Step by Step:
Revisiting Step Tolling in the
Bottleneck Model

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Abstract

In most dynamic traffic congestion models, congestion tolls must vary continuously over time to achieve the full optimum. This is also the case in Vickrey's (1969) 'bottleneck model'. To date, the closest approximations of this ideal in practice have so-called 'step tolls', in which the toll takes on different values over discrete time intervals, but is constant within each interval. Given the prevalence of step tolling schemes they have received surprisingly little attention in the literature. This paper compares two step-toll schemes that have been studied using the bottleneck model by Arnott, de Palma and Lindsey (1990) and Laih (1994). It also proposes a third scheme in which late in the rush hour drivers slow down or stop just before reaching a tolling point, and wait until the toll is lowered from one step to the next step. Such behaviour has indeed been observed in reality. Analytical derivations and numerical modelling show that the three tolling schemes have different optimal toll schedules and reduce total social costs by different percentages. These differences persist even in the limit as the number of steps approaches infinity. Braking lowers the welfare gain from tolling by 14% to 21% in the numerical example. Therefore, preventing or limiting braking seems important in designing step-toll systems.

1.   Introduction

In most dynamic traffic congestion models, congestion tolls must vary continuously over time to achieve the full social optimum. This is also true of Vickrey's (1969) 'bottleneck model' of the morning rush hour in which congestion takes the form of queuing at a bottleneck and drivers choose their departure times to minimize their travel cost. Yet no existing road pricing scheme has such sophisticated, continuously time-varying tolls. The closest approximations of this ideal have so-called 'step tolls' in which the toll takes on different values over discrete time intervals, but is constant within each interval.[1] Given the prevalence of step-tolling schemes, they have received surprisingly little attention in the scholarly literature.

Two step-tolling schemes for the bottleneck model have been studied: one by Arnott, de Palma and Lindsey (1990, 1993) and the other by Laih (1994, 2004). For brevity, we will refer to their models and treatments as the "ADL model" and the "Laih model" respectively. These models each have their limitations or drawbacks. There are three problems with the ADL model. First, it is limited to a single-step toll that is either "on" or "off". Second, equilibrium in the model exists only if a mass of drivers departs when the toll comes off, with individual positions in the mass determined randomly. Mass departures are problematic in practice, and empirical evidence for them is scant.

Third, and most fundamentally, the ADL model ignores the fact that as the end of the tolling period approaches, drivers have an incentive to stop before reaching the tolling point and wait until the toll ends. Waiting has indeed been observed on some tolled facilities. Singapore implemented step tolls in 1998 when it introduced Electronic Road Pricing, and for the first few years tolls were changed in half-hour intervals. Some of the toll schedules involved large changes, and motorists were observed slowing down before they reached a toll

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[1] Many road-pricing schemes feature step tolls. The London congestion pricing scheme has a fixed toll from 7 am to 6 pm on weekdays (see the scheme's website at www.tfl.gov.uk/tfl/roadusers/congestioncharge/whereandwhen/; accessed October 20, 2010). Singapore's Electronic Road Pricing system features multi-step toll schedules on expressways and arterials. For instance, at the Bugis-Marina Centre (Nicoll Highway), the toll has 7 steps during the weekday morning peak (rates for 3 May to 1 August 2010, at www.onemotoring.com.sg/publish/onemotoring/en/on_the_roads/ERP_Rates.html, accessed on 11 May 2010). The toll scheme in Stockholm has 10 steps between 6:30 am and 6:29 pm (http://en.wikipedia.org/wiki/Stockholm_congestion_tax, accessed 11 May 2010). And the westbound express-lane on State Route 91 in California has eight steps in the toll during the morning peak on Monday to Thursday with the toll ranging from $1.30 to $4.55 (toll schedule effective October 1, 2010, www.octa.net/91_schedules.aspx, accessed on 25 October 2010).

gantry to take advantage of a toll decrease.[2] In 2003, five-minute graduated rates were introduced between some half-hour periods to deter this behaviour.[3] More recently, slowing down occurred at the Golden Gate Bridge when step tolling was introduced in July 2010. A newspaper article reports how drivers slowed up considerably in the minutes before the toll was lowered from \$6 to \$4. One driver is quoted as saying: "You bet I waited, I'm saving a couple of bucks" (Mercury News, 2010). Finally, Fosgerau (2011) reports that such braking behaviour also occurs with the step toll scheme in Stockholm.

In both his papers, Laih (1994, 2004) considers multi-step tolling. He also avoids some of the problems of the ADL model, by not modelling mass departures. Laih (2004) indicates that he assumes that drivers who choose to pass the bottleneck after a certain tolling period can wait on a set of secondary lanes without impeding other drivers who do pass the bottleneck in that tolling period. Nevertheless, as in the ADL model, Laih overlooks the incentive for drivers who use the main lanes to delay passage through the bottleneck when the toll is about to drop. Laih's model is also limited in scope because many roads do not have separate lanes, or even shoulders, where motorists can pull out of the traffic stream. As explained later, without such separate queues, the ADL model would be appropriate, and Laih's analysis would be inconsistent with his parameter-value assumptions.

In this paper we make two contributions. First and foremost, we analyze step-tolling in the bottleneck model under the assumption that drivers do stop and wait for a toll to decrease if the cost of waiting is less than the amount of toll saved. Unlike in the ADL and Laih models, the equilibrium is behaviourally consistent. Unlike in the ADL model, the equilibrium is free of mass departures. And unlike in Laih's formulation, the equilibrium does not require the presence of two sets of traffic lanes. One could object that stopping is dangerous and poses the risk of a traffic citation or fine. However, stopping does occur as noted above. Moreover, in lieu of stopping drivers can slow down in the kilometres leading up to a toll point and time

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[2] David Boyce recalls seeing motorcycles parked on the shoulder waiting for a toll to drop (personal communication, November 12, 2010). Motorists were also prone to racing to reach toll gantries before a toll increase. We discuss this behaviour below.

[3] See Chew (2008) and the Electronic Road Pricing web site at http://www.lta.gov.sg/motoring_matters/motoring_erp.htm (accessed October 14, 2010).

their arrival just before the toll drops. The analytics of the model are the same whether drivers slow down or stop. To allow for both possibilities we will refer to the new model as the "braking model". A notable feature of equilibrium in the braking model is that the bottleneck is unused while drivers are stopped. This loss of effective capacity undercuts the efficiency gains from tolling in comparison with the ADL and Laih models.

Our second contribution is to extend the ADL model to multiple toll steps. Doing so generalizes the ADL model as well as making it more empirically relevant since a number of tolling schemes have toll schedules with multiple steps. In the equilibrium of the ADL model with multiple toll steps, a mass of drivers departs after each toll reduction. This equilibrium is arguably less problematic than the equilibrium with one step because the numbers of drivers in each mass departure is smaller, and the incentives for drivers to stop before each toll reduction are weaker. After developing the braking model, and extending the ADL model to multiple steps, we compare the equilibria in the ADL, Laih and braking models as a function of model parameters and the number of toll steps. We also assess the efficiency loss from step tolling in the braking model due to gaps in usage of the bottleneck.

While our analysis of step tolling builds significantly on the literature it has some limitations. One limitation is that it ignores the drivers' incentives to speed up in order to pass through the tolling point just before a toll increase. Incorporating speeding into the bottleneck model is not straightforward, and it would probably be better to use a flow congestion model.[4] Another limitation is that we consider only toll schedules with the same numbers of steps "up" and steps "down". We also allow the toll levels to vary freely rather than requiring the level of each step-up toll to match the level of the corresponding step-down toll.[5]

The paper is organized as follows. Section 2 begins by presenting the basic bottleneck model, and solving for the no-toll equilibrium and social optimum. It then presents a graphical treatment of the one-step toll in the ADL, Laih and braking models. Section 3 extends the analysis to multi-step tolls and provides a formal derivation of optimal toll schedules for each

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[4] Braking behaviour could also be studied using other models. Chu (1999) analyzes one-step tolling using a dynamic model with flow congestion although he does not consider braking. His one-step toll generally attains a larger fraction of the first-best efficiency gains than the one-step toll in the bottleneck model. The fraction depends on the specification of the travel time function and the number of users, whereas in the bottleneck model the fraction is constant as we will show.

[5] Optimal toll schedules in the Laih model and the braking model turn out to be symmetric anyway.

model. Section 4 provides a brief numerical comparison of the models and assesses the efficiency loss due to braking in the braking model. Section 5 concludes with a summary of the main results and ideas for future research.

2.   No-toll equilibrium, first-best optimum, and single-step tolls

2.1.   *No-toll equilibrium and first-best optimum*

Both the no-toll equilibrium and the first-best or social optimum of the bottleneck model are described in detail in Arnott, de Palma and Lindsey (1990, 1993, 1998), as well as in various textbooks such as Small and Verhoef (2007). We will therefore be brief, while keeping the exposition self-contained.

In the bottleneck model a continuum of N identical individuals travel alone by car from a common origin to a common destination connected by a single road that is subject to bottleneck queuing congestion. Free-flow travel time is normalized without loss of generality to zero. Without a queue, an individual thus departs 'from home', passes the bottleneck, and completes her trip by arriving 'at work' all at the same moment. If the driver leaves at time t and there is a queue, she encounters a queuing delay, $T(t)$, equal to

$$(1) \qquad T(t) = \frac{1}{s} \cdot \left(\int_{t_s}^{t} r(u)\,du - s(t - t_s) \right),$$

where t_s is the time at which the queue began, s is bottleneck capacity, and $r(\ )$ is the departure rate as a function of time. The driver arrives at time $t_a = t + T(t)$.

Travellers incur a unit cost of α from queuing delay. They also have a preferred arrival time of t^*, and incur a so-called 'schedule delay cost' if they arrive earlier or later. The unit cost of arriving early is β, and the unit cost of arriving late is γ. The 'generalized cost' of a trip, c, is the sum of queuing delay cost and schedule delay cost:

$$(2) \qquad c(t) = \alpha \cdot T(t) + \beta \cdot \mathrm{Max}\left(0, t^* - t_a\right) + \gamma \cdot \mathrm{Max}\left(0, t_a - t^*\right).$$

Let $\tau(t)$ denote the toll paid for a departure at time t, if any.[6] The generalized price (hereafter referred to as the "price" for brevity) includes the generalized cost and the toll:

(3) $p(t) = c(t) + \tau(t).$

In equilibrium, all drivers experience the same trip price. During intervals when there is either no toll or a toll that is constant, the queue must evolve so that $c(t)$ is constant. This requires a departure rate $s \cdot \alpha /(\alpha - \beta) > s$ for early arrivals, and $s \cdot \alpha /(\alpha + \gamma) < s$ for late arrivals.

Departures and arrivals take place over a time interval $[t_s, t_e]$. Since the bottleneck operates at capacity,

(4) $t_e = t_s + N/s.$

Because the first and last drivers do not encounter a queue, their schedule delay costs must be equal:

(5) $\beta \cdot (t^* - t_s) = \gamma \cdot (t_e - t^*).$

Equations (4) and (5) together yield

(6a,b) $t_s = t^* - \dfrac{\gamma}{\beta + \gamma} \cdot \dfrac{N}{s}, \quad t_e = t^* + \dfrac{\beta}{\beta + \gamma} \cdot \dfrac{N}{s}.$

The equilibrium generalized cost and price in the no-toll (NT) equilibrium are therefore

(7) $p_{NT} = c_{NT} = \delta \cdot \dfrac{N}{s} \quad \text{with} \quad \delta \equiv \dfrac{\beta \cdot \gamma}{\beta + \gamma}.$

The total private cost (*PC*) and total social cost (*TC*) for the *N* travellers are equal:

(8) $PC_{NT} = TC_{NT} = \delta \cdot \dfrac{N^2}{s}.$

Because the schedule delay cost is linear in arrival time, half of these costs are travel delay costs, and half are schedule delay costs.

