Duisenberg school of finance - Tinbergen Institute Discussion Paper

TI 12-057/2/DSF36

Aggregating Credit and Market Risk: The Impact of Model Specification

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Aggregating Credit and Market Risk: the Impact of Model Specification*

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May 29, 2012

Abstract

We investigate the effect of model specification on the aggregation of (correlated) market and credit risk. We focus on the functional form linking systematic credit risk drivers to default probabilities. Examples include the normal based probit link function for typical structural models, or the exponential (Poisson) link function for typical reduced form models. We first show analytically how model specification impacts 'diversification benefits' for aggregated market and credit risk. The specification effect can lead to Value-at-Risk (VaR) reductions in the range of 3% to 47%, particularly at high confidence level VaRs. We also illustrate the effects using a fully calibrated empirical model for US data. The empirical effects corroborate our analytic results.

Keywords: risk aggregation, credit risk, market risk, link function, diversification, reduced form models, structural models.

JEL classification: G32, G21, C58.

*Email: a.lucas@vu.nl or bastiaan.verhoef@rbs.com. We would like to thank Bernd Schwaab and Bastian Wemmenhove for their useful comments. Any remaining errors are our own. Andre Lucas thanks the Dutch National Science Foundation (NWO) for financial support. Bastiaan Verhoef thanks the Royal Bank of Scotland for providing the rating data.

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1 Introduction

In this paper we analyze the impact of credit risk model specification on the potential diversification benefits of integrated market and credit risk. The interaction between credit and market risk is a focal point of attention for current risk management research within financial institutions, regulatory agencies, supervisors, and academia. See for example the overview studies published by the Basle Committee on Banking Supervision (BCBS 2009; 2011).[1]

One important question is whether the interaction between market and credit risk results in a reduction or an increase in aggregate capital requirements. As both risk sources may be affected by the same risk drivers, the aggregated risk measure may be substantially different from the sum of the separate market and credit risk measures. The direction of the effect is not clear and may depend on the portfolio under consideration: the aggregate risk measure may be lower due to diversification benefits or higher as result of compounding effects; see also BCBS (2009).

We contribute to the existing literature in three ways. First, we provide a detailed comparative analysis of the effect of model specification on risk aggregation for a large bond portfolio. In particular, we focus on the effect of different credit risk model specifications, such as the normal based probit link function for typical structural models, or the exponential (Poisson) link function for typical reduced form models.[2] The effect of different (properly calibrated) model specifications on aggregate credit and market risk has to our knowledge not been investigated earlier. Second, we provide semi-analytic results for a special subclass of models. These analytic results help to characterize the main drivers of model specification effects and diversification benefits between market and credit risk. Third, we provide an in-depth analysis of all combined effects for a fully calibrated empirical model based on U.S. interest rate, default, and credit spread data. The results of this empirical analysis both confirm and extend the results from our analytical section.

Our model has a two-factor structure with a (latent) market and a (latent) credit cycle factor. These two factors impact physical default probabilities, discount factors, and credit spreads via the risk neutral default probabilities at the same time. For example, as the

[1] See also the special issue of the *Journal of Banking and Finance* (2010, volume 34), which published a selection of the papers summarized in BCBS (2009).

[2] For an overview, see for example Bluhm, Overbeck, and Wagner (2002), Lando (2004), and McNeil, Frey, and Embrechts (2005).

general state of the economy deteriorates, defaults will increase. At the same time, the fraction of bonds that does not default can still depreciate in value if credit spreads increase due to e.g. worsening of the credit cycle. As a result, credit risk and market risk become fully intertwined. Even for the special case where we assume discount rates to be constant, aggregating market and credit risk is a non-trivial exercise. For this special case, we show how to derive analytic results for an infinitely granular portfolio. The analytic results make clear that care should be taken in calibrating risk parameters across different model specifications in order to make a fair comparison. We solve this issue by calibrating the model to empirically feasible information, such as default probabilities and default correlations; see also Koyluoglu and Hickman (1998) and Gordy (2000).

For large defaultable bond portfolio, our results show that model specification matters. This is particularly true at high confidence Value-at-Risk (VaR) levels. The capital levels for probit link functions are typically substantially smaller than those of exponential or Poisson link functions, with the logit link function describing the intermediate case. This is surprising, as the shape of the link function is typically deemed to be less important for computing capital requirements. Our results show that though these link functions may behave similarly in the center of their respective supports, it is precisely their different tail behavior that becomes important at high confidence VaR levels.

In our empirical analysis, we fit the model to U.S. term structure data, Moody's default data for different rating categories, and market data for credit spreads. We carefully calibrate each of the different model specifications to the available data to obtain a fair comparison. Our empirical results confirm our earlier analytic results. Aggregate credit and market risk capital levels substantially deviate from the sum of the separate market and credit risk capital levels. In addition, exponential or Poisson based links between latent risk factors and default probabilities result in the most convervative capital levels, followed by the logit and probit link functions, respectively. The conclusions are robust across rating categories.

There are several related studies on the aggregation of market and credit risk, but none of these studies includes a systematic investigation of the impact of model specification on capital requirements. Breuer, Jandacka, Rheinberger, and Summer (2010) show that a top-down aggregation of credit and market risk can underestimate compounding effects in certain cases, e.g. foreign currency loans. Drehmann, Sorensen, and Stringa (2010) model the assets and liabilities of a representative bank. Their framework allows them to conduct a stress test to analyze the impact to the bank's market and credit risk of simultaneous shocks to property prices, inflation, and exchange rates. Alessandri and Drehmann (2010) modify the framework of Drehmann et al. (2010) to analyze the scope for diversification effects in the banking book. They find that capital requirements calculated from a stand-alone approach to credit and market risk is larger compared to an integrated approach. Hartmann (2010) notes that these findings should be taken with care, as many mitigating effects are left unmodeled.

Our model set-up is closest to that of Barnhill Jr and Maxwell (2002), Gründke (2005), Kupiec (2007), and Böcker and Hillebrand (2008). All these papers illustrate the diversification effects using a portfolio of zero-coupon bonds and a factor structure for the underlying risks. Again, however, none of these papers includes a systematic comparison of the model specification issues discussed in the current paper. The main difference with Böcker and Hillebrand (2008) is that they follow a top-down approach in modeling the joint loss distribution of aggregated market and credit losses, whereas we follow a bottom-up approach. The main difference with Kupiec (2007) is that we model credit spreads explicitly as a function of the unobservable default cycle and the detrended short-term interest rate via the risk-neutral default probabilities. Kupiec (2007), by contrast, models the correlation between default risk and market risk through a one-factor model. A related approach is followed by Gründke (2005), who directly relates default risk to a systematic credit and interest rate risk factor. In addition, the interaction between credit spreads and the short rate is imposed by correlated innovations. In our framework, we model the interaction between default risk and market risk directly through the interaction between the physical and risk-neutral default probabilities through the (latent) market and (latent) credit risk factor.

The remainder of the paper is set up as follows. We introduce the integrated model in Section 2. In section 3 we derive an analytic expression for the integrated loss distribution for an infinitely granular portfolio and deterministic riskfree discount rates. In section 4, we calibrate our full empirical model and provide aggragate and disaggregate loss distributions and risk measures based on Monte-Carlo simulation. Section 5 concludes.

2 Theoretical framework

The model consists of three main building blocks. The blocks are linked through a shared exposure to a common factor structure. We first discuss the credit risk module of the model, followed by the market risk modules for interest rate and credit spread risk.

2.1 Credit risk

For the credit risk part of the model, we use a Bernoulli-mixture model. Consider a *homogeneous* portfolio of n exposures at time t. We assume that the default of a specific exposure over the period $[t, t+h]$ depends on a vector of state variables $\psi_{t+h} \in \mathbb{R}^k$, see McNeil, Frey, and Embrechts (2005) and Gagliardini and Gourieroux (2005). The indicator variables $Y_{i,t+h}$ for $i = 1, \ldots, n$ take the value one if exposure i defaulted before time $t+h$ with $h > 0$, and zero otherwise. In the Bernoulli mixture model, the variables $Y_{i,t+h}$ are conditionally independent Bernoulli draws, where the conditioning is done with respect to the (vector of) state variables ψ_{t+h}. The conditional probability of default for exposure i can thus be written as

$$\Pr\left[Y_{i,t+h} = 1 | \psi_{t+h}\right] = p_j(\psi_{t+h}), \tag{1}$$

where j is the rating of firm i, and $p_j(\cdot)$ is a function $p_j : \mathbb{R}^k \rightarrow [0, 1]$ for $j = 1, \ldots, J$, with J the number of ratings. The common dependence of all exposures i on the same vector of state variables ψ_{t+h} causes defaults to be unconditionally dependent. As a result, the credit loss distribution at the portfolio level is non-degenerate, even if the number of exposures n grows indefinitely. Our interest lies in modeling these credit portfolio losses in relation to any simultaneous market risk that arises in the same portfolio.

The Bernoulli mixture model as presented above focusses on credit losses due to defaults only. In practice, of course, credit losses also arise due to re-rating activity. Regulations even require capital to be set aside for both default and re-rating risk simultaneously, see BCBS (2009b). In this paper, we concentrate on default risk for expositional purposes. The model can easily be complicated further to allow for rating changes, e.g., by replacing the assumption of a mixture Bernoulli model by a mixture ordered probit or mixture ordered logit model. Though this increases the absolute level of credit risk arising from the model, it provides no additional insight into the effect of *dependence* between credit and market risk or the effect of *aggregating* these two sources of risk. As the latter is the prime focus of the current paper, we do not unnecessarily complicate the model at this stage with an additional re-rating module.

