TI 2010-023/2
Tinbergen Institute Discussion Paper

Preopening and Equilibrium Selection

Riccardo Calcagno[1]
Stefano Lovo[2]

[1] *VU University Amsterdam;*
[2] *HEC, Paris.*

Tinbergen Institute

The Tinbergen Institute is the institute for economic research of the Erasmus Universiteit Rotterdam, Universiteit van Amsterdam, and Vrije Universiteit Amsterdam.

Tinbergen Institute Amsterdam
Roetersstraat 31
1018 WB Amsterdam
The Netherlands
Tel.: +31(0)20 551 3500
Fax: +31(0)20 551 3555

Tinbergen Institute Rotterdam
Burg. Oudlaan 50
3062 PA Rotterdam
The Netherlands
Tel.: +31(0)10 408 8900
Fax: +31(0)10 408 9031

Preopening and Equilibrium Selection*

Riccardo Calcagno[†] and Stefano Lovo[‡]

February 16, 2010

Abstract

We introduce a form of pre-play communication that we call "preopening". During the preopening, players announce their tentative actions to be played in the underlying game. Announcements are made using a posting system which is subject to stochastic failures. Posted actions are publicly observable and players payoffs only depend on the opening outcome, i.e. the action profile that is posted at the end of the preopening phase. We show that when the posting failures hit players idiosyncratically all equilibria of the preopening game lead to the same opening outcome that corresponds to the most "sensible" pure Nash equilibrium of the underlying game. By contrast preopening does not operate an equilibrium selection when posting failure hits players simultaneously.

Keywords: Preopening, equilibrium selection, bargaining, cheap talk.

JEL codes: C72, C73, C78, G1.

---

*We thank, Thomas Mariotti, Sebastien Pouget, Tristan Tomala, Nicolas Vieille, and, particularly, Johannes Hörner for useful comments and suggestions. We also thank seminar audiences at the GDR Conference in Luminy 2009, SAET Conference in Ischia 2009, Toulouse School of Economics for their comments. Stefano Lovo gratefully acknowledges financial support from the HEC Foundation.

[†]VU University Amsterdam and Tinbergen Institute, rcalcagno@feweb.vu.nl

[‡]HEC, Paris, lovo@hec.fr

1 Introduction

This paper studies an equilibrium model of pre-play communication that we call "preopening", a term we borrow from the financial markets. During the preopening phase players continuously submit tentative actions that are posted on a publicly observable screen. Players' actual payoffs only depend on the action profile that is posted at the end of the preopening phase, the "opening" date. If the posting system is perfect, tentative actions can be instantaneously changed during the preopening phase and the resulting game corresponds to a pure "cheap-talk" game (as in Aumann and Hart (2003)). We consider the case where the posting system is inefficient in the sense that an action submitted at time t will be posted with some exogenous random delay. Our objective is to identify the forms of posting inefficiency guaranteeing that all equilibria of the preopening game lead to the same opening outcome. We show that this happens when there is nil probability that players' submitted actions are posted simultaneously ("idiosyncratic" inefficiency). Since the resulting opening outcome is one of the pure Nash equilibrium of the underlying game, the preopening can be interpreted as an equilibrium selection device. Namely, in games of common interests, preopening leads to the Pareto dominant action profile. In two-action games of conflicting interests, preopening leads to select the underlying-game equilibrium preferred by the player with (i) the strongest preference over the different equilibria, (ii) the lower cost of miscoordination or (iii) the less efficient posting system. By contrast, when the posting failure is due to factors that simultaneously affect all players ("systemic" inefficiency), many opening outcomes can be observed in equilibrium.

Consider for example the coordination game of Figure 1. This game has two pure-strategy Nash equilibria, (U, L) which Pareto dominates (D, R), and one equilibrium in mixed strategies. Traditional refinements such as trembling hand (Selten (1975)) or properness (Myerson (1978)) are not effective to eliminate the inefficient equilibrium (D, R). Also, it is well known that (D, R) remains an equilibrium even if actual play is preceded by a communication phase where messages are neither costly nor binding (pure "cheap talk", Crawford and Sobel (1982)).[1] The

[1] There always exists an ("babbling") equilibrium of the cheap-talk game in which players end up playing the inefficient Nash equilibrium $\{D, R\}$ irrespective of what is "said" during the pre-play communication phase.

risk dominant equilibrium is (D, R), suggesting that uncertainty about the other player strategy might lead to coordinate on a Pareto inferior Nash equilibrium. By contrast, we show that when the posting inefficiency is idiosyncratic, all equilibria of a long enough preopening game lead players to play the Pareto dominant outcome (U, L) at the opening.

	L	R
U	$4, 4$	$0, 2$
D	$2, 0$	$3, 3$

Figure 1

Let the underlying game be any given finite normal form game. A preopening game unfolds as follows. The "preopening phase" lasts for a time interval $[0, T]$. In this phase, at any moment each player submits a tentative action chosen in the set of his actions in the underlying game. A submitted action is meant to be posted on a publicly observable screen. We assume that the posting system is imperfect in the following sense: if during the interval $[t, t + \Delta]$ a player consistently submits the same action, this action is posted on the screen before $t + \Delta$ with an exogenous probability that increases with Δ and, possibly, depends on the identity of the player. With the complement probability, at instant $t + \Delta$ the submitted action is not posted on the screen but the player's posted action is identical to that posted at instant t. Players' payoffs only depend on the action profile that is actually posted on the screen at date T and correspond to the underlying game payoffs for that action profile.

Our preopening game and its name are inspired from the daily practice of some financial markets, such as Nasdaq or Euronext for example, where half an hour before the opening of the market, participants are allowed to submit orders which can be continuously withdrawn and changed until opening time. These orders and/or the resulting (virtual) equilibrium trading price are publicly posted during the whole preopening period. Only orders that are still posted at opening time are binding and hence executed. In this framework it is natural to assume that traders do not always manage to withdraw and submit new orders instantaneously. Delays can be due either to technological failures (caused for example by a congestion or a temporary breakdown of the electronic communication system) or to human factors (as for instance: the time required to fill in the new order faultlessly, lack of attention, etc.). In all these situations, the time between the decision to submit an order and its actual posting is random. Our model can also be a stylized representation of strategic interaction where communication and implementation of actions occur through mediators who are subject to mistakes and/or delays. For example companies, governments or other institutions may announce their strategies through press releases or press conferences, see how other parties react and possibly revise their announcements. Another example could be the sequence of preparatory meetings used by government delegations to negotiate the terms of a treaty that will be signed by heads of state at an international summit. A further example could be the case of two armies deploying their forces on a battleground, knowing that redeployment time is uncertain and that the outcome of the battle depends on the location of each party's forces at the moment of impact.

A preopening game encompasses as special cases both the cheap-talk game, when the posting system is perfectly efficient, and the one shot game.[2] When the underlying game has multiple Nash equilibria, multiplicity of equilibria extends to both these specifications of the preopening game. This paper aims at understanding whether and under which restrictions on the inefficiency of the posting system the preopening game leads players to select one single outcome at the opening. We focus on two-player complete information normal form games. We show that when the posting system is systemic inefficient, any pure Nash equilibria of the underlying game can be the opening outcome of a sub-game perfect equilibrium of the preopening game. On the other hand, the preopening game is an effective equilibrium selection tool when the posting inefficiency is idiosyncratic. In this case, all equilibria of a sufficiently long preopening game will lead players to open at a one single pure Nash equilibrium of the underlying game. When in the underlying game there is a strategy profile $\widehat{x}$ Pareto dominating all other outcomes (as for example in Figure 1), then preopening leads to play $\widehat{x}$ at the opening. For underlying games that have no Pareto superior outcome, we focus on two-action two-player underlying games that have two pure Nash equilibria. In these games players have opposite preferences over the two pure Nash equilibria

[2] The latter can be obtained when, after simultaneously choosing the initial posted action profile, players cannot change their actions because of the extreme inefficiency of the posting system.

(see for instance the entry games of Figure 2, 3 and 4). Nevertheless, the opening outcome can be easily deduced from the underlying game payoffs matrix. For example, in Figures 2 and 3, the preopening game will lead to open at {out, in}, i.e. column player's preferred pure Nash equilibrium.

	in	out
out	$1,3$	$0,0$
in	$0,0$	$2,1$

Figure 2

	in	out
out	$1,2$	$-1,0$
in	$-1,0$	$2,1$

Figure 3

	in	out
out	$1,\alpha$	$0,0$
in	$0,0$	$\alpha,1$

Figure 4

$\alpha > 1$

In fact, in Figure 2 the column player is the one that has the strongest relative preference over the two pure Nash equilibria {out, in} and {in, out}.[3] In Figure 3, players' opposite preferences over the two pure Nash equilibria are symmetric; however the column player is the one suffering less in the event of miscoordination ({in,in} or {out,out}). In symmetric games with no payoff dominant equilibrium the payoff structure is not sufficient to determine which equilibrium prevails at the opening (see for example the game of Figure 4). However, we show that when the posting efficiencies are idiosyncratic but asymmetric, the preopening game leads to a single opening outcome. The player with the least efficient posting system is the one that can most credibly commit on playing his preferred equilibrium at the opening. In fact, this is the equilibrium that will be selected through the preopening game.

It is worth stressing that the emphasis of this paper is different from that which can be found in the literature on equilibrium refinements. In fact, we do not see the equilibrium selected through preopening as that most likely to be observed in the absence of preopening. We rather see the preopening game as a device that can help players to eliminate uncertainty regarding the other players' strategies and select the "most sensible" Nash equilibria of the underlying game.

[3]The column player's payoff in {out, in} is three times his payoff in {in,out} while for the row player, {in,out} is only twice as good as {out, in}.

1.1 Related literature

Whether pre-play communication can help players to coordinate their actions in games with multiple Nash equilibria is a question that has been extensively studied in economic theory over the last twenty years. Two approaches analyze this problem when communication is "cheap-talk". One approach pioneered by Farrell (1987) and further developed by Rabin (1994), among others, has assumed that rational players share a pre-existing language they can use in communication rounds which precede the actual play. This literature shows that there are equilibria where communication enhances coordination; however other equilibria (the so-called "babbling" equilibria) cannot be excluded. The second approach relates to communication games with non-equilibrium models where players' behavior is adaptive rather than rational. In a variety of settings (see for instance Kandori, Mailath and Rob (1993), Kim and Sobel (1995), Banerjee and Weibull (2000), Demichelis and Weibull (2008)) it has been shown that applying evolutionary stability to pre-play communication games tends to eliminate socially inefficient outcomes.[4] Our approach differs from these papers in three important aspects. First, while our pre-play communication is not costly, it is binding with some exogenous probability. Second, we present an equilibrium model of pre-play communication where players are fully rational and, third, we do not assume there is a common language with meaning outside the model that can be used in the pre-play stage.