Queuing delay is a pure deadweight loss in the bottleneck model: shortening the queue reduces travel time delays without raising schedule delays as long as flow through the bottleneck remains equal to *s*. First-best time-varying tolling, also called 'fine tolling',

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[6] The toll schedule can be defined as a function of either departure time or arrival time. For first-best time-varying pricing, these times are the same. We will define the toll as a function of departure time here, but later on we will define it as a function of the moment that the bottleneck is passed, which is the arrival time at the destination.

therefore eliminates queuing. To achieve this, the toll schedule is chosen to match the cost of travel delay, $\alpha \cdot Q(t)$, in the no-toll equilibrium. The travel period is still given by eqn. (6a,b). The private cost of a trip remains unchanged, but the social cost of a trip is reduced by half:

(9) $p_{FB} = \delta \cdot \dfrac{N}{s}, \quad c_{FB} = \tfrac{1}{2} \cdot \delta \cdot \dfrac{N}{s},$

where FB denotes the first-best optimum. The corresponding total costs are

(10) $PC_{FB} = \delta \cdot \dfrac{N^2}{s}, \quad TC_{FB} = \tfrac{1}{2} \cdot \delta \cdot \dfrac{N^2}{s}.$

In the next three subsections we discuss how single-step (coarse) tolling works in the ADL, Laih and braking models. We defer formal derivations of the equilibria until Section 3 where we provide derivations for multi-step tolls.

2.2.   Coarse tolling: the ADL model

Arnott, de Palma and Lindsey (1990, 1993) were the first to consider step tolling in the bottleneck model. They focused on a single-step toll, or coarse toll in their terminology. The coarse toll is defined by its level, ρ, and the times at which it is turned on (t^+) and off (t^-). The qualitative properties of the optimal coarse toll depend on the relative magnitudes of α and γ. If $\alpha \geq \gamma$, the coarse toll does not affect the timing of arrivals and the solution is straightforward. But if $\alpha < \gamma$, as Small (1982) and most later empirical studies have found, the arrival period is affected and the analytics are more complicated.

To see why this is the case, it is helpful to describe in detail the equilibrium with the coarse toll when $\alpha < \gamma$. Figure 1 does so by plotting cumulative departures and arrivals against time. The first driver travels at t'_s before t^+ when the toll is switched on. Departures continue at a rate $s \cdot \alpha / (\alpha - \beta)$ as in the equilibrium without toll as long as drivers pass the bottleneck before t^+. The first person to arrive after t^+ and pay the toll must experience a queuing time that is ρ / α shorter than the queuing time of the person arriving just before t^+ who avoids the toll. At some time, t_0, departures therefore halt. To maximize the reduction in queuing time, t^+ and ρ are jointly chosen so that the queue reaches zero at t^+.

Next, there is a period that lasts until the toll is lifted, at t^-, during which all travellers pay the toll. (The possibility that drivers can brake and wait until the toll is lifted is ignored.) The equilibrium departure rate is $s \cdot \alpha / (\alpha - \beta)$ for drivers who arrive early, and $s \cdot \alpha / (\alpha + \gamma)$ for those who arrive late, again the same as in the equilibrium without toll. Parameters t^- and ρ are jointly chosen so that the queue reaches zero at t^-.
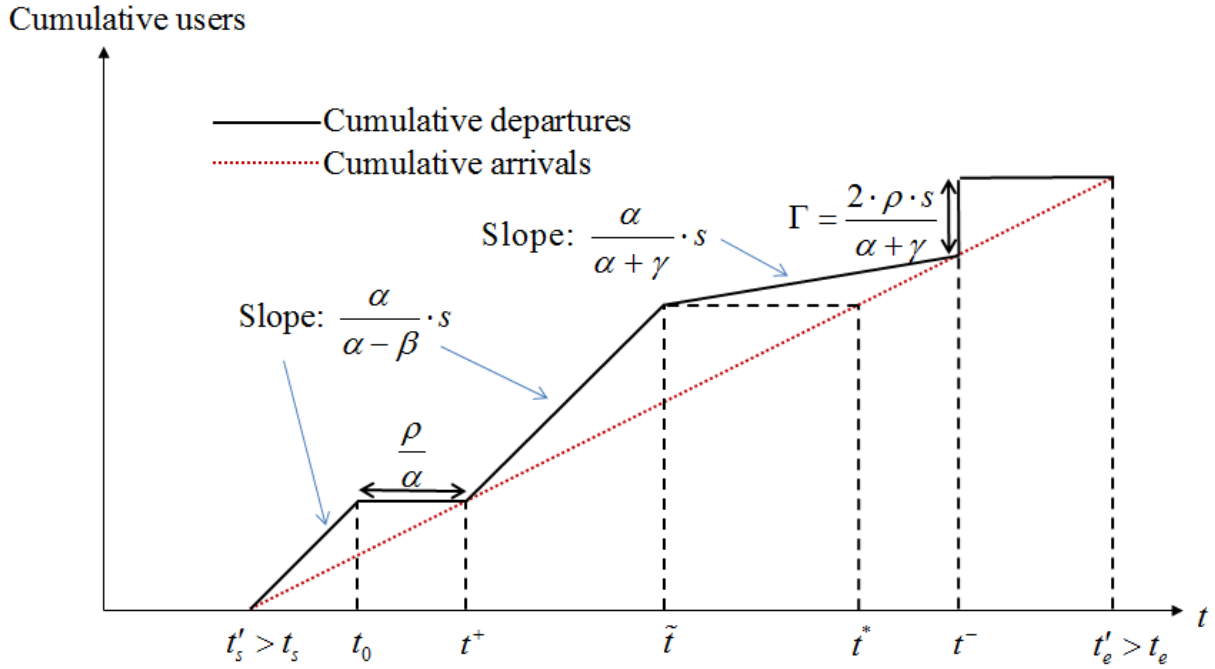

Figure 1. Equilibrium with a coarse toll: The ADL model for $\alpha < \gamma$

The question now arises how drivers who pass the bottleneck after t^-, and pay no toll, can be equally well off as earlier drivers. Arnott, de Palma and Lindsey (1990) recognized that this requires a mass of individuals to depart at t^-, with individual positions in the mass determined randomly. The number of drivers in the mass, Γ, is such that the expected generalized cost for someone in the mass equals the deterministic price paid by the last traveller just before t^-. The unit cost of time spent in the queue is $\alpha + \gamma$, and the random amount of time spent in the queue is uniformly distributed between 0 and Γ / s. The expected additional generalized cost for someone in the mass is therefore $\frac{1}{2} \cdot (\alpha + \gamma) \cdot \Gamma / s$. Setting this cost equal to ρ gives $\Gamma = 2 \cdot \rho \cdot s / (\alpha + \gamma)$.

We can now explain why the coarse toll delays the start of travel if $\alpha < \gamma$. Consider a driver who departs at time $t^- + \Delta$, and assume the queue from the mass departure has not yet dissipated so that $\Delta < \Gamma / s$. The driver incurs a generalized cost of $\gamma \cdot \left(t^- + \Delta - t^* \right) + \left(\alpha + \gamma \right) \cdot \left(\Gamma - \Delta \cdot s \right) / s$. Setting this cost equal to the expected generalized cost of being in the mass one obtains

$$\gamma \cdot \left(t^- + \Delta - t^* \right) + \left(\alpha + \gamma \right) \cdot \frac{\Gamma - \Delta \cdot s}{s} = \gamma \cdot \left(t^- - t^* \right) + \tfrac{1}{2} \cdot \frac{\Gamma}{s} \cdot \left(\alpha + \gamma \right),$$

which simplifies to

$$(11) \quad \Delta \cdot \alpha = \tfrac{1}{2} \cdot \frac{\Gamma}{s} \cdot \left(\alpha + \gamma \right).$$

Equation (11) is consistent with the assumption $\Delta < \Gamma / s$ only if $\alpha > \tfrac{1}{2} \cdot \left(\alpha + \gamma \right)$, or $\alpha > \gamma$.

If $\alpha < \gamma$, as empirical studies suggest, no one will depart after t^-, neither when the queue dissipates nor later.[7] The equilibrium depicted in Figure 1 then applies. Because the bottleneck operates at full capacity throughout the peak[8], the peak has the same duration as the no-toll equilibrium and social optimum. But the timing of the peak, $\left[t_s', t_e' \right]$, is now determined not by eqn. (5), but by the condition that the schedule delay cost of the first traveller equals the expected cost of a traveller in the mass:

$$(12) \quad \beta \cdot \left(t^* - t_s' \right) = \gamma \cdot \left(t^- - t^* \right) + \tfrac{1}{2} \cdot \left(\alpha + \gamma \right) \cdot \left(t_e' - t^- \right).$$

The solution works out to[9]

$$(13) \quad t_s' = t^* - \frac{\gamma}{\beta + \gamma} \cdot \frac{N}{s} + \frac{\left(\gamma - \alpha \right) \cdot \rho}{\left(\beta + \gamma \right) \cdot \left(\alpha + \gamma \right)}.$$

Given $\gamma > \alpha$, the peak begins later than with no toll. Since the first driver incurs only a schedule delay cost, and the expected price of a trip is the same for everyone, the expected price falls. The drop in price comes despite the fact that the shift in the peak causes total schedule delay costs to rise.

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[7] This argument applies for any coarse toll, whether it is optimal or not.

[8] We use the term "peak" for ease of reference although the bottleneck model does not have peak and off-peak periods, but rather a continuum of trip times that vary in their attractiveness.

[9] See Arnott et al. (1990, eqn. 14b).

Now consider the case $\alpha > \gamma$ which is shown in Figure 2.[10] Equation (11) then implies that $\Delta < \Gamma / s$. Contrary to the case with $\alpha < \gamma$, there is a final time interval after the mass departure during which drivers depart at rate $s \cdot \alpha /(\alpha + \gamma)$. This last cohort of drivers experiences a deterministic trip price, and the last driver in the cohort experiences no queuing delay. Therefore, in equilibrium, the schedule delay costs for the first and last drivers are equal, condition (5) applies, and the timing of the peak is given by eqns. (6a,b).
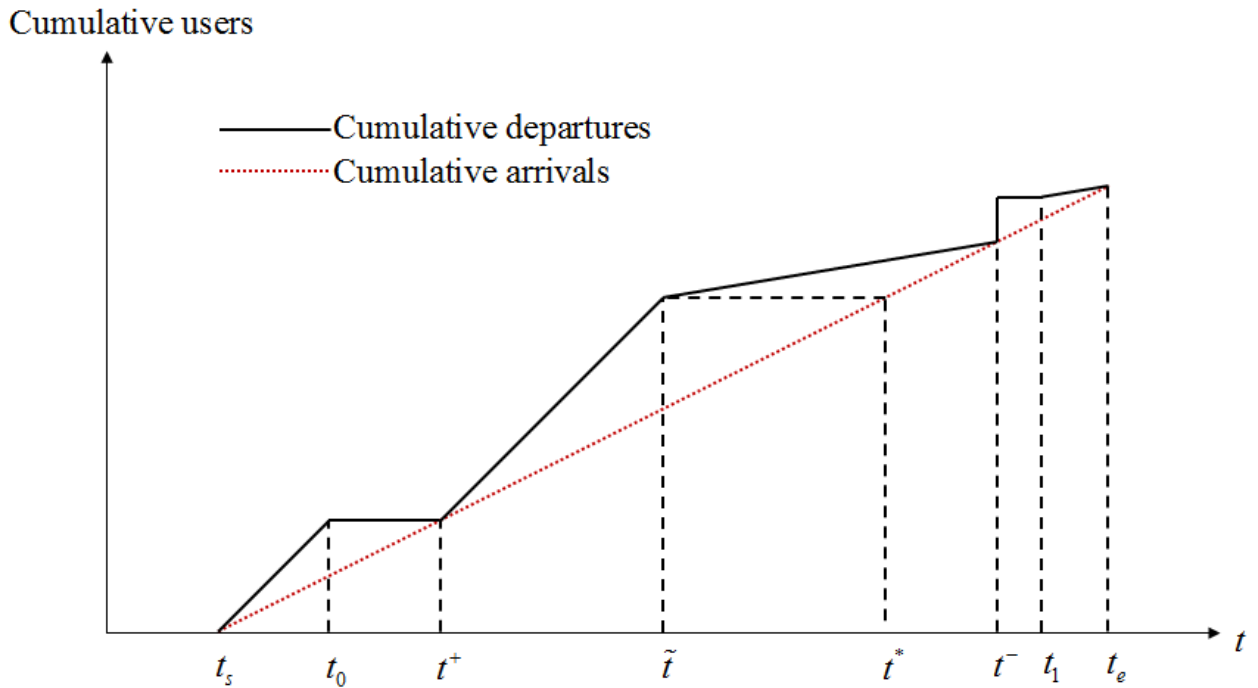
Cumulative users

Figure 2. Equilibrium with a coarse toll: the ADL model for α>γ

The optimal coarse toll will be derived in Section 3 as a special case of the optimal multi-step toll of the ADL model. We now turn to consider the single-step toll in the Laih model.

