(1) process for the output gap or marginal costs. In both applications, the law of motion of the economy is linear, but it is driven by an exogenous stochastic AR(1) process. Agents however are not fully aware of the exact linear structure of the economy, but use a simple univariate forecasting rule, to predict asset prices or inflation. In the asset pricing model a unique SCEE exists and it is stable under SAC-learning. An important feature of the SCEE is that it is characterized by high-persistence and excess volatility in asset prices, significantly higher than under rational expectations. In the New Keynesian model, multiple SCEE arise and a low and a high-persistence misspecification equilibrium coexist. The SAC-learning exhibits path dependence and it depends on the initial states whether the system converges to the low-persistence or the high-persistence inflation regime. In particular, when there are shocks – e.g. oil shocks – temporarily causing high inflation, SAC-learning may lock into the high-persistence inflation regime.

Are these behavioral learning equilibria empirically relevant or would smart agents

recognize their (second order) mistakes and learn to be perfectly rational? This empirical question should be addressed in more detail in future work, but we provide some arguments for the empirical relevance of our equilibrium concept. Firstly, in our applications the SCEE already explain some important stylized facts: (i) high persistence (close to unit root) and excess volatility in asset prices, (ii) high persistence in inflation and (iii) regime switching in inflation dynamics, which could explain a long phase of high US inflation in the 1970s and early 1980s as well as a long phase of low inflation in the 1990s and 2000s. Secondly, we stress the simplicity and *behavioral* interpretation of our learning equilibrium concept. The univariate AR(1) rule and the SAC-learning process are examples of simple forecasting heuristics that can be used without any knowledge of statistical techniques, simply by observing a time series and roughly "guesstimating" its sample average and its first-order persistence coefficient. Coordination on a behavioral forecasting heuristic that performs reasonably well to a first-order approximation seems more likely than coordination on more complicated learning or sunspot equilibria as, for example, in Woodford (1990). Even though some smart individual agents might be able to improve upon the best linear, univariate forecasting rule, a majority of agents might still stick to their simple univariate rule. It therefore seems relevant to describe aggregate phenomena by simple misspecification equilibria and behavioral learning processes. Our behavioral learning equilibrium concept also relates to the "natural expectations" in Fuster et al. (2010), emphasizing parsimonious forecasting rules giving much weight to recent changes to explain the long-run persistence of economic shocks. Our simple univariate AR(1) rule may be seen as the most parsimonious forecasting rule leading to long-run persistence. There is already some experimental evidence for the relevance of misspecification equilibria in Adam (2007). More recently Assenza et al. (2011) and Pfajfar and Zakelj (2010) ran learning to forecasting experiments with human subjects in a New Keynesian framework with expectations feedback from individual inflation and output gap forecasts. Coordination on simple linear univariate models explain a substantial part of individual inflation and output gap forecasting behavior.

In future work we plan to consider more general economic settings to study behavioral learning equilibria. An obvious next step is to apply our SCEE and SAC-learning framework to higher dimensional linear economic systems, with agents forecasting by univariate linear rules. In particular, the fully specified New Keynesian model of inflation and output dynamics would be an interesting (two-dimensional) application. Including asset prices in a New Keynesian model, as in Bernanke and Gertler (1999, 2001), provides

another interesting (three-dimensional) application. It is also interesting and challenging to study SCEE and misspecification under *heterogeneous expectations* and allow for switching between different rules. Branch (2004) and Hommes (2011) provide some empirical and experimental evidence on heterogeneous expectations, while Berardi (2007) and Branch and Evans (2006, 2007) have made some related studies on heterogeneous expectations and learning in similar settings. Future work should focus on the robustness and survival of behavioral forecasting rules, such as AR(1) and SAC-learning, in a heterogeneous expectations environment. In addition to theoretical work, it would be of interest to study coordination and learning of BLE in laboratory settings with multiple restricted perception and/or sunspot equilibria.

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# Appendix

## A Recursive dynamics of SAC-learning

The sample average is

$$\begin{aligned}
\alpha_t &= \frac{1}{t+1}[x_0 + x_1 + \cdots + x_t] \\
&= \frac{1}{t+1}[t\alpha_{t-1} + x_t] \\
&= \frac{1}{t+1}[(t+1)\alpha_{t-1} + x_t - \alpha_{t-1}] \\
&= \alpha_{t-1} + \frac{1}{t+1}[x_t - \alpha_{t-1}].
\end{aligned}$$

Let

$$\begin{aligned}
z_t &:= (x_0 - \alpha_t)(x_1 - \alpha_t) + \cdots + (x_{t-1} - \alpha_t)(x_t - \alpha_t) \\
&= (x_0 - \alpha_{t-1} - \frac{1}{t+1}(x_t - \alpha_{t-1}))(x_1 - \alpha_{t-1} - \frac{1}{t+1}(x_t - \alpha_{t-1})) + \\
&\quad \cdots + (x_{t-1} - \alpha_{t-1} - \frac{1}{t+1}(x_t - \alpha_{t-1}))(x_t - \alpha_{t-1} - \frac{1}{t+1}(x_t - \alpha_{t-1})) \\
&= (x_0 - \alpha_{t-1})(x_1 - \alpha_{t-1}) + \cdots + (x_{t-2} - \alpha_{t-1})(x_{t-1} - \alpha_{t-1}) \\
&\quad + \frac{x_t - \alpha_{t-1}}{t+1}(2\alpha_{t-1} - x_0 - x_1 + \cdots + 2\alpha_{t-1} - x_{t-2} - x_{t-1}) + \frac{t-1}{(t+1)^2}(x_t - \alpha_{t-1})^2 \\
&\quad + \frac{t}{t+1}(x_{t-1} - \alpha_{t-1})(x_t - \alpha_{t-1}) - \frac{t}{(t+1)^2}(x_t - \alpha_{t-1})^2 \\
&= z_{t-1} + \frac{1}{t+1}(x_t - \alpha_{t-1})[2(t-1)\alpha_{t-1} - x_0 - 2x_1 - \cdots - 2x_{t-2} - x_{t-1} + t(x_{t-1} - \alpha_{t-1})] \\
&\quad - \frac{1}{(t+1)^2}(x_t - \alpha_{t-1})^2, \\
&= z_{t-1} + \frac{1}{t+1}(x_t - \alpha_{t-1})[x_0 + (t+1)x_{t-1} - (t+2)\alpha_{t-1}] - \frac{1}{(t+1)^2}(x_t - \alpha_{t-1})^2 \\
&= z_{t-1} + (x_t - \alpha_{t-1})\left[x_{t-1} + \frac{x_0}{t+1} - \frac{t+2}{t+1}\alpha_{t-1} + \frac{1}{(t+1)^2}\alpha_{t-1} - \frac{1}{(t+1)^2}x_t\right] \\
&= z_{t-1} + (x_t - \alpha_{t-1})\Phi_4,
\end{aligned}$$