Beyond the literature on cheap-talk, our work is related to studies where pre-play communication is binding (van Damme and Hurkens (1996)) or is not cheap (Caruna and Einav (2008)). Van Damme and Hurkens (1996) study a "timing game" in which each player can choose whether to move in either of two periods; once a player has moved, however, he cannot change his action. They show that most mixed strategy equilibria of the underlying game cannot arise as equilibrium outcomes of this timing game; however, Pareto dominated pure strategy equilibria of the underlying game are also equilibria of the timing game. Caruna and Einav (2008) study

[4] A related method based on k-level thinking (Crawford (2007)) and dispersion of players' beliefs about other players' strategies (Crawford (1995)) makes it possible to explain patterns that are observed in experimental studies.

players' ability to credibly commit to actions and apply their model to entry deterrence games. In their setting players alternatively announce their intended final action and incur in exogenous switching costs when changing the action. The switching cost increases as the end of the game approaches. As opposite to van Damme and Hurkens (1996) and Caruna and Einav (2008), in our framework players can change their actions continuously over time at no cost. Also, the changes may not always be registered by the posting system, an event that is not contemplated in van Damme and Hurkens (1996) or Caruna and Einav (2008).

Two other papers are particularly relevant to our work. In relation to the preopening in financial markets our paper is related to Biais et al. (2008). They propose an experiment simulating trade in a financial market where the actual play is preceded by one round of pre-play communication. The underlying game is a two-player two-action game which represents a stylized model of trade in financial markets. A key parameter in their model allows to cover the analysis from a prisoner dilemma type of game to a coordination game (similar to the one in Figure 1). The set up of their experiment differs from our theoretical framework in two aspects. First, they allow for only one round of pre-play communication. Second, they analyze two cases: one where the pre-play actions are binding with certainty (as in Van Damme and Hurkens (1996)); and a second one where the pre-play actions are pure cheap-talk. In both specifications there are multiple Nash equilibria. They show that pre-play communication significantly improves the subjects ability to coordinate on Pareto superior equilibria only when the communication is binding. However, consistently with the existence of multiple equilibria, both Pareto superior and Pareto inferior Nash equilibria are observed in both formats of the experiment.

Finally, our model is also related to the "Revision game II" in Kamada and Kandori (2008). They study symmetric equilibria in N-player symmetric games where each player's action space is a closed interval and where the payoff function satisfies some regularity conditions. While the family of underlying games they consider is different from ours, the way they define a revision game is in many perspectives close to our preopening game with systemic posting inefficiency. Their emphasis however is opposite to ours: they assume systemic inefficiency and show that a revision game can expand the set of equilibria of the underlying game. We rather focus on the opposite problem and show that, when payoffs are generic, equilibrium selection can be obtained with idiosyncratic posting inefficiency.

The paper is organized as follows. Section 2 presents the model, the solution concept and how best reply correspondences depend on the type of posting inefficiency. In section 3 we show that systemic inefficiency does not help in selecting equilibria. In Section 4 we study the case of idiosyncratic inefficiency. Section 4.1 characterizes the equilibrium of short preopening phases. In section 4.2 we study games of common interest. Section 4.3 studies two-action games of conflicting interests. Section 5 presents some critical discussion and Section 6 concludes. All proofs are collected in the Appendix. Formal proofs of more intuitive results are provided in the supplementary material.

2 The model

The underlying game

Let the underlying game be a two-player finite game in normal form. Let X_i be the set of actions available to player i, $X := X_1 \times X_2$ the set of action profiles and $u_i : X \to \mathbb{R}$ player i's payoff function. Let $BR_i : \triangle X_{-i} \to \triangle X_i$ denote player i's best reply correspondence, N^* the set of Nash equilibria of the underlying game and $N \subseteq N^*$ the set of pure Nash equilibria of the underlying game (UGE henceforth). We will focus on games with $N \neq \varnothing$. We denote with player 1 (player 2) the row player (column player).

The preopening game

In the following we consider a situation where the underlying game is played over a continuous time interval $[0, T]$ but players' payoffs only depend on the action profile resulting at date T. More precisely, we study an extended game which consists of the following three phases:

- The *starting phase* occurs at date 0: an initial action profile of the underlying game is arbitrarily determined.[5] At instant 0 this initial action profile is posted on a screen that is commonly observed by both players.

- The *preopening phase* is played in the time interval $]0, T]$. Let $x_i(t) \in X_i$ be the action of player i which is *posted* on the screen at instant t and $x(t) \in X$ the action profile posted (or posted action profile, PAP henceforth) at instant t. In every instant t players commonly observe the current PAP $x(t) \in X$ and submit an action profile $y(t) = (y_1(t), y_2(t))$, where $y_i(t) \in X_i$ denotes the action that player i chooses to submit in instant t. Conditionally on players consistently submitting an action profile $y \in X$ between instant t and instant $t + \Delta \leq T$, let

$$Q(x(t), y, z, \Delta) = \Pr\left(x(t+\Delta) = z\right)$$

where $z \in X$ is any given action profile. In words, $Q(x(t), y, z, \Delta)$ is the probability that the PAP at instant $t + \Delta$ is z, given that the PAP at t is $x(t)$ and that during the time interval $[t, t+\Delta]$ players consistently submit the action profile y. These probabilities are exogenous and depend on the form of inefficiency in the posting system. We assume that $Q(., \Delta)$ is continuous in Δ and satisfies:

$$\lim_{\Delta \to 0} Q(x, y, x, \Delta) = 1, \forall x \in X, y \in X \tag{1}$$
$$\lim_{\Delta \to \infty} Q(x, y, y, \Delta) = 1, \forall x \in X, y \in X \tag{2}$$

Condition (1) implies that, when submitting a new action, no player can instantaneously change his current posted action. Condition (2) implies that an action profile consistently submitted for a sufficiently long time is bound to be eventually posted on the screen.

- The *opening phase*. Let $x(T)$ be the PAP at T, i.e. at the end of the preopening phase. At this instant, player i receives payoff $u_i\left(x(T)\right)$.

[5] For example the starting actions can be simultaneously chosen by each player. However we will show that the way the initial action profile is determined is strategically irrelevant.

Posting inefficiencies in the preopening game

We consider different forms of inefficiency of the posting system. In particular, the following ones are natural ways to describe delays between the submission and the posting time:

- Systemic inefficiency : the only source of posting inefficiency affects all players simultaneously. For example this can be due to a temporary slowdown (or breakdown) of the centralized posting system. Formally, at instant t, player i's submitted action is posted if and only if player $-i$'s submission is also posted.

- Idiosyncratic inefficiency : player 1 and player 2 posting delays are independently distributed. This happens, for example, when posting inefficiency can be due to idiosyncratic imperfections in the communication channels between each player and the centralized posting system or in the player posting ability.

- Limited processing capacity: This corresponds to a situation in which the posting system cannot process more than one submission at the time.

We formally define the posting inefficiency in a way that is tractable but general enough to encompass the different forms of inefficiency described above. To this purpose, we introduce two vectors q^1 and q^2 and two random arrival Poisson processes $\mathbf{p}_1$ and $\mathbf{p}_2$. Let λ_1 (resp. λ_2) denote the intensity of the Poisson process $\mathbf{p}_1$ (resp. $\mathbf{p}_2$) i.e. $\Pr(\mathbf{p}_i < t) = 1 - e^{-\lambda_i t}$. Let $q^i := \{q^i_{11}, q^i_{10}, q^i_{01}, q^i_{00}\}$ be vectors of non-negative numbers satisfying $q^i_{11} + q^i_{10} + q^i_{01} + q^i_{00} = 1$, for $i = 1, 2$.

Posting events are distributed as follows: conditional on arrival on process $\mathbf{p}_i$ at instant t, with probability q^i_{11} both players' submissions $y_1(t)$ and $y_2(t)$ are instantaneously posted; with probability q^i_{10} the action submitted by player 1, $y_1(t)$, is posted while that of player 2, $y_2(t)$, is not; with probability q^i_{01}, $y_2(t)$ is posted but $y_1(t)$ is not; with probability q^i_{00} neither $y_1(t)$ nor $y_2(t)$ are posted. Hence, if at instant t the PAP is $x(t) = (x_1, x_2)$ and from t to $t + \Delta$ players submit actions $(y_1, y_2) \neq (x_1, x_2)$, we have:

$$
\begin{aligned}
Q(x,y,y,\Delta) &= (1-e^{-\lambda_1\Delta})e^{-\lambda_2\Delta}q^1_{11} + (1-e^{-\lambda_2\Delta})e^{-\lambda_1\Delta}q^2_{11} + (1-e^{-\lambda_1\Delta})(1-e^{-\lambda_2\Delta})k_{11}(\Delta) \\
Q(x,y,(y_1,x_2),\Delta) &= (1-e^{-\lambda_1\Delta})e^{-\lambda_2\Delta}q^1_{10} + (1-e^{-\lambda_2\Delta})e^{-\lambda_1\Delta}q^2_{10} + (1-e^{-\lambda_1\Delta})(1-e^{-\lambda_2\Delta})k_{10}(\Delta) \\
Q(x,y,(x_1,y_2),\Delta) &= (1-e^{-\lambda_1\Delta})e^{-\lambda_2\Delta}q^1_{01} + (1-e^{-\lambda_2\Delta})e^{-\lambda_1\Delta}q^2_{01} + (1-e^{-\lambda_1\Delta})(1-e^{-\lambda_2\Delta})k_{01}(\Delta) \\
Q(x,y,x,\Delta) &= 1 - Q(x,y,y,\Delta) - Q(x,y,(y_1,x_2),\Delta) - Q(x,y,(x_1,y_2),\Delta)
\end{aligned}
$$

for some positive and bounded functions $k_{11}(\Delta)$, $k_{10}(\Delta)$ and $k_{01}(\Delta)$. Note that this specification of $Q(.)$ satisfies properties (1) and (2). We then introduce variable r which can be interpreted as the instantaneous probability of having *both* players submitted actions posted at t conditional on *at least one* of the two submitted actions being posted at t:

$$
\begin{aligned}
r \quad : &= \lim_{\Delta\to 0} \frac{Q(x,y,y,\Delta)}{Q(x,y,y,\Delta)+Q(x,y,(y_1,x_2),\Delta)+Q(x,y,(x_1,y_2),\Delta)} = \\
&= \frac{\lambda_1 q^1_{11} + \lambda_2 q^2_{11}}{\lambda_1(q^1_{11}+q^1_{10}+q^1_{01}) + \lambda_2(q^2_{11}+q^2_{10}+q^2_{01})}
\end{aligned}
$$

Note that $r \in [0,1]$ and $r = 0$ when the probability of simultaneous posting is nil while $r = 1$ when $q^i_{10} = q^i_{01} = 0$ for $i = 1, 2$.

