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# Consistency, Population Solidarity, and Egalitarian Solutions for TU-games

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**Abstract**

A (point-valued) solution for cooperative games with transferable utility, or simply TU-games, assigns a payoff vector to every TU-game. In this paper we discuss two classes of equal surplus sharing solutions, one consisting of all convex combinations of the equal division solution and the CIS-value, and its dual class consisting of all convex combinations of the equal division solution and the ENSC-value. We provide several characterizations using either population solidarity or a reduced game consistency in addition to other standard properties.



Keywords: TU-game, equal division solution, CIS-value, ENSC-value, population solidarity, consistency

JEL Classification: C71

# 1 Introduction

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, shortly TU-game. A (point-valued) solution on a class of TU-games assigns a payoff vector to every game in the class.

Recently, egalitarian or equal surplus sharing solutions gained attention in the literature. Three well-known equal surplus sharing solutions are the equal division solution which allocates the worth of the 'grand coalition' equally among all players, the *Center-of-gravity of the Imputation-Set value*, shortly denoted by *CIS-value* (see Driessen and Funaki (1991)) which first gives every agent its own singleton worth and distributes the remainder equally among all players, and the *Egalitarian Non-Separable Contribution value* (also known as Equal Allocation of Non-Separable costs or EANS-value), shortly denoted by *ENSC-value* being the dual of the CIS-value. In van den Brink and Funaki (2009) the class of all convex combinations of these three solutions is studied.

Chun and Park (2012) characterize the CIS-value by efficiency, covariance and *population solidarity*, the last property requiring that upon an arrival of a new player all the original players should be affected in the same direction, all weakly gain or all weakly lose. It turns out that all convex combinations of the equal division solution and the CIS-value satisfy population solidarity. We extend the characterization of the CIS-value given by Chun and Park (2012) to this class of solutions.

Besides axiomatizing all convex combinations of the equal division solution and the CIS-value using population solidarity, we reconsider the axiomatizations using consistency provided by van den Brink and Funaki (2009). Whereas they axiomatized the class of all convex combinations of the equal division solution, the CIS-value and the ENSC-value using a parametrized standardness for two-player games and a parametrized consistency, the convex combinations of the equal division solution and the CIS-value have a non-parametrized consistency in common which we use in an axiomatization together with $\alpha$-standardness for two-player games. For a fraction $\alpha \in [0,1]$, $\alpha$-standardness for two-player games states that for two-player games each player first receives a fraction $\alpha$ of its singleton worth, and what remains of the worth of the 'grand coalition' is split equally among the two players. In van den Brink and Funaki (2009) it is shown that any solution that satisfies efficiency, symmetry and linearity on the class of two-player games satisfies $\alpha$-standardness for two-player games for some $\alpha \in [0,1]$. Since linearity is only used for two-player games, we prefer to have a characterization of $\alpha$-standardness without linearity. In their characterization of the CIS-value, Chun and Park (2012) use covariance. However, the CIS-value is the only covariant solution in the class considered here. Therefore, we consider a weak covariance which requires that the payoffs of all players change the same if we add a constant times the sum of the unanimity games of all singletons. Although this can also be seen as a weakening of the fairness axiom in van den Brink (2001), we use another weak fairness axiom requiring that the payoffs of all players change by the same amount when we only change the worth of the 'grand coalition'. We show that any solution on the class of two-player games that satisfies these two properties together with efficiency, homogeneity and weak individual rationality (requiring that every player in any weakly essential game earns at least the minimum of its singleton worth and the per capita payoff), satisfies $\alpha$-standardnesss for some $\alpha \in [0,1]$. Requiring this lower bound on the payoffs of a player for any game (we call this the boundary condition), the equal division solution is the only solution satisfying this property together with efficiency and weak fairness.

To select a particular solution from the convex combinations of the equal division solution and the CIS-value, we use $\alpha$-individual rationality requiring that for appropriate games a player always earns at least a fraction $\alpha \in [0,1]$ from its singleton worth. For specific values of $\alpha$ (1, respectively 0), this axiom yields the usual individual rationality or nonnegativity. This axiom gives $\alpha$ an interpretation as some 'wealth taxation' parameter. Without taxation, every player can guarantee itself its own singleton worth. This singleton worth can be seen as the individual wealth of the single player which it can earn on its own without any cooperation with other players. Then individual rationality means that a player gets at least its own wealth. However, if there is some taxation on the wealth of players, then they cannot guarantee themselves their singleton worth, but only a fraction $\alpha$ (if $(1-\alpha)$ is the tax rate). Of course, these 'after tax' individual wealths can only be satisfied if the worth of the 'grand coalition' is large enough.

Finally, we introduce some dual version of population solidarity that is satisfied by the ENSC-value, the equal division solution and all their convex combinations.

The paper is organized as follows. Section 2 discusses some preliminaries on TU-games and solutions. In Section 3, we consider two-player games and characterize $\alpha$-standardness for two-player games. In Section 4, we extend these definitions to $n$-player games using a consistency. In Section 5, we give an axiomatic characterization using population solidarity. Finally, in Section 6 we consider the dual class consisting of all convex combinations of the equal division solution and the ENSC-value.

# 2 Preliminaries

A *cooperative game with transferable utility*, shortly TU-game, is a pair $(N, v)$, where $N \subset \mathbb{N}$ is a finite set of players with $|N| \geq 2$, and $v \colon 2^N \to \mathbb{R}$ is a characteristic function on $N$ such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, $v(S)$ is called the *worth* of coalition $S$. This is what the members of coalition $S$ can obtain by agreeing to cooperate. We denote the class of all TU-games by $\mathcal{G}$. For a fixed player set $N$, we denote the class of all

TU-games $(N, v)$ by $\mathcal{G}^N$.

A *payoff vector* of game $(N, v)$ is an $|N|$-dimensional real vector $x \in \mathbb{R}^N$, which represents a distribution of the payoffs that can be earned by cooperation over the individual players. A (point-valued) *solution* on a class of TU-games $\mathcal{C} \subseteq \mathcal{G}$ is a function $\psi$ which assigns a payoff vector $\psi(N, v) \in \mathbb{R}^N$ to every TU-game $(N, v) \in \mathcal{C}$. If a solution assigns to every game a payoff vector that exactly distributes the worth of the 'grand coalition' $N$ then the solution is *efficient*[1]. In this paper we discuss two classes of solutions for TU-games that all have some egalitarian flavour.

The *equal division solution* $ED$ distributes the worth of the 'grand coalition' equally among all players, i.e., for all $(N, v) \in \mathcal{G}$ and $i \in N$,

$$ED_i(N, v) = \frac{1}{|N|} v(N).$$

Instead, the Center-of-gravity of the Imputation-Set value *CIS*, shortly called CIS-value, first assigns to every player its individual worth, and distributes the remainder of the worth of the 'grand coalition' $N$ equally among all players, i.e., for all $(N, v) \in \mathcal{G}$ and $i \in N$,

$$CIS_i(N, v) = v(\{i\}) + \frac{1}{|N|}\left(v(N) - \sum_{j \in N} v(\{j\})\right).$$

In this paper, we are mainly interested in convex combinations of the equal division solution and the CIS-value, i.e. for every $\alpha \in [0, 1]$, the corresponding solution is defined by

$$\varphi^\alpha(N, v) = \alpha CIS(N, v) + (1 - \alpha) ED(N, v). \tag{2.1}$$

We denote the class of all solutions that are obtained in this way by $\Phi := \{\varphi^\alpha \mid \alpha \in [0, 1]\}$. It is straightforward to verify that for every $(N, v) \in \mathcal{G}$ and every $\alpha \in [0, 1]$ it holds that

$$\varphi_i^\alpha(N, v) = \alpha v(\{i\}) + \frac{1}{|N|}\left(v(N) - \sum_{j \in N} \alpha v(\{j\})\right). \tag{2.2}$$

Next we state some well-known properties of solutions for TU-games. Players $i, j \in N$ are *symmetric* in game $(N, v)$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. Player $i \in N$ is a *null player* in game $(N, v)$ if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$. For a game $(N, v) \in \mathcal{G}$ and a permutation $\pi \colon N \to N$, the permuted game $(N, \pi v)$ is defined by $\pi v(S) = v(\{\pi(i) \mid i \in S\})$ for all $S \subseteq N$. For $(N, v), (N, w) \in \mathcal{G}$ and $a, b \in \mathbb{R}$ the game $(N, av + bw) \in \mathcal{G}$ is defined by $(av + bw)(S) = av(S) + bw(S)$ for all $S \subseteq N$. Finally, a game $(N, v) \in \mathcal{G}$ is called *weakly essential* if $\sum_{i \in N} v(\{i\}) \leq v(N)$. A solution $\psi$

[1]Efficient solutions are often called *values*.