where  $\Phi_4 = x_{t-1} + \frac{x_0}{t+1} - \frac{t^2+3t+1}{(t+1)^2}\alpha_{t-1} - \frac{1}{(t+1)^2}x_t$ .

Write

$$\begin{aligned}
n_t &:= (x_0 - \alpha_t)^2 + (x_1 - \alpha_t)^2 + \cdots + (x_t - \alpha_t)^2 \\
&= (x_0 - \alpha_{t-1} - \frac{1}{t+1}(x_t - \alpha_{t-1}))^2 + \cdots + (x_t - \alpha_{t-1} - \frac{1}{t+1}(x_t - \alpha_{t-1}))^2 \\
&= (x_0 - \alpha_{t-1})^2 + (x_1 - \alpha_{t-1})^2 + \cdots + (x_{t-1} - \alpha_{t-1})^2 + \frac{t+t^2}{(t+1)^2}(x_t - \alpha_{t-1})^2 \\
&= n_{t-1} + \frac{t}{t+1}(x_t - \alpha_{t-1})^2.
\end{aligned}$$

All these results are consistent with those in Appendix 1 of Hommes, Sorger & Wagener (2004). Note that in our paper  $R_t$  is different from  $n_t$  in Hommes et al. (2004). In fact,

$$\begin{aligned}
R_t &= \frac{1}{t+1}n_t \\
&= \frac{1}{t+1}\left[n_{t-1} + \frac{t}{t+1}(x_t - \alpha_{t-1})^2\right] \\
&= \frac{t}{t+1}\frac{n_{t-1}}{t} + \frac{t}{(t+1)^2}(x_t - \alpha_{t-1})^2 \\
&= \frac{t}{t+1}R_{t-1} + \frac{t}{(t+1)^2}(x_t - \alpha_{t-1})^2 \\
&= R_{t-1} + \frac{1}{t+1}\left[\frac{t}{t+1}(x_t - \alpha_{t-1})^2 - R_{t-1}\right].
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\beta_t &= \frac{z_t}{n_t} \\
&= \beta_{t-1} + \left[\frac{z_t}{n_t} - \frac{z_{t-1}}{n_{t-1}}\right] \\
&= \beta_{t-1} + \frac{1}{n_t n_{t-1}}[z_t n_{t-1} - z_{t-1} n_t] \\
&= \beta_{t-1} + \frac{1}{n_t n_{t-1}}\left[(z_{t-1} + (x_t - \alpha_{t-1})\Phi_4)n_{t-1} - z_{t-1}\left(n_{t-1} + \frac{t}{t+1}(x_t - \alpha_{t-1})^2\right)\right] \\
&= \beta_{t-1} + \frac{1}{n_t n_{t-1}}\left[(x_t - \alpha_{t-1})\Phi_4 n_{t-1} - z_{t-1}\frac{t}{t+1}(x_t - \alpha_{t-1})^2\right] \\
&= \beta_{t-1} + \frac{1}{n_t}\left[(x_t - \alpha_{t-1})\Phi_4 - \beta_{t-1}\frac{t}{t+1}(x_t - \alpha_{t-1})^2\right] \\
&= \beta_{t-1} + \frac{R_{t-1}^{-1}}{t+1}\left[(x_t - \alpha_{t-1})\left(x_{t-1} + \frac{x_0}{t+1} - \frac{t^2+3t+1}{(t+1)^2}\alpha_{t-1} - \frac{x_t}{(t+1)^2}\right) - \frac{t}{t+1}\beta_{t-1}(x_t - \alpha_{t-1})^2\right].
\end{aligned}$$

## B Rational expectations equilibrium of $x$

Under the assumption that the transversality condition  $\lim_{k \rightarrow \infty} b_1^k E_t(x_{t+k}^*) = 0$  holds, the REE  $x_t^*$  can be computed as

$$\begin{aligned}
x_t^* &= b_0 + b_1 E_t x_{t+1}^* + b_2 y_t + u_t \\
&= b_0 + b_1 E_t [b_0 + b_1 E_{t+1} x_{t+2}^* + b_2 y_{t+1} + u_{t+1}] + b_2 y_t + u_t \\
&= b_0(1 + b_1) + b_1^2 E_t x_{t+2}^* + b_1 b_2 E_t y_{t+1} + b_2 y_t + u_t \\
&= b_0(1 + b_1) + b_1^2 E_t x_{t+2}^* + b_1 b_2 (a + \rho y_t) + b_2 y_t + u_t \\
&= b_0(1 + b_1 + \dots + b_1^{n-1}) + b_1^n E_t x_{t+n}^* + \sum_{k=1}^{n-1} [b_1^k b_2 (a + \rho a + \rho^{k-1} a + \rho^k y_t)] + b_2 y_t + u_t \\
&= b_0 \sum_{k=0}^{n-1} b_1^k + b_1^n E_t x_{t+n}^* + \sum_{k=1}^{n-1} \frac{b_2 a}{\rho - 1} b_1^k (\rho^k - 1) + b_2 y_t \sum_{k=0}^{n-1} b_1^k \rho^k + u_t \\
&= \dots \\
&= \frac{b_0}{1 - b_1} + \frac{ab_1 b_2}{(1 - b_1 \rho)(1 - b_1)} + \frac{b_2}{1 - b_1 \rho} y_t + u_t.
\end{aligned} \tag{B.1}$$