Then systemic inefficiency is obtained for $q^1_{10} + q^1_{01} = q^2_{10} + q^2_{01} = 0$ and implies $r = 1$; idiosyncratic inefficiency results for $q^1_{11} = q^2_{11} = 0$ and implies $r = 0$; limited processing capacity obtains for $q^1_{10} + q^1_{01} = q^2_{10} + q^2_{01} = 1$ implying $q^1_{11} = q^2_{11} = 0$ and leading to $r = 0$.[6]

2.1 Solution concept

In this section we introduce the concepts necessary to solve the preopening game, given the characteristics of the underlying game and of the posting imperfections described above. For

[6]Further examples are the limit cases mentioned in the introduction: the one shot game can be obtained by assuming that the starting PAP is determined by the simultaneous choice of the players and then fixing $\lambda_i = 0$ or $q^i_{00} = 1$ for $i = 1, 2$ implying that the PAP cannot subsequently be changed. For $\lambda_i = \infty$, and $q^i_{11} = 1$, $i = 1, 2$, we obtain the continuous time version of a long cheap-talk game, in which all submissions before time T are not binding and can be changed instantaneously. Note that one can reproduce a discrete time format by focusing on strategies where the interval $[0, T]$ is partitioned in n sub intervals and $y(t)$ does not change within a sub interval.

expositional clarity, we shall focus on sub-game perfect Markov equilibria where players strategies at date t only depend on the time remaining before the opening and on the current PAP. In Section 6 however we will consider the wider class of Nash equilibria and analyze a form of posting inefficiency under which the set of Nash equilibria of the preopening game coincides with the set of its Markov SPE.

Let $\tau := T - t$ denote the time remaining before the opening. To avoid confusion, in the following we shall refer to t as the calendar *date* and to τ as the *time* left before the opening. With an abuse of notation, we can denote with $x(\tau)$ the PAP at time τ to the opening. Henceforth $x(0)$ is the PAP at the opening. A Markov strategy for player i is a Borel-measurable mapping $\sigma_i : X \times [0, T] \to \triangle X_i$. Then for any given Markov strategy profile σ and any PAP $x(\tau)$ at time τ, we denote with,

$$\pi_i(x(\tau), \tau) := E\left[u_i(x(0)) | x(\tau), \sigma\right] \tag{3}$$

player i's expected continuation payoff given the Markov strategy profile σ and conditional on the PAP at time τ being $x(\tau)$. Condition (1) implies that when the opening is close there is little chance that the PAP will change. Hence, independently of the players' strategies, toward the end of the preopening phase, it results that

$$\lim_{\tau \to 0} \pi_i(x, \tau) = u_i(x). \tag{4}$$

Endowed with the definition of the expected continuation payoffs in (3) we can now compute the best reply for each player i at each time τ in a Markov equilibrium. Consider time τ to the opening, and suppose that player $-i$'s currently posted action is x_{-i} while he is submitting action y_{-i}. Denote with $x_i^*(x_{-i}, \tau)$ the best reply for player i to x_{-i} in a one shot game where his reward function is given by $\pi_i(x, \tau)$, $x \in X$ as in (3):

$$x_i^*(x_{-i}, \tau) \in \arg\max_{x_i \in X_i} \pi_i(x_i, x_{-i}, \tau)$$

In the following lemma we show that for extreme values of r the best reply of player i is to submit either $x^*(x_{-i}(\tau), \tau)$ (when r is close enough to zero) or $x^*(y_{-i}(\tau), \tau)$ (when r is close enough to one), i.e. his best reply to the other player's currently posted action $x_{-i}(\tau)$ or to action $y_{-i}(\tau)$

which the other player is currently submitting, respectively. Notice that the optimal action submitted at time τ from the opening can be interpreted as the limit of the optimal action submitted in a discrete-time version of the game as the time interval between two submissions converges to zero. Then we have:

Lemma 1 *: Let $x(\tau)$ and $y_{-i}(\tau)$ be time τ PAP and player $-i$'s submitted action, respectively. There exist $\underline{r}_i(\tau)$ and $\overline{r}_i(\tau)$, with $0 \leq \underline{r}_i(\tau) \leq \overline{r}_i(\tau) \leq 1$ such that player i best reply in a Markov equilibrium is*

$$\beta(x, y_{-i}, \tau) := \begin{cases} x_i^*(x_{-i}(\tau), \tau) \text{ if } r \leq \underline{r}_i(\tau) \\ x_i^*(y_{-i}(\tau), \tau) \text{ if } r \geq \overline{r}_i(\tau) \end{cases}$$

The intuition of Lemma 1 is as follows. If r is close to 1 (resp. to 0), each player knows that whenever his submitted action is posted, it is very likely (resp. unlikely) that his opponent's submission is also posted at the same time. In the case of r close to 1, each player applies a conjectural argument which leads them to play the best reply to the opponent's submitted action. Quite to the opposite, when r is close to zero, such a conjectural argument is not correct: each player knows that his opponent cannot change his own posted action from the one posted on the current PAP if he himself is allowed to do so. Hence, he submits his best reply to the opponent's *currently* posted action. It is indeed this different logic behind the construction of each player's best reply during the preopening game that may or may not allow them to select a single Nash equilibrium at the opening.

3 Systemic posting inefficiency and multiple equilibria

In this section we show that when the source of posting inefficiency affects all players simultaneously, the preopening game has at least as many equilibria as the underlying game.

Proposition 2 *: For any pure Nash equilibrium of the underlying game $x^N \in N$ and any $\varepsilon > 0$, there exist $\overline{r}(\varepsilon) \in]0, 1]$ and $T(\varepsilon)$ such that if $r \geq \overline{r}(\varepsilon)$ and $T > T(\varepsilon)$, it is a SPE of the preopening* *game to always submit the action profile x^N during the whole length of the preopening phase. This leads players to play x^N at the opening with probability no smaller than $1 - \varepsilon$.*

The intuition of the result is as follows. Suppose the posting inefficiencies of the two players are perfectly correlated (i.e. $r = 1$). At time τ, when choosing the action to be submitted, player i knows that she can affect her expected payoff only if her own action is posted. However, this event occurs only when player $-i$'s submitted action is posted as well. Now fix any UGE $x^N = (x_1^N, x_2^N)$. If at all time τ and in all PAPs player $-i$ submits x_{-i}^N, then at the end of the preopening only two outcome are possible: either none of the two players' submitted action is ever posted,[7] or both players' submitted actions are posted and player $-i$ opening action is x_{-i}^N. Hence the best player i can do is to maximize $u_i(y(0), x_{-i}^N)$ always submitting x_i^N.

4 Idiosyncratic posting inefficiency and equilibrium selection

In this section we analyze preopening games where the inefficiency of the communication system is such that r is equal to zero. In this case $q_{11}^1 = q_{11}^2 = 0$: the probability that the action of player i is posted at some time τ conditional on the action submitted by $-i$ being posted at τ is zero.

4.1 Short preopening phases

The preopening game has only one SPE when the opening date is sufficiently close. In such equilibrium, at any τ close enough to 0, each player submits the action that is the underlying game best reply to the other player's currently *posted* action. When both players adopt these strategies they tend to stick to any of the pure Nash equilibria of the underlying game as soon as the PAP forms one. For this reason we call such play in the preopening game the "equilibrating scenario". Formally,

[7] In this instance players' payoffs do not depend on their submissions during the preopening phase.

Definition 1: *We say that player i adopts the "equilibrating strategy" at time τ if*

$$y_i(\tau) \in BR_i(x_{-i}(\tau))$$

The "equilibrating scenario" emerges when both players adopt the equilibrating strategy.

Then we have:

Proposition 3 *: If $r = 0$ and the underlying game has more than one Nash equilibrium and generic payoffs, then there exists a finite $\tau^* > 0$ such that in all SPE:*

(i) At any time $\tau < \tau^*$ the equilibrating strategy is strictly optimal for both players and players' equilibrium continuation payoff only depends on τ and $x(\tau)$;

(ii) At time $\tau > \tau^*$ the equilibrating strategy is not optimal for at least one player.

Interestingly, the equilibrating scenario has the flavour of the communication symmetric equilibrium in Farrell (1987). In Farrell's communication phase players agree on a language stipulating that, first, as long as the message profile in the communication phase does not form a UGE, players keep changing their messages with positive probability. Second, as soon as the message profile in the communication game does form a UGE, players commit to that action profile until the underlying game is played. The equilibrating scenario of our preopening game fully reflects this second feature. However, when the current PAP does not form a UGE, players do not use mixed strategies as in Farrell (1987). They both submit different actions from the one currently posted i.e. they try to move away from PAPs that are not in N. It is worth stressing that the equilibrating scenario does not emerge because of an ex-ante agreement between the two players on some language, but it results from strictly dominant strategies of the preopening game for τ small and $r = 0$.