2.3. Coarse tolling: the Laih model

Laih (1994, 2004) considers multi-step tolls, under the assumption that tolling does not affect the timing of arrivals. In both papers Laih assumes that $\gamma > \alpha > \beta > 0$.[11] However, the

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[10] Daniel (2009) discusses equilibrium with multi-step tolls with $\alpha > \gamma$ in the context of airport congestion.

[11] Laih (1994) states this assumption as inequality (5) on page 199. Laih (2004) makes the assumption in the text on page 61.

solution he presents in Laih (1994) is valid only if $\gamma \leq \alpha$, unless there are separate queuing facilities. Indeed, he does not discuss mass departures. Rather, he uses a graphical approach to infer how much queuing time costs are reduced as a function of the number of toll steps. Laih's (2004) solution is valid regardless of the relative magnitudes of γ and α, because he circumvents the problem of mass departures by assuming that users who arrive after t^-, and do not pay the toll, can wait on a set of secondary lanes or shoulders without impeding other drivers who arrive before t^- and use the main set of lanes. There are therefore separate queues and no mass departures occur. And because all travel times are deterministic, the timing of the peak and the equilibrium price are unaffected by tolling.
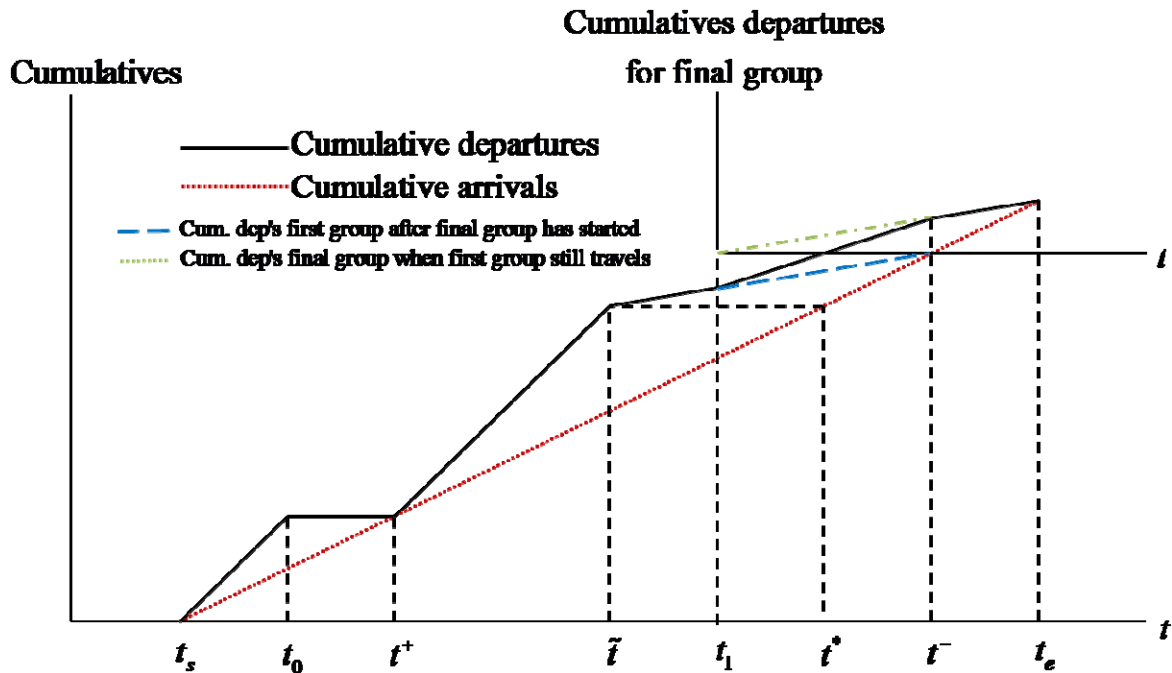

Figure 3. Equilibrium with a coarse toll: the Laih model

Figure 3 shows equilibrium in the Laih model with a single-step toll. Travellers who pay the toll arrive between t^+ and t^-. Between t_1 and t^- their cumulative departures are shown by the blue dashed line. Travellers who arrive late and do not pay the toll depart between t_1 and t^-, and arrive between t^- and t_e. Their cumulative departures are shown, using a second

set of axes, by the green dot-dashed line. Cumulative departures for the two groups combined are identified by the black solid line.

2.4. Coarse tolling: the braking model

Both the ADL model and the Laih model overlook that drivers have an incentive to delay reaching the tolling point when the toll is about to drop if the waiting cost they incur is outweighed by the money they will save. The braking model takes this incentive into account. The equilibrium is depicted in Figure 4.
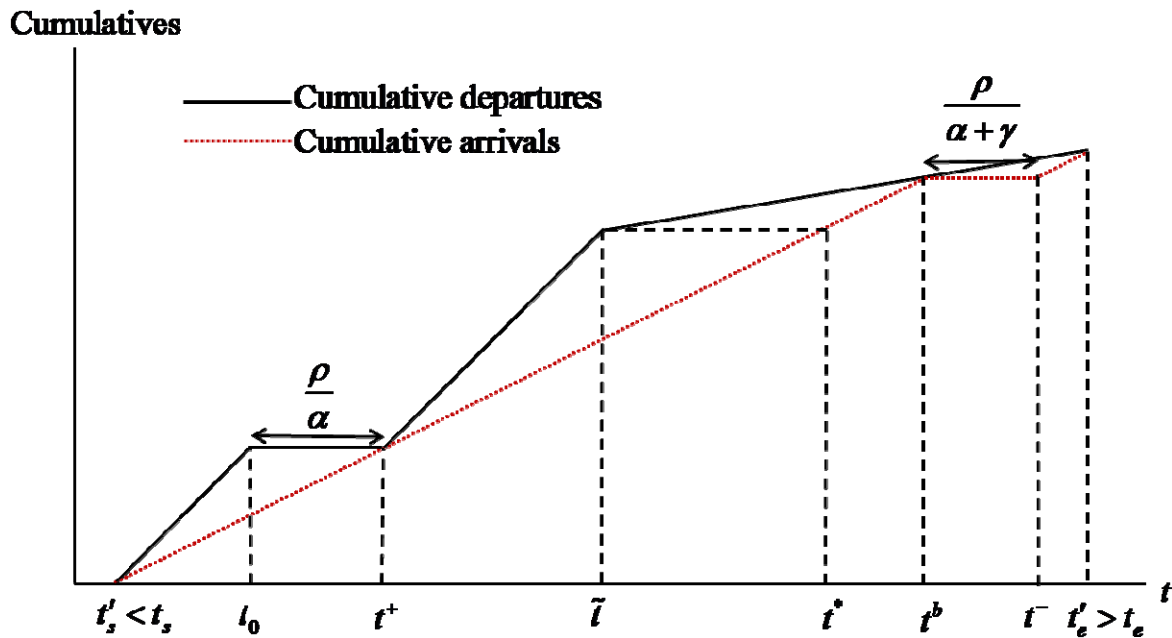

Figure 4. Equilibrium with a coarse toll: the braking model

The equilibrium is qualitatively the same as in the ADL model (cf. Figure 2) up to time t^b when the last driver pays the toll. The bottleneck is then idle for a period $t^- - t^b = \rho/(\alpha+\gamma)$: a queue is building up, but no-one passes the tolling point. Consequently, the arrival period $[t'_s, t'_e]$ is longer than in the no-toll equilibrium and social optimum. Since all travel times are deterministic, the schedule delay costs of the first and last drivers are equal. Departures start earlier, and end later, than in the no-toll equilibrium and social optimum (i.e. $t'_s < t_s$ and $t'_e > t_e$), and the equilibrium price is therefore higher. This contrasts with both the ADL

model in which the coarse toll causes private costs to fall, and the Laih model in which private costs are unchanged by the toll. The cost rises in the braking model because the bottleneck goes unused for a certain time interval before the toll ends.

The braking model avoids the difficulties in the ADL and Laih models. First, it is behaviourally consistent since it accounts for the possibility that drivers can reduce their travel costs by braking. Second, it avoids the problematic nature of mass departures in the ADL model. And third, it avoids the requirement of the Laih model for parallel traffic lanes.

3.   Multi-step tolls

3.1.   Overview

The coarse tolls described in Section 2 are a simple case of multi-step tolls. The same behavioural assumptions and solution concepts apply for tolls with multiple steps in each of the ADL, Laih and braking models. But the analytics become more complicated, and formulas for the optimal timing and levels of the step tolls in the ADL model are complex and rather opaque.

Before the analytics of the three models are presented, some additional terminology and notation need to be introduced. The toll schedules are assumed to have m levels with a series of increasing 'step-up' tolls while drivers arrive early, and a series of decreasing 'step-down' tolls while drivers arrive late. Tolling periods are indexed so that period 1 is the central period spanning a time interval $\left(t_1^+, t_1^-\right)$ that contains t^*, and with a toll level ρ_1 as shown in Figure 5. Period 2 spans earlier and later time intervals on either side of the central period. Period 2 is bracketed by period 3, and so on through period m. The early arrival segment of period i spans period $\left[t_i^+, t_{i-1}^+\right)$ during which the 'step-up' toll ρ_i^+ applies. Similarly, the late arrival segment of period i spans period $\left[t_{i-1}^-, t_i^-\right)$ during which the 'step-down' toll ρ_i^- applies. Since period 1 forms a single connected time interval, $\rho_1^+ = \rho_1^- = \rho_1$. The toll is assumed to be zero at the beginning (t_s') and end (t_e') of the travel period; a harmless assumption since demand is perfectly inelastic.
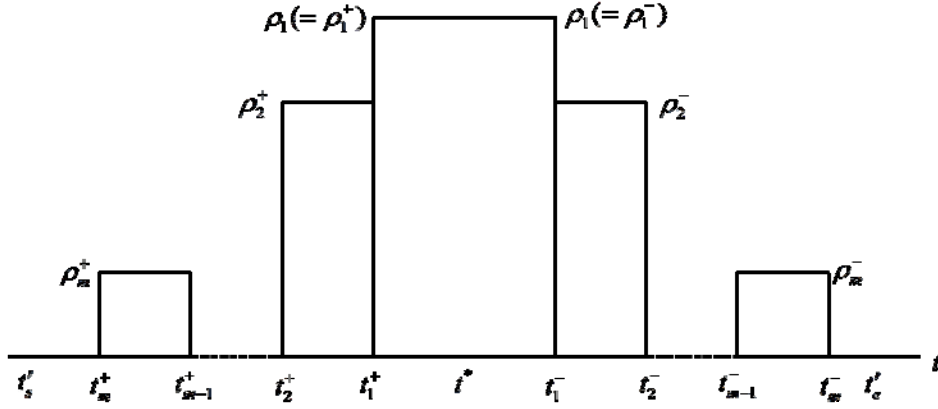
Figure 5. Notation in the multi-step toll models

The efficiency of the optimal *m*-step toll of each model can be measured as a fraction of the welfare gain derived from the fine toll using the index

$$\omega_r = \frac{TC_{NT} - TC_r}{TC_{NT} - TC_{FB}},$$

where *r* indexes the model or regime (ADL, Laih, or braking). We begin with the Laih model since it is the easiest to analyze.