## C First-order autocorrelation coefficient of $x$

We rewrite model (3.12) as

$$\begin{cases} x_t - \bar{x} = b_1 \beta^2 (x_{t-1} - \bar{x}) + b_2 (y_t - \bar{y}) + u_t, \\ y_t - \bar{y} = \rho (y_{t-1} - \bar{y}) + \varepsilon_t. \end{cases} \tag{C.1}$$

That is,

$$\begin{cases} x_t - \bar{x} = b_1 \beta^2 (x_{t-1} - \bar{x}) + b_2 \rho (y_{t-1} - \bar{y}) + b_2 \varepsilon_t + u_t, \\ y_t - \bar{y} = \rho (y_{t-1} - \bar{y}) + \varepsilon_t. \end{cases} \tag{C.2}$$

$$\begin{aligned}
&E[(x_t - \bar{x})(x_{t-1} - \bar{x})] \\
&= E\left[b_1 \beta^2 (x_{t-1} - \bar{x})^2 + b_2 \rho (x_{t-1} - \bar{x})(y_{t-1} - \bar{y}) + b_2 (x_{t-1} - \bar{x}) \varepsilon_t + (x_{t-1} - \bar{x}) u_t\right] \\
&= b_1 \beta^2 \text{Var}(x_t) + b_2 \rho E[(x_{t-1} - \bar{x})(y_{t-1} - \bar{y})] + b_2 E[(x_{t-1} - \bar{x}) \varepsilon_t] + (x_{t-1} - \bar{x}) u_t \\
&= b_1 \beta^2 \text{Var}(x_t) + b_2 \rho E[(x_{t-1} - \bar{x})(y_{t-1} - \bar{y})] \\
&= b_1 \beta^2 \text{Var}(x_t) + b_2 \rho E[(x_t - \bar{x})(y_t - \bar{y})].
\end{aligned} \tag{C.3}$$



$$\begin{aligned}
& \text{Var}(x_t) \\
&= E(x_t - \bar{x})^2 \\
&= E\left[b_1\beta^2(x_t - \bar{x})(x_{t-1} - \bar{x}) + b_2\rho(x_t - \bar{x})(y_{t-1} - \bar{y}) + b_2(x_t - \bar{x})\varepsilon_t + (x_t - \bar{x})u_t\right] \\
&= b_1\beta^2 E[(x_t - \bar{x})(x_{t-1} - \bar{x})] + b_2\rho E[(x_t - \bar{x})(y_{t-1} - \bar{y})] + b_2(x_t - \bar{x})\varepsilon_t + (x_t - \bar{x})u_t \\
&= b_1\beta^2 E[(x_t - \bar{x})(x_{t-1} - \bar{x})] + b_2\rho E[(x_t - \bar{x})(y_{t-1} - \bar{y})] + b_2^2\sigma_\varepsilon^2 + \sigma_u^2, \tag{C.4}
\end{aligned}$$

where the last equation is based on the fact that  $E[(x_t - \bar{x})\varepsilon_t] = E\left[b_1\beta^2(x_{t-1} - \bar{x})\varepsilon_t + b_2\rho(y_{t-1} - \bar{y})\varepsilon_t + b_2\varepsilon_t^2 + u_t\varepsilon_t\right] = b_2\sigma_\varepsilon^2$  and  $E[(x_t - \bar{x})u_t] = E\left[b_1\beta^2(x_{t-1} - \bar{x})u_t + b_2\rho(y_{t-1} - \bar{y})u_t + b_2\varepsilon_t u_t + u_t^2\right] = \sigma_u^2$ .

Based on (C.3) and (C.4),

$$\begin{aligned}
\text{Var}(x_t) &= b_1\beta^2 E[(x_t - \bar{x})(x_{t-1} - \bar{x})] + b_2\rho E[(x_t - \bar{x})(y_{t-1} - \bar{y})] + b_2^2\sigma_\varepsilon^2 + \sigma_u^2 \\
&= b_1\beta^2 \left[ b_1\beta^2 \text{Var}(x_t) + b_2\rho E[(x_t - \bar{x})(y_t - \bar{y})] \right] + b_2\rho E[(x_t - \bar{x})(y_{t-1} - \bar{y})] + b_2^2\sigma_\varepsilon^2 + \sigma_u^2 \\
&= b_1^2\beta^4 \text{Var}(x_t) + b_1\beta^2 b_2\rho E[(x_t - \bar{x})(y_t - \bar{y})] + b_2\rho E[(x_t - \bar{x})(y_{t-1} - \bar{y})] + b_2^2\sigma_\varepsilon^2 + \sigma_u^2.
\end{aligned}$$

That is,

$$\text{Var}(x_t) = \frac{b_1\beta^2 b_2\rho E[(x_t - \bar{x})(y_t - \bar{y})] + b_2\rho E[(x_t - \bar{x})(y_{t-1} - \bar{y})] + b_2^2\sigma_\varepsilon^2 + \sigma_u^2}{1 - b_1^2\beta^4}. \tag{C.5}$$