Note that while the equilibrating scenario characterizes the play when τ is sufficiently close to 0, Proposition 3 does not tell us which of the UGEs is played at the opening. In fact, when using the equilibrating strategies, players do not move away from any of the PAPs that form a UGE. Proposition 3 also states that the equilibrating scenario is *not* an equilibrium when

$\tau > \tau^*$, provided that the underlying game has generic payoffs and more than one UGE. To understand why, consider for example the game in Figure 5:

	L	R
U	$3,3$	$0,0$
D	$0,0$	$1,1$

$\Longrightarrow$

Equilibrating dynamics

	L	R
U	$\circlearrowright$	$\longleftarrow \downarrow$
D	$\uparrow \longrightarrow$	$\circlearrowright$

Figure 5

In this game $N = \{(U, L), (D, R)\}$ are the only sinks of the dynamics induced by the equilibrating scenario. Suppose for the sake of contradiction that the equilibrating scenario is used during the whole length of a long preopening game. If at some time τ the PAP $x(\tau)$ belongs to N, it will not change and the opening will be at $x(0) = x(\tau)$. Hence, $\pi_1(x, \tau) = u_i(x)$ for $x \in N$. When the equilibrating scenario is expected to be used for enough time, the probability that before the opening the PAP moves from any $x \notin N$ to some $x \in N$ converges to one because of property (2). Hence, for $x \notin N$ in the game in Fig. 5 we have $\lim_{\tau \to \infty} \pi_i(x, \tau) \in]1, 3[$. That is to say that player i's expected continuation payoff in $x(\tau) \notin N$ is larger than his continuation payoff for $x(\tau) = (D, R)$. When this happens, player 1 prefers to move from (D, R) to $x(\tau) \notin N$ as this would increases the chances of opening at his preferred UGE (U, L) and gain 3 instead of 1. Thus, the equilibrating strategy stipulating $y((D, R), \tau) = (D, R)$ is not optimal for τ large.

4.2 Long preopening phases and Pareto optimal outcomes

Note that the dynamics of the posted action profile induced by the equilibrating scenario can be non-trivial when the starting PAP is not in N. For instance the game in Figure 6 has only one pure Nash equilibrium $N = (D, R)$ that is a stable point in the dynamics of PAP induced by the equilibrating scenario. However the cycle $\mathcal{C} := (U, L) \to (U, C) \to (M, C) \to (M, L) \to (U, L)$ is also stable under the equilibrating scenario dynamics, while PAPs $(D, L), (D, C), (U, R)$ and (M, R) are not stable in the sense that players will submit actions that will eventually move the PAP either to N or to a point in $\mathcal{C}$. This example shows that it can be hard to characterize the optimal strategies preceding the equilibrating scenario for a generic game where players have more than two actions.

	L	C	R
U	$3,2$	$2,3$	$0,0$
M	$2,3$	$3,2$	$0,0$
D	$0,0$	$0,0$	$4,4$

$\Longrightarrow$

Equilibrating dynamics

	L	C	R
U	$\longrightarrow$	$\downarrow$	$\leftarrow\downarrow$
M	$\uparrow$	$\longleftarrow$	$\leftarrow\downarrow$
D	$\uparrow\longrightarrow$	$\uparrow\rightarrow$	$\circlearrowright$

Figure 6

However, we are able to provide a strong result in a special class of games, which we define as common interest games:

Definition 2: *A two-player normal form game is said to be of common interest if there is a single Pareto optimal action profile* $\widehat{x}$.

We conclude this section by showing that for games of common interest, all SPE equilibria of a sufficiently long preopening phase with idiosyncratic posting efficiency almost surely open at $\widehat{x}$.

Theorem 4 : *Consider a two-player common interest game and let* $\widehat{x} \in X$ *be such that* $u_i(\widehat{x}) > u_i(x)$ *for all* i *and* $x \neq \widehat{x}$. *Then for any* $\varepsilon > 0$, *there exists* $T(\varepsilon)$ *such that if* $r = 0$ *and* $T > T(\varepsilon)$, *then all equilibria of the preopening game lead players to open at* $\widehat{x}$ *with probability larger than* $1 - \varepsilon$.

Theorem 4 proves that in games of common interests, for a sufficiently long preopening game with idiosyncratic posting inefficiency, all SPE in the preopening game lead players to coordinate on the Pareto-optimal outcome at the opening. Differently from what happens in most of the "cheap-talk" literature, this result does not rely on the possibility of communication through an external language nor on the presence of non-rational/adaptive behavior. Also it is worth pointing out that our result is independent from risk dominance considerations. For instance in the game in Figure 1, action profile $\{U, L\}$ forms the Pareto dominant, payoff maximizing pure

Nash equilibrium, whereas action profile $\{D, R\}$ is Pareto dominated but it is a risk dominant equilibrium. Under the hypothesis of Theorem 4, the preopening phase will lead players to coordinate on $\{U, L\}$. This is quite natural since the risk dominance criterion fits to situations where players are uncertain of other player's strategies, whereas the preopening game is meant to indeed eliminate such uncertainty.

4.3 Idiosyncratic posting inefficiency in 2×2 games

In this subsection we focus on two-player two-action games which have two pure strategy equilibria, as illustrated in Figure 7. Player 1 chooses the row, $X_1 = \{\text{up,down}\}$, while player 2 chooses the column, $X_2 = \{\text{left,right}\}$, and payoffs are given by the following matrix:

	left	right
up	a_1, a_2	b_1, b_2
down	c_1, c_2	d_1, d_2

Figure 7

It is convenient to denote with A, B, C and D the action profiles {up,left}, {up,right}, {down,left} and {down,right} respectively; then $x \in \{A, B, C, D\}$.

Assumption 1: $a_1 > c_1$, $d_1 > b_1$, $a_2 > b_2$, $d_2 > c_2$.

Under Assumption 1 the underlying game has three Nash equilibria: two in pure strategies (A and D) and one in mixed strategies. Within this class of games and assuming idiosyncratic posting inefficiency, we show that the preopening is a powerful equilibrium selection mechanism. Considering generic payoffs, we can distinguish games of common interest (as in Definition 2) from games of conflicting interests:

Definition 3: *In a conflicting interests game,* $a_1 < d_1$ *and* $a_2 > d_2$. *In a common interest game,* $a_i > d_i$, $i = 1, 2$.

In games of common interest the two pure Nash equilibria are Pareto ranked implying that both players prefer the same equilibrium. In these games $a_i > \max\{b_i, c_i, d_i\}$, hence Theorem 4 applies.

In the rest of the paper we focus on conflicting interests games, i.e. games where player 1 (player 2) prefers UGE D to UGE A (resp. UGE A to UGE D). With an abuse of notation, for a given Markov strategy profile σ of the preopening game, we denote $a_i(\tau) := \pi_i(A, \tau)$ as player i's expected payoff given that at time τ to the opening the PAP is $x(\tau) = A$. We shall denote with $\dot{a}_i(\tau) := \partial a_i(\tau)/\partial\tau$ the variation of $a_i(\tau)$ occurred by increasing the time to the opening by an infinitesimal amount $\partial\tau$. Quantities $b_i(\tau)$, $c_i(\tau)$, $d_i(\tau)$ and $\dot{b}_i(\tau)$, $\dot{c}_i(\tau)$, $\dot{d}_i(\tau)$ are defined in the same way. Finally, notice that condition (4) becomes

$$a_i(0) = a_i,\ b_i(0) = b_i,\ c_i(0) = c_i,\ d_i(0) = d_i \tag{5}$$

Throughout this session we will assume idiosyncratic inefficiency in the posting system.

Assumption 2: $q^1_{11} = q^2_{11} = 0$*;* $q^1_{10} = 1$*,* $q^2_{01} = 1$*, hence* $r = 0$*.*

Under Assumption 2 the posting times of the two players are independent, in the sense that the time at which the action submitted by player i is posted is distributed according to a Poisson process of intensity λ_i. We shall say that at time τ player i *keeps* her action if she submits $y_i(\tau) = x_i(\tau)$ and she *moves* if she submits $y_i(\tau) \neq x_i(\tau)$. Since we are considering a two-action game, this is sufficient to completely describe players strategies at τ in any given PAP.

4.3.1 Preopening game equilibrium in conflicting interest games

We know from Proposition 3 that when the opening time approaches and r is close to zero players adopt the equilibrating strategy. Under Assumption 1 this means that players do not move from PAPs A and D, while they try to move away from PAPs B and C as illustrated by the following picture.

$x(\tau) = A$ $keep; keep$	$x(\tau) = B$ $move; move$
$x(\tau) = C$ $move; move$	$x(\tau) = D$ $keep; keep$

$\Longrightarrow$

Equilibrating dynamics

$x(\tau) = A$ $\circlearrowright$	$x(\tau) = B$ $\longleftarrow \downharpoonleft$
$x(\tau) = C$ $\uparrow \longrightarrow$	$x(\tau) = D$ $\circlearrowright$

Figure 8

Given Assumption 2, we can compute the law of motion of the expected continuation payoff $\pi_i(x(\tau), \tau)$ when both players adopt the equilibrating strategies. Obviously $\dot{a}(\tau) = \dot{d}(\tau) = 0$. Suppose that at time $\tau + \Delta$ the current PAP is C. If players use the equilibrating strategies between time $\tau + \Delta$ and τ, then

$$\begin{aligned} c_i(\tau + \Delta) &= a_i(\tau)(1 - e^{-\lambda_1 \Delta})e^{-\lambda_2 \Delta} + d_i(\tau)(1 - e^{-\lambda_2 \Delta})e^{-\lambda_1 \Delta} \\ &\quad + c_i(\tau)e^{-(\lambda_1 + \lambda_2)\Delta} + k_i(\tau)(1 - e^{-\lambda_2 \Delta})(1 - e^{-\lambda_1 \Delta}) \end{aligned}$$

where $k_i(\tau)$ is some weighted average of $a_i(\tau)$, $b_i(\tau)$, $c_i(\tau)$ and $d_i(\tau)$. Compute the difference $c_i(t - \Delta) - c_i(t)$ and take $\lim_{\Delta \to 0} \frac{c_i(t-\Delta) - c_i(t)}{\Delta}$ to obtain

$$\dot{c}_i(\tau) = \lambda_1 \left(a_i(\tau) - c_i(\tau)\right) + \lambda_2 \left(d_i(\tau) - c_i(\tau)\right)$$

Applying the same method for PAP B we have

$$\dot{b}_i(\tau) = \lambda_2 \left(a_i(\tau) - b_i(\tau)\right) + \lambda_1 \left(d_i(\tau)\lambda_1 - b_i(\tau)\right)$$