3.2. *Step-tolling in the Laih model*

Optimal toll levels and tolling periods for the Laih model are described in Laih (1994, 2004). Appendix A summarizes the derivations. The optimal tolls turn out to be symmetric between early and late segments of each tolling period so that $\rho_i^+ = \rho_i^- \equiv \rho_i$ for $i = 1,...,m$, and the $^+$ and $^-$ superscripts can be omitted. The *i*th step-up toll begins at

$$t_i^+ = t_s + \frac{\rho_i}{\beta},$$

and the *i*th step-down toll ends at

$$t_i^- = t_e - \frac{\rho_i}{\gamma}.$$

Optimal toll levels are

(14) $$\rho_1^{Laih} = \frac{m}{m+1} \cdot \delta \cdot \frac{N}{s}, \quad \rho_i^{Laih} = \frac{m+1-i}{m} \cdot \rho_1^{Laih}, \quad i = 2,...,m.$$

The central toll, ρ_1^{Laih}, approaches the peak level of the fine toll, $\delta \cdot N/s$, as $m \to \infty$. Because the toll steps up and down in uniform increments, and the time intervals between steps are also uniform, the Laih toll approaches the optimal fine toll as $m \to \infty$. Total social costs are

$$(15) \quad TC_{Laih} = \delta \cdot \frac{N^2}{s} \cdot \left(1 - \frac{1}{2} \cdot \frac{m}{m+1}\right).$$

Using eqns. (8), (10) and (15), it follows that the Laih toll has a relative efficiency of

$$(16) \quad \omega_{Laih} = \frac{m}{1+m}.$$

The Laih toll asymptotically approaches the efficiency of the first-best (fine) toll as $m \to \infty$.

3.3.   Step-tolling in the ADL model

The ADL model is easy to solve if $\alpha \geq \gamma$ since it is then equivalent to the Laih model in terms of the toll schedule, private costs, social costs and relative efficiency. But if $\alpha < \gamma$, the ADL model is tedious to solve. In part, this is because the optimal tolls are not symmetric in levels: the ith step-up toll ρ_i^+ is lower than the corresponding step-down toll ρ_i^-.[12] Derivations are provided in Appendix B. The optimal tolls have simple recursive formulas:

$$(17) \quad \rho_i^+ = \frac{m+1-i}{m} \cdot \rho_1, \quad i = 1,...,m, \quad \rho_i^- = (i-2) \cdot \rho_3^- - (i-3) \cdot \rho_2^-, \quad i = 2,...,m.$$

As in the Laih model, the step-up tolls grow in uniform increments. And the step-down tolls follow a simple recursion relation. However, for $m > 1$, ρ_1, ρ_2^- and ρ_3^- have complicated expressions.[13] Some noteworthy properties are nevertheless discernible. First, the size of the time shift toward later departures decreases with m, and approaches zero as $m \to \infty$. Second, the ADL toll approaches the fine toll as $m \to \infty$ so that $\omega_{ADL} \to 1$. Third, for $1 < m < \infty$, each ADL toll is larger than the corresponding Laih toll. Fourth, for finite m, $\omega_{ADL} > \omega_{Laih}$. This last inequality is consistent with the fact that the generalized cost of travel decreases with tolling in the ADL model, but not the Laih model.

3.4.   Step-tolling in the braking model

When multiple toll steps are introduced in the braking model, the bottleneck is idle for a time interval before each step down in the toll. The total time idle turns out to be $\rho_1 / (\alpha + \gamma)$, and

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[12] Symmetric tolls are actually even harder to solve than asymmetric tolls and we have been unable to derive closed-form solutions. Imposing symmetry would have no advantage with electronic toll collection technology. And it is unlikely that drivers would gain a lot from toll symmetry in terms of ease of recall or trip planning.

[13] Appendix B provides explicit formulae for m=1 and m=2.

therefore depends only on the level of the central toll. The number of toll steps only determines the number of intervals into which total idle time is divided. As shown immediately below, ρ_1 is an increasing function of m so that total idle time actually increases as the toll structure becomes more refined. Hence, with braking the step toll does not asymptotically approach the first-best optimum in the limit $m \to \infty$.

The timing and levels of the optimal step toll in the braking model are derived in Appendix C. The toll levels work out to

$$(18) \quad \rho_1^{Braking} = \frac{m}{m+1} \cdot \delta \cdot \frac{N}{s}, \quad \rho_i^{Braking} = \frac{m+1-i}{m} \cdot \rho_1^{Braking}, \quad i = 2,...,m,$$

The toll levels are the same as in eqn. (14) for the Laih model. However, there is a distinction between the time, t_i^-, when the ith step-down toll ρ_i^- is lifted, and the time, t_i^b, when ρ_i^- is last paid. Formulae for the t_i^+ and t_i^b can be written as functions of the beginning or end of the travel period:

$$(19) \quad t_i^+ = t_s' + \frac{\rho_i}{\beta}, \quad i = 1,...,m,$$

$$(20) \quad t_i^b = t_e' - \frac{\rho_i}{\gamma}, \quad i = 1,...,m.$$

Total social cost is

$$(21) \quad TC_{Braking} = \delta \cdot \frac{N^2}{s} \cdot \left(1 - \frac{1}{2} \cdot \frac{m}{m+1} \cdot \left(1 - \frac{\beta \cdot \gamma}{(\beta+\gamma) \cdot (\alpha+\gamma)} \right) \right).$$

The relative efficiency of the braking toll is derived using eqn. (21) and eqn. (10):

$$(22) \quad \omega_{Braking} = \frac{m}{m+1} \cdot \left(1 - \frac{\beta \cdot \gamma}{(\beta+\gamma) \cdot (\alpha+\gamma)} \right).$$

The relative efficiency of the braking toll can be compared with the efficiency in the Laih and ADL models. The comparison is straightforward for the Laih model. From eqn. (22) and eqn. (16) one has

$$(23) \quad r_{BL} \equiv \frac{\omega_{Braking}}{\omega_{Laih}} = 1 - \frac{\beta \cdot \gamma}{(\beta+\gamma) \cdot (\alpha+\gamma)}.$$

Unless $\beta = 0$ or $\gamma = 0$ (in which case travel is costless in all three models because rescheduling to avoid queuing is costless) tolling is less efficient in the braking model than in the Laih model. The ratio r_{BL} converges to 1 in the limit $\alpha \to \infty$ or $\gamma \to \infty$ because total idle time for the bottleneck, $\rho_1 /(\alpha + \gamma)$, then approaches zero. Furthermore, r_{BL} is a decreasing function of α. Since $\alpha > \beta$,[14] the relevant limit is

$$\lim_{\alpha \to \beta} r_{BL} = 1 - \frac{\beta\gamma}{(\beta+\gamma)^2}.$$

The ratio r_{BL} reaches a minimum value of 3/4 with $\beta = \gamma$. Thus, relative to tolling in the Laih model the efficiency loss from tolling due to braking cannot exceed 25%.

Comparison of the braking and ADL models is intractable for general values of *m* because the formula for ω_{ADL} is very messy. But the comparison is easy for the coarse toll. Setting $m = 1$ in eqn. (22) one has

$$\omega_{Braking} = \frac{1}{2} \cdot \left(1 - \frac{\beta \cdot \gamma}{(\beta + \gamma) \cdot (\alpha + \gamma)} \right).$$

For the coarse toll in the ADL model (see Appendix B)

$$\omega_{ADL} = \frac{1}{2} \cdot \left(1 + \frac{\beta \cdot (\gamma - \alpha)}{(\beta + \gamma) \cdot (\alpha + \gamma)} \right).$$

The relative efficiency of tolling in the two models is therefore

$$r_{BA}\big|_{m=1} \equiv \frac{\omega_{Braking}}{\omega_{ADL}} = \frac{(\beta + \gamma) \cdot (\alpha + \gamma) - \beta\gamma}{(\beta + \gamma) \cdot (\alpha + \gamma) + \beta \cdot (\gamma - \alpha)}.$$

Since the ADL model is solved here for the case $\gamma \geq \alpha$, $r_{BA} < 1$ as long as $\beta > 0$. Similar to the comparison with the Laih model, $r_{BA} \to 1$ in the limit as $\gamma \to \infty$. A minimum value of $r_{BA} \cong 0.6991$ is reached with $\beta = \alpha$ and $\gamma / \alpha = (1 + \sqrt{7})/2$. Relative to tolling in the ADL model, the efficiency loss from tolling due to braking cannot exceed about 30%.

4.   Numerical results

To get a better idea of how much the three tolling models differ quantitatively, and how braking affects the efficiency of tolling, we turn to a brief numerical analysis. We use the same values for the unit cost parameters as Arnott, de Palma and Lindsey (1990): α=6.4,

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[14] The opposite case, $\alpha<\beta$, would imply that early arriving drivers would prefer to extend their trip.

β=3.9, and γ=15.21.[15] We set N=9000 and s= 3600 vehicles per hour so that travel lasts for 2½ hours in all regimes except the braking model. Optimal toll schedules were computed (in Mathematica) using the analytical results presented in the appendices, and various numerical consistency checks were performed to make sure that we indeed consider second-best equilibria.

4.1.   Tolls and equilibrium prices

Figure 6 shows the three step-tolling schemes for m=5 along with the first-best toll for comparison. Consistent with the analytical results in Section 3, toll levels are the same in the braking model and the Laih model, but the timing differs. The early-arrival tolling periods have the same duration, but the late-arrival periods are slightly longer in the braking model. Braking prolongs travel by 18 minutes, which is nearly 12% of the peak duration without braking.[16] In the ADL model the early-arrival tolls differ only slightly from the other two models. The timing of the steps also differs only slightly from the Laih model because of the small departure delay. For late arrivals, the difference in toll levels between the ADL model and the Laih model is larger due to the asymmetry in the ADL toll schedule. The braking toll exceeds the other two tolls as well as the fine toll over most of the travel period (but recall that the braking toll is not paid during intervals when the bottleneck is idle).

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[15] The results of interest depend only on the ratios β / α and γ / α.

[16] For m=1, the increase is about 11 minutes, and as $m \rightarrow \infty$ the increase approaches 21 minutes.
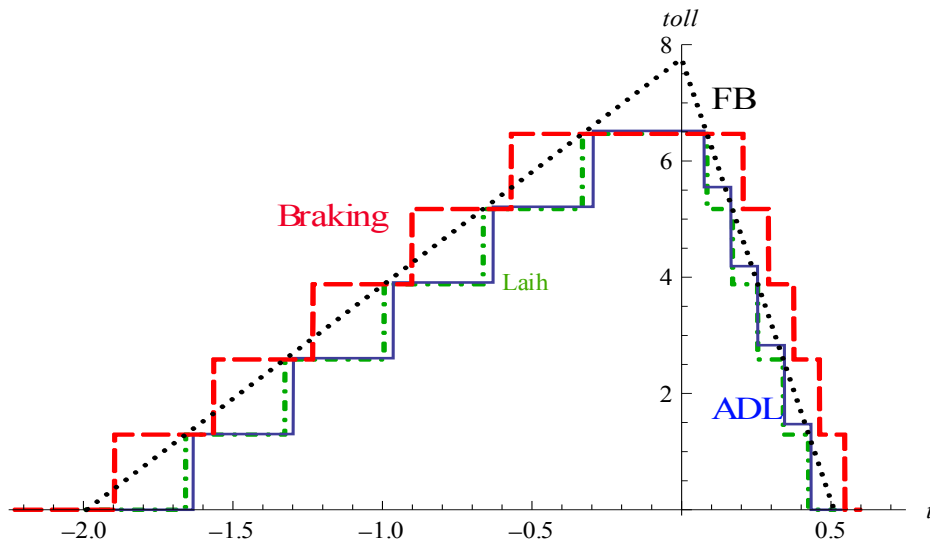
Figure 6. Toll levels with step tolls and first-best (fine) tolls (m=5)