- satisfies *efficiency* on $\mathcal{C} \subseteq \mathcal{G}$ if $\sum_{i \in N} \psi_i(N,v) = v(N)$ for all $(N,v) \in \mathcal{C}$;
- satisfies *symmetry* on $\mathcal{C} \subseteq \mathcal{G}$ if $\psi_i(N,v) = \psi_j(N,v)$ whenever $i$ and $j$ are symmetric players in $(N,v)$ for all $(N,v) \in \mathcal{C}$;
- satisfies *linearity* on $\mathcal{C} \subseteq \mathcal{G}$ if $\psi(N, av+bw) = a\psi(N,v)+b\psi(N,w)$ for all $(N,v), (N,w) \in \mathcal{C}$ and all $a, b \in \mathbb{R}$ such that $(N, av+bw) \in \mathcal{C}$;
- satisfies *individual rationality* on $\mathcal{C} \subseteq \mathcal{G}$ if $\psi_i(N,v) \geq v(\{i\})$ for all $i \in N$ and all weakly essential games $(N,v) \in \mathcal{C}$;
- is *nonnegative* on $\mathcal{C} \subseteq \mathcal{G}$ if $\psi_i(N,v) \geq 0$ for all $i \in N$ and all $(N,v) \in \mathcal{C}$ satisfying $v(N) \geq 0$;
- satisfies *$\alpha$-standardness for two-player games* on $\mathcal{C} \subseteq \mathcal{G}$ if for every $(N,v) \in \mathcal{C}$ with $N = \{i,j\},\ i \neq j$, it holds that

$$\psi_i(N,v) = \alpha v(\{i\}) + \frac{1}{2}[v(N) - \alpha(v(\{i\}) + v(\{j\}))].$$

The last property is used by Joosten (1996) to characterize the class of $\alpha$-egalitarian Shapley values. Standardness for two-player games coincides with $\alpha = 1$, and egalitarian standardness coincides with $\alpha = 0$.

# 3 Characterizations for two-player games

In the following sections we use $\alpha$-standardness to characterize solutions in the class $\Phi$. In order to have axiomatizations on the class $\Phi$ with no parameterized axiom, in this section we first support $\alpha$-standardness by showing how $\alpha$-standardness can be characterized on the class of two-player games by axioms that do not depend on $\alpha$. We denote the class of all two-player TU-games by $\mathcal{G}^2$.

In van den Brink and Funaki (2009, Proposition 4.2) it is shown that any solution that satisfies efficiency, symmetry and linearity on the class of two-player games also satisfies $\alpha$-standardness for some $\alpha \in [0,1]$. Since we do not need linearity in the following sections, in this section we characterize $\alpha$-standardness for two-player games without linearity. Note that on the class of two-player games, a solution satisfying $\alpha$-standardness for some $\alpha \in [0,1]$ is equivalent to saying that the solution belongs to $\Phi$.

First, we impose a weak fairness axiom,[2] stating that changing only the worth of the 'grand coalition' changes the payoffs of all players by the same amount. This is a rather

[2]A solution $\psi$ satisfies fairness on $\mathcal{G}$ if $\psi_i(N,v+w) - \psi_i(N,v) = \psi_j(N,v+w) - \psi_j(N,v)$ whenever $i$ and $j$ are symmetric players in $(N,w)$. For further discussion, see van den Brink (2001).

weak axiom which is satisfied by many solutions such as all equal surplus division solutions in van den Brink and Funaki (2009) and the Shapley value (Shapley, 1953).

**Axiom 3.1** *A solution $\psi$ satisfies weak fairness on $\mathcal{C} \subseteq \mathcal{G}$ if for every pair of games $(N,v),(N,w) \in \mathcal{C}$ such that $v(S) = w(S)$ for all $S \subsetneq N$, there exists $c^* \in \mathbb{R}$ such that $\psi_i(N,v) - \psi_i(N,w) = c^*$ for all $i \in N$.*

Next, we state a boundary condition saying that a player always earns at least the minimum of its own worth and the per capita worth of the grand coalition. It provides each player an incentive to participate by guaranteeing at least the minimum of the two numbers.

**Axiom 3.2** *A solution $\psi$ satisfies the boundary condition on $\mathcal{C} \subseteq \mathcal{G}$ if for every game $(N,v) \in \mathcal{C}$ and $i \in N$ it holds that $\psi_i(N,v) \geq \min\left\{v(\{i\}), \frac{v(N)}{|N|}\right\}$.*

As it turns out, these two axioms together with efficiency characterize the equal division solution. We note that this theorem holds for all $n$-player games.

**Theorem 3.3** *A solution $\psi$ on $\mathcal{G}$ satisfies efficiency, weak fairness and the boundary condition if and only if it is the equal division solution.*

Proof It is obvious that the equal division solution satisfies efficiency, weak fairness and the boundary condition. To show uniqueness, suppose that a solution $\psi$ satisfies efficiency, weak fairness and the boundary condition, and let $(N,v)$ be an $n$-player game. Suppose without loss of generality that $v(\{1\}) = \min_{i \in N} v(\{i\})$. First, consider a game $(N,w)$ given by $w(S) = v(S)$ for all $S \subsetneq N$, and $w(N) = |N|v(\{1\})$. The boundary condition implies that $\psi_i(N,w) \geq \min\left\{w(\{i\}), \frac{w(N)}{|N|}\right\} = \min\{v(\{i\}), v(\{1\})\} = v(\{1\})$. Efficiency for game $w$ implies that $\sum_{i \in N} \psi_i(N,w) = |N|v(\{1\})$, and thus $\psi_i(N,w) = v(\{1\})$ for all $i \in N$. Weak fairness then implies that there is a $c^* \in \mathbb{R}$ such that $\psi_i(N,v) = v(\{1\}) + c^*$ for all $i \in N$. Efficiency for $(N,v)$ then determines $c^* = \frac{v(N) - |N|v(\{1\})}{|N|}$, and thus $\psi_i(N,v) = \frac{v(N)}{|N|} = ED_i(N,v)$ for all $i \in !N$. □

It is obvious that $\varphi^\alpha$ does not satisfy the boundary condition if $\alpha \in [0,1)$. However, it does satisfy this property if we require it to hold only for weakly essential games. Since this property is implied by individual rationality, we refer to it as *weak individual rationality*.

**Axiom 3.4** *A solution $\psi$ satisfies weak individual rationality on $\mathcal{C} \subseteq \mathcal{G}$ if for every weakly essential game $(N,v) \in \mathcal{C}$ and all $i \in N$ it holds that $\psi_i(N,v) \geq \min\left\{v(\{i\}), \frac{v(N)}{|N|}\right\}$.*

Although from the class of solutions $\varphi^\alpha \in \Phi$, the CIS-value ($\alpha = 1$) is the only solution satisfying individual rationality, all solutions in this class satisfy weak individual rationality. Adding homogeneity and a weak covariance property characterizes the class of solutions $\Phi$, i.e. $\alpha$-standardness for two-player games.

**Axiom 3.5** *A solution $\psi$ satisfies homogeneity (of degree 1) on $\mathcal{C} \subseteq \mathcal{G}$ if for every game $(N, v) \in \mathcal{C}$ and $c \in \mathbb{R}$ such that $(N, cv) \in \mathcal{C}$, it holds that $\psi(N, cv) = c\psi(N, v)$.*

**Axiom 3.6** *A solution $\psi$ satisfies weak covariance on $\mathcal{C} \subseteq \mathcal{G}$ if for every pair of games $(N, v), (N, w) \in \mathcal{C}$ such that there is $c \in \mathbb{R}$ with $w(S) = v(S) + |S|c$ for all $S \subset N$, it holds that $\psi_i(N, w) = \psi_i(N, v) + c$ for all $i \in N$.*

Note that $w$ in Axiom 3.6 can be written as $v + c\sum_{i \in N} u_{(\{i\})}$, where $u_{(\{i\})}$ is defined by $u_{(\{i\})}(S) = 1$ if $S \ni i$, $u_{(\{i\})}(S) = 0$ otherwise. Also note that both axioms are weaker than covariance.