Considering the transversality condition (5), it follows that if both players adopt the equilibrating strategy from time τ until the opening, then at time τ their expected payoffs for each of the four PAPs are

$$a_i(\tau) = a_i \tag{6}$$

$$b_i(\tau) = \frac{a_i \lambda_2 + d_i \lambda_1}{\lambda_1 + \lambda_2} + \left(b_i - \frac{a_i \lambda_2 + d_i \lambda_1}{\lambda_1 + \lambda_2}\right) e^{-(\lambda_1 + \lambda_2)\tau} \tag{7}$$

$$c_i(\tau) = \frac{a_i \lambda_1 + d_i \lambda_2}{\lambda_1 + \lambda_2} + \left(c_i - \frac{a_i \lambda_1 + d_i \lambda_2}{\lambda_1 + \lambda_2}\right) e^{-(\lambda_1 + \lambda_2)\tau} \tag{8}$$

$$d_i(\tau) = d_i \tag{9}$$

Let us introduce τ_1^* and τ_2^* defined as

$$\begin{aligned} \tau_1^* &: = \min_{\tau} \{c_1(\tau) = a_1\} \\ \tau_2^* &: = \min_{\tau} \{c_2(\tau) = d_2\} \end{aligned}$$

From the definition of conflicting interests games we have

$$\tau_1^* = \frac{1}{\lambda_1 + \lambda_2} \ln \left[\frac{\lambda_1(a_1 - c_1) + \lambda_2(d_1 - c_1)}{\lambda_2(d_1 - a_1)} \right] \tag{10}$$

$$\tau_2^* = \frac{1}{\lambda_1 + \lambda_2} \ln \left[\frac{\lambda_1(a_2 - c_2) + \lambda_2(d_2 - c_2)}{\lambda_1(a_2 - d_2)} \right] \tag{11}$$

Finally, let

$$\begin{aligned} \tau^* &: = \min\{\tau_1^*, \tau_2^*\} \\ i^* &: = \arg \min_{i \in \{1,2\}} \tau_i^* \\ x^{i^*} &: = \arg \max_{x \in \{A,D\}} u_{i^*}(x) \end{aligned}$$

In terms of Proposition 3, time τ^* is the closest time to the opening when *both* players adopt the equilibrating strategy. They will hold to this strategy until the opening. However for $\tau > \tau^*$, player i^* will not use the equilibrating strategy. In other words, i^* is the last player adopting the equilibrating strategy, i.e. at the date closest to the opening. Action profile x^{i^*} is the UGE preferred by player i^*.

Theorem 5 *: Consider a* 2×2 *game of conflicting interests satisfying Assumption 1 and suppose that the posting inefficiency satisfies Assumption 2. Then for any* $\varepsilon > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$ *finite, there exists a* $T(\varepsilon)$ *such that if* $T > T(\varepsilon)$ *all Markov SPE of the preopening game lead players to open at* x^{i^*} *with probability larger than* $1 - \varepsilon$.

In words, Theorem 5 states that if the preopening phase is long enough and $r = 0$, then at the opening the two players coordinate on the UGE preferred by player i^*, the player who is last in adopting the equilibrating strategy. This result has a natural economic interpretation. In games of conflicting interests, the preopening game can be reinterpreted as a period in which players bargain in order to determine which UGE they shall play at the opening. When a player starts adopting the equilibrating strategy he is basically conceding the bargain, as in fact he is ready to accept either of the two UGE as soon as it appears as the PAP. In this logic, the strongest player is the one that waits the longest in conceding the bargaining, i.e. the one adopting the equilibrating strategy at the latest date. Theorem 5 predicts that it is the UGE preferred by this player that will prevail at the opening.

The intuition behind the proof of Theorem 5 goes as follows. When time to the opening is equal to τ^* player i^* stops following the equilibrating strategy. More precisely, at any equilibrium, for $\tau > \tau^*$ player i^* targets the UGE she prefers, i.e. irrespective of the PAP she submits the action that is consistent with her preferred UGE. On the other hand, for some time player $-i^*$ continues to play the equilibrating strategy. However, when τ increases, at some point player $-i^*$ also modifies his behavior. The proof shows that at sufficiently distant times from the opening, both players submit the actions that form UGE x^{i^*}. It is only when the opening is close enough that they will adopt other strategies and eventually, the equilibrating strategy. This dynamic behavior in turn assures that when T is large players have enough time to coordinate on x^{i^*} in the early phase of the preopening. Once this happens, they will not change their action afterwards, leading them to x^{i^*} as the opening outcome. We can now use Theorem 5 to study some classic games of conflicting interests.

4.3.2 Asymmetric payoffs and symmetric inefficiency

Consider the case where players' payoffs are not symmetric while $\lambda_1 = \lambda_2 = \lambda > 0$ and $q^1_{10} = q^2_{01} = 1$. This implies that players communicate with the posting system with instruments that bear the same degree of inefficiency; hence, the posting times are independently distributed according to two identically distributed Poisson processes with intensity λ. In this case i^* does not depends on λ since (10)-(11) imply that $\tau^*_1 \leq \tau^*_2$ if and only if

$$\frac{a_1 + d_1 - 2c_1}{d_1 - a_1} \leq \frac{a_2 + d_2 - 2c_2}{a_2 - d_2}$$

The numerators of these expressions can be interpreted as each player's average gain from playing either A or D rather than playing action profile C. Notice that profile C would arise at the opening if each player were continuously submitting the action corresponding to his preferred UGE. The denominators reflect each player's gain from playing her preferred UGE rather than the other UGE. Thus, ceteris paribus, when player i's preferences over A and D are more pronounced than those of player $-i$, and/or the cost of mis-coordination is lower for i than for $-i$, it is the UGE preferred by i that will be selected during the preopening.

Suppose for instance that player 2's preferences over the two UGEs are more pronounced than those of player 1, as for example in the game in Figure 9

	left	right
up	$1, \alpha$	$0, 0$
down	$0, 0$	$\beta, 1$

$\alpha > \beta > 1$

Figure 9

In this case $\tau_1^* > \tau_2^*$ and according to Theorem 5 the preopening phase will lead players to coordinate on PAP $\{\text{up}, \text{left}\}$ as it seems natural to expect. Similarly, suppose that the miscoordination cost for player 1 is larger than the miscoordination cost for player 2 as in the game of Figure 10

	left	right
up	$1, \alpha$	$0, 0$
down	$-\gamma, 0$	$\alpha, 1$

$\alpha > 1, \gamma > 0$

Figure 10

Then player 1 will tend to engage in the equilibrating strategy earlier than player 2; as a result, a sufficiently long preopening phase will lead almost surely to PAP $\{\text{up}, \text{left}\}$ at the opening.

Another example of game of conflicting interest is the "chicken game" illustrated below.

	Straight	Swerve
Swerve	$-1, 1+\delta$	$0, 0$
Straight	$-10, -10$	$1, -1$

Figure 11

In this example i^* is the player that is going to swerve last when the PAP is {Straight, Straight}. If δ is positive, this player will be player 2, i.e. the player benefitting more from winning the race versus crashing. In this case players will coordinate on {Swerve, Straight} since the start of the game.

When players differ in both their miscoordination costs and the strength of their preferences over action profiles A and D, the prediction on the opening PAP is less intuitive. However, the comparison between τ_1^* and τ_2^* as given by expressions (10) and (11) readily provides the PAP that shall be observed at the opening.

4.3.3 Symmetric payoffs and asymmetric inefficiency

When the underlying game payoff structure is perfectly symmetric (as for instance in Figure 4) but the communication systems of the two players differ in their posting inefficiency, i.e. $\lambda_1 \neq \lambda_2$, the preopening equilibrium leads to a single UGE. Namely, during the preopening, the PAP will converge to the UGE preferred by the player with the lowest posting efficiency λ_i. The economic intuition for this result is simple. A less efficient posting system has the same role as an exogenous commitment device and provides the player who is using it with a strategic advantage. As a result players coordinate on the UGE preferred by the most committed player. Consider for example the symmetric game in Figure 4 similar to those studied in Farrell (1987) and recently revisited in Crawford (2007). For this example, (10) and (11) become:

$$\begin{aligned}
\tau_1^* &= \frac{1}{\lambda_1 + \lambda_2} \ln \left[\frac{\lambda_1 + \alpha\lambda_2}{(\alpha - 1)\lambda_2} \right] \\
\tau_2^* &= \frac{1}{\lambda_1 + \lambda_2} \ln \left[\frac{\alpha\lambda_1 + \lambda_2}{(\alpha - 1)\lambda_1} \right]
\end{aligned}$$

and $\tau_1^* > \tau_2^*$ if and only if $\lambda_1 > \lambda_2$: the player with the lower instantaneous efficiency in his posting system is the one adopting the equilibrating strategy at the latest moment. Another example is the chicken game of Figure 11 when $\delta = 0$. The winner of the game is the player who is known to have a less efficient or slower "steering system".

4.3.4 Symmetric payoffs and symmetric inefficiency

In conflicting interest games with perfectly symmetric payoffs when players use equally (in)efficient communication systems, i.e. $\lambda_1 = \lambda_2$, a preopening game is not a perfect coordination tool even if $r = 0$. In such a case[8], the preopening game has multiple equilibria leading to different opening outcome. Consider for instance the preopening game with $\lambda_1 = \lambda_2 = \lambda$ where the underlying game is that of Figure 4. In this case $\tau^* = \frac{1}{2\lambda} \ln \left[\frac{\alpha+1}{\alpha-1}\right]$ and the preopening game has three equilibria. One leading to open at A, another one leading to open at D and a third in which players submit C until date $T - \tau^*$ and then adopt the equilibrating strategy until the opening. The latter is the only symmetric equilibrium of the preopening game. In this equilibrium, players are ready to signal to their opponent that they want to stick to their "best" action until few moments (more precisely, τ^*) before the opening.[9] In this symmetric equilibrium the probability that players do not manage to coordinate on either A or D is $(\alpha - 1)/(\alpha + 1)$, for any λ finite. Interestingly, this is exactly the failure rate with abundant communication in Farrell (1987) (see also Crawford (2007) page 10). Formally,

Proposition 6 *Consider the underlying game of Figure 4. Let* $\tau^* := \frac{1}{2\lambda} \ln \left[\frac{\alpha+1}{\alpha-1}\right]$ *with* $\lambda = \lambda_1 = \lambda_2$. *Then*

(i) At any time $\tau < \tau^*$, *the equilibrium of the preopening round is unique, in strictly dominant strategies and consists of both players using the equilibrating strategy.*

(ii) There exists a symmetric equilibrium in the preopening game such that at any time $\tau \geq \tau^*$, *players submits the action profile* $\{in, in\}$

.