Thus, in order to obtain  $E[(x_t - \bar{x})(x_{t-1} - \bar{x})]$  and  $\text{Var}(x_t)$ , we need calculate  $E[(x_t - \bar{x})(y_t - \bar{y})]$  and  $E[(x_t - \bar{x})(y_{t-1} - \bar{y})]$ .

$$\begin{aligned}
E[(x_t - \bar{x})(y_t - \bar{y})] &= E\left[b_1\beta^2(x_{t-1} - \bar{x})(y_t - \bar{y}) + b_2\rho(y_{t-1} - \bar{y})(y_t - \bar{y}) + b_2\varepsilon_t(y_t - \bar{y}) + u_t(y_t - \bar{y})\right] \\
&= b_1\beta^2 E\{(x_{t-1} - \bar{x})[\rho(y_{t-1} - \bar{y}) + \varepsilon_t]\} + b_2\rho E[(y_{t-1} - \bar{y})(y_t - \bar{y})] \\
&\quad + b_2 E\{\varepsilon_t[\rho(y_{t-1} - \bar{y}) + \varepsilon_t]\} + E[u_t(y_t - \bar{y})] \\
&= b_1\beta^2 \rho E[(x_{t-1} - \bar{x})(y_{t-1} - \bar{y})] + 0 + \frac{b_2\rho^2\sigma_\varepsilon^2}{(1 - \rho^2)} + b_2\sigma_\varepsilon^2 + 0.
\end{aligned}$$

Thus

$$E[(x_t - \bar{x})(y_t - \bar{y})] = \frac{b_2\sigma_\varepsilon^2}{(1 - \rho^2)(1 - b_1\beta^2\rho)}. \tag{C.6}$$

Hence based on (C.6),

$$\begin{aligned}
& E[(x_t - \bar{x})(y_{t-1} - \bar{y})] \\
&= E\left[b_1\beta^2(x_{t-1} - \bar{x})(y_{t-1} - \bar{y}) + b_2\rho(y_{t-1} - \bar{y})^2 + b_2\varepsilon_t(y_{t-1} - \bar{y}) + u_t(y_{t-1} - \bar{y})\right] \\
&= b_1\beta^2 E[(x_{t-1} - \bar{x})(y_{t-1} - \bar{y})] + b_2\rho E(y_{t-1} - \bar{y})^2 + 0 + 0 \\
&= b_1\beta^2 \cdot \frac{b_2\sigma_\varepsilon^2}{(1-\rho^2)(1-b_1\beta^2\rho)} + b_2\rho \cdot \frac{\sigma_\varepsilon^2}{1-\rho^2} \\
&= \frac{b_2\sigma_\varepsilon^2}{(1-\rho^2)} \left[ \frac{b_1\beta^2}{1-b_1\beta^2\rho} + \rho \right] \\
&= \frac{b_2\sigma_\varepsilon^2}{(1-\rho^2)} \cdot \frac{b_1\beta^2(1-\rho^2) + \rho}{1-b_1\beta^2\rho} \\
&= \frac{b_2\sigma_\varepsilon^2}{(1-b_1\beta^2\rho)} \left[ b_1\beta^2 + \frac{\rho}{1-\rho^2} \right]. \tag{C.7}
\end{aligned}$$

Therefore, based on (C.5), (C.6) and (C.7),

$$\begin{aligned}
Var(x_t) &= \frac{1}{1-b_1^2\beta^4} \left\{ b_1\beta^2 b_2\rho E[(x_t - \bar{x})(y_t - \bar{y})] + b_2\rho E[(x_t - \bar{x})(y_{t-1} - \bar{y})] + b_2^2\sigma_\varepsilon^2 + \sigma_u^2 \right\} \\
&= \frac{1}{1-b_1^2\beta^4} \left\{ \frac{b_1\beta^2 b_2^2\rho\sigma_\varepsilon^2}{(1-\rho^2)(1-b_1\beta^2\rho)} + \frac{b_2^2\rho\sigma_\varepsilon^2}{(1-b_1\beta^2\rho)} \left[ b_1\beta^2 + \frac{\rho}{1-\rho^2} \right] + b_2^2\sigma_\varepsilon^2 + \sigma_u^2 \right\} \\
&= \frac{\sigma_\varepsilon^2}{1-b_1^2\beta^4} \left\{ \frac{b_2^2\rho [b_1\beta^2(2-\rho^2) + \rho]}{(1-\rho^2)(1-b_1\beta^2\rho)} + b_2^2 + \frac{\sigma_u^2}{\sigma_\varepsilon^2} \right\} \\
&= \frac{\sigma_\varepsilon^2}{1-b_1^2\beta^4} \left\{ \frac{b_2^2(b_1\beta^2\rho + 1)}{(1-\rho^2)(1-b_1\beta^2\rho)} + \frac{\sigma_u^2}{\sigma_\varepsilon^2} \right\}. \tag{C.8}
\end{aligned}$$

According to (C.3),

$$\begin{aligned}
E[(x_t - \bar{x})(x_{t-1} - \bar{x})] &= b_1\beta^2 Var(x_t) + b_2\rho E[(x_t - \bar{x})(y_t - \bar{y})] \\
&= b_1\beta^2 Var(x_t) + \frac{b_2^2\rho\sigma_\varepsilon^2}{(1-\rho^2)(1-b_1\beta^2\rho)}. \tag{C.9}
\end{aligned}$$