We prove the characterization of $\alpha$-standardness for two-player games by a series of lemmas. We first fix the player set $N = \{1, 2\}$. A game $(N, v) \in \mathcal{G}$ is *inessential* (or additive) if $\sum_{i \in N} v(\{i\}) = v(N)$. In the following, let $\mathcal{G}_A^{\{1,2\}}$ be the class of all inessential (additive) games on $N = \{1, 2\}$, and let $\mathcal{G}_0^{\{1,2\}}$ be the class of all inessential games on $N = \{1, 2\}$ such that $v(\{1\}) = 0$ (and thus $v(\{2\}) = v(N)$).

**Lemma 3.7** *A solution $\psi$ on $\mathcal{G}_0^{\{1,2\}}$ satisfies efficiency, homogeneity, and weak individual rationality if and only if for some $\alpha \in [0, 1]$, $\psi$ satisfies $\alpha$-standardness on $\mathcal{G}_0^{\{1,2\}}$.*

Proof It is obvious that each $\varphi^\alpha \in \Phi$ satisfies efficiency, homogeneity and weak individual rationality on $\mathcal{G}_0^{\{1,2\}}$. Conversely, let $\psi$ be a solution satisfying efficiency, homogeneity, and weak individual rationality. Let $(N, v) \in \mathcal{G}_0^{\{1,2\}}$ such that $v(\{2\}) = 1$. By weak individual rationality, $\psi_1(N, v) \geq 0$ and $\psi_2(N, v) \geq \frac{1}{2}$. By efficiency, there exists an $\alpha \in [0, 1]$ such that $\psi_1(N, v) = \frac{1-\alpha}{2}$ and $\psi_2(N, v) = \frac{\alpha+1}{2}$. For any $(N, w) \in \mathcal{G}_0^{\{1,2\}}$, since there exists $c = w(\{2\}) \in \mathbb{R}$ such that $\psi(N, w) = \psi(N, cv)$, by homogeneity, we have $\psi(N, w) = c\psi(N, v) = (\frac{w(\{2\})(1-\alpha)}{2}, \frac{w(\{2\})(\alpha+1)}{2}) = \varphi^\alpha(N, w)$, as desired. □

Without homogeneity, we can prove that the solution assigns to every game a convex combination of the equal division solution and the CIS-value, but the solution does need to belong to the class $\Phi$ since the weights given to the equal division solution and the CIS-value need not be the same for different games.

Next, we show that adding weak covariance characterizes the $\alpha$-standard solutions on the class of all inessential (additive) games on $N = \{1, 2\}$.

**Lemma 3.8** *A solution $\psi$ on $\mathcal{G}_A^{\{1,2\}}$ satisfies efficiency, homogeneity, weak individual rationality and weak covariance if and only if for some $\alpha \in [0,1]$, $\psi$ satisfies $\alpha$-standardness on $\mathcal{G}_A^{\{1,2\}}$.*

PROOF It is obvious that solution $\varphi^\alpha$ satisfies efficiency, homogeneity, weak individual rationality and weak covariance on $\mathcal{G}_A^{\{1,2\}}$. Conversely, let $\psi$ be a solution satisfying efficiency, homogeneity, weak individual rationality and weak covariance, and $(N,v) \in \mathcal{G}_A^{\{1,2\}}$. Consider a game $(N,w)$ given by $w(\{i\}) = v(\{i\}) - v(\{1\})$ for $i \in \{1,2\}$, and $w(N) = v(N) - 2v(\{1\})$. By Lemma 3.7, $\psi(N,w) = \varphi^\alpha(N,w)$ for some fixed $\alpha \in [0,1]$. But then weak covariance implies that $\psi_i(N,v) = \psi_i(N,w) + v(\{1\}) = \varphi_i^\alpha(N,w) + v(\{1\}) = \varphi_i^\alpha(N,v)$ for $i \in \{1,2\}$. $\square$

Since $N$ is the only coalition with more than one player, adding weak fairness implies a characterization of $\alpha$-standardness for all games on $N = \{1,2\}$.

**Theorem 3.9** *A solution $\psi$ on $\mathcal{G}^{\{1,2\}}$ satisfies efficiency, homogeneity, weak individual rationality, weak covariance and weak fairness if and only if for some $\alpha \in [0,1]$, $\psi$ satisfies $\alpha$-standardness for all two-player games.*

Next we show logical independence of the five axioms in Theorem 3.9 by the following five alternative solutions that do not satisfy $\alpha$-standardness for any $\alpha \in [0,1]$:

1. The solution $\psi$ given by $\psi_i(N,v) = v(\{i\})$ for all $i \in N$ and $(N,v) \in \mathcal{G}^{\{1,2\}}$ satisfies the axioms of Theorem 3.9 except efficiency.

2. The solution $\psi$ given by $\psi(N,v) = \mathit{ED}(N,v)$ if $|v(\{1\}) - v(\{2\})| \leq 10$, and $\psi(N,v) = \mathit{CIS}(N,v)$ if $|v(\{1\}) - v(\{2\})| > 10$, satisfies the axioms of Theorem 3.9 except homogeneity.

3. The solution $\psi$ given by $\psi_i(N,v) = 2v(\{i\}) + \frac{v(N) - \sum_{j \in N} 2v(\{j\})}{|N|}$ for all $i \in N$ and $(N,v) \in \mathcal{G}^{\{1,2\}}$ satisfies the axioms of Theorem 3.9 except weak individual rationality.

4. The solution $\psi$ given by $\psi(N,v) = \mathit{ED}(N,v)$ if $\frac{v(\{1\})}{v(\{2\})} \leq 10$, and $\psi(N,v) = \mathit{CIS}(N,v)$ if $\frac{v(\{1\})}{v(\{2\})} > 10$ satisfies the axioms of Theorem 3.9 except weak covariance.

5. The solution $\psi$ given by $\psi_1(N,v) = v(\{1\})$, and $\psi_2(N,v) = v(N) - v(\{1\})$ for all $(N,v) \in \mathcal{G}^{\{1,2\}}$ satisfies the axioms of Theorem 3.9 except weak fairness.

Note that as corollaries from Theorem 3.9 we obtain characterizations of the equal division solution and the CIS-value, which extend to $n$-player games with the properties of consistency and population solidarity which are discussed in the next sections. As shown before, the equal division solution is the only solution in the class $\Phi$ that satisfies the boundary condition or nonnegativity, and the CIS-value is the only solution in $\Phi$ that satisfies covariance or individual rationality. With Theorem 3.9 this immediately yields axiomatizations of the equal division solution and the CIS-value as corollaries. Further, it is obvious that adding anonymity characterizes $\alpha$-standardness on the class of all two-player games.

Next, we characterize specific solutions $\varphi^\alpha$ from the class $\Phi$ using a parametrized axiom that has nonnegativity and individual rationality as special cases. For $\alpha \in [0,1]$ we call a game *$\alpha$-essential* if $\sum_{i \in N} \alpha v(\{i\}) \leq v(N)$. Clearly, for $\alpha = 0$ this boils down to $v(N) \geq 0$, while for $\alpha = 1$ this is weak essentiality.

**Axiom 3.10** *Let $\alpha \in [0,1]$. A solution $\psi$ satisfies $\alpha$-individual rationality on $\mathcal{C} \subseteq \mathcal{G}$ if for every $\alpha$-essential game $(N,v) \in \mathcal{C}$ it holds that $\psi_i(N,v) \geq \alpha v(\{i\})$ for all $i \in N$.*

Clearly $\alpha = 1$ yields individual rationality, while $\alpha = 0$ yields nonnegativity. This axiom gives $\alpha$ an interpretation as some 'wealth taxation' parameter. Without taxation, every player $i$ can guarantee itself its own singleton worth $v(\{i\})$. This singleton worth can be seen as the individual wealth of the single player which it can earn on its own without any cooperation with other players. Then, individual rationality means that a player gets at least its own wealth. However, if there is some taxation on the wealth of players, then they cannot guarantee themselves their singleton worth, but only a fraction $\alpha$ if $(1-\alpha)$ is the tax rate. Of course, these 'after tax' individual wealths can only be satisfied if the game is $\alpha$-essential.