Figure 7 plots equilibrium prices for the three models against the number of steps. Consistent with the analytical results, the price in the Laih model is independent of the number of steps, and remains equal to the price for the no-toll equilibrium and the social optimum. Also as expected, the price in the ADL model is lower than in the Laih model but approaches it as the number of steps increases. By contrast, in the braking model the price is higher than in the Laih model and the difference increases with the number of steps because the total idle time increases. For a single step the price increase is 7% of the no-toll equilibrium price, for 5 steps it is 12%, and as $m \to \infty$ the increase approaches 14%.
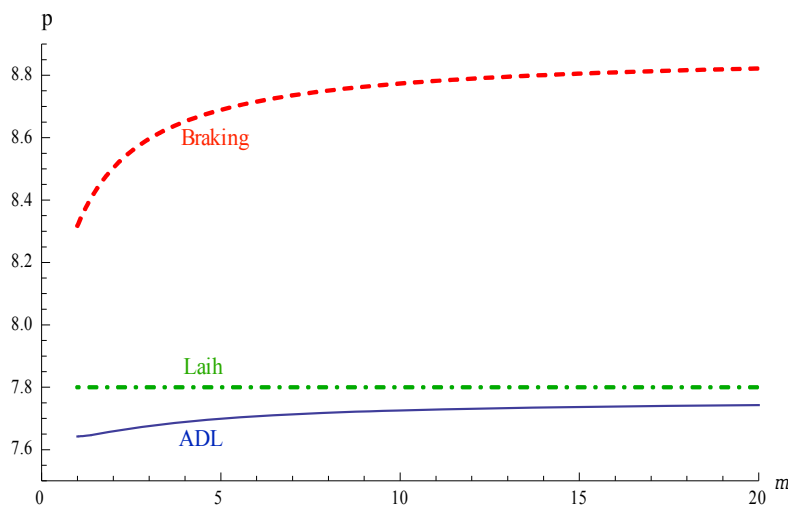

Figure 7. Prices with step tolls as a function of m

4.2. Relative cost reductions

Figure 8 shows the percentage reductions in total social costs for the three models as a function of the number of steps. The highest possible reduction, obtained with first-best pricing, is 50%. As m increases, the cost reductions for the ADL and Laih models rapidly approach the 50% upper bound. The relative efficiency of the Laih toll is $m/(1+m)$ so that the cost reduction in percentage terms is $50 \cdot m/(1+m)$. The ADL toll does only slightly better. The fact that it does better suggests that if mass departures are observed with step tolls in reality, it need not be welfare enhancing to try and avoid this by opening a parallel queuing facility, *e.g.* a shoulder, to allow for a Laih equilibrium to replace the ADL equilibrium.

The braking toll produces appreciably smaller cost reductions. As the number of steps approaches infinity, the total cost reduction approaches an upper limit of only 43%. As per eqn. (23), the efficiency gain from the braking toll is 14% lower than the gain from the Laih toll regardless of the number of steps. For the ADL toll the difference is slightly larger: 21% with one step, 16% with 5 steps and asymptotically approaches 14% as $m \to \infty$.

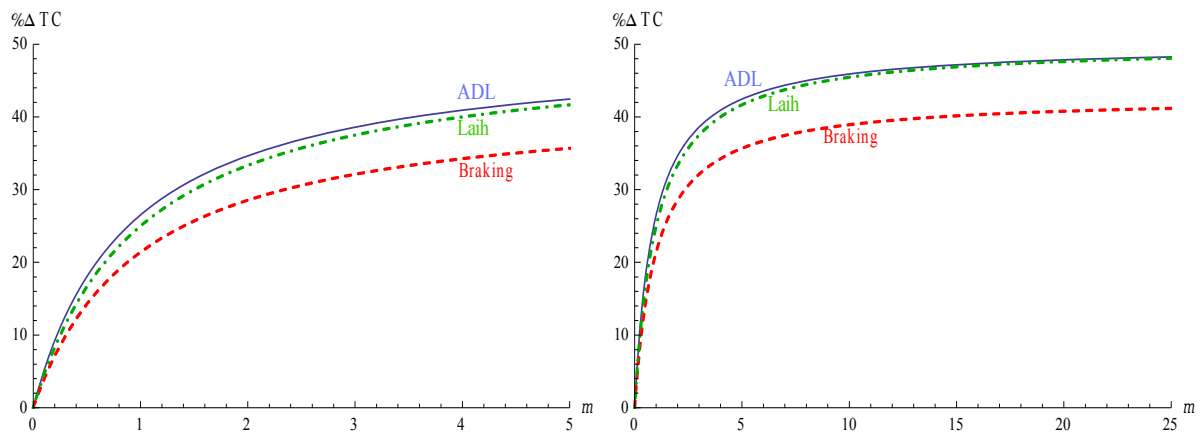*Figure 8. Percentage reduction in social cost as a function of m (up to m=5 in the left panel, and m=25 in the right panel)*

4.3.   Sensitivity analysis for relative efficiency gains

The numerical results in Sections 4.1 and 4.2 are derived for a single set of parameter values and it is useful to test the robustness of these results by conducting some sensitivity analysis. Arguably of greatest interest is the efficiency of the braking toll compared to the toll in the other two models, and attention is limited to this here. The relative efficiency of the braking toll and the Laih toll, r_{BL}, given in eqn. (23) depends only on the ratios β/α and γ/α. Figure 9 plots r_{BL} over the range $\beta/\alpha \in [0,1)$ and $\gamma/\alpha \in [0,10]$. As noted earlier, $r_{BL} = 1$ with $\beta/\alpha = 0$ or $\gamma/\alpha = 0$; and r_{BL} reaches a minimum value of 0.75 with $\beta/\alpha = 1$ and $\gamma/\alpha = 1$; see the front corner of Figure 9. Over an appreciable portion of the surface, the braking toll underperforms the Laih toll by 10-15%.
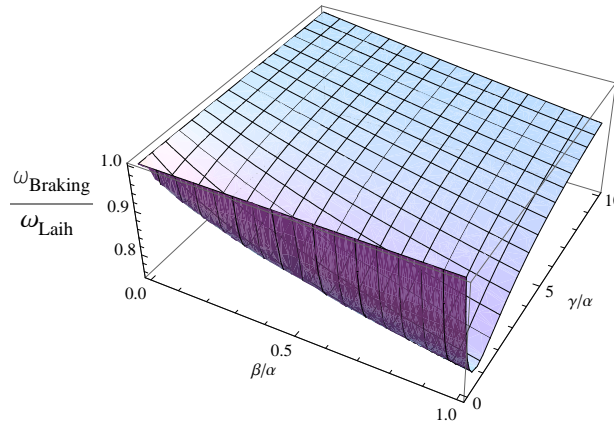

Figure 9. Efficiency gain from braking toll relative to Laih toll

The relative efficiency of the braking toll and the ADL toll, r_{BA}, depends on β/α, γ/α and *m*. Figure 10 plots r_{BA} for *m*=1 in the left panel, and *m*=5 in the right panel. The two surfaces are very similar. Consistent with the analytical results in Section 3 for the coarse toll, r_{BA} reaches a minimum value with $\beta/\alpha = 1$ and an intermediate value of γ/α of around 1.8. Under these conditions, a large fraction of travellers arrive late so that the harmful effects of braking are magnified.
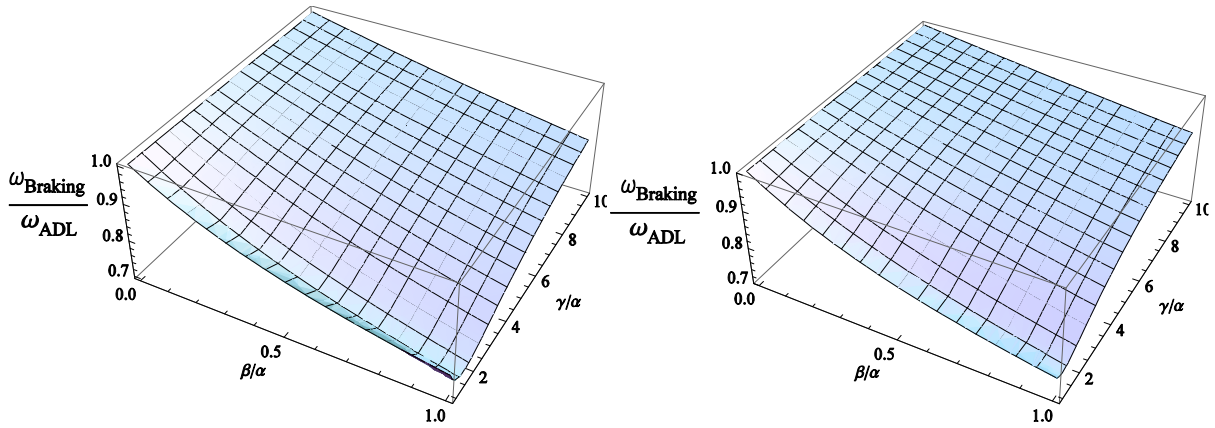
Figure 10. Efficiency gain from braking toll relative to ADL toll (m=1 in the left panel, m=5 in the right panel)

5. Conclusions

This paper has reconsidered step tolling in the bottleneck model. Its main contribution is to remedy an inconsistency in earlier studies by Arnott, de Palma and Lindsey (1990, 1993) and Laih (1994, 2004). These studies ignored that when a toll is about to drop, drivers have an incentive to slow down or stop before reaching the tolling point and wait. We develop a "braking model" in which drivers wait if the toll payment saved outweighs the travel time and schedule delay cost incurred by waiting. Unlike in the ADL and Laih models, equilibrium in the braking model is based on internally consistent traveller behaviour. The equilibrium is also consistent with observed cases of waiting on roads with step tolls. An additional advantage of the braking model is that the equilibrium does not require either mass departures—as in the ADL model—or parallel sets of traffic lanes—as in the Laih model.

A second contribution of the paper is to extend the ADL model from a single-step (coarse) toll to multi-step tolls. Comparison of the braking model with the ADL and Laih models reveals that braking can materially reduce the efficiency gains from tolling because the bottleneck is unused while drivers are waiting. The percentage reduction in efficiency gain depends on the unit costs of travel time and schedule delay, and (in the comparison with the ADL model) the number of steps in the toll schedule. For some parameter value combinations the loss exceeds 20%.

The fact that braking occurs in practice, and can be quite costly, raises the question how it can be prevented. Stopping in the middle of traffic lanes or parking on the shoulder can be deterred by enforcing traffic laws. Slowing down is more difficult to prevent although it can be discouraged by imposing minimum speed limits. In either case, enforcement is costly. The problems associated with braking arise when tolls are levied at specific locations. They could be avoided by replacing location-based toll collection schemes with more sophisticated systems with charges proportional to distance driven.[17]

The analysis in the paper could be extended in various directions. One is to model drivers' incentives to speed in order to reach a tolling point before a toll increase. Although speeding is the mirror image of braking, the bottleneck model is not well-designed to handle speeding because it does not feature a smooth trade-off between travel speed and accident risks.

A second possible extension is to account for the inconvenience or risk incurred by drivers in slowing down or stopping. To the extent that these costs are fixed, drivers may choose not to slow down or stop if the toll payment avoided is small. Multi-step tolling schemes with several, small toll adjustments may therefore be less vulnerable to braking than coarse schemes with tolls that are on or off.

In the standard bottleneck model travellers are assumed to care about when they arrive at their destination, but not when they leave the origin. This may be a reasonable approximation for the morning commute, but not for the evening commute or some other types of trips where early departure may be heavily penalized. The bottleneck model with departure-time preferences has been studied by a few authors including Vickrey (1973), Fargier (1983), and de Palma and Lindsey (2002a, 2002b). However, to our knowledge step tolling has not been considered in the variant of the bottleneck model with departure-time preferences.

Acknowledgments

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[17] Comprehensive, per-kilometre road-pricing schemes have other advantages, including more complete network coverage and greater spatial/geographical equity.