Thus, the correlation coefficient  $Corr(x_t, x_{t-1})$  satisfies

$$\begin{aligned}
Corr(x_t, x_{t-1}) &= E[(x_t - \bar{x})(x_{t-1} - \bar{x})] / Var(x_t) \\
&= b_1\beta^2 + \frac{\frac{b_2^2\rho\sigma_\varepsilon^2}{(1-\rho^2)(1-b_1\beta^2\rho)}}{\frac{\sigma_\varepsilon^2}{1-b_1^2\beta^4} \left\{ \frac{b_2^2(b_1\beta^2\rho + 1)}{(1-\rho^2)(1-b_1\beta^2\rho)} + \frac{\sigma_u^2}{\sigma_\varepsilon^2} \right\}} \\
&= b_1\beta^2 + \frac{b_2^2\rho(1-b_1^2\beta^4)}{b_2^2(b_1\beta^2\rho + 1) + (1-\rho^2)(1-b_1\beta^2\rho) \frac{\sigma_u^2}{\sigma_\varepsilon^2}} \\
&= \frac{b_2^2(b_1\beta^2 + \rho) + b_1\beta^2(1-\rho^2)(1-b_1\beta^2\rho) \frac{\sigma_u^2}{\sigma_\varepsilon^2}}{b_2^2(b_1\beta^2\rho + 1) + (1-\rho^2)(1-b_1\beta^2\rho) \frac{\sigma_u^2}{\sigma_\varepsilon^2}}.
\end{aligned}$$

## D Proof of uniqueness of $\beta^*$ (Proposition 2)

Using the first-order autocorrelation  $F(\beta)$  in (3.9), it can be calculated that

$$F''(\beta) = \frac{2b_1(1-\rho^2)}{(\rho b_1\beta^2+1)^2} - \frac{8\rho b_1^2\beta^2(1-\rho^2)}{(\rho b_1\beta^2+1)^3} = \frac{2b_1(1-\rho^2)(1-3\rho b_1\beta^2)}{(\rho b_1\beta^2+1)^3}.$$

In the case  $b_1 > 0$ , if  $\rho \leq \frac{1}{3b_1}$ , then  $1 - 3\rho b_1\beta^2 \geq 1 - \beta^2 > 0$ . Thus  $G''(\beta) = F''(\beta) > 0$ . Note that  $G(0) > 0$ ,  $G'(0) = -1 < 0$  and  $G(1) < 0$ ,  $G'(1) = \frac{2b_1(1-\rho^2)}{(b_1\rho+1)^2} - 1$ . Hence if  $G'(1) \leq 0$ , then  $G'(\beta^*) < 0$ . If  $G'(1) > 0$ , then there exists a minimal point  $\beta_1$  such that  $G'(\beta_1) = 0$ . Moreover, since  $G(1) < 0$ , then  $G(\beta_1) < 0$  (otherwise,  $G(1) \geq G(\beta_1) \geq 0$ , which is contradictory to  $G(1) < 0$ ). Hence  $\beta^* (\in (0, \beta_1))$  is unique and  $G'(\beta^*) < 0$ , hence  $0 < F'(\beta^*) < 1$ .

If  $\rho > \frac{1}{3b_1}$ , then  $G''(\beta)|_{\beta=\sqrt{1/(3b_1\rho)}} = F''(\beta)|_{\beta=\sqrt{1/(3b_1\rho)}} = 0$  and  $G'(\beta)|_{\beta=\sqrt{1/(3b_1\rho)}}$  is maximal. Thus in the case that  $\rho > \frac{1}{3b_1}$ ,

$$\begin{aligned} G'(\beta) &= F'(\beta) - 1 \\ &= \frac{2b_1\beta(1-\rho^2)}{(\rho b_1\beta^2+1)^2} - 1 \\ &\leq \frac{2b_1\frac{1}{\sqrt{3b_1\rho}}(1-\rho^2)}{(b_1\rho\frac{1}{3b_1\rho}+1)^2} - 1 \\ &= \frac{3\sqrt{3}\sqrt{b_1}(1-\rho^2)}{8\sqrt{\rho}} - 1 \\ &< \frac{3\sqrt{3}\sqrt{b_1}(1-\frac{1}{9b_1^2})}{8\frac{1}{\sqrt{3b_1}}} - 1 \\ &= \frac{-(1-b_1)(1+9b_1)}{8b_1} < 0. \end{aligned}$$

Furthermore, it is easy to see that  $F(\beta)$  only depends on  $\beta^2$  and  $F(\beta) > 0$ . Hence  $G(\beta) = F(\beta) - \beta > 0$  for  $\beta \in [-1, 0]$ . So for  $b_1 > 0$  there is a unique  $\beta^*$  satisfying  $0 < F'(\beta^*) < 1$ .

In the case  $b_1 \leq 0$ , since  $F'(\beta) = \frac{2b_1\beta(1-\rho^2)}{(\rho b_1\beta^2+1)^2} \leq 0$  for  $\beta \in [0, 1]$ , then  $G'(\beta) = F'(\beta) - 1 < 0$ . Thus  $G(\beta)$  is monotonically decreasing and hence  $\beta^*$  is unique within the interval  $(0, 1)$  satisfying  $F'(\beta^*) < 1$ . Moreover, for  $\beta \in [-1, 0)$ ,  $G''(\beta) = F''(\beta) \leq 0$ . It is easy to see further  $G(-1) = F(-1) + 1 > 0$ ,  $G(0) > 0$  and  $G'(0) = -1$ . For  $\beta \in [-1, 0)$ ,  $G(\beta)$  is decreasing or first increasing and then decreasing. In any case there is no solution for  $G(\beta) = 0$  within the interval  $[-1, 0]$ . So for  $b_1 \leq 0$   $\beta^*$  is unique satisfying  $\beta^* \in (0, 1)$  and  $F'(\beta^*) < 1$ .

Therefore  $\beta^*$  is unique, which is within the interval  $(0, 1)$  and satisfies  $F'(\beta^*) < 1$ .