Before, we introduced weak fairness, and gave a characterization of the class of solutions $\Phi$ for two-player games using axioms that do not depend on $\alpha$ in Theorem 3.9. For a specific $\alpha \in [0,1]$, $\alpha$-standardness for two-player games is characterized by efficiency, weak fairness and the corresponding $\alpha$-individual rationality.

**Theorem 3.11** *Let $\alpha \in [0,1]$. A solution $\psi$ on $\mathcal{G}^2$ satisfies efficiency, weak fairness and $\alpha$-individual rationality if and only if it satisfies $\alpha$-standardness for two-player games.*

Proof It is obvious that on $\mathcal{G}^2$, $\varphi^\alpha$ satisfies efficiency, weak fairness and $\alpha$-individual rationality. Conversely, let $\psi$ be a solution satisfying efficiency, weak fairness and $\alpha$-individual rationality for some $\alpha \in [0,1]$. Let $(N,v)$ be a two-player game with $N = \{i,j\}$, $i \neq j$. First, consider a game $(N,w)$ given by $w(\{i\}) = v(\{i\})$, $w(\{j\}) = v(\{j\})$ and $w(N) = \alpha(v(\{i\}) + v(\{j\}))$. Since $(N,w)$ is an $\alpha$-essential game, $\alpha$-individual rationality implies that $\psi_i(N,w) \geq \alpha w(\{i\}) = \alpha v(\{i\})$ and $\psi_j(N,w) \geq \alpha w(\{j\}) = \alpha v(\{j\})$. Efficiency then implies that these inequalities are equalities. But then weak fairness implies that $\psi_i(N,v) - \alpha v(\{i\}) = \psi_j(N,v) - \alpha v(\{j\})$. With efficiency it follows that $\psi(N,v) = \varphi^\alpha(N,v)$. □

Taking $\alpha = 0$ and $\alpha = 1$ we obtain the following corollaries.[3]

**Corollary 3.12** *(i) A solution $\psi$ satisfies efficiency, weak fairness and nonnegativity if and only if it satisfies egalitarian standardness for two-player games.*
*(ii) A solution $\psi$ satisfies efficiency, weak fairness and individual rationality if and only if it satisfies standardness for two-player games.*

# 4 Consistency and egalitarian solutions

In this section we consider the extension of the solutions in the previous section to $n$-player games. We can use the reduced game introduced by van den Brink and Funaki (2009, Section 5), but since we consider only convex combinations of the equal division solution and the CIS-value, we can make a simplification of their reduced game.

Take a game $(N, v) \in \mathcal{G}$, a payoff vector $x \in \mathbb{R}^N$, and a player $j \in N$. The player set of a *reduced game* is obtained by removing player $j$ from the original player set $N$. The worths of the coalitions in this reduced game reflect what these coalitions can earn if player $j$ has left the game with its payoff $x_j$. The worth of the coalition $N \setminus \{j\}$ (the 'grand coalition') in the reduced game is equal to the worth of $N$ minus the payoff $x_j$ assigned to player $j$. Clearly, this is what is left to be allocated to the players in $N \setminus \{j\}$ after removing player $j$ from the game with payoff $x_j$. For the other coalitions $S \subset N \setminus \{j\}$ we assume that they simply earn their worth $v(S)$ in the original game[4].

**Definition 4.1** *Given a game $(N, v) \in \mathcal{G}$ with $|N| \geq 3$, a player $j \in N$, and a payoff vector $x \in \mathbb{R}^N$, the* **reduced game with respect to** *$j$ and $x$ is the game $(N \setminus \{j\}, v^x)$ given by*

$$v^x(S) = \begin{cases} v(N) - x_j & \text{if } S = N \setminus \{j\} \\ v(S) & \text{if } S \subset N \setminus \{j\}. \end{cases}$$

Although we can allow for different worths for coalitions of size strictly between 1 and $|N| - 1$, we consider only the game given above.[5] We are ready to give a definition of the

[3]Recall that van den Brink and Funaki (2009) characterized the equal division solution (respectively CIS-value) by efficiency, symmetry, linearity and a weaker nonnegativity requiring nonnegative payoffs only if the worths of all coalitions are nonnegative (respectively individual rationality).

[4]In van den Brink and Funaki (2009) it is assumed that in the reduced game a coalition has the participation of the leaving player $j$ (but must pay $x_j$ to $j$) or not. Also, because of the simplification we do not have to consider the case $|N| = 3$ different from the case $|N| \geq 4$, as done in van den Brink and Funaki (2009).

[5]Here we only consider the class $\mathcal{G}$ of all TU-games. If one considers subclasses $\mathcal{C} \subset \mathcal{G}$, then in the definition of consistency one should additionally require that the reduced game $(N \setminus \{j\}, v^x)$ in this definition also belongs to $\mathcal{C}$.

consistency property of a solution associated with this reduced game.[6]

**Definition 4.2** *A solution $\psi$ satisfies* **consistency** *if and only if for every $(N,v) \in \mathcal{G}$ with $|N| \geq 3$, $j \in N$, and $x = \psi(N,v)$ it holds that $\psi_i(N \setminus \{j\}, v^x) = \psi_i(N,v)$ for all $i \in N \setminus \{j\}$.*

Consistency implies that given a game $(N,v)$, if $x$ is a solution payoff vector for $(N,v)$, then for every player $j \in N$, the payoff vector $x_{N\setminus\{j\}}$ with payoffs for the players in $N\setminus\{j\}$, must be a solution payoff vector of the reduced game $(N \setminus \{j\}, v^x)$. It is a kind of internal consistency requirement to guarantee that players respect the recommendations made by the solution.

**Proposition 4.3** *For every $\alpha \in [0,1]$ the solution $\varphi^\alpha$ satisfies consistency on the class of all games $\mathcal{G}$.*

PROOF[7] Take any $\alpha \in [0,1]$, and any $(N,v) \in \mathcal{G}$ with $|N| \geq 3$. For $x = \varphi^\alpha(N,v)$ and $i \in N \setminus \{j\}$ we have

$$\varphi_i^\alpha(N \setminus \{j\}, v^x) = \alpha v^x(\{i\}) + \frac{1}{|N|-1}\left(v^x(N \setminus \{j\}) - \sum_{k\in N\setminus\{j\}} \alpha v^x(\{k\})\right)$$

$$= \alpha v(\{i\}) + \frac{1}{|N|-1}\left(v(N) - x_j - \sum_{k\in N\setminus\{j\}} \alpha v(\{k\})\right)$$

$$= \alpha v(\{i\}) + \frac{1}{|N|-1}\left(v(N) - \alpha v(\{j\}) - \frac{1}{|N|}\left(v(N) - \sum_{k\in N} \alpha v(\{k\})\right) - \sum_{k\in N\setminus\{j\}} \alpha v(\{k\})\right)$$

$$= \alpha v(\{i\}) + \frac{1}{|N|-1}\left(\frac{|N|-1}{|N|}\left(v(N) - \sum_{k\in N} \alpha v(\{k\})\right)\right) = \varphi_i^\alpha(N,v).$$

□

Adding $\alpha$-standardness for two-player games characterizes[8] the solution $\varphi^\alpha$.

[6]The equal division solution satisfies several well-known consistency axioms that are also satisfied by the Shapley value such as those of Sobolev (1973) and Hart and Mas-Colell (1988, 1989), which are not satisfied by the CIS-value. In van den Brink, Funaki and Ju (2011) it is shown that all convex combinations of the Shapley value and equal division solution as introduced in Joosten (1996), satisfy Sobolev (1073)'s consistency. Ju, Borm and Ruys (2007) consider the convex combinations of the Shapley value and the CIS-value.