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Appendix A: Derivation of optimal multi-step tolls in the Laih model

At the beginning (t_s) and the end (t_e) of the travel period in the Laih model there is neither a queue nor a toll. Schedule delay costs at t_s and t_e must therefore be equal and eqn. (5) in the text holds. Combined with condition (4) this implies:

(A.1) $t_s = t^* - \dfrac{\gamma}{\beta + \gamma}\left(\dfrac{N}{s}\right),$

(A.2) $t_e = t^* + \dfrac{\beta}{\beta + \gamma}\left(\dfrac{N}{s}\right).$

The price, *p*, does not change due to step tolling, and total private cost is given by eqn. *(8)* in the text:

(A.3) $PC \equiv p \cdot N = \delta \dfrac{N^2}{s}.$

In the Laih model there are separate queues during late arrivals for drivers who pay toll ρ_i and those who pay toll ρ_{i+1}. The first driver to arrive in each toll period faces no queue. The price paid at time t_i^+ is $\beta\left(t^* - t_i^+\right) + \rho_i$ which must match the price paid at time t_s, $\beta\left(t^* - t_s\right)$. Accordingly, toll ρ_i starts at time:

(A.4) $t_i^+ = t_s + \dfrac{\rho_i}{\beta}, \; i = 1, ..., m.$

Similarly, for late arrivals the first driver to arrive in each toll period faces no queue and incurs a price $\gamma\left(t_i^- - t^*\right) + \rho_i$. Setting this price equal to the price for the last driver, $\gamma\left(t_e - t^*\right)$, one obtains:

(A.5) $t_i^- = t_e - \dfrac{\rho_i}{\gamma}, \; i = 1, ..., m.$

Total social costs equal total private costs minus toll revenue (*TR*). Let TR_i denote the toll revenue derived from the combined early and late arrival intervals for toll period *i*. Total social costs can then be written

(A.6) $TC = p \cdot N - TR = PC - TR_1 - \sum_{i=2}^{m} TR_i.$

Toll revenues are

(A.7)

$$TR = TR_1 + \sum_{i=2}^{m} TR_i = \rho_1 \cdot s\left(t_1^- - t_1^+\right) + \left(\sum_{i=2}^{m} \rho_i s\left(t_{i-1}^+ - t_i^+\right)\right) + \left(\sum_{i=2}^{m} \rho_i s\left(t_i^- - t_{i-1}^-\right)\right)$$

$$= \rho_1 \cdot s\left(N/s - \rho_1/\beta - \rho_1/\gamma\right) + \left(\sum_{i=2}^{m} s \cdot \rho_i \left(\rho_{i-1} - \rho_i\right)/\beta\right) + \left(\sum_{i=2}^{m} s \cdot \rho_i \left(\rho_{i-1} - \rho_i\right)/\gamma\right)$$

$$= \rho_1\left(N - s\rho_1/\delta\right) + \left(\frac{s}{\delta}\sum_{i=2}^{m} \rho_i \left(\rho_{i-1} - \rho_i\right)\right).$$

Inserting *(A.7)* into (A.6), and taking derivatives gives the following first-order conditions:

(A.8) $N - 2 \cdot s\left(\rho_1/\delta\right) + s \cdot \rho_2/\delta = 0,$ for $i = 1$

(A.9) $\rho_{i-1} - 2\rho_i + \rho_{i+1} = 0,$ for $i = 2, ..., m-1$

(A.10) $\rho_{m-1} - 2\rho_m = 0,$ for $i = m$

Equation *(A.10)* implies $\rho_{m-1} = 2\rho_m$. Solving *(A.9)* iteratively gives:

(A.11) $\rho_i = \dfrac{m+1-i}{m}\rho_1, \quad i = 1, ..., m.$

Inserting (A.11) into (A.8) gives:

(A.12) $\rho_1 = \dfrac{m}{m+1}\delta\dfrac{N}{s}.$

Inserting (A.7), (A.11), and *(A.12)* into (A.6) implies an equilibrium total social cost of:

(A.13) $TC = \delta\dfrac{N^2}{s}\left(1 - \dfrac{1}{2}\dfrac{m}{m+1}\right).$

Appendix B: Derivation of optimal multi-step tolls in the ADL model with α>γ

B.1. Overview of solution

The solution presented here is based on heuristic reasoning rather than a rigorous derivation of the general structure of the toll. The timing of the step toll is derived on the following assumptions:

- During the early arrival period there is an interval prior to each step up in the toll when no user departs. The queue drops to zero at the instant when the toll steps up.

- During the late arrival period a mass of drivers departs each time the toll steps down. With $\gamma \geq \alpha$, the queue drops to zero at the instant when the toll takes its next step down and the

next mass of drivers departs. All drivers who arrive late depart in masses; unlike with $\gamma < \alpha$ there are no time intervals during which drivers depart at a finite rate.

B.2. Derivation of solution

The first step is to derive the timing of the toll schedule for given toll levels. For early arrivals the solution is the same for $\gamma \geq \alpha$ and $\gamma < \alpha$. The first user to depart incurs only an early arrival cost and hence pays a price of:

(B.1)　$p(t_s) = \beta(t^* - t_s)$.

The user who departs at t_i^+ immediately after step-up toll ρ_i^+ is imposed incurs an early arrival cost, and pays the toll, but faces no queue. The user's cost is therefore:

(B.2)　$p(t_i^+) = \beta(t^* - t_i^+) + \rho_i^+, \ i = 1,...,m.$

Time t_i^+ is solved by equating prices in eqn. (B.1) and eqn. *(B.2)*:

(B.3)　$t_i^+ = t_s + \dfrac{\rho_i^+}{\beta}, \ i = 1,...,m.$

The user who departs at t_i^- just before the central-toll interval ends incurs a price:

(B.4)　$p(t_1^-) = \rho_1 + \gamma(t_1^- - t^*)$.

A mass of M_i users departs at time t_i^-. These users incur an expected price of

(B.5)　$E \cdot p(t_i^-) = \rho_{i+1}^- + \gamma(t_i^- - t^*) + \dfrac{1}{2}\dfrac{M_i}{s}(\alpha + \gamma), \ i = 1,...,m.$

M_i is solved by equating prices in eqn. (B.4) and eqn. *(B.5)*:

(B.6)　$M_i = \dfrac{2s}{\alpha + \gamma}\left(\gamma(t_1^- - t_i^-) + \rho_1 - \rho_{i+1}^-\right), \ i = 1,...,m.$

Using eqn. (B.6),

(B.7)　$t_{i+1}^- = t_i^- + \dfrac{M_i}{s} = \dfrac{\alpha - \gamma}{\alpha + \gamma}t_i^- + \dfrac{2\gamma}{\alpha + \gamma}t_1^- + \dfrac{2}{\alpha + \gamma}\left(\rho_1 - \rho_{i+1}^-\right), \ i = 1,...,m-1.$

Define $R \equiv (\alpha - \gamma)/(\alpha + \gamma) < 0$. Applying eqn. *(B.7)* iteratively, summing the component terms, and simplifying expressions one obtains

(B.8) $t_i^- = t_1^- + \dfrac{1}{\gamma}\left(1 - R^{i-1}\right)\rho_1 - \dfrac{2}{\alpha+\gamma}\sum_{j=2}^{i} R^{i-j}\rho_j^-$, $i = 2,\dots,m.$

Equilibrium price as a function of tolls

The expected price for users in the last mass to depart is:

(B.9) $E \cdot p\left(t_m^-\right) = \gamma\left(t_m^- - t^*\right) + \dfrac{1}{2}\left(t_e - t_m^-\right)\left(\alpha + \gamma\right).$

Equate prices in eqn. (B.9) and (B.1), and use $t_e = t_s + N/s$:

(B.10) $\gamma\left(t_m^- - t^*\right) + \dfrac{1}{2}\left(t_s + N/s - t_m^-\right)\left(\alpha + \gamma\right) = \beta\left(t^* - t_s\right).$

Equate prices in eqn. (B.4) and eqn. (B.1), and solve for t_1^-:

(B.11) $t_1^- = \dfrac{\beta+\gamma}{\gamma}t^* - \dfrac{\beta t_s + \rho_1^-}{\gamma}.$

Substitute eqn. (B.11) into eqn. (B.8), set $i = m$, substitute the resulting formula for t_m^- into eqn. (B.10), and solve for t_s:

(B.12) $t_s = t^* - \dfrac{\gamma}{\beta+\gamma}\dfrac{N}{s} - \dfrac{R^m}{\beta+\gamma}\left(\rho_1 + \dfrac{2\gamma}{\alpha+\gamma}R\sum_{j=2}^{m}R^{-j}\rho_j^-\right).$

Substitute eqn. (B.12) into eqn. (B.11):

(B.13) $t_1^- = t^* + \dfrac{\beta}{\beta+\gamma}\dfrac{N}{s} + \dfrac{\beta}{\gamma(\beta+\gamma)}R^m\left(\rho_1 + \dfrac{2\gamma}{\alpha+\gamma}R\sum_{j=2}^{m}R^{-j}\rho_j^-\right) - \dfrac{\rho_1}{\gamma}.$

Equilibrium travel prices can now be expressed as a function of toll levels alone by substituting eqn. (B.13) into eqn. (B.4):

(B.14) $p = \dfrac{\beta\gamma}{\beta+\gamma}\left(\dfrac{N}{s} + R^m\left(\dfrac{\rho_1}{\gamma} + \dfrac{2R}{\alpha+\gamma}\sum_{j=2}^{m}R^{-j}\rho_j^-\right)\right).$

Toll revenues

The toll revenue generated by the central toll, ρ_1, is

(B.15) $TR_1 = \rho_1 s\left(t_1^- - t_1^+\right).$

Substitute eqn. (B.3) with $i = 1$ and eqn. (B.13) into eqn. (B.15):

(B.16) $TR_1 = \rho_1 s\left(\dfrac{N}{s} - \dfrac{1}{\gamma}\left(\dfrac{\beta+\gamma}{\beta} - R^m\right)\rho_1 + \dfrac{2}{\alpha+\gamma}R^{m+1}\sum_{j=2}^{m}R^{-j}\rho_j^-\right).$

Revenue from step-up toll ρ_i^+ is:

$$TR_i^+ = \rho_i^+ s\left(t_{i-1}^+ - t_i^+\right), \ i = 2,...,m,$$

or, using eqn. (B.3),

(B.17) $TR_i^+ = \rho_i^+ s \dfrac{\rho_{i-1}^+ - \rho_i^+}{\beta}, \ i = 2,...,m.$

Revenue from step-down toll ρ_i^- is:

$$TR_i^- = \rho_i^- s\left(t_i^- - t_{i-1}^-\right), \ i = 2,...,m,$$

or, using eqn. (B.8),

(B.18) $TR_i^- = \rho_i^- s\left(\dfrac{(1-R)R^{i-2}}{\gamma}\rho_1 - \dfrac{2}{\alpha+\gamma}\rho_i^- + \dfrac{2(1-R)R^{i-1}}{\alpha+\gamma}\sum_{j=2}^{i-1}R^{-j}\rho_j^-\right), \ i = 2,...,m.$

Total social costs

Total social costs are

$$TC = C\square N - \sum_{i=1}^m TR_i = C\square N - \left(TR_1 + \sum_{i=2}^m TR_i^+ + \sum_{i=2}^m TR_i^-\right).$$

Using eqns. (B.6), (B.17) and (B.18), *TC* can be written as:

(B.19)

$$TC = \delta\frac{N^2}{s} + \left(\frac{\delta N R^m}{\gamma} - N\right)\rho_1 + \frac{s}{\gamma}\left(\frac{\beta+\gamma}{\beta} - R^m\right)\rho_1^2$$

$$+ \frac{2}{\alpha+\gamma}R^{m+1}\left(\delta N - s\rho_1\right)\sum_{i=2}^m R^{-i}\rho_i^- - \frac{s}{\beta}\sum_{i=2}^m \rho_i^+\left(\rho_{i-1}^+ - \rho_i^+\right) + \frac{2s}{\alpha+\gamma}\sum_{i=2}^m R^{-i}\left(\rho_i^-\right)^2$$

$$- \frac{(1-R)s}{\gamma}R^{-2}\rho_1\sum_{i=2}^m R^i\rho_i^- - \frac{2(1-R)R^{-1}}{\alpha+\gamma}s\sum_{i=2}^m R^i\rho_i^- \sum_{j=2}^{i-1}R^{-j}\rho_j^-.$$