So far, we have left the functional form of the link function $p_j(\psi_{t+h})$ unspecified. The literature suggests two directions based on either structural or reduced form credit risk models. Structural models build on the seminal paper of Merton (1974), who uses a brownian motion process for the asset value of the firm. The link function then becomes a probit link function, i.e. $p_j(\psi_{t+h}) = \Phi(\theta_{0j} + \theta_j' \psi_{t+h})$, with $\Phi(\cdot)$ the standard normal distribution function, and $\theta_{0j} \in \mathbb{R}$ and $\theta_j \in \mathbb{R}^k$ fixed, unknown parameters for $j = 1, \ldots, J$. This link function also underlies industry models such as CreditMetrics, see Gupton, Finger, and Bhatia (1997).

A direct alternative for the probit specification is the logit link function as used in for example Koopman and Lucas (2008). The logit and probit specifications are generally deemed to be close, except possibly in the tails. In our current context of default rate modeling, this might be precisely the relevant area. We therefore include the logit specification in our comparison when studying the effect of model specification on aggregating market and credit risk.

The final specification considered here emanates from the reduced form credit risk literature and industry models such as Creditrisk$^+$, see for example McNeil, Frey, and Embrechts (2005). In this literature, default times are typically modelled as a doubly stochastic Poisson process with stochastic default intensity λ. We assume λ is fixed until the time of default, but depends on the hidden state vector ψ_{t+h}. The link function for the probability of default then becomes $p_j(\psi_{t+h}) = 1 - \exp(-\lambda_j(\psi_{t+h}))$, where the default intensity is modelled as $\lambda_j(\psi_{t+h}) = \exp(\theta_{0j} + \theta_j' \psi_{t+h})$ to ensure a non-negative intensity. Examples of this approach are Koopman, Lucas, and Monteiro (2008), and Duffie, Eckner, Horel, and Saita (2009).

All three link functions above fit the general Bernoulli-mixture model, but with different mixing distributions. However, Koyluoglu and Hickman (1998), Gordy (2000), and McNeil et al. (2005, Ch. 8) show that different mixing distributions can result in substantially different joint default behaviour, even if the alternative model specifications are calibrated on the same data. This holds particularly if one is interested in the extreme tail behaviour of the aggregate (portfolio) loss distribution. This is precisely the region of interest for risk management purposes. The differences can, moreover, also be important for assessing the diversification benefits between market and credit risk, and more general for assessing model risk.

2.2 Market risk: interest rates and spreads

In this paper we follow Gründke (2005) and Kupiec (2007), and focus on a large portfolio of corporate bonds to illustrate the key pattern of market and credit risk interaction. Bond, loan, and mortgage portfolios are typically the largest parts of the trading and banking books. Other forms of market risk, however, can also be incorporated in our framework and treated in a similar way to what is presented below.

Consider a portfolio of zero-coupon bonds. Recall that we consider homogeneity within each rating cohort j. The present value of a risky zero-coupon bond issued by firm i with credit rating j is then defined as

$$v_j(t,T) = e^{-(R(t,T)+s_j(t,T))(T-t)}, \tag{2}$$

where $R(t,T)$ is the yield to maturity at time t of a riskless zero-coupon bond with maturity $T-t$ and $s_j(t,T)$ is the credit spread at tenor $T-t$ for a firm with credit rating j. Note that the price of a riskless zero-coupon bond is given by

$$P(t,T) = \mathrm{E}\left[\exp\left(-\int_t^T r_s \mathrm{d}s\right)\right] = e^{-R(t,T)(T-t)}, \tag{3}$$

given some stochastic process for the risk-free short-rate r_t.

We assume that the term structure of interest rates $R(t,T)$ follows a mean reverting one-factor Vasicek (1977) model. Alternative short-rate models could also be used, see for example Shreve (2004), Baxter and Rennie (2007), or Hull (2009). As our focus in this paper is more on the effect of different model specifications in the credit risk module of the analysis, we stick to the current simple short-rate model for illustration purposes.

The short-rate follows the mean-reverting process

$$\mathrm{d}r(t) = \kappa\left(\bar{r} - r(t)\right)\mathrm{d}t + \sigma_r \mathrm{d}W_r(t), \tag{4}$$

where κ is the mean-reversion rate, $\bar{r}$ is the long-term average of the short-rate, σ_r is the diffusion parameter, and $W_r(t)$ is a Brownian motion under the real world probability measure. Under standard no-arbitrage conditions, the term structure of interest rates is given by

$$R(t,T) = \frac{B(t,T)}{T-t} r(t) - \frac{A(t,T)}{T-t}, \tag{5}$$

where

$$
\begin{aligned}
B(t,T) &= \frac{1-e^{-\kappa(T-t)}}{\kappa}, \\
A(t,T) &= \bar{R}\big(B(t,T)-(T-t)\big)-\frac{\sigma_r^2}{4\kappa}B(t,T)^2, \\
\bar{R} &= \bar{r}+\lambda\frac{\sigma_r}{\kappa}-\frac{1}{2}\frac{\sigma_r^2}{\kappa^2},
\end{aligned}
$$

with λ the market price of risk, i.e., the risk premium per unit of volatility, and $\bar{r}$, κ, and λ non-negative scalar parameters.

The short-rate $r(t+h)$ at time $t+h$ is put into the vector of state variables ψ_{t+h} as defined above equation (1), together with a factor reflecting the general state of the default cycle. We thus consider a two-factor ($k=2$) structure throughout the remainder of this paper. Credit risk and market risk can now interact. For example, the (physical) default probability can depend on the short-rate. At the same time the price of a defaultable bond may depend on the general default cycle through the interaction of the credit spread with the general state of the default cycle as a part of ψ_{t+h}.

Credit spreads $s_j(t+h,T)$ vary over time and depend on both elements of ψ_{t+h} through the risk-neutral default probability. We assume that the one-period risk-neutral default probability at time $t+h$ is given by $q_{j,t+h}=q(\eta_{0j}+\eta_j'\psi_{t+h})$, with $\eta_{0j}\in\mathbb{R}$, $\eta_j\in\mathbb{R}^k$, and q a function such that $q:\mathbb{R}^{k+1}\to[0,1]$, where $q(\cdot)$ has the same functional form as the real world link function $p(\cdot)$. The difference between the risk-neutral and the physical default probabilities is then given by the difference in parameters: (θ_{0j},θ_j) versus (η_{0j},η_j), respectively.

To compute the value of the risky zero-coupon bonds at time $t+h$ given the available information about the state of the economy and the history of default information up to $t+h$, we proceed as follows. If no default occurs before expiration of the bond, the investor receives the face value of 1 dollar. In case of default, the holder of the defaultable bond receives a fixed recovery of $1-\delta$ dollars.

Theorem 1 *Given above assumptions and, in addition, assuming constant risk-neutral (one-period) default probabilities from time $t+h$ onwards, i.e., $q_j(t+\tilde{h})\equiv q_{j,t+h}$ for $\tilde{h}=$*

$h, h+1, \ldots, T-h$, the value of the defaultable bond at time $t+h$ is given by

$$
\begin{aligned}
v_j(t+h,T) &= \mathrm{E}_{t+h}\left[e^{-\int_{t+h}^{T} r_s \mathrm{d}s}\left(I_{j,\{\tau>T\}} + (1-\delta) I_{j,\{\tau\leq T\}}\right)\right] \\
&= P(t+h,T)\left[1-\delta\cdot\left(1-\widetilde{Q}_j(t+h,T)\right)\right] \qquad (6)
\end{aligned}
$$

where $\mathrm{E}[\cdot]$ is the expectation under the risk-neutral measure, τ is the default time, T is the bond's maturity date, $\widetilde{Q}_j(t+h,T) = (1-q_{j,t+h})^{T-t-h}$ is the risk-neutral survival probability over the period $t+h$ to T, and δ is the risk-neutral loss given default rate.