[7]Since we slightly changed the reduced game of van den Brink and Funaki (2009) for $\beta = 1$, we give the short proof, which follows similar lines as that of their Proposition 5.3. Their case $|N| \geq 4$ with $\beta = 1$ now also holds for the case $|N| = 3$.

[8]Note that, compared to van den Brink and Funaki (2009), we do not need efficiency.

**Theorem 4.4** *Let $\alpha \in [0,1]$. A solution $\psi$ satisfies $\alpha$-standardness for two-player games and consistency on the class of all games $\mathcal{G}$ if and only if $\psi = \varphi^\alpha$.*

Proof The solution $\varphi^\alpha$ satisfying $\alpha$-standardness for two-player games is straightforward. Consistency follows from Proposition 4.3. Here we prove the 'only if' part. Take $\alpha \in [0,1]$, and let $\psi$ be a solution which satisfies $\alpha$-standardness for two-player games and consistency. If $|N| = 2$ then $\psi(N,v) = \varphi^\alpha(N,v)$ follows from $\alpha$-standardness for two-player games.

Proceeding by induction, for $|N| \geq 3$, suppose that $\psi(N',v') = \varphi^\alpha(N',v')$ whenever $|N'| = |N| - 1$. We will show that $\psi(N,v) = \varphi^\alpha(N,v)$. Take any $i,j \in N$ such that $i \neq j$. Let $x = \psi(N,v)$ and $y = \varphi^\alpha(N,v)$. For the two reduced games $(N \setminus \{j\}, v^x)$ and $(N \setminus \{j\}, v^y)$, by consistency of $\varphi^\alpha$ and $\psi$, and the induction hypothesis we have

$$x_i - y_i = \psi_i(N \setminus \{j\}, v^x) - \varphi_i^\alpha(N \setminus \{j\}, v^y) = \varphi_i^\alpha(N \setminus \{j\}, v^x) - \varphi_i^\alpha(N \setminus \{j\}, v^y). \tag{4.3}$$

By definition of $\varphi^\alpha$ and the reduced game, we have

$$\begin{aligned}
&\varphi_i^\alpha(N \setminus \{j\}, v^x) - \varphi_i^\alpha(N \setminus \{j\}, v^y) \\
&= \alpha v^x(\{i\}) + \frac{1}{|N|-1}\left(v(N \setminus \{j\}) - \sum_{k \in N \setminus \{j\}} \alpha v^x(k)\right) - \alpha v^y(\{i\}) \\
&\quad - \frac{1}{|N|-1}\left(v(N \setminus \{j\}) - \sum_{k \in N \setminus \{j\}} \alpha v^y(k)\right) \\
&= \alpha v(\{i\}) - \alpha v(\{i\}) - \frac{1}{|N|-1}\sum_{k \in N \setminus \{j\}} (\alpha v(\{k\}) - \alpha v(\{k\})) = 0.
\end{aligned}$$

With (4.3) this implies that $x_i - y_i = 0$ for all $i \in N$. This shows that $\psi(N,v) = \varphi^\alpha(N,v)$.

This completes the proof. □

By Theorem 3.11, and Theorem 4.4, we obtain the following corollary.

**Corollary 4.5** *Let $\alpha \in [0,1]$. A solution $\psi$ satisfies efficiency, weak fairness, $\alpha$-individual rationality and consistency on the class of all games $\mathcal{G}$ if and only if $\psi = \varphi^\alpha$.*

# 5 Population solidarity and egalitarian solutions

Now we consider another extension of $\alpha$-standardness to $n$-player games by imposing the axiom of population solidarity. Population solidarity requires that upon an arrival of a new player all the original players should be affected in the same direction, all weakly gain or all weakly lose. Its implications have been studied in various contexts (Thomson, 1983; Chun 1986), and for TU-games by Chun and Park (2012).

**Axiom 5.1** *A solution $\psi$ satisfies population solidarity if for all $(N, v), (N', w) \in \mathcal{G}$ satisfying $N \subset N'$ and $v(S) = w(S)$ for all $S \subseteq N$, it holds that either $\psi_j(N, v) \geq \psi_j(N', w)$ for all $j \in N$, or $\psi_j(N, v) \leq \psi_j(N', w)$ for all $j \in N$.*

It is easy to check that the convex combinations of the equal division solution and the CIS-value are the only ones in the class of equal surplus division solutions considered in van den Brink and Funaki (2009) that satisfy population solidarity. Chun and Park (2012, Theorem 1) shows that standardness for two-player games, efficiency and population solidarity characterize the CIS-value. In a similar way we can show the following theorem, and refer to the appendix for the proof.

**Theorem 5.2** *Let $\alpha \in [0, 1]$. A solution $\psi$ satisfies efficiency, $\alpha$-standardness for two-player games and population solidarity if and only if $\psi = \varphi^\alpha$.*

Logical independence of the axioms in Theorem 5.2 follows from the following three alternative solutions:

1. The solution $\psi^\alpha$, given by $\psi_i^\alpha(N, v) = \alpha v(\{i\}) + \frac{1}{2} \sum_{\substack{T \subseteq N \\ |T|=2}} \left(v(T) - \alpha \sum_{k \in T} v(\{k\})\right)$ for all $(N, v) \in \mathcal{G}$ and $i \in N$, satisfies $\alpha$-standardness for two-player games and population solidarity. It does not satisfy efficiency.

2. For all $i \in \mathbb{N}$, let $t_i$ be a number assigned to player $i$, such that these numbers are distinct. The solution $\psi$, for all $(N, v) \in \mathcal{G}$ given by $\psi(N, v) = \varphi^\alpha(N, w)$, where $(N, w)$ is a game such that $w(\{i\}) = t_i v(\{i\})$ for all $i \in N$ and $w(S) = v(S)$ for any other $S$, satisfies efficiency and population solidarity. It does not satisfy $\alpha$-standardness for two-player games.

3. The Shapley value satisfies efficiency and $\alpha$-standardness for two-player games. It does not satisfy population solidarity.

By Theorem 3.11, and Theorem 5.2, we obtain the following corollary.

**Corollary 5.3** *Let $\alpha \in [0, 1]$. A solution $\psi$ satisfies efficiency, weak fairness, $\alpha$-individual rationality, and population solidarity if and only if $\psi = \varphi^\alpha$.*

Whereas Theorem 5.2 characterizes each solution $\varphi^\alpha$, $\alpha \in [0, 1]$, by Theorem 3.9 and Theorem 5.2 it follows straightforwardly that adding anonymity, a solution $\psi$ on $\mathcal{G}$ belongs to $\Phi$ if and only if it satisfies efficiency, homogeneity, weak individual rationality, weak covariance, weak fairness, anonymity and population solidarity. It turns out that we can even do without anonymity which follows from the following lemma which states that under efficiency and population solidarity, a solution that satisfies $\alpha$-standardness for games on a player set $N = \{i, j\}$, $i \neq j$, must satisfy the same $\alpha$-standardness on any class of games on player sets $N' \subset \mathbb{N}$ with $|N'| = 2$. In other words, $\alpha$ does not depend on the choice of the player set $N'$.

To prove the following lemma, we use this notation: For any $(N, v) \in \mathcal{G}$ and $T \subset N$, the *subgame* $(T, v_T)$ is given by $v_T(S) = v(S)$ for all $S \subseteq T$.