[8] This is clearly a non generic game in the space of preopening games.

[9] Recall that PAP C corresponds to the combination of actions that each player would play in his own preferred UGE .

5 Discussion

Theorem 4 and Theorem 5 allow us to conclude that when $r = 0$ the a generic preopening game has a unique Markov SPE selecting a single UGE as the opening PAP. However, in order to claim that the preopening can indeed be effective in selecting equilibria, an additional step is required. We still need to argue that the preopening game has no other Nash equilibria leading to an opening PAP that differs from the one selected by the Markov SPE. The objective of this section is to prove that, for an appropriate family of posting inefficiencies, all Nash equilibria of the preopening game are observational equivalent to the Markov SPE we have characterized above.

In the preopening game a player's strategy in principle may depend on the entire history of past PAPs. Thus, at any date $t \in]0, T]$ a history $h(t)$ of the preopening game consists of two elements: the sequence of the PAPs observed until t, each of them associated to the instant at which these profiles were posted. Formally, $h(t) = \{(x(t_k), t_k)\}_{k=0,1,...,n(t)}$, with $t_{n(t)} \leq t$, where for any $k \leq n(t)$, $x(t') = x(t'')$ for all $t', t'' \in [t_k, t_{k+1}[$ and $x(t_k) \neq x(t_{k+1})$. Let $H(t)$ be the set of all possible histories of length t, while we denote with H the set of all possible histories of the entire preopening game. A strategy σ_i for player i is any Borel-measurable function that maps any history $h(t) \in H(t)$ into the possibly mixed action submitted by player i at date t. For a given strategy profile σ player i' expected continuation payoff after history $h(t)$ is then[10]:

$$E\left[u_i(x(T))|h(t), \sigma\right]$$

where now $x(T)$ denotes the PAP at the opening.

First note that focusing on sub-game perfect equilibria of the preopening game does not artificially restricts the set of equilibria as long as we can choose the form of posting inefficiency. Namely, if posting inefficiency is such that all Borel-measurable histories in H occur with positive probability, then the set of Nash equilibria of the preopening game coincides with the set of sub-game perfect equilibria. In order to obtain this result we generalize the transition probabilities

[10]This definition of expected continuation payoff for player i generalizes our previous definition (3) for any strategy σ, including also non-Markov strategies.

$Q(.)$ described in Section 2.

Assumption 3: *Let*

$$
\begin{aligned}
Q(x,y,z,\Delta) \quad &: \quad = (1-e^{-\lambda_1\Delta})e^{-\lambda_2\Delta}q^1_{xyz} + (1-e^{-\lambda_2\Delta})e^{-\lambda_1\Delta}q^2_{xyz} \\
&+ \quad (1-e^{-\lambda_1\Delta})(1-e^{-\lambda_2\Delta})k_{xyz}(\Delta) \text{ for } z \neq x, \\
Q(x,y,x,\Delta) \quad &: \quad = 1 - \sum_{z\neq x} Q(x,y,z,\Delta)
\end{aligned}
$$

where $Q(.)$ *has the same interpretation as in Section 2,* $k_{xyz}(\Delta)$ *is a bounded function and* $\{q^i_{xyz}\}_{z\in X} \in \triangle^X$.

Moreover, for any triple $(x,y,z) \in X \times X \times X$ *and any* $i = 1,2$, *we assume*

1. $q^i_{xyz} = 0$ *if* $x_1 \neq z_1$ *and* $x_2 \neq z_2$.
2. $q^i_{xyz} > 0$ *otherwise.*
3. *There exists* $\varepsilon > 0$ *arbitrary small such that* $q^i_{xyz} \leq \varepsilon$ *if* $z_1 \neq y_1$ *or* $z_2 \neq y_2$.

The term $q^i_{xyz} \geq 0$ can be interpreted as the conditional probability that, upon arrival of the Poisson process $\mathbf{p}_i$, the PAP changes from x into z when players submit the action profile y. Note that conditions (1) and (2) are verified under Assumption 3. The second part of Assumption 3 imposes additional restrictions on parameters q^i so that 1. the posted actions of both players cannot change simultaneously; 2. the posted action of each player can change into any action including those which are different from the submitted one; however 3. if a player's posted action changes, it is much more likely that the new PAP will reflect the players' submitted actions rather than another action.

If the transition probabilities satisfy Assumption 3 then, for any PAP $x \in X$ and any date $t > 0$, the PAP at t is equal to x with a strictly positive probability, independently of players' strategies. Notice that this would not be true if $r = 0$[11]. Hence, if Assumption 3 holds the set

[11] However, the transition dynamics allowed under Assumption 3 may include those obtained with $r = 0$, namely when $\varepsilon = 0$.

of SPE equilibria and the set of Nash equilibria of the preopening game coincide. Property 1. of Assumption 3 is crucial in order to prove that all SPE of the preopening game have a Markov structure but for a finite number of instants. The formal statement of this result is contained in our last Proposition.

Proposition 7 *: Under Assumptions 1 and 3, every Nash equilibrium of the preopening game is observationally equivalent to the Markov SPE of Theorem 5.*

6 Conclusions

A preopening game is a model of pre-play communication inspired by a widely existing mechanism in financial markets. During the preopening phase, players announce through a posting system the action they intend to play in the underlying game. At the end of the preopening phase (the opening) players are bound to play the last action profile posted on the system. We study the case of a posting system that suffers from certain degree of inefficiency, affecting either both players simultaneously or the two players independently.

We show that when posting system failures hit the two players idiosyncratically, in games with common interest, the preopening selects the Pareto dominant Nash equilibrium. Thus a sufficiently long preopening phase allows players to coordinate on Pareto dominant equilibrium avoiding sub optimal equilibria that could arise for example due to strategy uncertainty (see for instance risk dominant equilibria). With idiosyncratic inefficiency in a conflicting interest game, the preopening leads to the underlying game equilibrium preferred by the player that suffers the least from miscoordination and/or has the sharpest preference over alternative equilibria and/or has the least efficient posting system.

On the contrary, when the posting system is affected by systemic failures, the preopening operates no equilibrium selection and the opening can occur at any of the Nash equilibria of the underlying game. Our results have clear policy implications for the preopening in financial markets. For example on Xetra, the German computerized stock exchange, the exact time of the opening can be random. In terms of our model, this corresponds to a situation in which all players might not be able to change their posted action after some unknown period. Such a form of posting inefficiency affects all participants simultaneously. According to our result, this form of posting inefficiency does not prevent market participants to coordinate on Pareto inferior equilibria. The selection of a Pareto superior equilibrium (when this exists) would rather be achieved by introducing some idiosyncratic delay in the posting system or by reducing the system ability to simultaneously process changes in the tentative orders submitted during the preopening. While in this paper we focus on two-player games with complete information, there are two natural directions of future research. First, consider the robustness of our result to the case of more than two players. Second, consider the case in which there is incomplete information regarding players' underlying game payoffs matrix. In the latter case, players might want to delay adopting the equilibrating strategy in order to signal sharper preference over the equilibrium outcome.

References

Aumann, R. and Hart S. (2003) "Long Cheap Talk", *Econometrica*, Vol. 71, No. 6. pp. 1619-1660.

Banerjee A. and Weibull J. (2000) "Neutrally stable outcomes in Cheap-talk coordination games", *Games and Economic Behavior*, 32, 1-24.

Biais B., Bisière C. and Pouget S. (2008) "Equilibrium discovery and preopening mechanisms in an experimental market", mimeo.

Caruana, G. and L. Einav (2008) "A Theory of Endogenous Commitment", *Review of Economic Studies,* 75, 99–116.

Crawford V. (1995) "Adaptive dynamics in coordination games", *Econometrica*, 63, 103-143.

Crawford V. (2007) "Let's talk it over: coordination via preplay communication with level k thinking", mimeo.

Crawford, V. P. and J. Sobel (1982): “Strategic Information Transmission,” *Econometrica*, 50, 1431-1451.

van Damme, E. and S. Hurkens (1996) "Commitment Robust Equilibria and Endogenous Timing", *Games and Economic Behavior*, 15, 290-311.

Demichelis S. and Weibull J. "Language, meaning and games: a model of communication, coordination and evolution", *American Economic Review*, forthcoming.

Kamada Y. and Kandori M. (2008) "Revision Games" mimeo.

Kandori M., Mailath G. and Rob R. (1993) "Learning, mutation and long run equilibria in games", *Econometrica*, 61, 29-56.

Kim Y. and Sobel J., "An evolutionary approach to pre-play communication", *Econometrica*, 63, 1181-1193.

Farrell J. (1987) "Cheap talk coordination and entry", *RAND Journal of Economics*, 18, 34-39.

Myerson, R. (1978) "Refinements of the Nash equilibrium concept", *International Journal of Game Theory*, 7, 73-80.

Rabin M. (1994) "A model of pre-game communication", *Journal of Economic Theory*, 63, 370-391.

Selten, R. (1975) "Reexamination of the perfectness concept for equilibrium points in extensive games". *International Journal of Game Theory*, 4, 25-55.