Proof: It follows directly from Lemma 9.9 and Corollary 9.10 in McNeil, Frey, and Embrechts (2005) that

$$
\begin{aligned}
v_j(t+h,T) &= \mathrm{E}_{t+h}\left[e^{-\int_{t+h}^{T} r_s \mathrm{d}s}\left(I_{j,\{\tau>T\}} + (1-\delta) I_{j,\{\tau\leq T\}}\right)\right] \\
&= (1-\delta)\mathrm{E}\left[e^{-\int_{t+h}^{T} r_s ds}\middle|\mathcal{F}_{t+h}\right] \\
&+ \delta I_{j,\{\tau>t+h\}}\mathrm{E}\left[e^{-\int_{t+h}^{T} r_s ds}\frac{\Pr\left(\tau>T\middle|\mathcal{F}_{t+h}\right)}{\Pr\left(\tau>t+h\middle|\mathcal{F}_{t+h}\right)}\middle|\mathcal{F}_{t+h}\right].
\end{aligned}
$$

Now assume that the risk-neutral default probability has a constant term structure, i.e. $q_j(t+h,s) = q_{j,t+h}$, then the survival probability is given as

$$
\Pr(\tau > s|\mathcal{F}_t + h) := \widetilde{Q}_j(t+h,s) = (1-q_{j,t+h})^{s-t-h}
$$

and is known at time $t+h$ given the realizations of r_{t+h} and ψ_{t+h}, i.e. $\widetilde{Q}_j(t+h,s)$ is $(\mathcal{F}_{t+h})$-measurable. This implies that the price of the defaultable bond follows as

$$
\begin{aligned}
v_j(t+h,T) &= (1-\delta)\mathrm{E}\left[e^{-\int_{t+h}^{T} r_s ds}\middle|\mathcal{F}_{t+h}\right] + \delta\cdot\widetilde{Q}_j(t+h,T)\mathrm{E}\left[e^{-\int_{t+h}^{T} r_s ds}\middle|\mathcal{F}_{t+h}\right] \\
&= P(t+h,T)\left[1-\delta\cdot\left(1-\widetilde{Q}_j(t+h,T)\right)\right]
\end{aligned}
$$

with $P(t+h,T) := \mathrm{E}\left[\exp\left[-\int_{t+h}^{T} r_s ds\right]\middle|\mathcal{F}_{t+h}\right]$.

■

The theorem states that even if the risk-neutral default probability depends on the short-rate via ψ_{t+h}, the simplifying assumption of a constant term risk-neutral default probability from $t+h$ onwards ensures that the price of the risky zero-coupon bond can be decomposed as the product of a risk free bond and one minus the risk neutral expected loss, see the standard result under a recovery of treasury assumption in for example Jarrow and Turnbull (1995) and Schönbucher (2003). The key to this result is that at time $t+h$, ψ_{t+h} is known. At time t, however, both the applicable default rate over $[t, t+h]$ and the default probability term structure at time $t+h$ are unknown and depend on ψ_{t+h}. This allow us to study the interaction between these two types of risks in a tractable manner.

To operationalize the dependence of the credit spread $s_j(t+h, T)$ on the risk-neutral default probability $q_{j,t+h}$, and thus the interest rate, and the default cycle, we combine (2), (3) and (6) into

$$s_j(t+h, T) = \frac{-\ln\left[1 - \delta \cdot \left(1 - \widetilde{Q}_j(t+h, T)\right)\right]}{T - t - h}. \tag{7}$$

This equation is used later on the calibrate the model's parameters empirically.

The current model structure is built on the simplifying assumption of a constant term structure of the risk-neutral default probabilities. This, however, is already sufficiently detailed to study the effect of functional form specification on the interaction between market and credit risk. It allows us to obtain all key insights at minimum level of analytical complexity.[3] Moreover, the current framework is in a certain sense even more flexible by allowing different functional forms $p(\cdot)$ and $q(\cdot)$ for the relation between default probabilities and state variables ψ_{t+h}. Previous papers typically built on a continuous time framework for credit risk, thus restricting the attention to either the Poisson or the probit link function. In our current framework, each of these link functions can easily be included, as can alternative specifications.

3 Asymptotic loss distribution

The full model as introduced in Section 2 is too complicated to solve analytically. We therefore first derive the analytic loss distribution for a special case of the general model. Next, in Section 4 we obtain our results for the fully calibrated empirical model using simulations. The analytical results obtained in the current section help us to understand the main drivers of the empirical results in Section 4.

Consider a setting with a fixed short-rate $r(t) \equiv \bar{r}$. The vector of state variables ψ_{t+h} is now one-dimensional ($k = 1$) and can be interpreted as a scalar reflecting the uncertainty

[3] Alternative, more complicated specifications to link market and credit risk are found in seminal papers like Jarrow, Lando, and Turnbull (1997), Lando (1998), and Jarrow and Turnbull (2000).

about the future state of the default cycle. Note that the model still includes both a credit risk and a market risk component through the random nature of credit spreads, see equation (7). We assume that the latent factor $\psi_{t+h} \in \mathbb{R}$ has distribution function $F_\psi(\cdot)$, and we consider the loss distribution of an infinitely granular homogeneous portfolio ($n \to \infty$) of unit-notional zero-coupon bonds that mature at time T.

Using the arguments in Lucas, Klaassen, Spreij, and Straetmans (2001), the total loss on this portfolio of n names as a percentage of the notional is given by $\hat{L}_{t+h,n} = n^{-1} \sum_{i=1}^{n} L_{t+h,i}$, where $L_{t+h,i}$ is the loss on counter party i at time $t+h$. Considering the limiting case $n \to \infty$ of an infinitely granular portfolio, we obtain under the recovery of treasury assumption that

$$\begin{aligned} L(\psi_{t+h}) &= \lim_{n\to\infty} \hat{L}_n = \lim_{n\to\infty} \mathrm{E}[L_{t+h,i}|\psi_{t+h}] \\ &= \sum_{j=1}^{J} \pi_j \cdot \Big[v_j(t,T) - (1 - p_j(\psi_{t+h})) \cdot v_j(t+h,T) \\ &\qquad - p_j(\psi_{t+h}) \cdot (1-\delta) \cdot P(t+h,T) \Big], \end{aligned} \tag{8}$$

where J is the number of rating categories, and π_j the percentage of firms in the portfolio with rating j for $j = 1, \ldots, J$. We assume that both p_j and q_j are decreasing in the state variable ψ_{t+h}, as is also the case based on our empirical estimates in Section 4.1. The limiting loss $L(\psi_{t+h})$ is then a monotonically decreasing function of ψ_{t+h}. Values of the state variable that cause a high default probability over the period $[t, t+h]$ also cause a high value of the risk-neutral default probability at time $t+h$ and therefore of the default spreads $s_j(t+h,T)$. Combined, these two effects result in large credit risk induced losses for the firms that default over the period $[t, t+h]$, as well as large market risk induced losses for the counter parties that survive till time $t+h$. The monotonicity can be exploited directly to derive the total loss distribution. The next result follows immediately.

Theorem 2 *Given the assumptions above, the distribution of the aggregated market and credit risk loss is given by*

$$F_L(\ell) = \Pr\left[L(\psi_{t+h}) < \ell\ \right] = \Pr\left[\psi_{t+h} < L^{-1}(\ell)\right] = F_\psi\left(L^{-1}(\ell)\right), \tag{9}$$

where $F_L(\cdot)$ and $F_\psi(\cdot)$ are the distribution functions of $L = L(\psi_{t+h})$ and ψ_{t+h}, respectively, and $L^{-1}(\cdot)$ is the inverse function $L^{-1}(L(\psi_{t+h})) = \psi_{t+h}$. The probability density function of

L is given by

$$f_L(\ell) = \frac{\partial}{\partial \ell} F_L(\ell) = \frac{\partial}{\partial \ell} F_\psi(L^{-1}(\ell)) = \frac{f_\psi\left(L^{-1}(\ell)\right)}{|L'(L^{-1}(\ell))|}, \tag{10}$$

where $L'(z) = \partial L(z)/\partial z$.

Evaluating expression (10) in the current one-factor case with monotonic loss function $L(\psi_{t+h})$ is straightforward. Using a grid of values $\psi_{t+h}^{(1)} \leq \ldots \leq \psi_{t+h}^{(m)}$, we plot $f_\psi(\psi_{t+h}^{(g)})/|L'(\psi_{t+h}^{(g)})|$ against $L(\psi_{t+h}^{(g)})$ for $g = 1, \ldots, m$. The crucial ingredient in all these computations is the functional dependence of L on ψ, which is the focus of the present paper.

Using (8), the derivative $L'(\cdot)$ is given by

$$\begin{aligned} L'(\psi_{t+h}) \quad &= \quad \sum_{j=1}^{J} \pi_j \cdot \Bigg[(T-t-h) \cdot (1 - p_j(\psi_{t+h})) \cdot v_j(t+h,T) \frac{\partial s_j(t+h,T)}{\partial \psi_{t+h}} \\ &\qquad + \Big(v_j(t+h,T) - (1-\delta) P(t+h,T) \Big) \cdot \frac{\partial p_j(\psi_{t+h})}{\partial \psi_{t+h}} \Bigg], \end{aligned} \tag{11}$$

where both $\partial s_j / \partial \psi_{t+h}$ and $\partial p_j / \partial \psi_{t+h}$ depend on the functional specification chosen for the link function in the credit risk part of the model. Recall that $P(t+h,T)$ is deterministic since we keep the term structure of riskfree interest rates fixed. We can now study the effect of market risk and credit risk separately, as well as their interaction. In the case of only market risk, we have $p_j(\psi_{t+h}) = 0$ and $\partial p_j / \partial \psi_{t+h} = 0$, such that (11) simplifies to

$$L'(\psi_{t+h}) = \sum_{j=1}^{J} \pi_j \cdot (T-t-h) \cdot v_j(t+h,T) \frac{\partial s_j(t+h,T)}{\partial \psi_{t+h}}. \tag{12}$$

An analogous result holds if we only consider credit risk, in which case $v_j(t+h,T) = \bar{v}_{j,t+h}$ is non-stochastic and $\partial s_j / \partial \psi_{t+h} = 0$. Equation (11) then simplifies to

$$L'(\psi_{t+h}) = \sum_{j=1}^{J} \pi_j \cdot \Big(\bar{v}_{j,t+h} - (1-\delta) P(t+h,T) \Big) \cdot \frac{\partial p_j(\psi_{t+h})}{\partial \psi_{t+h}}. \tag{13}$$

If both types of risk are present, (11) cannot be simplified further.