**Lemma 5.4** *Suppose that a solution $\psi$ satisfies efficiency and population solidarity. If $\psi$ satisfies $\alpha_{ij}$-standardness for two-player games on $\mathcal{G}^{\{i,j\}}$ for any $i, j \in \mathbb{N}$, then $\alpha_{ij} = \alpha_{i'j'}$ for any $i, j, i', j' \in \mathbb{N}$.*

Proof Let $\psi$ be a solution that satisfies efficiency, population solidarity, and $\alpha_{ij}$-standardness for two-player games on $\mathcal{G}^{\{i,j\}}$ for any $i, j \in \mathbb{N}$. Suppose by contradiction that there exist $i, j, k \in \mathbb{N}$ such that $\alpha_{ij} \neq \alpha_{ik}$. For simplicity, we assume that $i = 1$, $j = 2$, $k = 3$ and $\alpha_{12} > \alpha_{13}$. Let $\delta$ and $\epsilon$ be positive real numbers such that

$$\alpha_{12} - \alpha_{13} < \delta < 2(\alpha_{12} - \alpha_{13})$$
$$0 < \epsilon < 2\alpha_{12} - 2\alpha_{13} - \delta.$$

Then it follows that

$$3\alpha_{12} + 2\alpha_{13} + \delta < 5\alpha_{12} - \epsilon.$$

Now, consider the following game $(N, v)$ where $N = \{1, 2, 3\}$ and $v \in \mathcal{G}^N$ defined as follows:

| $S$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{2,3\}$ | $\{1,3\}$ |
|---|---|---|---|---|---|---|
| $v(S)$ | 1 | 2 | 2 | $3\alpha_{12}$ | $4\alpha_{12} - 2\epsilon$ | $3\alpha_{13} + 2\delta$ |

and $v(N)$ is any real number satisfying

$$3\alpha_{12} + 2\alpha_{13} + \delta < v(N) < 5\alpha_{12} - \epsilon. \tag{5.4}$$

Note that, by $\alpha_{ij}$-standardness for two-player games,

$$\psi_1(\{1,2\}, v_{\{1,2\}}) = \alpha_{12} + \frac{1}{2}(3\alpha_{12} - 3\alpha_{12}) = \alpha_{12}$$
$$\psi_2(\{1,2\}, v_{\{1,2\}}) = 2\alpha_{12} + \frac{1}{2}(3\alpha_{12} - 3\alpha_{12}) = 2\alpha_{12}$$
$$\psi_2(\{2,3\}, v_{\{2,3\}}) = 2\alpha_{23} + \frac{1}{2}(4\alpha_{12} - 2\epsilon - 4\alpha_{23}) = 2\alpha_{12} - \epsilon$$
$$\psi_3(\{2,3\}, v_{\{2,3\}}) = 2\alpha_{23} + \frac{1}{2}(4\alpha_{12} - 2\epsilon - 4\alpha_{23}) = 2\alpha_{12} - \epsilon$$
$$\psi_1(\{1,3\}, v_{\{1,3\}}) = \alpha_{13} + \frac{1}{2}(3\alpha_{13} + 2\delta - 3\alpha_{13}) = \alpha_{13} + \delta$$
$$\psi_3(\{1,3\}, v_{\{1,3\}}) = 2\alpha_{13} + \frac{1}{2}(3\alpha_{13} + 2\delta - 3\alpha_{13}) = 2\alpha_{13} + \delta.$$

We distinguish the following three cases.

1. Suppose that $\psi_1(N,v) < \alpha_{12}$. Applying population solidarity to $(\{1,2\}, v_{\{1,2\}})$ and $(N,v)$, yields $\psi_2(N,v) \leq 2\alpha_{12}$. Since $\alpha_{12} < \alpha_{13} + \delta$, it follows that $\psi_1(N,v) < \alpha_{13} + \delta$. Applying population solidarity to $(\{1,3\}, v_{\{1,3\}})$ and $(N,v)$, yields $\psi_3(N,v) \leq 2\alpha_{13} + \delta$. Then,

$$v(N) = \psi_1(N,v) + \psi_2(N,v) + \psi_3(N,v) < \alpha_{12} + 2\alpha_{12} + 2\alpha_{13} + \delta = 3\alpha_{12} + 2\alpha_{13} + \delta,$$

a contradiction to (5.4).

2. Suppose that $\psi_1(N,v) > \alpha_{12}$. Applying population solidarity to $(\{1,2\}, v_{\{1,2\}})$ and $(N,v)$, yields $\psi_2(N,v) \geq 2\alpha_{12}$. Since $\psi_2(N,v) > 2\alpha_{12} - \epsilon$, applying population solidarity to $(\{2,3\}, v_{\{2,3\}})$ and $(N,v)$, yields $\psi_3(N,v) \geq 2\alpha_{12} - \epsilon$. Then,

$$v(N) = \psi_1(N,v) + \psi_2(N,v) + \psi_3(N,v) > \alpha_{12} + 2\alpha_{12} + 2\alpha_{12} - \epsilon = 5\alpha_{12} - \epsilon,$$

a contradiction to (5.4).

3. Suppose that $\psi_1(N,v) = \alpha_{12}$. Then, $\psi_1(N,v) < \alpha_{13} + \delta$. Applying population solidarity to $(\{1,3\}, v_{\{1,3\}})$ and $(N,v)$, yields $\psi_3(N,v) \leq 2\alpha_{13}+\delta$. Since $2\alpha_{13}+\delta < 2\alpha_{12}-\epsilon$, it holds that $\psi_3(N,v) < 2\alpha_{12} - \epsilon$. Applying population solidarity to $(\{2,3\}, v_{\{2,3\}})$ and $(N,v)$, yields $\psi_2(N,v) \leq 2\alpha_{12}-\epsilon$. Then, since $\psi_3(N,v) \leq 2\alpha_{13}+\delta$ and $\psi_2(N,v) \leq 2\alpha_{12}-\epsilon$,

$$v(N) = \psi_1(N,v) + \psi_2(N,v) + \psi_3(N,v) \leq \alpha_{12} + 2\alpha_{12} - \epsilon + 2\alpha_{13} + \delta < 3\alpha_{12} + 2\alpha_{13} + \delta,$$

a contradiction to (5.4). □

Now, by Theorem 3.9, Theorem 5.2 and Lemma 5.4, we obtain the following corollary.

**Corollary 5.5** *A solution $\psi$ on $\mathcal{G}$ belongs to $\Phi$ if and only if it satisfies efficiency, homogeneity, weak individual rationality, weak covariance, weak fairness and population solidarity.*

Note that in this corollary, homogeneity, weak individual rationality, weak covariance and weak fairness can be replaced by symmetry[9] and linearity.

[9]Instead of symmetry we can also use *local monotonicity* meaning that $\psi_i(N,v) \geq \psi_j(N,v)$ whenever $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for all $i,j \in N$, all $S \subseteq N \setminus \{i,j\}$, and all $(N,v) \in \mathcal{C} \subseteq \mathcal{G}$. This axiom used in Levinský and Silársky (2004) is also known as desirability, see Peleg and Sudhölter (2003).

# 6 The ENSC-value

The *dual game* $(N, v^*)$ of a game $(N, v)$ is the game that assigns to each coalition $S \subseteq N$ the worth that is lost by the 'grand coalition' $N$ if coalition $S$ leaves $N$, i.e.,

$$v^*(S) = v(N) - v(N \setminus S) \text{ for all } S \subseteq N.$$

The ENSC-value assigns to every game $(N, v)$ the CIS-value of its dual game, i.e.,

$$\begin{aligned} ENSC_i(N,v) &= CIS_i(N,v^*) = v^*(\{i\}) + \frac{1}{|N|}\left(v^*(N) - \sum_{j\in N} v^*(\{j\})\right) \\ &= v(N) - v(N\setminus\{i\}) + \frac{1}{|N|}\left(v(N) - \sum_{j\in N}(v(N) - v(N\setminus\{j\}))\right) \\ &= -v(N\setminus\{i\}) + \frac{1}{|N|}\left(v(N) + \sum_{j\in N} v(N\setminus\{j\})\right) \text{ for all } i \in N. \end{aligned}$$

Thus, the ENSC-value assigns to every player in a game its marginal contribution to the 'grand coalition' and distributes the (positive or negative) remainder equally among the players.

It is known that the ENSC-value is the dual of the CIS-value and has several dual properties. In this section we present two properties for values that are defined as a convex combination of the ENSC-value and the equal division solution. These properties are induced from the theorems in the previous two sections by its duality. Thus we omit the proofs of the two theorems. Note that for two-player games, both solutions coincide (with any standard solution).

For each $\alpha \in [0, 1]$, let $\overline{\varphi}^\alpha$ be given by

$$\overline{\varphi}^\alpha(N, v) = \alpha ENSC(N, v) + (1 - \alpha)ED(N, v). \tag{6.5}$$

It is easy to check that the solution $\overline{\varphi}^\alpha$ is the dual of $\varphi^\alpha$, that is, $\overline{\varphi}^\alpha(N, v) = \varphi^\alpha(N, v^*)$.