First-order conditions for tolls

The first-order conditions for optimal tolls are derived by differentiating eqn. (B.19). The first-order condition for ρ_1 is:

(B.20) $\left(1 + \dfrac{\gamma}{\beta} - R^m\right)\left(2\rho_1 - \delta\dfrac{N}{s}\right) = (1-R)R^{-2}\sum_{i=2}^m R^i\rho_i^- + (1-R)R^{m+1}\sum_{i=2}^m R^{-i}\rho_i^- + \dfrac{\gamma}{\beta}\rho_2^+.$

The first-order conditions for the step-up tolls, ρ_i^+, are

(B.21) $\quad\begin{aligned}&\rho_{i-1}^+ - 2\rho_i^+ + \rho_{i+1}^+ = 0, \quad i = 2,...,m-1,\\ &\rho_{m-1}^+ - 2\rho_m^+ = 0, \qquad\qquad \text{for } i = m.\end{aligned}$

Eqn. (B.21b) yields

(B.22) $\quad \rho_m^+ = \dfrac{1}{2}\rho_{m-1}^+.$

The remaining tolls can be derived using the recursive equation:

(B.23) $\quad \rho_i^+ = \dfrac{m+1-i}{m+2-i}\rho_{i-1}^+, \quad i = 2,...,m.$

Eqn. (B.23) is proved by induction. Given eqn. (B.22) it holds for $i = m$. Assume it holds for general i. For $i-1$, eqn. (B.21a) and eqn. (B.23) imply

$$\rho_{i-2}^+ - 2\rho_{i-1}^+ + \rho_i^+ = \rho_{i-2}^+ - 2\rho_{i-1}^+ + \frac{m+1-i}{m+2-i}\rho_{i-1}^+ = \rho_{i-2}^+ - \frac{m+3-i}{m+2-i}\rho_{i-1}^+ = 0,$$

or

$$\rho_{i-1}^+ = \frac{m+2-i}{m+3-i}\rho_{i-2}^+.$$

Eqn. (B.23) therefore holds for $i-1$ ☐ .

Applying eqn. (B.23) iteratively one obtains, finally,

(B.24) $\quad \rho_i^+ = \dfrac{m+1-i}{m}\rho_1, \quad i = 1,...,m.$

Setting $i = 2$ in eqn. (B.24), substituting the expression for ρ_2^+ into eqn. (B.20), and collecting terms gives

(B.25)
$$\left(\frac{m+1}{m}\frac{\gamma}{\beta} + 2\left(1-R^m\right)\right)\rho_1 - \left(1-R\right)R^{-2}\sum_{i=2}^m R^i\rho_i^- + \left(1-R\right)R^{m+1}\sum_{i=2}^m R^{-i}\rho_i^-$$
$$= \left(1 + \frac{\gamma}{\beta} - R^m\right)\delta\frac{N}{s}.$$

The first-order conditions for the step-down tolls, ρ_i^-, are

(B.26)
$$\left(R^{m+1-i} + R^{i-2}\right)\rho_1 - 2\rho_i^- + \left(1-R\right)\left(R^{i-1}\sum_{j=2}^{i-1} R^{-j}\rho_j^- + R^{-1-i}\sum_{j=i+1}^m R^j\rho_j^-\right)$$
$$= R^{m+1-i}\delta\frac{N}{s}, \quad i = 2,...,m-1,$$

and for $i = m$,

(B.27) $\left(R+R^{m-2}\right)\rho_1 - 2\rho_m^- + (1-R)R^{m-1}\sum_{j=2}^{m-1}R^{-j}\rho_j^- = R\delta\dfrac{N}{s}.$

(Note that setting $i=m$ in eqn. (B.26) gives eqn. (B.27).) Replace i with $i-1$ in eqn. (B.26), multiply by R, and subtract the result from eqn. (B.26) to get

(B.28) $R^{m+1-i}(1-R)\rho_1 - (2-R)\rho_i^- + \rho_{i-1}^- + (1-R)^2 R^{-1-i}\sum_{j=i+1}^{m}R^j\rho_j^- = R^{m+1-i}(1-R)\delta\dfrac{N}{s}.$

Replace i with $i+1$ in eqn. (B.28), multiply by R, and subtract the result from eqn. (B.28) to get

$$\rho_{i-1}^- - 2\rho_i^- + \rho_{i+1}^- = 0, \quad i = 3,...,m-1.$$

Rewrite these equations as

(B.29) $\rho_i^- = 2\rho_{i-1}^- - \rho_{i-2}^-, \quad i = 4,...,m.$

We now prove by induction that

(B.30) $\rho_i^- = (i-2)\rho_3^- - (i-3)\rho_2^-, \quad i = 2,...,m.$

Eqn. (B.30) holds trivially for $i=2$ and $i=3$, and by eqn. (B.29) it also holds for $i=4$. Assume it holds for general i. For $i+1$, eqn. (B.29) gives

$$\rho_{i+1}^- = 2\rho_i^- - \rho_{i-1}^- = 2\left((i-2)\rho_3^- - (i-3)\rho_2^-\right) - \left((i-3)\rho_3^- - (i-4)\rho_2^-\right)$$
$$= (i-1)\rho_3^- - (i-2)\rho_2^-.$$

This concludes the proof.□

Substitute eqn. (B.30) into eqn. (B.27) to get

(B.31)
$$\left(R+R^{m-2}\right)\rho_1 - 2\left((m-2)\rho_3^- - (m-3)\rho_2^-\right)$$
$$+(1-R)R^{m-1}\sum_{j=2}^{m-1}R^{-j}\left((j-2)\rho_3^- - (j-3)\rho_2^-\right) = R\delta\dfrac{N}{s}.$$

Working out the summation terms in (B.31), and rearranging components, yields a linear equation in ρ_1, ρ_2^- and ρ_3^-:

(B.32)
$$(1-R)(R+R^{m-2})\rho_1 + \left(m-2-(m-3)R-2R^{m-2}+R^{m-1}\right)\rho_2$$
$$+\left(1-m+(m-2)R+R^{m-2}\right)\rho_3 = R(1-R)\delta N/s.$$

Substitute eqn. (B.30) into eqn. (B.25)

$$\left(\frac{m+1}{m}\frac{\gamma}{\beta}+2\left(1-R^m\right)\right)\rho_1 -\left(1-R\right)R^{-2}\sum_{i=2}^{m}R^i\left(\left(i-2\right)\rho_3^- -\left(i-3\right)\rho_2^-\right)$$

(B.33)

$$-\left(1-R\right)R^{m+1}\sum_{i=2}^{m}R^{-i}\left(\left(i-2\right)\rho_3^- -\left(i-3\right)\rho_2^-\right)=\left(1+\frac{\gamma}{\beta}-R^m\right)\delta\frac{N}{s}.$$

Combining the summation terms in (B.33), and rearranging components, yields a second linear equation in ρ_1, ρ_2^- and ρ_3^-:

$$\left(\frac{m+1}{m}\frac{\gamma}{\beta}+2\left(1-R^m\right)\right)\rho_1 +\left(-1+\left(m-2\right)R-\left(m-2\right)R^{m-1}+R^m\right)\rho_2^-$$

(B.34)

$$-R\left(m-1\right)\left(1-R^{m-2}\right)\rho_3^- =\left(1+\frac{\gamma}{\beta}-R^m\right)\delta\frac{N}{s}.$$

Substitute eqn. (B.30) into eqn. (B.26) with $i=2$:

$$(B.35)\left(1+R^{m-1}\right)\rho_1 -2\rho_2^- +\left(1-R\right)\rho_3^- +\left(1-R\right)R^{-3}\sum_{j=4}^{m}R^j\left(\left(j-2\right)\rho_3^- -\left(j-3\right)\rho_2^-\right)=R^{m-1}\delta\frac{N}{s}.$$

Combining the summation terms in (B.35), and rearranging components, yields a third, nonlinear, equation in ρ_1, ρ_2^- and ρ_3^-:

$$\left(1+R^{m-1}\right)\rho_1 -2\rho_2^- +\left(1-R\right)\rho_3^-$$

(B.36)

$$+\left(1-R\right)\left[R\frac{1-R^{m-3}}{1-R}\left(3\rho_2^- -2\rho_3^-\right)+\frac{R\left(4-3R-\left(1+m\right)R^{m-3}+mR^{m-2}\right)}{\left(1-R\right)^2}\right]\left(\rho_3^- -\rho_2^-\right).$$

$$=R^{m-1}\delta\frac{N}{s}.$$

Optimal tolls

Equations (B.32), (B.34) and (B.36) can be solved simultaneously for ρ_1, ρ_2^- and ρ_3^-. The solutions are homogeneous of degree one in parameters $\left(\alpha,\beta,\gamma\right)$. Introducing new variables $b\equiv\beta/\alpha$ and $c\equiv\gamma/\alpha$ one obtains:

$$(B.37)\ \ \rho_1=\delta\frac{N}{s}\frac{m\left(1+2b+c\left(1+m+\left(2m-1\right)\left(2b+c\right)+\left(m-1\right)c^2\right)\right)}{1+m+4bm+c\left(\left(m+1\right)^2+4bm^2+c\left(m-1-c+2m^2+m^2c\right)\right)},$$

$$(B.38)\ \ \rho_2=\delta\frac{N}{s}\frac{m-1+c\left(c^2+c-1-m+2m^2+8m\left(m-1\right)b+2m^2c^2-3c^2m-5mc+4m^2c\right)}{2\left(1+m+4bm+c\left(\left(m+1\right)^2+4bm^2+c\left(m-1-c+2m^2+m^2c\right)\right)\right)},$$

(B.39) $\rho_3 = \delta \dfrac{N}{s} \dfrac{m-1+c\left(2m^2-3m-3+8m(m-2)b+\left(4m^2-9m+1\right)c+\left(2m^2-5m+3\right)c^2\right)}{2\left(1+m+4bm+c\left((m+1)^2+4bm^2+c\left(m-1-c+2m^2+m^2c\right)\right)\right)}.$

The remaining tolls, $\rho_4^-, ..., \rho_m^-$, can then be solved by repeated application of eqn. (B.30). The formula for the equilibrium cost, *TC*, is cumbersome.

The one-step toll has a rather simple solution:

$$\rho_1 = \delta \frac{N}{2s},$$

$$TC = \left(1 - \frac{\gamma(\alpha + 2\beta + \gamma)}{4(\alpha + \gamma)(\beta + \gamma)}\right)\delta \frac{(N)^2}{s}.$$

However, the two-step toll is already rather complex:

$$\rho_1 = \frac{2}{3}\delta \frac{N}{s}\left(1 + \frac{2\alpha\beta(-\alpha + \gamma)}{3\alpha^3 + 3\gamma^3 + \alpha^2(8\beta + 9\gamma) + \alpha\gamma(16\beta + 9\gamma)}\right),$$

$$\rho_2^- = \rho_1 \frac{\alpha^3 + 5\alpha^2\gamma + 3\gamma^3 + \alpha\gamma(16\beta + 7\gamma)}{4(\alpha^3 + \gamma^3 + 3\alpha\gamma(2\beta + \gamma) + \alpha^2(2\beta + 3\gamma))},$$

$$TC = \left(1 - \frac{\gamma(2\alpha^3 + \gamma^2(3\beta + 2\gamma) + \alpha^2(11\beta + 6\gamma) + 2\alpha(8\beta^2 + 9\beta\gamma + 3\gamma^2))}{2(\beta + \gamma)(3\alpha^3 + 3\gamma^3 + \alpha^2(8\beta + 9\gamma) + \alpha\gamma(16\beta + 9\gamma))}\right)\delta \frac{(N)^2}{s}.$$

B.3. Properties of the solution

(a) Limiting behaviour as $\gamma \to \alpha$

Equation (B.37) can be rewritten

$$\rho_1 = \delta \frac{N}{s}\frac{m\left(4m(1+b)+(c-1)\left(8m-4+2b(2m-1)+(c-1)(4m-2)+(c-1)^2\right)\right)}{4m(1+b)(m+1)+(c-1)\left(4\left(2m^2+bm^2+m-1\right)+(c-1)\left(5m^2+m-4\right)+(c-1)^2\left(m^2-1\right)\right)}.$$

In the limit $\gamma \to \alpha$, $c \to 1$ and $\rho_1 \to m/(m+1)$. Similarly, it is readily shown that $\rho_2 \to (m-1)/(m+1)$ and $\rho_3 \to (m-2)/(m+1)$. Using eqn. (B.30) for $i > 3$, one finds $\rho_i \to (m+1-i)/(m+1)$.