As an illustration, we plot the asymptotic loss densities and the tail of the loss distributions for credit risk, market risk, and aggregated risk in Figures 1 and 2. In Figure 1, we consider a credit portfolio of low quality and assume that the real world and risk-neutral default probabilities are 18%, thus abstracting from any market risk premium. The real world and risk-neutral default correlations are set to 4%, implying a joint default probability

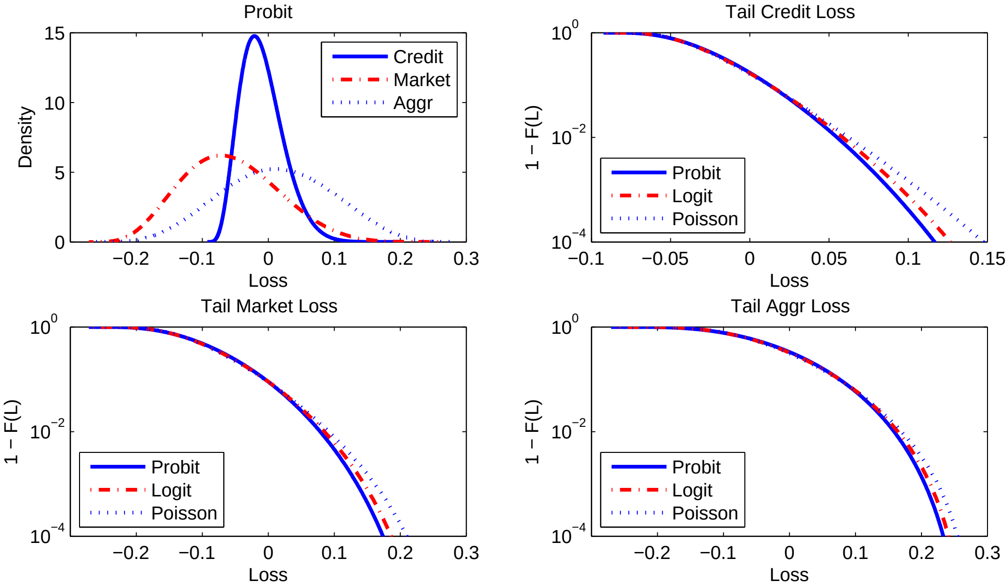

Figure 1: Loss distributions for a low-quality portfolio
The upper left panel shows the asymptotic loss density of credit, market and aggregated risk in a credit portfolio for the probit-mixture distribution. The other panels show the tail of the distribution functions of credit, market and aggregated risk for the probit-, logit- and Poisson-mixture distributions. In this example, it is assumed that the unconditional real world and risk-neutral default probability is 18%. The assumed default correlation is set to 4% implying a joint default probability of 5%.

of 5%. In Figure 2, we consider a portfolio of medium credit quality. The real world and risk-neutral default probabilities are set to 5% and 10%, respectively, implying a risk premium factor of 2. The real world and risk-neutral default correlations are 2.5% and 1.25%, respectively, implying joint default probabilities of 0.4% and 1.1%.[4]

To make the different model specifications mutually comparable, the model parameters are calibrated as follows. For each model, i.e., probit, logit, or Poisson link function, we set θ and η such that the unconditional default probability, default correlation, and the pairwise default probability are the same; see the appendix for further details. The resulting parameters are shown in Table 1. We assume a portfolio of zero-coupon bonds with $T = 3$ years to maturity, a risk horizon of $h = 1$ year, a constant and flat term structure of interest rates at $\bar{r} = 4\%$, and a loss given default rate of $\delta = 60\%$. The credit spread specified in equation (7) is a function of the risk-neutral default probability. At time $t = 0$ the risk-neutral default probability is set to its expected value, i.e. 18% and 10% in examples one and two respectively. For measuring credit risk (without market risk), the same term structure of credit spreads is used at time $t = 0$ and at the risk horizon $t = h$.

[4] Hull, Predescu, and White (2005) show that the market risk premium increases with credit quality.

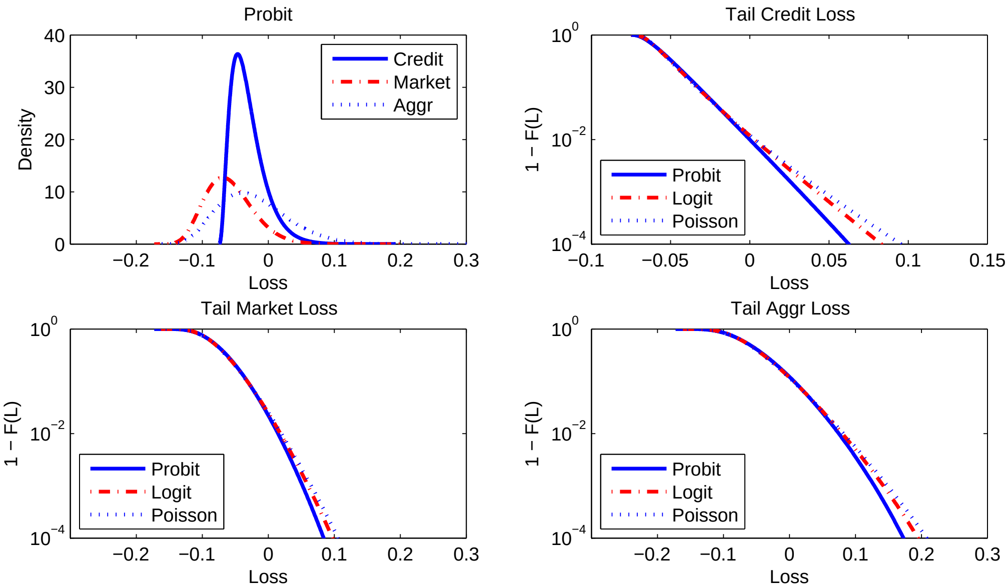

Figure 2: Loss distributions for a medium-quality portfolio

The upper left panel shows the asymptotic loss density of credit, market and aggregated risk in a credit portfolio for the probit-mixture distribution. The other panels show the tail of the distribution functions of credit, market and aggregated risk for the probit-, logit- and Poisson-mixture distributions. In this example, the real world and risk-neutral default probabilities and correlations are different. The unconditional real world and risk-neutral default probabilities are respectively 5% and 10% implying a risk premium of 2. The real world and risk-neutral default correlations are respectively 2.5% and 1.25% implying joint default probabilities of 0.4% and 1.1% respectively.

Table 1: Calibrated model parameters

Parameters of real world (θ_0, θ_1) and risk-neutral (η_0, η_1) link functions. The parameters are calibrated such that the unconditional (joint) default probabilities and default correlations are the same for the probit, logit and Poisson mixture distributions. Two examples are considered. In Example I, it is assumed that $p = q = 18\%$ and $\rho_p = \rho_q = 4\%$. In Example II, it is assumed that $p = 5\%$, $q = 10\%$, $\rho_p = 2.5\%$ and $\rho_q = 1.25\%$. See the main text for further details on the parameters.

	Example I		Example II			
	$\theta_0 = \eta_0$	$\theta_1 = \eta_1$	θ_0	θ_1	η_0	η_1
Probit	-0.956	-0.301	-1.732	-0.330	-1.305	-0.192
Logit	-1.603	-0.529	-3.150	-0.684	-2.251	-0.370
Poisson	-1.703	-0.469	-3.171	-0.654	-2.304	-0.348

Both in Figure 1 and 2, the loss density for aggregated market and credit risk has a fatter tail compared to the loss densities for credit or market risk only. The effect is more pronounced for the low-quality portfolio in Figure 1. It is also clear that loss quantiles for combined credit and market risk are not simply the sum of the separate credit risk and market risk quantiles. This is due to the dependence between the two types of risk, and the non-linear relationship between the common risk factor and the aggregate loss expression.

The remaining panels in Figures 1 and 2 present the differences in tail behaviour for the different model specifications. For all cases considered, the loss quantiles of the Poisson distribution are farther out than those of the logit specification, which in turn are farther out than the probit loss quantiles. Differences increase the further we go into the tail. This holds for market, credit, and aggregated risk alike. Up to 90% or even 99% confidence levels, the effect of the different model specifications is modest. At the 99% confidence level, the VaR increases by 5%-10% if the probit specification is replaced by the Poisson specification for the low-quality portfolio (example I), and by about 5% for the medium-quality portfolio (example II), see Table 2. If we move further out into the tails, the impact of the functional form increases substantially. At the 99.99% level, the credit loss VaR is 29% higher for the Poisson than for the probit specification. For market and aggregated risk, the difference is somewhat smaller at 14% or 15%, respectively, but still substantial, particularly because the magnitude of the aggregated VaR is larger than that of the credit loss quantile.

In Table 3 we consider the effect of model specification on VaR reductions from aggregating market and credit risk. The further we move into the tail, the larger the VaR reductions become. These reductions are the largest for the Poisson mixture distribution. For a low-quality credit portfolio the benefits of aggregating market and credit risk vary between 16% to 45% for the Poisson link function compared to 16% to 39% for the probit link function. The benefits of the Poisson specification are 5% to 14% larger compared to the probit specification. For the medium-quality credit portfolio we obtain similar results. Aggregation benefits range between the 5% to 21% for the probit specification and 5% to 26% for the Poisson specification, where the Poisson specification results in 5% to 24% larger benefits. The results for the logit specification are in between those of the probit and the Poisson specification. Summarizing, model specification has a clear impact on the VaR reductions of aggregating market and credit risk, and that the differences become more pronounced the further we look into the tail of the loss distribution.