## E Proof of Proposition 3

$$\begin{aligned}
& \text{A straightforward computation yields } G(\beta) = F(\beta) - \beta = \\
&= \frac{b_2^2(b_1\beta^2 + \rho) + b_1\beta^2(1 - \rho^2)(1 - b_1\beta^2\rho)R_v}{b_2^2(b_1\beta^2\rho + 1) + (1 - \rho^2)(1 - b_1\beta^2\rho)R_v} - \beta \\
&= \frac{-b_1^2\rho(1 - \rho^2)R_v\beta^4 - b_1\rho[b_2^2 - (1 - \rho^2)R_v]\beta^3 + b_1[b_2^2 + (1 - \rho^2)R_v]\beta^2 - [b_2^2 + (1 - \rho^2)R_v]\beta + b_2^2\rho}{b_1\rho[b_2^2 - (1 - \rho^2)R_v]\beta^2 + [b_2^2 + (1 - \rho^2)R_v]},
\end{aligned}$$

where  $R_v = \frac{\sigma_u^2}{\sigma_\varepsilon^2}$ .

If  $b_1\rho[b_2^2 - (1 - \rho^2)R_v] \geq 0$ , then  $b_1\rho[b_2^2 - (1 - \rho^2)R_v]\beta^2 + [b_2^2 + (1 - \rho^2)R_v] > 0$  for any  $\beta$ . Thus  $G(\beta) = 0$  is equivalent to the 4-th order polynomial equation  $\bar{G}(\beta) = -b_1^2\rho(1 - \rho^2)R_v\beta^4 - b_1\rho[b_2^2 - (1 - \rho^2)R_v]\beta^3 + b_1[b_2^2 + (1 - \rho^2)R_v]\beta^2 - [b_2^2 + (1 - \rho^2)R_v]\beta + b_2^2\rho = 0$ . Hence, there are at most four real solutions. Since  $\bar{G}(-1) = b_2^2(1 + b_1)(1 + \rho) + (1 + b_1)(1 - b_1\rho)(1 - \rho^2)R_v > 0$  and  $\bar{G}(\beta) \rightarrow -\infty$  as  $\beta \rightarrow -\infty$  due to negative coefficient of  $\beta^4$ , there exists one solution within the interval  $(-\infty, -1)$ . So there are at most three solutions for  $\bar{G}(\beta) = 0$ , i.e.  $G(\beta) = 0$ , within the interval  $[-1, 1]$ .

If  $b_1\rho[b_2^2 - (1 - \rho^2)R_v] < 0$ , then there are two singularities of  $G(\beta)$ , i.e., two solutions for  $b_1\rho[b_2^2 - (1 - \rho^2)R_v]\beta^2 + [b_2^2 + (1 - \rho^2)R_v] = 0$ , given by  $\beta_{1,2} = \pm \sqrt{\frac{b_2^2 + (1 - \rho^2)R_v}{b_1\rho[(1 - \rho^2)R_v - b_2^2]}}$ . It is easy to see that  $|\beta_{1,2}| > 1$ . For  $\beta \in (\beta_2, \beta_1)$ , if  $\beta \rightarrow \beta_2$ , then  $\bar{G}(\beta) \rightarrow b_2^2\rho(1 - b_1^2\beta_2^4)$ . Note that  $b_2^2\rho(1 - b_1^2\beta_2^4) < 0$ . Thus  $G(\beta) \rightarrow -\infty$  as  $\beta \rightarrow \beta_2$  for  $\beta \in (\beta_2, \beta_1)$ . As discussed above,  $G(-1) > 0$ . Hence there exists one solution for  $G(\beta) = 0$  within  $(\beta_2, -1)$ . Furthermore, in the interval  $(\beta_2, \beta_1) \supset [-1, 1]$ ,  $G(\beta) = 0$  is equivalent to  $\bar{G}(\beta) = 0$ . So there are at most three solutions for  $G(\beta) = 0$  within the interval  $[-1, 1]$ .

Therefore there are at most three zeros for  $G(\beta) = F(\beta) - \beta$  within the interval  $[-1, 1]$ . That is, at most three first-order stochastic consistent expectations equilibrium (SCEE)  $(\alpha^*, \beta^*)$  coexist.

## F Proof of Proposition 4

Set  $\gamma_t = (1 + t)^{-1}$ . For the state dynamics equations in (3.12) and (2.8)<sup>19</sup>, since all functions are smooth, the SAC-learning rule satisfies the conditions (A.1-A.3) of Section 6.2.1 in Evans and Honkapohja (2001, p.124).

In order to check the conditions (B.1-B.2) of Section 6.2.1 in Evans and Honkapohja

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<sup>19</sup>For convenience of theoretical analysis, one can set  $S_{t-1} = R_t$ .

(2001, p.125), we rewrite the system in matrix form by

$$X_t = A(\theta_{t-1})X_{t-1} + B(\theta_{t-1})W_t,$$

where  $\theta'_t = (\alpha_t, \beta_t, R_t)$ ,  $X'_t = (1, x_t, x_{t-1}, y_t)$  and  $W'_t = (1, u_t, \varepsilon_t)$ ,

$$A(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_0 + b_1\alpha(1 - \beta^2) + b_2a & b_1\beta^2 & 0 & b_2\rho \\ 0 & 1 & 0 & 0 \\ a & 0 & 0 & \rho \end{pmatrix},$$

$$B(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b_2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As shown in Evans and Honkapohja (2001, p.186),  $A(\theta)$  and  $B(\theta)$  clearly satisfy the Lipschitz conditions and  $B$  is bounded. Since  $u_t$  and  $\varepsilon_t$  are assumed to have bounded moments, condition (B.1) is satisfied. Furthermore, the eigenvalues of matrix  $A(\theta)$  are 0 (double),  $\rho$  and  $b_1\beta^2$ . According to the assumption  $|\beta| \leq 1$ ,  $|b_1| < 1$  and  $0 < \rho < 1$ , all eigenvalues of  $A(\theta)$  are less than 1 in absolute value. Then it follows that there is a compact neighborhood including the SCEE solution  $(\alpha^*, \beta^*)$  on which the condition that  $|A(\theta)|$  is bounded strictly below 1 is satisfied.