Appendix

Proof of Lemma 1: The optimal action submitted at time τ can be interpreted as the limit as Δ goes to 0 of the optimal action submitted in a discretized-time version of the preopening game in which players are restricted to submit constant actions over intervals of length Δ. Suppose at some time $\tau+\Delta$ the PAP is $x(\tau+\Delta)=(x_1,x_2)$ and from time $\tau+\Delta$ until τ player 2 consistently submits action y_2. Then, in a Markov equilibrium, player 1's expected payoff from submitting action y_1 in the same time interval is equal to

$$Q(x,(y_1,y_2),(y_1,x_2),\Delta)\pi_1(y_1,x_2,\tau)+Q(x,(y_1,y_2),(y_1,y_2),\Delta)\pi_1(y_1,y_2,\tau)+$$
$$+Q(x,(y_1,y_2),(x_1,y_2),\Delta)\pi_1(x_1,y_2,\tau)+Q(x,(y_1,y_2),x,\Delta)\pi_1(x_1,x_2,\tau).$$

where $\pi_1(.,\tau)$ is player 1's equilibrium continuation payoff if at time τ. Note that since $Q(x,(y_1,y_2),(x_1,y_2),$ and $Q(x,(y_1,y_2),x,\Delta)$ do not depend on y_1, maximizing the previous expression with respect to y_1 reduces to determine

$$y_1^*(x,y_2,\tau,\Delta):=\arg\sup_{y_1\in X_1} Q(x,(y_1,y_2),(y_1,x_2),\Delta)\pi_1(y_1,x_2,\tau)+Q(x,(y_1,y_2),(y_1,y_2),\Delta)\pi_1(y_1,y_2,\tau)$$

When $\Delta\to 0$ the right hand side of the previous expression converges to zero because of property (1); however it is easy to see that $\lim_{\Delta\to 0} y_1^*(x,y_2,\tau,\Delta)$ is the y_1 maximizing the following expression:

$$\lim_{\Delta\to 0}\frac{Q(x,(y_1,y_2),(y_1,x_2),\Delta)\pi_1(y_1,x_2,\tau)+Q(x,(y_1,y_2),(y_1,y_2),\Delta)\pi_1(y_1,y_2,\tau)}{\Delta}=$$
$$=(\lambda_1 q_{10}^1+\lambda_2 q_{10}^2)\pi_1(y_1,x_2,t)+(\lambda_1 q_{11}^1+\lambda_2 q_{11}^2)\pi_1(y_1,y_2,t) \tag{12}$$

Finally, note that $r=1$ implies $q_{10}^i=q_{01}^i=0$ while, $r=0$ implies $\lambda_1 q_{11}^1+\lambda_2 q_{11}^2=0$ for $i=1,2$. Hence expression (12) is maximized for $y_1=x_1^*(y_2,t)$ if $r=1$, and for $y_1=x_1^*(x_2,t)$ if $r=0$. Since (12) is continuous in q^i the same maximizers apply when r is close to 1 and to 0, respectively. □

Proof of Proposition 2: See supplementary material.

Proof of Proposition 3: (i) Note first that property (4) implies that for any $\varepsilon > 0$, there exists some $\tau' > 0$ small, such that for $\tau \leq \tau'$, no matter the actions submitted by players from τ until the opening, it results

$$|u_i(x) - \pi(x, \tau)| < \varepsilon$$

We can apply Lemma 1 and deduce that for ε small enough and generic payoffs of the underlying game, at time τ and PAP $x(\tau)$, player i strictly dominant action is to submit $y_i = BR_i(x_{-i}(\tau))$. Thus, when the opening date is close enough the equilibrium play is uniquely defined in a way that only depends on the current PAP and corresponds to the equilibrating scenario. Hence in all SPE we can express players i's continuation payoff as a function $\pi_i(x(\tau), \tau)$. (ii) Let x^{N1} and x^{N2} be two UGE. Generic payoffs implies $u_1(x^{N1}) > u_1(x^{N2})$, without loss of generality. The proof is by contradiction. Suppose that there is a SPE equilibrium where the equilibrating scenario is played during the whole preopening game independently of its length. Then, $\pi_i(x^{N2}, \tau) = u_i(x^{N2})$ for all i, τ. Suppose now that at τ the PAP is $x' = (x_1^{N1}, x_2^{N2})$. According to the equilibrating strategy, player 1 submits $y_1(\tau) = BR_1(x_2^{N2}) = x_1^{N2}$ while player 2 submits $y_2(\tau) = BR_2(x_1^{N2}) = x_2^{N1}$. If $r = 0$, then the probability that the two submissions are posted simultaneously is nil. This implies that PAP x' will either evolve into PAP x^{N1} or into PAP x^{N2}. Once either x^{N1} or x^{N2} is reached, players will stick to the posted action profile until the opening. Therefore, if the equilibrating scenario is expected to be played for a sufficiently long period the probability that the PAP during the preopening will eventually evolve from x' to either x^{N1} or x^{N2} tends to one; consequently $\pi_1(x', \tau)$ converges to an average between $u_1(x^{N1})$ and $u_1(x^{N2})$. This in turn implies that for τ sufficiently large:

$$\pi_1(x^{N1}, \tau) = u_1(x^{N1}) > \pi_1(x', \tau) > u_1(x^{N2}) = \pi_1(x^{N2}, \tau).$$

Recall that Lemma 1 shows that if r is sufficiently close to 0 player 1's optimal submitted action in τ when the PAP is x' is $x_1^*(x_2^{N2}, \tau) = \arg\max_{y_1} \pi_1(y_1, x_2^{N2}, \tau)$. However $\pi_1(x', \tau) > \pi_1(x^{N2}, \tau)$ implies that $x_1^*(x_2^{N2}, \tau) \neq x_1^{N2}$ contradicting the claim that the equilibrating strategy is used at any τ (as this would imply that at PAP x' player 1 submits x_1^{N2}). ■

Proof of Theorem 4: To begin with we recall some properties of continuous time stochastic processes. Consider a continuous process $\mathbb{D}$ defining the stochastic transition dynamics of X into itself. A subset $\mathcal{S} \subset X$ is said to be stable under $\mathbb{D}$ if a trajectory reaching $\mathcal{S}$ cannot exit this set and if each point in $\mathcal{S}$ is reached infinitely many times. Formally, $\mathcal{S} \subset X$ is stable under $\mathbb{D}$ if: (S1) $\Pr(x(t) \in \mathcal{S} | \, x(0) \in \mathcal{S}) = 1$ and (S2) for any $x \in \mathcal{S}$, $\exists t' > 0$ and $\varepsilon > 0$ such that for all $t' > t$, $\Pr(x(t) = x | \, x(0) \in \mathcal{S}) > \varepsilon$. For stable sets $\mathcal{S}$ we have that given a bounded function $f : X \to \mathbb{R}$, there exists $\widehat{f}_{\mathbb{D}}(S) \in (\min_{x\in\mathcal{S}} f(x), \max_{x\in\mathcal{S}} f(x))$ satisfying

$$\lim_{t\to\infty} E[f(\mathbb{D}(x,t))] = \widehat{f}_{\mathbb{D}}(S) \text{ for all } x \in \mathcal{S} \tag{13}$$

Let $r = 0$, then action profile $\widehat{x}$ is stable under all equilibrium dynamics of the preopening game. In fact Lemma 1 implies that player i at time τ from the opening will submit action $y_i(\tau) = x^*(x_{-i}(\tau), \tau)$. When τ is close to 0 condition (4) guarantees that $x^*(\widehat{x}_{-i}(\tau), \tau) = \widehat{x}_i$ implying that $\pi_i(\widehat{x}, \tau) = u_i(\widehat{x})$ for τ small. However, as long as $\pi_i(\widehat{x}, \tau) = u_i(\widehat{x})$, since $u_i(\widehat{x}) > \pi_i(x, \tau)$ for any $x \neq \widehat{x}$, we have that $x^*(\widehat{x}_{-i}(\tau), \tau) = \widehat{x}_i$ and $\pi_i(\widehat{x}, \tau) = u_i(\widehat{x})$ for all τ. In other words, once the PAP $\widehat{x}$ is reached, players will never submit actions different from $\widehat{x}$.

Lemma 8 **:** *Under the assumptions of Theorem 4, there exists τ' finite such that at $\tau > \tau'$ from the opening, $\widehat{x}$ is the only stable set of the equilibrium dynamics of PAPs.*

Proof: The proof is by contradiction. Take any finite $\tau' \geq 0$ and suppose there exists a set $\mathcal{C} \subset X$ with $\widehat{x} \notin \mathcal{C}$ that is stable under the equilibrium dynamics for all $\tau \geq \tau'$. In other words, for any $\tau > \tau'$, whenever the PAP is in the set $\mathcal{C}$, players submit actions that keep the following PAP in $\mathcal{C}$ and as τ goes to infinity all PAPs in $\mathcal{C}$ are reached infinitely many times between time τ and time τ' (see for example the cycle depicted in Figure 6). Property (13) implies that for any $\alpha > 0$ there is $\tau(\alpha) > \tau'$ such that

$$\max_{x\in\mathcal{C}} \pi_i(x, \tau(\alpha)) - \min_{x\in\mathcal{C}} \pi_i(x, \tau(\alpha)) < \alpha \tag{14}$$

In other words the expected continuation payoffs from different PAPs in $\mathcal{C}$ converge to the same value as $\tau(\alpha)$ goes to infinity.

Notice that for any $(x_1, x_2) \in X$, we have $(x_1, \widehat{x}_2) \notin \mathcal{C}$ and $(\widehat{x}_1, x_2) \notin \mathcal{C}$: in fact, $x_i^*(\widehat{x}_{-i}(\tau), \tau) = \widehat{x}_i$ implies that when the PAP is $(x_i, \widehat{x}_{-i})$ players submit $y(\tau) = \widehat{x} \notin \mathcal{C}$, hence $(x_i, \widehat{x}_{-i})$ cannot belong to $\mathcal{C}$ as it does not satisfy condition (S1) of a stable set.