Table 2: Asymptotic loss distribution characteristics

This Table shows the expected loss, EL; and the $VaR^{1-\alpha}$ at different confidence levels α for credit, market and aggregated risk and different mixing distributions. Two examples are considered. In Example I, a low-quality credit portfolio, it is assumed that $p = q = 18\%$ and $\rho_p = \rho_q = 4\%$. In Example II, a medium-quality credit portfolio, it is assumed that $p = 5\%$, $q = 10\%$, $\rho_p = 2.5\%$ and $\rho_q = 1.25\%$. See the main text for further details on the parameters.

	Example I - Low quality					Example II - Medium quality				
	Risk metric in %			Ratio to probit		Risk metric in %			Ratio to probit	
	probit	logit	Poisson	logit	Poisson	probit	logit	Poisson	logit	Poisson
	Credit Risk									
EL	-2.6	-2.6	-2.6	1.00	1.00	-5.3	-5.3	-5.3	1.00	1.00
$VaR^{10^{-1}}$	3.9	3.9	3.8	1.00	0.98	2.0	1.9	1.9	0.96	0.94
$VaR^{10^{-2}}$	8.1	8.5	8.9	1.04	1.09	5.3	5.6	5.6	1.05	1.06
$VaR^{10^{-3}}$	11.5	12.2	13.4	1.06	1.16	8.4	9.5	9.9	1.12	1.18
$VaR^{10^{-4}}$	14.3	15.3	17.4	1.07	1.21	11.5	13.6	14.8	1.18	1.29
$VaR^{10^{-5}}$	16.7	17.8	20.8	1.07	1.25	14.5	17.6	20.0	1.21	1.38
	Market Risk									
EL	-9.7	-9.7	-9.7	1.00	1.00	-7.6	-7.6	-7.6	1.00	1.00
$VaR^{10^{-1}}$	9.2	9.2	9.1	1.00	0.99	4.4	4.4	4.4	1.00	0.99
$VaR^{10^{-2}}$	17.5	18.2	18.8	1.04	1.07	9.1	9.4	9.6	1.04	1.06
$VaR^{10^{-3}}$	23.1	24.2	25.8	1.05	1.12	12.8	13.6	14.1	1.07	1.10
$VaR^{10^{-4}}$	27.1	28.3	30.7	1.05	1.13	16.0	17.3	18.2	1.08	1.14
$VaR^{10^{-5}}$	29.9	31.1	33.8	1.04	1.13	18.8	20.6	22.0	1.10	1.17
	Aggregated Risk									
EL	-3.4	-3.4	-3.4	1.00	1.00	-5.4	-5.4	-5.4	1.00	1.00
$VaR^{10^{-1}}$	11.0	11.0	10.9	1.00	0.99	6.1	6.0	6.0	0.99	0.98
$VaR^{10^{-2}}$	19.3	19.8	20.4	1.03	1.06	12.9	13.4	13.6	1.04	1.05
$VaR^{10^{-3}}$	23.9	24.7	25.9	1.03	1.08	18.3	19.7	20.3	1.07	1.11
$VaR^{10^{-4}}$	26.7	27.5	28.9	1.03	1.08	22.7	24.9	26.2	1.10	1.15
$VaR^{10^{-5}}$	28.5	29.1	30.3	1.02	1.07	26.4	29.2	31.2	1.11	1.18

Table 3: Tail area reductions in VaR

This table presents the 'diversification benefits' (in %) at different confidence levels α, measured as the percentage by which the aggregate loss VaR is smaller than the sum of the separate credit risk and market risk VaRs. The results are shown for two examples: a low-quality and a medium-quality credit portfolio. See for further details the caption of Table 2 and the main text.

	Benefits (in %)			Ratio to probit	
$1-\alpha$	probit	logit	Poisson	logit	Poisson
Example I - Low quality					
10^{-1}	15.8	15.8	15.7	1.00	0.99
10^{-2}	24.9	25.6	26.3	1.03	1.05
10^{-3}	30.9	32.1	33.9	1.04	1.10
10^{-4}	35.4	36.8	39.9	1.04	1.13
10^{-5}	38.9	40.5	44.5	1.04	1.14
Example II - Medium quality					
10^{-1}	4.8	4.7	4.7	0.99	0.98
10^{-2}	9.8	10.1	10.3	1.04	1.05
10^{-3}	13.8	14.9	15.4	1.08	1.12
10^{-4}	17.4	19.3	20.6	1.11	1.18
10^{-5}	20.7	23.5	25.6	1.14	1.24

4 Empirical Results

In this section we consider the portfolio loss distribution for an empirically calibrated model with all three risk factors operating simultaneously: interest rate risk, default risk, and spread risk. The state variable $\psi_{t+h} \in \mathbb{R}^2$ now contains both the default cycle and the short-rate. Note that the (one-factor) term structure is now fully stochastic and possibly correlated with the default cycle. Due to its complexity, the full model is too complicated to handle analytically. Therefore, we derive the loss distributions using Monte Carlo simulation.

4.1 Parameter Calibration

4.1.1 Term structure of interest rates

The parameters of a one-factor Vasicek model are estimated using a state space representation of equations (4) and (5). See Durbin and Koopman (2001) for an advanced treatment of state space models, and Bolder (2001) for an implementation of the state space framework for multi-factor term-structure models. The observation equation for the one-factor Vasicek

Table 4: Interest rate model 1996–2010: empirical estimates

The superscript a, b, and c denotes significant at the 1%, 5%, 10% significance level.

	One-factor Vasicek model	
	Estimate	Standard error
$\bar{r}$	0.011^a	0.000
κ	0.180^a	0.008
σ_r	0.012^a	0.001
λ	0.795^a	0.000
Log-likelihood	2294.32	
Observations	540	

model is given by

$$\begin{bmatrix} R(t,T_1) \\ R(t,T_2) \\ \vdots \\ R(t,T_m) \end{bmatrix} = \begin{bmatrix} \frac{-A(t,T_1)}{T_1-t} \\ \frac{-A(t,T_2)}{T_2-t} \\ \vdots \\ \frac{-A(t,T_m)}{T_m-t} \end{bmatrix} + \begin{bmatrix} \frac{B(t,T_1)}{T_1-t} \\ \frac{B(t,T_2)}{T_2-t} \\ \vdots \\ \frac{B(t,T_m)}{T_m-t} \end{bmatrix} r(t) + \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix},$$

with state equation

$$r(t+h) = \bar{r}(1-e^{-\kappa h}) + e^{-\kappa h} r(t) + \varepsilon_r(t),$$

where the m-dimensional vector $u(t) \sim \mathrm{N}(0, H)$, $H = \mathrm{diag}(\sigma_1^2, ..., \sigma_j^2)$, and the random variable $\varepsilon_r(t) \sim \mathrm{N}(0, \sigma_\varepsilon^2)$, where $\sigma_\varepsilon^2 = \frac{\sigma_r^2}{2\kappa}\left(1-e^{-2\kappa\Delta t}\right)$. The parameters $\bar{r}$, κ, λ, σ_r^2, σ_ε^2, and the covariance matrix H can be estimated by standard maximum likelihood using the Kalman filter. The distribution of the initial state is set equal to its unconditional distribution. Once the parameters are estimated, the estimates $\widehat{r}(t)$ and $\widehat{\varepsilon}_r(t)$ of the short-rate and its disturbance, respectively, follow directly from the Kalman smoother.

The data used for this part of the model are 1996–2010 US treasury yields for maturities 3m, 6m, 1y, 2y, 3y, 5y, 7y, 10y, and 20y. The data are downloaded from the Federal Reserve Economic Data (FRED) database. Table 4 shows the main results. All parameters are significant at the 1% level. The long run average short-rate $\bar{r}$ is estimated at around 1.1%. This clearly reflects the downward trend in interest rates over the sample. The average rate under the risk neutral measure is $\bar{r} + \lambda\sigma_r/\kappa \approx 6.4\%$.

Figure 3 plots the smoothed estimates of the short-rate together with the 3-month and the 20-year Treasury yield. The short-rate follows the 3-month Treasury yield closely. The figure also illustrates one of the short-comings of the Vasicek model, which is the probability to generate negative short-rates. In our sample, the short-rate was negative and close to zero during 2010 as result of the ongoing uncertainty in the financial markets and sluggish

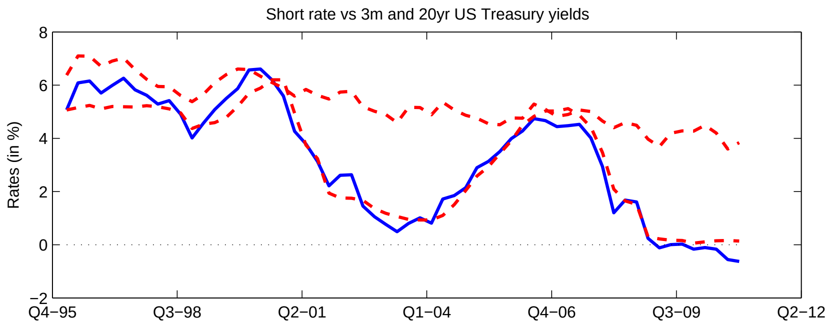

Figure 3: short-rate estimates
Estimated short-term interest rate (solid blue line) for the one-factor Vasicek model and the observed 3m and 20y US Treasury yields (dashed red lines).

economic growth prospects.