From Funaki (1998) it follows that, for any game $(N, v) \in \mathcal{G}$ with $|N| \geq 3$, a player $j \in N$, and a payoff vector $x \in \mathbb{R}^N$, the dual of the reduced game defined in Definition 4.2 (we call this the *dual reduced game with respect to* $j$ and $x$) is the game $(N \setminus \{j\}, (v^x)^*)$ given by

$$(v^x)^*(S) = \begin{cases} v(S \cup \{j\}) - x_j & \text{if } S \subseteq N \setminus \{j\}, S \neq \emptyset \\ 0 & \text{if } S = \emptyset. \end{cases}$$

The consistency property related to this reduced game is called *projection consistency* and is defined similar as Definition 4.2. Together with $\alpha$-standardness for two-player games it characterizes the corresponding convex combination of the equal division solution and the CIS-value.

**Theorem 6.1** *Let $\alpha \in [0,1]$. A solution $\phi$ satisfies $\alpha$-standardness for two-player games and projection consistency on the class of all games $\mathcal{G}$ if and only if $\phi = \overline{\varphi}^{\alpha}$.*

Now we consider the dual property of population solidarity. It is obtained by replacing $(N, v)$ by $(N, v^*)$ in the original property. For a game $(N', w)$ and $N \subset N'$, let $\bar{w}(S) = \bar{w}(S \cup (N' \setminus N))$ for all $S \subseteq N$.

**Axiom 6.2** *A solution $\psi$ satisfies the dual of population solidarity if for all $(N, v), (N', w) \in \mathcal{G}$ satisfying $N \subset N'$ and $v(N) - v(N \setminus S) = \bar{w}(N) - \bar{w}(N \setminus S)$ for all $S \subseteq N$, it holds that either $\psi_j(N, v) \geq \psi_j(N', w)$ for all $j \in N$, or $\psi_j(N, v) \leq \psi_j(N', w)$ for all $j \in N$.*

The dual of population solidarity requires the following. Consider two games $(N, v)$ and $(N', w)$ such that $N \subset N'$. We compare the worth of the coalitions in the two games for the player set $N$. Then we consider $\bar{w}$ instead of $w$. If for any $S \subset N$, the contributions of $S$ to $N$ in both games coincide, all the original players in $N$ should be affected in the same direction.

**Theorem 6.3** *Let $\alpha \in [0,1]$. A solution $\phi$ satisfies efficiency, $\alpha$-standardness for two-player games and the dual of population solidarity if and only if $\phi = \overline{\varphi}^{\alpha}$.*

Since $\overline{\varphi}^{\alpha}$ satisfies $\alpha$-standard for two-player games, we can formulate similar corollaries as at the end of Section 4.

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# Appendix: Proof of Theorem 5.2

We present a proof of Theorem 5.2 which generalizes that of Chun and Park (2012). It is obvious that $\varphi^\alpha$ satisfies efficiency, $\alpha$-standardness for two-player games and population solidarity.

To prove uniqueness, let $\psi$ be a solution satisfying efficiency, $\alpha$-standardness for two-player games and population solidarity. We assume that $N = \{1, \dots, n\}$. For any game $(N, v) \in \mathcal{G}$ and $\alpha \in [0, 1]$, we define the real number $x^\alpha(N, v)$ by

$$x^\alpha(N, v) = \frac{v(N) - \alpha \sum_{k \in N} v(k)}{|N|},$$

and we also define a vector $\theta^\alpha(N, v) \in \mathbb{R}^N$ by $\theta_i^\alpha(N, v) = \psi_i(N, v) - \alpha v(i)$ for all $i \in N$. For notational convenience, in case there is no confusion we will often omit the superscript $\alpha$ and shortly write $x(N, v)$ and $\theta(N, v)$. Note that from efficiency,

$$\sum_{i \in N} \theta_i(N, v) = \sum_{i \in N} \psi_i(N, v) - \alpha \sum_{i \in N} v(\{i\}) = v(N) - \alpha \sum_{i \in N} v(\{i\}) = x(N, v) \cdot |N|. \quad (6.6)$$

If $|N| = 2$, then $\psi(N, v) = \varphi^\alpha(N, v)$ follows from the assumption that $\psi$ satisfies $\alpha$-standardness for two-player games. Let $(N, v) \in \mathcal{G}$ with $|N| \geq 3$. For simplicity, let $N = \{1, 2, 3, \dots, n\}$. It is sufficient to show that $\theta_i(N, v) = x(N, v)$ for all $i \in N$. We will show this by contradiction. Suppose that there is a player $k \in N$ such that $\theta_k(N, v) \neq x(N, v)$. Since $\sum_{i \in N} \theta_i(N, v) = x(N, v) \cdot |N|$, there exists a player $j \in N$ such that $\theta_j(N, v) > x(N, v)$. Without loss of generality, we may assume that $\theta_1(N, v) - x(N, v) > 0$.

Let $\delta$ be a positive real number defined by

$$\delta \equiv \frac{\theta_1(N, v) - x(N, v)}{2(n+1)^2}.$$

Let $N' \equiv \{1, 2, \dots, n+1\}$ and $N'' \equiv \{1, 2, \dots, n+1, n+2\}$. We consider a game $(N'', w)$ such that for all $S \subset N$, $w(S) = v(S)$, and $w$ satisfies the following:

| $S$ | $w(S)$ |
|---|---|
| $\{i, n+1\}$ for $i \in N \setminus \{1\}$ | $\alpha w(i) + \alpha w(n+1) + 2(x(N, v) + \delta)$ |
| $\{i, n+2\}$ for $i \in N \setminus \{1\}$ | $\alpha w(i) + \alpha w(n+2) + 2(x(N, v) + 2\delta)$ |
| $\{1, n+1\}$ | $\alpha w(1) + \alpha w(n+1) + 2(x(N, v) + 2(n+1)^2\delta)$ |
| $\{1, n+2\}$ | $\alpha w(1) + \alpha w(n+2) + 2(x(N, v) + (2n+3)\delta)$ |
| $\{n+1, n+2\}$ | $\alpha w(n+1) + \alpha w(n+2) + 2(x(N, v) + \frac{3}{2}\delta)$ |
| $N'$ | $\alpha \sum_{i=1}^{n+1} w(i) + (n+1)(x(N, v) + (2n+2)\delta)$ |
| $N''$ | $\alpha \sum_{i=1}^{n+2} w(i) + (n+2)(x(N, v) + (2n+2)\delta)$ |

For simplicity, for any $S \subseteq N''$, the subgame of $(N'', w)$ induced by $S$ is denoted by $(S, w)$.

From the previous table and from the fact that $\psi$ is equal to $\varphi^\alpha$ for all two-player games, we have the following: for $i \in N \setminus \{1\}$,

$$\theta_i(\{i,n{+}1\},w) = \theta_{n+1}(\{i,n{+}1\},w) = x(\{i,n{+}1\},w) = x(N,v) + \delta \tag{6.7}$$

$$\theta_i(\{i,n{+}2\},w) = \theta_{n+2}(\{i,n{+}2\},w) = x(\{i,n{+}2\},w) = x(N,v) + 2\delta \tag{6.8}$$

and for $i \in \{n+1, n+2\}$,

$$\theta_i(\{n{+}1,n{+}2\},w) = x(\{n{+}1,n{+}2\},w) = x(N,v) + \frac{3\delta}{2}. \tag{6.9}$$

In addition, it holds that

$$\theta_1(\{1,n{+}1\},w) = \theta_{n+1}(\{1,n{+}1\},w) = x(\{1,n{+}1\},w) = x(N,v) + 2(n+1)^2\delta \tag{6.10}$$

$$\theta_1(\{1,n{+}2\},w) = \theta_{n+2}(\{1,n{+}2\},w) = x(\{1,n{+}2\},w) = x(N,v) + (2n+3)\delta \tag{6.11}$$

$$x(N',w) = x(N'',w) = x(N,v) + 2(n+1)\delta. \tag{6.12}$$

Together with (6.6) and (6.12),

$$\sum_{i=1}^{n+1} \theta_i(N',w) = (n+1)x(N',w) = (n+1)x(N,v) + 2(n+1)^2\delta \tag{6.13}$$

$$\sum_{i=1}^{n+2} \theta_i(N'',w) = (n+2)x(N'',w) = (n+2)x(N,v) + 2(n+1)(n+2)\delta. \tag{6.14}$$

Next, we prove several claims.