Thus the technical conditions for Section 6.2.1 of Chapter 6 in Evans and Honkapohja (2001) are satisfied. Moreover, since  $x_t$  is stationary under the condition  $|\beta| \leq 1$ ,  $|b_1| < 1$  and  $0 < \rho < 1$ , then the limits

$$\sigma^2 := \lim_{t \rightarrow \infty} E(x_t - \alpha)^2, \quad \sigma_{xx-1}^2 := \lim_{t \rightarrow \infty} E(x_t - \alpha)(x_{t-1} - \alpha)$$

exist and are finite. Hence according to Section 6.2.1 of Chapter 6 in Evans and Honkapohja (2001, p.126), the associated ODE is

$$\begin{cases} \frac{d\alpha}{d\tau} = \bar{x}(\alpha, \beta) - \alpha, \\ \frac{d\beta}{d\tau} = R^{-1}[\sigma_{xx-1}^2 - \beta\sigma^2], \\ \frac{dR}{d\tau} = \sigma^2 - R. \end{cases}$$

That is,

$$\begin{cases} \frac{d\alpha}{d\tau} = \frac{b_0 + b_1\alpha(1 - \beta^2) + b_2\bar{y}}{1 - b_1\beta^2} - \alpha = \frac{b_0 + \alpha(b_1 - 1) + b_2\bar{y}}{1 - b_1\beta^2}, \\ \frac{d\beta}{d\tau} = F(\beta) - \beta = \frac{b_2^2(b_1\beta^2 + \rho) + b_1\beta^2(1 - \rho^2)(1 - b_1\beta^2\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}}{b_2^2(b_1\beta^2\rho + 1) + (1 - \rho^2)(1 - b_1\beta^2\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}} - \beta. \end{cases} \quad (\text{F.1})$$

Furthermore,

$$JG(\alpha^*, \beta^*) = \begin{pmatrix} \frac{-(1-b_1)}{1-b_1\beta^2} & 0 \\ 0 & F'(\beta^*) - 1 \end{pmatrix}.$$

Hence a SCEE corresponds to a fixed point of the ODE (F.1). Furthermore, the SAC-learning  $(\alpha_t, \beta_t)$  converges to the stable SCEE  $(\alpha^*, \beta^*)$  as time  $t$  tends to  $\infty$ . In the special case  $\sigma_u = 0$  or  $b_2 \rightarrow \infty$ , based on Proposition 2 and Appendix D, the SCEE  $(\alpha^*, \beta^*)$  is unique and stable with  $F'(\beta^*) - 1 < 0$ . Thus the SAC-learning  $(\alpha_t, \beta_t)$  converges to the unique (locally) stable SCEE  $(\alpha^*, \beta^*)$  as time  $t$  tends to  $\infty$ .

## G Dependence of F on parameters

In this appendix we show that the partial derivatives of the first-order autocorrelation coefficient  $F(\beta)$  of the implied ALM (4.25) satisfy (4.26).

Based on (4.25),  $F(\beta) = \delta\beta^2 + \frac{\rho(1-\delta^2\beta^4)}{(\delta\beta^2\rho+1)+(1-\rho^2)(1-\delta\beta^2\rho)\frac{1}{\gamma^2}\frac{\sigma_u^2}{\sigma_\varepsilon^2}} > 0$ . As shown in the first paragraph in the Subsection 4.2.4,

$$\frac{\partial F}{\partial \gamma} = \frac{2\rho(1 - \delta^2\beta^4)(1 - \rho^2)(1 - \delta\beta^2\rho)\frac{\sigma_u^2}{\sigma_\varepsilon^2}}{\gamma^3[(\delta\beta^2\rho + 1) + (1 - \rho^2)(1 - \delta\beta^2\rho)\frac{1}{\gamma^2}\frac{\sigma_u^2}{\sigma_\varepsilon^2}]^2} > 0.$$

Denote  $\frac{\sigma_u^2}{\sigma_\varepsilon^2}$  by  $\xi$ .

$$\frac{\partial F}{\partial \xi} = \frac{-\rho(1 - \delta^2\beta^4)(1 - \rho^2)(1 - \delta\beta^2\rho)\frac{1}{\gamma^2}}{[(\delta\beta^2\rho + 1) + (1 - \rho^2)(1 - \delta\beta^2\rho)\frac{1}{\gamma^2}\frac{\sigma_u^2}{\sigma_\varepsilon^2}]^2} < 0.$$

Now consider the parameter  $\rho$ . It can be calculated that

$$\frac{\partial F}{\partial \rho} = \frac{(1 - \delta^2\beta^4)\left[1 + (1 + \rho^2 - 2\delta\beta^2\rho^3)\frac{1}{\gamma^2}\frac{\sigma_u^2}{\sigma_\varepsilon^2}\right]}{\left[(\delta\beta^2\rho + 1) + (1 - \rho^2)(1 - \delta\beta^2\rho)\frac{1}{\gamma^2}\frac{\sigma_u^2}{\sigma_\varepsilon^2}\right]^2}.$$

In the following we will show that

$$1 + \rho^2 - 2\delta\beta^2\rho^3 > 0$$

for any given  $\delta \in (0, 1)$ ,  $\rho \in [0, 1)$  and  $\beta \in [-1, 1]$ . Thus  $\frac{\partial F}{\partial \rho} > 0$ .