Another important feature that emerges from Figure 3 is the structural decline in the short-rate. To prevent any spurious relation between default probabilities and short-rates due to such trends, we use the detrended short-rate $\tilde{r}_t = \hat{r}_t - \hat{r}_t^{HP}$ when modeling the dependence between short-rates and default probabilities. The trend $\hat{r}_t^{HP}$ is computed using the Hodrick-Prescott (HP) filter, where the HP parameter is set to its usual value of 1,600 for quarterly data. The filtered trend and cyclical component of the short-rate are shown in Figure 4.

4.1.2 Physical default probabilities

To estimate the parameter vector θ_j in (1), we use Moody's rating data of 13,229, predominantly US, firms covering 16 years from January 1995 to December 2010. At a quarterly frequency we count the number of defaults per rating cohort during the previous year. We construct a quarterly time-series of annual default frequencies using time-series for the annual number of defaults $y_{t,j}$ and the number of exposures $n_{t,j}$ for rating cohort j at time t. These time-series are used in estimating the parameters of the physical annualized conditional default probabilities $p_{j,t} = p_j(\psi_t)$. To account for the problem of overlapping observations, we compute Newey-West corrected standard errors based on four lags, see Andrews (1991). The results are robust to using twelve lags as well. The conditional default probabilities directly determine the unconditional default probabilities π_j, the pairwise joint default probabilities $\pi_{j,2}$, and the default correlation $\rho_{\pi,j}$.

We estimate θ_j using a conditional maximum likelihood procedure, where we condition on the estimate of the (detrended) short-rate $\psi_{2,t} = \tilde{r}_t$ obtained from Section 4.1.1. We

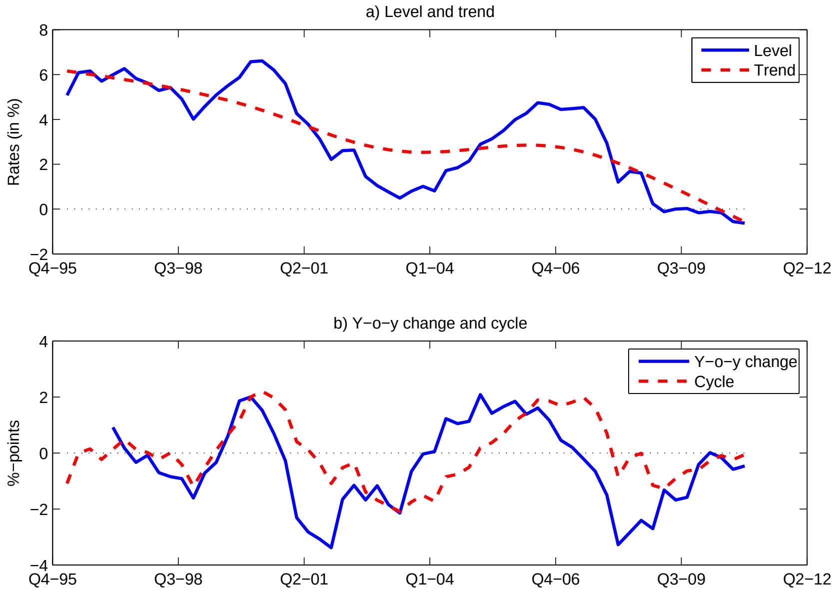

Figure 4: Interest rate components

Trend, cyclical component, and year-on-year change of the short-term interest rate. Panel a) shows the level and the trend of the short-rate. Panel b) shows the year-on-year change (solid blue line) in the short rate, and the detrended short-rate (dashed red line).

maximize

$$L(\theta_j|\hat{\psi}_{2,t}) = \prod_t \int \prod_j^k \binom{n_{j,t}}{y_{j,t}} p_{j,t}^{y_{j,t}} (1-p_{j,t})^{n_{j,t}-y_{j,t}} \phi(\psi_{1,t}) \mathrm{d}\psi_{1,t}, \tag{14}$$

such that all rating cohorts j are subject to the same latent factor $\psi_{1,t}$. Furthermore, the latent factor $\psi_{1,t}$ is assumed to be independently and identically standard normally distributed. The latter may not hold in practice due to the cyclical nature of systematic default risk. However, in our current setting we are more interested in the unconditional variation of systematic risk factors, for which (14) provides a sensible estimation strategy. Concentrating on the unconditional variation in credit and market risk factors is also sensible given the regulatory focus in Basel III on through-the-cycle credit risk for capital requirements.

Due to the limited number of investment grade defaults we group all investment grade ratings Aaa–Baa into one rating category IG. Similarly, we group the ratings Caa–C into one rating category because long time-series for credit spreads are only available for this broader category. Aggregated annual default rates for the four rating categories are shown in Figure 5. The figure clearly shows the familiar bursts in default activity after the collapse of the dotcom bubble and in the aftermath of the recent financial crisis.

Table 5 shows the estimated parameters together with the estimated unconditional de-

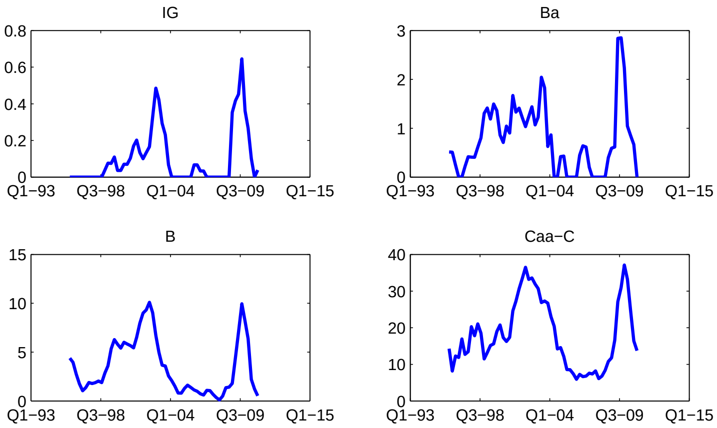

Figure 5: Default rates per rating cohort

Default rate (in percentages) per rating cohort for the period 1996–2010. IG includes the investment grade ratings Aaa to Baa.

fault probability $\widehat{\pi}$, the pairwise joint default probability $\widehat{\pi}_2$, and the default correlation $\widehat{\rho}$ for the different link functions under consideration. The unconditional (joint) default probabilities are computed by numerical integration of (A2) in the Appendix with respect to the bivariate latent factor ψ_{t+h}. The default correlations can then be computed given the unconditional (joint) default probabilities according to equation (A1) in the Appendix.

The last columns of the table show that the different link functions all produce similar unconditional (joint) default probabilities for the different rating cohorts. As seen earlier, the clustering appears somewhat stronger for the Poisson than for the probit specification. We come back to these results in Section 4.1.3 when we compare risk-neutral and physical probabilities of default.

The estimated coefficients for the latent factor and the detrended short-rate have the expected negative sign and are all significant at the 1% or 5% level. An increase in the latent factor $\psi_{1,t}$ reflects better economic and business conditions and therefore lower default probabilities. The (detrended) short-rate factor $\psi_{2,t}$ has a negative marginal effect on the probability of default. A possible explanation for this is the opportunity cost of capital: an increase in the risk-free rate increases the expected rate of return on the investments of a firm, and thus lowers the probability of default, see the arguments in Jarrow and Turnbull (2000).

Table 5: Parameter estimates: physical probabilities

Estimated parameters for the link functions of the real world default probabilities p_t specified as $p_{t+h} = p(\theta_0 + \theta_1\psi_{1,t+h} + \theta_2\psi_{2,t+h})$, where $\psi_{1,t+h}$ is the default cycle, and $\psi_{2,t+h}$ is the detrended short-rate. Standard errors are based on Newey-West correction using 4 lags. The superscript a, b, and c denotes significant at the 1%, 5%, and 10% significance level.