**Claim 1.** $\theta_1(N',w) = x(N,v) + 2(n+1)^2\delta$.

*Proof of Claim 1.* Suppose that $\theta_1(N',w) > x(N,v){+}2(n{+}1)^2\delta$. Since $x(N,v){+}2(n{+}1)^2\delta = \theta_1(N,v)$ by the definition of $\delta$, $\theta_1(N',w) > \theta_1(N,v)$. Applying population solidarity to $(N,v)$ and $(N',w)$, it holds that $\theta_i(N',w) \geq \theta_i(N,v)$ for all $i \in N$. Then

$$\sum_{i=1}^{n} \theta_i(N',w) > \sum_{i=1}^{n} \theta_i(N,v). \tag{6.15}$$

Then by (6.13) and (6.6),

$$\sum_{i=1}^{n} \theta_i(N',w) = (n+1)x(N,v) + 2(n+1)^2\delta - \theta_{n+1}(N',w),$$

and by (6.6), $\sum_{i=1}^{n} \theta_i(N,v) = nx(N,v)$. It follows by (6.15) that $(n{+}1)x(N,v){+}2(n{+}1)^2\delta -$ $\theta_{n+1}(N',w) > nx(N,v)$ or $\theta_{n+1}(N',w) < x(N,v) + 2(n+1)^2\delta$. By (6.10), this is equivalent to $\theta_{n+1}(N',w) < \theta_{n+1}(\{1,n{+}1\},w)$. Applying population solidarity to $(\{1,n{+}1\},w)$ and $(N',w)$, it holds that with (6.10) that

$$\theta_1(N',w) \leq \theta_1(\{1,n{+}1\},w) = x(N,v) + 2(n+1)^2\delta,$$

which is a contradiction to the assumption that $\theta_1(N', w) > x(N, v) + 2(n+1)^2\delta$. If $\theta_1(N', w) < x(N, v) + 2(n+1)^2\delta$, we can reach a contradiction similarly. Therefore, Claim 1 holds.

**Claim 2.** *$\theta_{n+1}(N', w) \leq x(N, v) + \delta$.*

*Proof of Claim 2.* Suppose that $\theta_{n+1}(N', w) > x(N, v) + \delta$. By (6.7), we have $\theta_{n+1}(N', w) > \theta_{n+1}(\{i, n+1\}, w)$ for all $i \in N \setminus \{1\}$. Applying population solidarity to $(\{i, n+1\}, w)$ and $(N', w)$ for each $i \in N \setminus \{1\}$, we obtain $\theta_i(N', w) \geq \theta_i(\{i, n+1\}, w)$ for each $i \in N \setminus \{1\}$. By (6.7), $\theta_i(N', w) \geq x(N, v) + \delta$ for all $i \in N \setminus \{1\}$. All together with Claim 1, we have

$$\begin{aligned}
\theta_1(N', w) &= x(N, v) + 2(n+1)^2\delta \\
\theta_i(N', w) &\geq x(N, v) + \delta \quad \text{for all } i \in N \setminus \{1\} \\
\theta_{n+1}(N', w) &> x(N, v) + \delta,
\end{aligned}$$

and so $\sum_{i=1}^{n+1} \theta_i(N', w) > (n+1)x(N, v) + 2(n+1)^2\delta + n\delta$, which contradicts (6.13). Therefore, Claim 2 holds.

**Claim 3.** *$\theta_i(N'', w) > \theta_i(N', w)$ for some $i \in N'$.*

*Proof of Claim 3.* Suppose that $\theta_i(N'', w) \leq \theta_i(N', w)$ for all $i \in N'$. Then

$$\theta_{n+1}(N'', w) \leq \theta_{n+1}(N', w) \leq x(N, v) + \delta < x(N, v) + \frac{3}{2}\delta = \theta_{n+1}(\{n+1, n+2\}, w),$$

where the second inequality follows from Claim 2, and the last equality follows from (6.9). Applying population solidarity to $(\{n+1, n+2\}, w)$ and $(N'', w)$, we have $\theta_{n+2}(N'', w) \leq \theta_{n+2}(\{n+1, n+2\}, w)$. Since, from (6.8) and (6.9), for all $i \in N \setminus \{1\}$,

$$\theta_{n+2}(N'', w) \leq \theta_{n+2}(\{n+1, n+2\}, w) = x(N, v) + \frac{3}{2}\delta < x(N, v) + 2\delta = \theta_{n+2}(\{i, n+2\}, w),$$

it follows that for all $i \in N \setminus \{1\}$, $\theta_{n+2}(N'', w) < \theta_{n+2}(\{i, n+2\}, w)$. Applying population solidarity to $(\{i, n+2\}, w)$ and $(N'', w)$, it holds that $\theta_i(N'', w) \leq \theta_i(\{i, n+2\}, w)$ for all $i \in N \setminus \{1\}$. By (6.8), $\theta_i(N'', w) \leq x(N, v) + 2\delta$ for all $i \in N \setminus \{1\}$. All together with Claim 1, we have

$$\begin{aligned}
\theta_1(N'', w) &\leq \theta_1(N', w) = x(N, v) + 2(n+1)^2\delta \\
\theta_i(N'', w) &\leq x(N, v) + 2\delta \quad \text{for all } i \in N \setminus \{1\} \\
\theta_{n+1}(N'', w) &\leq x(N, v) + \delta \\
\theta_{n+2}(N'', w) &\leq x(N, v) + \frac{3}{2}\delta.
\end{aligned}$$

Then $\sum_{i=1}^{n+2}\theta_i(N'',w) \le (x(N,v)+2(n+1)^2\delta)+(n-1)(x(N,v)+2\delta)+(x(N,v)+\delta)+(x(N,v)+\frac{3}{2}\delta)$, and so

$$\sum_{i=1}^{n+2}\theta_i(N'',w) \le (n+2)x(N,v)+(2n^2+6n+\frac{5}{2})\delta < (n+2)x(N,v)+2(n+1)(n+2)\delta,$$

which yields a contradiction to (6.14). Therefore, Claim 3 holds.

**Claim 4.** *$\theta_i(N'',w) < \theta_i(N',w)$ for some $i \in N'$.*

*Proof of Claim 4.* Suppose that $\theta_i(N'',w) \ge \theta_i(N',w)$ for all $i \in N'$. Then

$$\theta_1(N'',w) \ge \theta_1(N',w) = x(N,v)+2(n+1)^2\delta > x(N,v)+(2n+3)\delta = \theta_1(\{1,n{+}2\},w),$$

where the second equality follows from Claim 1, and the last equality follows from (6.11). Applying population solidarity to $(\{1,n{+}2\},w)$ and $(N'',w)$, we have $\theta_{n+2}(N'',w) \ge \theta_{n+2}(\{1,n{+}2\},w)$. Then

$$\theta_{n+2}(N'',w) \ge \theta_{n+2}(\{1,n{+}2\},w) = x(N,v)+(2n+3)\delta > x(N,v)+2(n+1)\delta,$$

where the second equality follows from (6.11). Since $\theta_i(N'',w) \ge \theta_i(N',w)$ for all $i \in N'$,

$$\sum_{i=1}^{n+1}\theta_i(N'',w)+\theta_{n+2}(N'',w) > \sum_{i=1}^{n+1}\theta_i(N',w)+(x(N,v)+2(n+1)\delta).$$

By (6.13) and (6.14),

$$(n+2)x(N,v)+2(n+1)(n+2)\delta > (n+1)x(N,v)+2(n+1)^2\delta+(x(N,v)+2(n+1)\delta),$$

where the right side is $(n{+}2)x(N,v){+}2(n{+}1)(n{+}2)\delta$, which is a contradiction. Therefore, Claim 4 holds.

From Claim 3 and Claim 4, we reach a contradiction to population solidarity applied to $(N',w)$ and $(N'',w)$. It completes the proof of Theorem 5.2. □