(b) Sizes of toll steps

Given eqn. (B.24) the size of the step between consecutive step-up tolls is constant. The size of the step between consecutive step-down tolls is constant for tolls ρ_2^- through ρ_m^-. To see this, apply eqn. (B.30) for i and $i-1$:

$$\rho_i^- = (i-2)\rho_3^- - (i-3)\rho_2^-,$$

$$\rho_{i-1}^- = (i-3)\rho_3^- - (i-4)\rho_2^-.$$

The difference is

$$\rho_i^- - \rho_{i-1}^- = \rho_3^- - \rho_2^-, \quad i = 4,...,m.$$

The step down from ρ_1 to ρ_2^- is smaller than the other toll steps. Using eqns. ((B.37), (B.38) and (B.39) one obtains after some algebra:

$$\rho_1 - \rho_2^- - \left(\rho_2^- - \rho_3^-\right) \overset{s}{=} (c-1)\big((c+1)\big(m(c+1)-(c-1)\big)+4mb\big) > 0.$$

(c) Asymptotic behaviour as the number of toll steps increases

Value of the peak toll

Taking the limit $m \to \infty$, all terms in the numerator and denominator of eqn. (B.37) drop out except those involving m^2, and $\lim_{m\to\infty}\rho_1 = \delta N / s$. The peak-period toll thus approaches the peak value of the continuous-time (fine) toll in the limit as $m \to \infty$. Since all steps between consecutive step-up tolls are constant, and all steps between consecutive step-down tolls are constant except for $\rho_1 - \rho_2^-$ (which is smaller than the other down-steps), the optimal m-step toll converges to the optimal fine toll in the limit as $m \to \infty$. By continuity, the efficiency gain from the optimal m-step toll converges to the efficiency gain from the optimal fine toll as well.

Timing of departures

Using equation (B.12) it is straightforward but tedious to show that the start time of the peak, t_s, is a monotonically decreasing function of m. In the limit as $m \to \infty$, the timing of departures converges to the timing with no toll. From eqn. (B.12) the limiting value of t_s is:

$$\lim_{m\to\infty} t_s = t^* - \frac{\gamma}{\beta+\gamma}\frac{N}{s} - \frac{\rho_1}{\beta+\gamma}\underbrace{\lim_{m\to\infty} R^m}_{=0} - \frac{2\gamma R}{(\alpha+\gamma)(\beta+\gamma)}\lim_{m\to\infty}\underbrace{\sum_{j=2}^{m} R^{m-j}\rho_j^-}_{Z}.$$

Term Z can be written $Z = \sum_{k=0}^{m-2} R^k \rho_{m-k}^-$. Since the ρ_i^- are monotonically decreasing with i, and have uniform decrements after $i=2$, for any given k:

$$Z < \left(1 + R + ... + R^k\right)\rho_{m-k}^- + R^{k+1}\left(1 + R + ... + R^{m-k-3}\right)\rho_2^- = \underbrace{\frac{1-R^{k+1}}{1-R}}_{(i)}\underbrace{\rho_{m-k}^-}_{(ii)} + \underbrace{R^{k+1}}_{(iii)}\underbrace{\frac{1-R^{m-k-2}}{1-R}}_{(iv)}\underbrace{\rho_2^-}_{(v)}.$$

Now set $k = \lceil \sqrt{m} \rceil$: the smallest integer not less than $\sqrt{m}$. In the limit $m \to \infty$, terms (*i*), (*iv*) and (*v*) approach finite positive values. Terms (*ii*) and (*iii*) converge to zero. Hence $\lim_{m\to\infty} Z = 0$,

and $\lim_{m\to\infty} t_s = t^* - \dfrac{\gamma}{\beta+\gamma}\dfrac{N}{s}$. $\square$

Comparison of ADL tolls with Laih tolls

As noted in the text, if $m > 1$ and $\gamma > \alpha$, each ADL toll is larger than the corresponding Laih toll. Formally,

(a) $\rho_1^{ADL} > \rho_1^{Laih}$,

(b) $\rho_i^{ADL+} > \rho_i^{Laih}$ for $i = 2, ..., m$.

(c) $\rho_i^{ADL-} > \rho_i^{Laih}$ for $i = 2, ..., m$.

Proof of (a):

Toll ρ_1^{Laih} is given in eqn. (14), and toll ρ_1^{ADL} is given in eqn. (B.37). Taking the difference one obtains:

$$\rho_1^{ADL} - \rho_1^{Laih} \overset{s}{=} (m-1)(c-1)$$

where $c \equiv \gamma/\alpha$.

Proof of (b):

From eqn. (14), $\rho_i^{Laih} = \dfrac{m+1-i}{m}\cdot\rho_1^{Laih}$, $i = 2, ..., m$, and from eqn. (17),

$\rho_i^{ADL+} = \dfrac{m+1-i}{m}\cdot\rho_1^{ADL}$, $i = 2, ..., m$. Result (b) then follows immediately from result (a).

Proof of (c):

From eqn. (B.30), $\rho_i^- = (i-2)\rho_3^- - (i-3)\rho_2^-$ for $i = 2, ..., m$. Using eqn. (B.38), eqn. (B.39) and eqn. (B.14) one obtains after some algebraic manipulations:

$$\rho_i^{ADL-} - \rho_1^{Laih} \overset{s}{=} (c-1)\left[(m^2-1)(c^2-1) + 2((m+1)(1+c)+4mb)(m+1-i)\right]$$

where $b \equiv \beta/\alpha$. Given $m > 1$ and $c > 1$, every term inside the square brackets is strictly positive.

Appendix C: Derivation of the optimal multi-step tolls in the braking model

In the braking model, the last driver to arrive during the *i*th toll period arrives at t_i^b, and faces no queue. The next driver waits before the tolling point for a period $\Delta t_i \equiv t_i^- - t_i^b$, until the toll is lowered at t_i^-. In equilibrium, the price for the driver who arrives at t_i^b and pays a toll of ρ_i equals the price for the first driver who waits and pays ρ_{i-1} instead. Therefore, $\Delta t_i = (\rho_{i-1} - \rho_i)/(\alpha + \gamma)$. The cumulative amount of time that bottleneck capacity is unused is equal to:

$$(C.1) \quad \Delta t = \sum_{i=1}^n \Delta t_i = \frac{\rho_1}{\alpha + \gamma}.$$

The total slack time, Δt, does not depend directly on *m*. Yet, because ρ_1 increases with *m*, Δt also increases. At the start (t_s') and end (t_e') of the peak, travel delays are zero, and the schedule delay costs must be equal. All users must pass the bottleneck during the time that it is used, $t_e' - t_s' - \Delta t$. Hence:

$$(C.2) \quad t_s = t^* - \frac{\gamma}{\beta + \gamma}\left(\frac{N}{s} + \Delta t\right) = t^* - \frac{\gamma}{\beta + \gamma}\left(\frac{N}{s} + \frac{\rho_1}{\alpha + \gamma}\right),$$

$$(C.3) \quad t_e = t^* + \frac{\beta}{\beta + \gamma}\left(\frac{N}{s} + \Delta t\right) = t^* + \frac{\beta}{\beta + \gamma}\left(\frac{N}{s} + \frac{\rho_1}{\alpha + \gamma}\right).$$

The first driver to pay the step-up toll ρ_i faces no queue. The price paid at time t_i^+ is $\beta\left(t^* - t_i^+\right) + \rho_i$ which must match the price at time t_s', $\beta\left(t^* - t_s'\right)$. The toll therefore starts at time:

$$(C.4) \quad t_i^+ = t_s' + \frac{\rho_i}{\beta}, \quad i = 1, \ldots, m.$$

Similarly, the last driver to pay the step-down toll ρ_i arrives at time:

$$(C.5) \quad t_i^b = t_e' - \frac{\rho_i}{\gamma}, \quad i = 1, \ldots, m.$$

Total social costs equal total private costs minus toll revenue:

$$(C.6) \quad TC = PC - TR = PC - TR_1 - \sum_{i=2}^m TR_i.$$

The component terms are:

$$(C.7) \quad PC = \delta N\left(\frac{N}{s} + \Delta t\right) = \delta N\left(\frac{N}{s} + \frac{\rho_1}{\alpha + \gamma}\right).$$

(C.8) $\quad TR_1 = s\rho_1\left(t_1^- - t_1^+\right) = s\rho_1\left(\dfrac{N}{s} + \dfrac{\rho_1}{\alpha+\gamma} - \dfrac{\rho_1}{\beta} - \dfrac{\rho_1}{\gamma}\right) = s\rho_1\left(\dfrac{N}{s} + \dfrac{\rho_1}{\alpha+\gamma} - \dfrac{\rho_1}{\delta}\right).$

$$TR_i = s\rho_i\left[\left(t_{i-1}^+ - t_i^+\right)\left(t_i^b - t_{i-1}^b - \Delta t_{i-1}\right)\right]$$

(C.9) $\quad = s\rho_i\left[\left(\dfrac{\rho_{i-1}-\rho_i}{\beta}\right)\left(\dfrac{\rho_{i-1}-\rho_i}{\gamma} - \dfrac{\rho_{i-1}-\rho_i}{\alpha+\gamma}\right)\right] \qquad i = 2,...,m.$

$$= s\rho_i\left(\rho_{i-1}-\rho_i\right)\left(\dfrac{1}{\delta} - \dfrac{1}{\alpha+\gamma}\right)$$

The first-order conditions for minimizing *TC* with respect to the tolls are determined by inserting eqns. (C.2) through (C.5) into (C.1), and taking derivatives:

(C.10) $\begin{array}{ll} \dfrac{\delta}{\alpha+\gamma}N - N + s\rho_1\left(\dfrac{2}{\alpha+\gamma} - \dfrac{2}{\delta}\right) - s\rho_2\left(\dfrac{1}{\delta} - \dfrac{1}{\alpha+\gamma}\right) & \text{for } i=1, \\[3mm] \rho_{i-1} - 2\rho_i + \rho_{i+1} = 0, & \text{for } i = 2,...,m-1, \\[2mm] \rho_{m-1} - 2\rho_m = 0 & \text{for } i = m. \end{array}$

Eqn. (C.10) for $i = 1,...,m-1$ gives the same first-order conditions as in the Laih model. Hence:

(C.11) $\quad \rho_i = \dfrac{m+1-i}{m}\rho_1, \quad i = 1,...,m.$

Inserting (C.11) for ρ_2 into (C.10a) gives:

(C.12) $\quad \rho_1 = \dfrac{m}{m+1}\delta\dfrac{N}{s},$

which is the same formula as in the Laih model. The formulas are also the same for the time at which the first driver arrives during the *i*th early toll (t_i^+), and the time when the last driver arrives during the ith late toll period (t_i^- in the Laih model, and t_i^b in the braking model). However, the timing of the peak differs because bottleneck capacity is unused for a time Δt_i after t_i^b. Inserting eqns. (C.11) and (C.12) for the tolls into the total-cost function gives:

(C.13) $\quad TC_{Braking} = \delta\dfrac{N^2}{s}\left(1 - \dfrac{1}{2}\dfrac{m}{m+1}\left(1 - \dfrac{\beta\gamma}{(\beta+\gamma)(\alpha+\gamma)}\right)\right),$

(C.14) $\quad \omega_{Braking} = \dfrac{TC_{NT} - TC_{Braking}}{TC_{NT} - TC_{FB}} = \dfrac{m}{1+m}\left(1 - \dfrac{\beta\gamma}{(\beta+\gamma)(\alpha+\gamma)}\right).$