	θ_0		θ_1		θ_2		Implied (in %)		
	Est.	S.e.	Est.	S.e.	Est.	S.e.	$\widehat{\pi}$	$\widehat{\pi}_2$	$\widehat{\rho}_\pi$
Probit		LogLik	-709.61						
IG	-3.27^a	0.09	-0.28^a	0.07	-0.23^a	0.04	0.11	0.00	0.25
Ba	-2.52^a	0.04	-0.27^a	0.04	-0.12^a	0.03	0.78	0.01	0.56
B	-1.93^a	0.02	-0.36^a	0.01	-0.10^a	0.01	3.49	0.21	2.48
Caa-C	-0.96^a	0.03	-0.27^a	0.02	-0.18^a	0.02	18.09	3.96	4.68
Logit		LogLik	-709.74						
IG	-7.59^a	0.32	-1.00^a	0.27	-0.82^a	0.13	0.12	0.00	0.40
Ba	-5.16^a	0.12	-0.75^a	0.12	-0.35^a	0.07	0.79	0.01	0.68
B	-3.65^a	0.04	-0.85^a	0.04	-0.25^a	0.02	3.53	0.23	3.06
Caa-C	-1.62^a	0.05	-0.49^a	0.03	-0.33^a	0.03	18.07	4.00	4.99
Poisson		LogLik	-709.48						
IG	-7.60^a	0.32	-1.01^a	0.28	-0.82^a	0.13	0.12	0.00	0.43
Ba	-5.17^a	0.12	-0.75^a	0.12	-0.35^a	0.07	0.79	0.01	0.71
B	-3.67^a	0.04	-0.84^a	0.04	-0.25^a	0.02	3.56	0.24	3.27
Caa-C	-1.72^a	0.05	-0.44^a	0.03	-0.30^a	0.02	18.07	4.05	5.29

4.1.3 Risk-neutral default probabilities

We estimate the parameter vector η_j linking the risk-neutral default probabilities $q_j(\psi_t)$ to the state variable ψ_t by non-linear least squares using the relation

$$s_j(t,T) = -\frac{\ln\left[1 - \delta \cdot \left(1 - (1 - q_{j,t})^{T-t}\right)\right]}{T-t}, \tag{15}$$

where the state vector ψ_t includes the estimated (detrended) short-rate and default cycle as obtained from the interest rate model and default rate model in Sections 4.1.1 and 4.1.2, respectively.

The data consist of daily US credit spreads for ratings Aaa to Caa as obtained from the FRED database for 1996–2010. We observe so-called option adjusted credit spreads as constructed by the Bank of America Merrill Lynch. These are defined as the spread at which a benchmark security would be trading if it had no embedded optionality. Daily data are transformed into quarterly averages. For the investment grade (IG) spread, we take a weighted average of Aaa-Baa credit spreads using the number of firms in each rating category as weights. The differences between using unweighted versus weighted averages are small.

The advantage of option-adjusted credit spreads is that reasonably long time-series are available for parameter estimation. To use these data in the non-linear regression (15), however, we need to make an assumption about the average maturity in each rating bucket. We therefore subtract the spreads from the corporate bond yields, which are also provided in the FRED database by Bank of America Merrill Lynch. We matched the resulting estimate of the bond-implied risk free rates to different maturity US Treasury yields. The best fit was obtained for five to seven year maturities. Therefore, we set the maturity T in the non-linear regressions (15) to six years. Moreover, we set the loss given default rate δ to 60%, which is broadly in line with the average rates observed during 1996 to 2010: averages of 62.6%, 61.1%, 57.4%, and 56.8% for rating categories IG, Ba, B and Caa-C, respectively, see Moody's (2011).

The results are shown in Table 6. The implied (joint) default probabilities and default correlation are again similar across all model specifications due to the calibration on the same data set. Furthermore, the coefficients for the credit cycle and the short-rate all have the expected negative sign. The negative short-rate coefficient η_2 implies a negative correlation between the credit spreads and the short-rate, which is in line with other earlier empirical studies (see Jarrow and Turnbull (2000)). The coefficients η_1 and η_2 for the credit cycle and short-rate, respectively, become larger in absolute value for lower credit rating categories. This is very interesting, as it is entirely opposite to the result obtained for the physical probabilities in Table 5. There, the higher rating categories obtained the higher systematic risk loadings, similar to the approach taken in the Pillar I Basel regulatory framework. The result implies that defaults of high-quality firms have a larger systematic component, but the price of default risk has a larger systematic component for low-quality firms.

Table 7 shows the implied risk premia by computing the ratio of the unconditional risk neutral and real world probabilities of default from Tables 5 and 6. The ratios are similar across the different link functions for the rating cohorts B and Caa-C, but differ substantially between rating cohorts. The ratio declines from approximately 23 for rating cohort IG to 2 for rating cohort Caa-C. Not surprisingly, we find somewhat larger ratios compared to earlier empirical results due to the financial crisis being part of our data set. For example, Hull, Predescu, and White (2005) found ratios ranging from 16.8 for credit rating Aaa to 1.3 for credit rating Caa and lower based on a sample from December 1996 to July 2004, see Hull (2009).

The real world and risk neutral default correlations reported in Tables 5 and 6 are sum-

Table 6: Parameter estimates: risk neutral probabilities

Estimated parameters for the link functions of the real world default probabilities q_t specified as $q_{t+h} = q(\eta_0 + \eta_1\psi_{1,t+h} + \eta_2\psi_{2,t+h})$, where $\psi_{1,t+h}$ is the default cycle, and $\psi_{2,t+h}$ is the detrended short-rate. The parameters are estimated assuming a loss given default rate of 60% and a maturity of 6 years. Standard errors are based on a Newey-West correction using 4 lags. The superscript a, b, and c denotes significant at the 1%, 5%, and 10% significance level.

	η_0		η_1		η_2		Implied (in %)		
	Est.	S.e.	Est.	S.e.	Est.	S.e.	$\widehat{q}$	$\widehat{q}_2$	$\widehat{\rho}_q$
Probit									
IG	-1.95^a	0.07	-0.04	0.05	-0.08^c	0.05	2.62	0.07	0.12
Ba	-1.50^a	0.07	-0.08^c	0.05	-0.12^a	0.05	6.86	0.51	0.66
B	-1.25^a	0.07	-0.11^b	0.05	-0.13^a	0.05	10.88	1.30	1.21
Caa-C	-0.49^b	0.27	-0.41^b	0.18	-0.28^c	0.17	32.85	13.52	12.37
Logit									
IG	-3.63^a	0.17	-0.08	0.12	-0.18^c	0.12	2.62	0.07	0.12
Ba	-2.65^a	0.15	-0.16^c	0.11	-0.25^a	0.09	6.86	0.51	0.67
B	-2.15^a	0.14	-0.22^b	0.11	-0.27^a	0.10	10.88	1.31	1.24
Caa-C	-0.82^b	0.44	-0.69^a	0.29	-0.47^c	0.31	32.76	13.55	12.79
Poisson									
IG	-3.65^a	0.17	-0.08	0.12	-0.18^c	0.12	2.62	0.07	0.12
Ba	-2.68^a	0.14	-0.16^c	0.11	-0.24^a	0.09	6.86	0.51	0.68
B	-2.20^a	0.13	-0.21^b	0.10	-0.24^a	0.09	10.89	1.31	1.26
Caa-C	-1.02^a	0.37	-0.61^a	0.24	-0.41^c	0.28	33.64	14.80	15.61

Table 7: Ratio and difference physical and risk neutral default probabilities

The table presents the ratios of the unconditional risk-neutral default probability over the real world default probability as computed in Tables 5 and 6 and the difference between both.

	Ratio			Difference (in bps)		
	Probit	Logit	Poisson	Probit	Logit	Poisson
IG	24.1	22.5	22.2	251	250	250
Ba	8.8	8.7	8.6	608	607	607
B	3.1	3.1	3.1	739	735	735
Caa-C	1.8	1.8	1.9	1476	1469	1556

Table 8: Estimated default correlations (in %) for different rating cohorts

This Table summarizes the results for the real world and risk-neutral default correlations reported in Tables 5 and 6.

	Real world			Risk-neutral		
	Probit	Logit	Poisson	Probit	Logit	Poisson
IG	0.25	0.40	0.43	0.12	0.12	0.12
Ba	0.56	0.68	0.71	0.66	0.67	0.68
B	2.48	3.06	3.27	1.21	1.24	1.26
Caa-C	4.68	4.99	5.29	12.37	12.79	15.61

marized in Table 8. For the real world correlations, the probit specification departs from the other two model specifications. In particular, by imposing a probit link, less clustering is implied for all rating categories considered. This is in line with the lower levels of capital for the probit model when using the simplified analytic model in Section 3. The relative differences are largest for the rating cohort IG. The difference for the different model specifications is absent for the risk neutral correlations. We also see that the risk neutral default correlations are in most cases lower than their real world counterparts. This implies there would be less of a premium for default clustering. The effect of lower default correlations, however, has to be off-set against the higher values of risk neutral default probabilities (compared to the physical probabilities) to compute the total effect on pricing. Also, note that we have not used basket products to calibrate risk neutral default correlations directly, as is done in for example Hull and White (2006). The difference should therefore be interpreted with some care.

4.2 Monte Carlo Results for the Complete Model

We now combine all parts of the model and obtain loss distributions using Monte Carlo simulation for the different model specifications. We assume that each rating cohort consists of 1,000 defaultable zero-coupon bonds with a maturity of three years ($T = 3$). The risk horizon at which we compute the loss distributions, the expected losses, and the Value-at-Risk is set at one year ($h = 1$), leaving a remaining maturity of the defaultable zero-coupon bonds of two years at the risk horizon. The loss given default rate is set to $\delta = 60\%$. At time $t = 0$, we assume that the default cycle and detrended short-rate are both zero: $\psi_{1,t} = \psi_{2,t} = 0$. Default losses are calculated under the recovery of treasury assumption. We simulate 100,000 scenarios for computing the loss distribution at the risk horizon $t + h$. The VaR is calculated for the confidence levels 97.5%, 99.0%, 99.9%, and 99.99%. The results are shown in Tables 9 and 10.

Table 9: Loss distribution characteristics

The table shows the expected and median losses, and the 99% Value-at-Risk (VaR) for credit risk, market risk and aggregated risk. These statistics are computed using different link functions for the real world and risk-neutral default probabilities.