Let  $h(\rho)$  denote the 3<sup>rd</sup> order polynomial  $-2\delta\beta^2\rho^3 + \rho^2 + 1$ . It is easy to see that  $h(0) = 1 > 0$ ,  $h(-\infty) \rightarrow +\infty$ ,  $h(+\infty) \rightarrow -\infty$ . Moreover  $h'(\rho) = 2\rho(1 - 3\delta\beta^2\rho)$ . That is, there are two values  $\rho = 0, \frac{1}{3\delta\beta^2}$  such that  $h'(0) = 0$  and  $h'(\frac{1}{3\delta\beta^2}) = 0$ . Moreover,  $h''(0) = 2 > 0$  and  $h''(\frac{1}{3\delta\beta^2}) = -2 < 0$ . Hence within the interval  $[0, 1]$ ,  $h(\rho)$  is monotonically increasing or first increasing and then decreasing. In any case since  $h(0) = 1 > 0$  and  $h(1) = 2(1 - \delta\beta^2) > 0$ , then  $h(\rho) > 0$  for any  $\rho \in [0, 1]$ . Hence  $\frac{\partial F}{\partial \rho} > 0$ .

Finally, for  $\delta$ , it can be calculated that

$$\frac{\partial F}{\partial \delta} = \frac{\beta^2 [\eta(\eta - 1)\rho^2\delta^2\beta^4 - 2\eta(1 + \eta)\rho\delta\beta^2 + (1 + \eta)^2 + \rho^2(\eta - 1)]}{[(\delta\beta^2\rho + 1) + \eta(1 - \delta\beta^2\rho)]^2},$$

where  $\eta = (1 - \rho^2)\frac{1}{\gamma^2}\frac{\sigma_u^2}{\sigma_\varepsilon^2} \geq 0$ . If  $\eta = 0$ , it is easy to get  $\frac{\partial F}{\partial \delta} = \frac{\beta^2[1 - \rho^2]}{(\delta\beta^2\rho + 1)^2} > 0$  for  $\beta \in (0, 1)$  and  $\rho \in [0, 1)$ . In the following we assume  $\eta > 0$ . Let  $g(\delta)$  denote the 2<sup>nd</sup> order polynomial  $\eta(\eta - 1)\rho^2\delta^2\beta^4 - 2\eta(1 + \eta)\rho\delta\beta^2 + (1 + \eta)^2 + \rho^2(\eta - 1)$ . We will show  $g(\delta) > 0$  for any  $\delta \in [0, 1)$ . If  $\eta = 1$ , then  $g(\delta) = 4(1 - \delta\beta^2\rho) > 0$  for  $\rho \in [0, 1)$ ,  $\beta \in (0, 1)$  and  $\delta \in [0, 1)$ . If  $\eta > 1$ , then the symmetric axis of the 2<sup>nd</sup> order polynomial  $g(\delta)$  is  $\delta = \frac{\eta+1}{(\eta-1)\rho\beta^2} > 1$  and the coefficient  $\eta(\eta - 1)\rho^2\beta^4 > 0$ . If  $\eta < 1$ , then the symmetric axis of the 2<sup>nd</sup> order polynomial  $g(\delta)$  is  $\delta = \frac{\eta+1}{(\eta-1)\rho\beta^2} < 0$  and the coefficient  $\eta(\eta - 1)\rho^2\beta^4 < 0$ . Hence no matter if  $\eta > 1$  or  $\eta < 1$ ,  $g(\delta)$  decreases within the interval  $[0, 1)$ . That is, if  $g(1) \geq 0$ , then  $g(\delta) > 0$  for any  $\delta \in [0, 1)$ .

Note that  $g(1) = \eta(\eta - 1)\rho^2\beta^4 - 2\eta(1 + \eta)\rho\beta^2 + (1 + \eta)^2 + \rho^2(\eta - 1) := \tilde{g}(\beta^2)$ . This is a 2<sup>nd</sup> order polynomial with respect to  $\beta^2$ . Similarly since the symmetric axis of the 2<sup>nd</sup> order polynomial  $\tilde{g}(\beta^2)$  is  $\beta^2 = \frac{\eta+1}{(\eta-1)\rho} > 1$  for  $\eta > 1$  and  $\beta^2 = \frac{\eta+1}{(\eta-1)\rho} < 0$  for  $\eta < 1$ , then  $\tilde{g}(\beta^2)$  decreases within the interval  $[0, 1]$  no matter if  $\eta > 1$  or  $\eta < 1$ . Thus we just need to prove  $\tilde{g}(1) \geq 0$ . Note that  $\tilde{g}(1) = \eta(\eta - 1)\rho^2 - 2\eta(1 + \eta)\rho + (1 + \eta)^2 + \rho^2(\eta - 1) = (\eta^2 - 1)\rho^2 - 2\eta(1 + \eta)\rho + (1 + \eta)^2 := \hat{g}(\rho)$ . Similarly since the symmetric axis of the 2<sup>nd</sup> order polynomial  $\hat{g}(\rho)$  is  $\rho = \frac{\eta+1}{\eta-1} > 1$  for  $\eta > 1$  and  $\rho = \frac{\eta+1}{\eta-1} < 0$  for  $\eta < 1$ . Hence  $\hat{g}(\rho)$  decreases within the interval  $[0, 1)$  no matter if  $\eta > 1$  or  $\eta < 1$ . That is, if  $\tilde{g}(1) \geq 0$ , then  $\hat{g}(\rho) > 0$  for any  $\rho \in [0, 1)$ . In fact,  $\tilde{g}(1) = (\eta^2 - 1) - 2\eta(1 + \eta) + (1 + \eta)^2 = 0$ . Thus based on the above analysis, for any  $\rho \in [0, 1)$ , we have  $\tilde{g}(1) = \hat{g}(\rho) > 0$ , and hence  $g(1) = \tilde{g}(\beta^2) > \tilde{g}(1) > 0$  for any  $\beta^2 \in (0, 1)$ . That is, for any  $\delta \in [0, 1)$ ,  $g(\delta) > 0$ . Therefore  $\frac{\partial F}{\partial \delta} > 0$ .

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