	Credit Risk			Market Risk			Aggr. Risk		
	Mean	Median	VaR	Mean	Median	VaR	Mean	Median	VaR
	Probit link function								
IG	-4.0	-4.1	0.5	-4.0	-4.1	3.1	-4.0	-4.0	3.1
Ba	-5.4	-5.5	1.4	-5.6	-5.7	3.5	-5.2	-5.4	4.7
B	-5.6	-6.0	5.2	-6.8	-7.1	7.4	-5.3	-5.9	11.1
Caa-C	-5.2	-5.6	7.0	-10.2	-10.3	23.1	-6.3	-5.7	20.5
	Logit link function								
IG	-4.0	-4.1	0.7	-4.0	-4.1	3.1	-4.0	-4.0	3.1
Ba	-5.4	-5.5	1.7	-5.5	-5.6	3.8	-5.1	-5.4	5.3
B	-5.5	-6.0	6.5	-6.7	-7.1	7.9	-5.1	-5.9	12.5
Caa-C	-5.1	-5.6	7.7	-9.8	-10.3	23.4	-5.9	-5.7	20.9
	Poisson link function								
IG	-4.0	-4.1	0.7	-4.0	-4.1	3.1	-4.0	-4.0	3.1
Ba	-5.4	-5.5	1.8	-5.5	-5.6	3.9	-5.1	-5.3	5.4
B	-5.5	-6.0	6.8	-6.6	-7.1	8.3	-5.1	-5.9	13.1
Caa-C	-5.0	-5.6	8.7	-9.2	-10.3	25.2	-5.4	-5.7	21.9
	Ratio Logit to Probit link function								
IG	1.00	1.00	1.29	1.00	1.00	1.01	1.00	1.00	1.01
Ba	1.00	1.00	1.21	0.99	0.99	1.09	0.98	0.99	1.12
B	0.99	1.00	1.23	0.98	0.99	1.08	0.97	1.00	1.13
Caa-C	0.98	1.00	1.10	0.96	1.00	1.02	0.94	1.00	1.02
	Ratio Poisson to Probit link function								
IG	1.00	1.00	1.29	1.00	1.00	1.01	0.99	1.00	1.01
Ba	0.99	1.00	1.24	0.98	0.99	1.13	0.98	0.99	1.15
B	0.98	1.00	1.30	0.98	0.99	1.12	0.96	1.00	1.18
Caa-C	0.97	1.00	1.24	0.90	1.00	1.09	0.86	1.00	1.07

Table 9 reports the expected and median losses and the 99% quantile for credit, market, and aggregated risk. At first glance, the results for the different link functions are similar. Both the expected losses due to credit and market risk are negative, i.e., a gain is expected instead of a loss. In both cases the losses are offset by the upward drift in the price of the defaultable zero-coupon bond as the remaining maturity decreases between time t ane time $t + h$. This upward effect diminishes for the higher grade bonds.

Looking at the 99% VaRs for the aggregated risk measure in Table 9, we see that the Poisson link function results in somewhat larger VaRs compared to the other two link functions. Comparing the probit 99% VaR to the Poisson one, we see that the VaR is slightly higher for the Poisson case by about 1% for investment grade bonds to 18% for B rated bonds.

Table 10: VaR and VaR reductions of aggregating credit and market risk.

The Table shows the Value-at-Risk (VaRs) and VaR reductions of aggregating credit and market risk at different confidence levels. These statistics are computed for different link functions.

	Value-at-Risk (in %)				VaR Reductions (in %)			
	97.50%	99.00%	99.90%	99.99%	97.50%	99.00%	99.90%	99.99%
	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)
	Probit link function							
IG	2.5	3.1	4.1	4.8	12.8	16.2	22.7	31.8
Ba	3.6	4.7	7.4	10.0	2.9	3.9	5.8	10.2
B	8.9	11.1	16.4	20.6	9.4	11.9	15.4	22.0
Caa-C	19.3	20.5	21.9	22.6	28.2	31.7	37.4	41.1
	Logit link function							
IG	2.6	3.1	4.1	5.3	14.3	19.6	32.0	47.3
Ba	3.9	5.3	8.8	12.7	3.3	4.5	7.9	13.6
B	9.7	12.5	19.6	25.3	9.7	13.3	19.0	24.5
Caa-C	19.8	20.9	22.1	22.8	29.3	33.1	39.2	42.7
	Poisson link function							
IG	2.6	3.1	4.1	5.6	14.2	19.6	33.1	47.1
Ba	4.0	5.4	9.2	13.5	3.6	4.7	7.8	14.8
B	10.0	13.1	21.3	28.1	9.9	13.5	19.7	26.5
Caa-C	21.1	21.9	22.8	23.4	31.3	35.4	42.0	46.6
	Ratio Logit to Probit link function							
IG	1.01	1.01	1.01	1.10	1.11	1.21	1.41	1.49
Ba	1.08	1.12	1.18	1.26	1.12	1.14	1.35	1.34
B	1.09	1.13	1.19	1.23	1.04	1.12	1.23	1.11
Caa-C	1.02	1.02	1.01	1.01	1.04	1.04	1.05	1.04
	Ratio Poisson to Probit link function							
IG	1.01	1.01	1.01	1.15	1.11	1.21	1.46	1.48
Ba	1.10	1.15	1.24	1.34	1.22	1.21	1.34	1.45
B	1.13	1.18	1.30	1.37	1.06	1.14	1.27	1.20
Caa-C	1.09	1.07	1.04	1.04	1.11	1.12	1.12	1.13

For market risk, the differences are smaller, averaging at around 10% across non-investment grade ratings. For credit risk, however, the functional form again has a substantial effect, with the VaR of the Poisson link function being 24% (Caa-C) to 30% (IG) higher than its probit counterpart. This underlines the typical defect of credit risk portfolio models based on the normal distribution if we are interested in tail events: the normal distribution implies too little clustering in general, which causes an under-estimation of risk quantiles.

Table 10 reports the VaRs and VaR reductions at different confidence levels from aggregating market and credit risk. The reductions in the VaR from aggregating risks vary from 3% to 47% depending on the rating cohort, confidence level, and link function. The VaR reductions are U-shaped across rating categories, with the smallest reductions observed for rating class Ba, and the largest for Caa-C. In terms of the effect of model specification, the largest benefits are obtained under the Poisson link function, followed by the logit and probit link, respectively. The observed differences between the diversification benefits computed for the three link functions are most pronounced when we move further out into the tails of the loss distribution. For quantiles such as 97.5%, percentage benefits range from 3% to 31% whilst the percentage benefits range from 6% to 42% at the 99.99% confidence level. This is in line with our analytical results shown in Section 3. The largest differences are observed for the investment grade (IG) rating class. This result is largely explained by the observed differences for the real world default correlations, which are largest for credit rating IG (see Table 8). Because of the relatively larger default correlations found for the Poisson and logit mixtures, the scope for VaR reductions is larger.

5 Conclusion

Recent studies have analyzed the size of diversification benefits from an aggregated approach towards modeling and quantifying risk in a bond portfolio. Most studies assume a probit link function within an exchangeable Bernoulli framework for modeling credit risk. The question we raise in this paper is whether the model choice for the link function affects the diversification benefits and whether substantial differences can be observed when other popular link functions such as the logit and Poisson link functions are used instead of the probit link function.

To answer this question, we presented a comprehensive framework including a stochastic default cycle and a risk-free short-rate as the the main risk drivers for credit and market risk. We implemented a consistent calibration approach for comparing the potential diversification benefits resulting from the different possible link functions. In addition, we provided an analytic approach for a special case of the general model. In both settings, we found that the different link functions result in differ loss distributions and risk measures, particularly at high confidence levels (such as 99.9% and higher). On average, the logit and the Poisson link functions result in larger diversification benefits than the probit link function. The differences are largest for the investment grade rating class.

The current analysis clearly shows that details in the specification of the model, such as the specification of link functions, can have substantial effects on risk measures and risk aggregation benefits, particularly in the tail areas. The current model can easily be generalized further, for example, by taking a more dynamic perspective on the evolution of state variables. We do not expect such changes to materially affect our results, however, and therefore leave them for further research.

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Appendix: Moment Matching

In Section 3 the parameters θ and η of the real world and risk-neutral link functions are calibrated such that the unconditional default probability π and the default correlation ρ_π are the same for all three mixing distributions; compare Section 8.4.1 of McNeil, Frey, and Embrechts (2005). The default correlation is given as

$$0 \leq \rho_\pi = \frac{\pi_2 - \pi^2}{\pi - \pi^2} \leq 1, \tag{A1}$$

where $\pi_2 = \rho_\pi \pi + (1 - \rho_\pi)\pi^2$ is the probability of a joint default of two counterparties. We compute θ_0 and θ_1 for fixed π and π_2 (and thus for fixed ρ_π by solving the two-equations system

$$\begin{aligned} \pi &= E[p(\theta_0 + \theta_1 \psi_{t+h})] \qquad \text{(A2)} \\ \pi_2 &= E[p(\theta_0 + \theta_1 \psi_{t+h})^2]. \end{aligned}$$

This can be done numerically. A similar approach is taken vor η_0 and η_1, replacing $p(\cdot)$ by $q(\cdot)$ and π and π_2 by their risk neutral equivalents.