Consider next PAP $(x_i, \widehat{x}_{-i})$ where x_i is chosen such that there exists a x_{-i} for which $(x_i, x_{-i}) \in \mathcal{C}$ (see for example the action profile (U, R) in Figure 6: $x_i = U$, $\widehat{x}_{-i} = R$ and x_{-i} is L or C). Then we have $(x_i^*(\widehat{x}_{-i}, \tau), \widehat{x}_{-i}) = \widehat{x}$ and $(x_i, x_{-i}^*(x_i, \tau)) \in \mathcal{C}$.[12] Since $r = 0$, the probability that two submissions are posted simultaneously is nil. Hence, between an arbitrarily large time $\tau \gg \tau'$ and time τ', PAP $(x_i, \widehat{x}_{-i})$ evolves with probability arbitrarily close to 1 into either PAP $\widehat{x}$ or into an element of $\mathcal{C}$ before τ'. Let

$$\theta := \lim_{\tau \to \infty} \Pr\left(x\left(\tau'\right) = \widehat{x} | x(\tau) = (x_i, \widehat{x}_{-i})\right) > 0.$$

Now fix $\alpha > 0$ such that $\alpha < \theta(u_i(\widehat{x}) - \max_{x \in \mathcal{C}} \pi_i(x, \tau'))$ and let $\tau(\alpha) > \tau'$ satisfies inequality (14). Note that for any $\tau'' > \tau(\alpha)$,

$$\max_{x \in \mathcal{C}} \pi_i(x, \tau'') \leq \max_{x \in \mathcal{C}} \pi_i(x, \tau(\alpha)) < \alpha + \min_{x \in \mathcal{C}} \pi_i(x, \tau(\alpha)) \tag{15}$$

where the first inequality follows from the fact that $\mathcal{C}$ is stable and $\tau'' > \tau(\alpha)$, while the second inequality follows from inequality (14). Note that there exists $\tau'' \gg \tau(\alpha)$ such that the probability that within time τ'' and $\tau(\alpha)$, PAP $x(\tau'') = (x_i, \widehat{x}_{-i})$ either evolves into $\widehat{x}$ or reaches $\mathcal{C}$ is arbitrarily close to 1. Namely for τ'' large enough

$$\begin{aligned}
\pi_i(x_i, \widehat{x}_{-i}, \tau'') &\geq \theta u_i(\widehat{x}) + (1-\theta) \min_{x \in \mathcal{C}} \pi_i(x, \tau(\alpha)) \\
&= \theta \left(u_i(\widehat{x}) - \min_{x \in \mathcal{C}} \pi_i(x, \tau(\alpha)) \right) + \min_{x \in \mathcal{C}} \pi_i(x, \tau(\alpha)) \\
&\geq \theta \left(u_i(\widehat{x}) - \max_{x \in \mathcal{C}} \pi_i(x, \tau(\alpha)) \right) + \min_{x \in \mathcal{C}} \pi_i(x, \tau(\alpha)) \\
&\geq \theta \left(u_i(\widehat{x}) - \max_{x \in \mathcal{C}} \pi_i(x, \tau') \right) + \min_{x \in \mathcal{C}} \pi_i(x, \tau(\alpha)) > \alpha + \min_{x \in \mathcal{C}} \pi_i(x, \tau(\alpha))
\end{aligned}$$

where the last inequality follows from the definition of α, and the relation before the last equality follows from the fact that $\mathcal{C}$ is stable and $\tau(\alpha) > \tau'$. Hence from equation (15) we have

[12] In fact, $(x_i, x_{-i}^*(x_i, \tau)) \notin \mathcal{C}$ would imply that $(x_i, x_{-i}) \in \mathcal{C}$ would possibly evolve into $(x_i, x_{-i}^*(x_i, \tau)) \notin \mathcal{C}$, contradicting either that that $\mathcal{C}$ is a stable or that $(x_i, x_{-i}) \in \mathcal{C}$.

$\pi_i(x_i, \widehat{x}_{-i}, \tau'') > \max_{x \in \mathcal{C}} \pi_i(x, \tau'') \geq \pi_i(x_i, x_{-i}, \tau'')$ as $(x_i, x_{-i}) \in \mathcal{C}$. This implies that if $x(\tau'') = (x_i, x_{-i})$, player $-i$ will submit $x^*(x_i(\tau), \tau) = \widehat{x}_{-i}$. This in turn means that for τ large enough the equilibrium dynamics of submissions can lead $(x_i, x_{-i}) \in \mathcal{C}$ into $(x_i, \widehat{x}_{-i}) \notin \mathcal{C}$ contradicting the assumption of $\mathcal{C}$ being stable for all $\tau > \tau'$. Hence for τ large enough the only possible stable set is $\widehat{x}$. □

Lemma 8 implies that for some τ' finite, during the first $T - \tau'$ part of the preopening the equilibrium play is such that the players' optimal submissions lead the PAP to converge to $\widehat{x}$. Once $\widehat{x}$ is reached the PAP will not change until the opening. Hence to obtain the result of the theorem it is sufficient to choose $T(\varepsilon)$ large enough to guarantee that the PAP at time τ' is $\widehat{x}$ with probability larger than $1 - \varepsilon$. ■

Proof of Theorem 5:

We shall provide the proof for the case $i^* = 2$ implying $x^{i^*} = A$. The case $i^* = 1$ can be obtained applying a symmetric argument.

Proposition 3 implies that the equilibrating strategy is strictly dominant for both players starting from time τ^* until the opening. At time τ^* the equilibrating strategy is already optimal for player 1, while player 2 keeps (moves) if $x(\tau^*) = A$ (resp. $x(\tau^*) = D$) while he is indifferent between moving and keeping if $x(\tau^*) = C$ or $x(\tau^*) = D$. Hence at time τ^* from the opening the following relations hold:

$$a_1(\tau^*) - c_1(\tau^*) > 0, \quad (16)$$

$$d_1(\tau^*) - b_1(\tau^*) > 0, \quad (17)$$

$$a_2(\tau^*) - b_2(\tau^*) > 0, \quad (18)$$

$$c_2(\tau^*) - d_2(\tau^*) = 0. \quad (19)$$

where the expected payoffs are obtained from equations (6)-(9) and the definition of τ^*. Consider now the equilibrium strategies at time $\tau^* + \epsilon$ from the opening. For $\epsilon > 0$ and arbitrarily small, it must be that the strategies in the sub-game starting at $\tau^* + \epsilon$ reflect either the equilibrating scenario or one of the following three scenarios:

Scenario 1:

A $keep; keep$	B $move; move$
C $move; keep$	D $keep; keep$

A $\circlearrowleft$	B $\leftharpoonup\downharpoonleft$
C $\uparrow$	D $\circlearrowleft$

(20)

Scenario 2:

$S_t = A$ $keep; keep$	$S_t = B$ $move; move$
$S_t = C$ $move; move$	$S_t = D$ $keep; move$

$S_t = A$ $\circlearrowleft$	$S_t = B$ $\leftharpoonup\downharpoonleft$
$S_t = C$ $\uparrow\rightharpoonup$	$S_t = D$ $\longleftarrow$

(21)

Scenario 3:

A $keep; keep$	B $move; move$
C $move; keep$	D $keep; move$

A $\circlearrowleft$	B $\leftharpoonup\downharpoonleft$
C $\uparrow$	D $\longleftarrow$

(22)

The following Lemma shows that only Scenario 3 is an equilibrium at time $\tau^* + \epsilon$.

Lemma 9 *There exists a finite $\tau' > 0$ such that Scenario 3 represents the unique equilibrium play for time $\tau^* + \tau'$ until time τ^*.*

Proof: From the previous discussion we know that an instant before time τ^*, either the equilibrating scenario, Scenario 1, 2, or 3 is played. Consider time $\tau = \tau^* + \epsilon$ to the opening with $\epsilon > 0$ and arbitrarily small. Suppose Scenario 1 is played at time τ, then

$$\begin{aligned} \dot{c}_i(\tau) &= \lambda_1(a_i(\tau) - c_i(\tau)) \\ \dot{d}_i(\tau) &= 0 \end{aligned}$$

Note that $a_2(\tau^*) - c_2(\tau^*) = a_2 - c_2(\tau^*) > 0$ where the inequality holds since a_2 is player 2's maximum payoff in the underlying game. Thus, $\dot{c}_i(\tau^*) > 0$ and $\dot{d}_i(\tau^*) = 0$ and therefore, using (19) we obtain $c_2(\tau^* + \epsilon) > d_2(\tau^* + \epsilon)$. Hence it is not optimal for player 2 to "keep" at $x(\tau^* + \epsilon) = D$. This in turn implies that Scenario 1 cannot be an equilibrium at time $\tau^* + \epsilon$.

Suppose now that Scenario 2 is played at time τ. In this case

$$\begin{aligned}
\dot{c}_i(\tau) &= \lambda_1(a_i(\tau) - c_i(\tau)) + \lambda_2(d_i(\tau) - c_i(\tau)) \\
\dot{d}_i(\tau) &= \lambda_2(c_i(\tau) - d_i(\tau))
\end{aligned}$$

Since $a_2(\tau^*) - c_2(\tau^*) > 0$ and $d_2(\tau^*) = c_2(\tau^*)$ by definition of τ^*; also in this case $\dot{c}_2(\tau^*) > 0$ and $\dot{d}_2(\tau^*) = 0$ implying that $c_2(\tau^* + \epsilon) > d_2(\tau^* + \epsilon)$. Again, it is not optimal for player 2 to "move" if $x(\tau^* + \epsilon) = C$. As a consequence Scenario 2 cannot be an equilibrium at time $\tau^* + \epsilon$.

Suppose then that the equilibrating scenario is played at time τ. In this case

$$\begin{aligned}
\dot{c}_i(\tau) &= \lambda_1(a_i(\tau) - c_i(\tau)) + \lambda_2(d_i(\tau) - c_i(\tau)) \\
\dot{d}_i(\tau) &= 0
\end{aligned}$$

As in the previous cases case $\dot{c}_2(\tau^*) > 0$ and $\dot{d}_2(\tau^*) = 0$ implying that $c_2(\tau^* + \epsilon) > d_2(\tau^* + \epsilon)$. Then, it is not optimal for player 2 to "keep" if $x(\tau^* + \epsilon) = D$. Thus, the equilibrating scenario cannot be an equilibrium at time $\tau^* + \epsilon$. Suppose finally that Scenario 3 is played; then:

$$\begin{aligned}
\dot{a}_i(\tau) &= 0 \\
\dot{b}_i(\tau) &= \lambda_2(a_i(\tau) - b_i(\tau)) + \lambda_1(d_i(\tau) - b_i(\tau)) \\
\dot{c}_i(\tau) &= \lambda_1(a_i(\tau) - c_i(\tau)) \\
\dot{d}_i(\tau) &= \lambda_2(c_i(\tau) - d_i(\tau))
\end{aligned}$$

which in turn imply that $\dot{c}_i(\tau^*) > 0$ and $\dot{d}_2(\tau^*) = 0$. Hence it must result that for $\epsilon > 0$ and sufficiently small we have

$$\begin{aligned}
a_1(\tau^*+\epsilon) - c_1(\tau^*+\epsilon) &> 0, \\
d_1(\tau^*+\epsilon) - b_1(\tau^*+\epsilon) &> 0, \\
a_2(\tau^*+\epsilon) - b_2(\tau^*+\epsilon) &> 0 \\
c_2(\tau^*+\epsilon) - d_2(\tau^*+\epsilon) &> 0
\end{aligned}$$

guaranteeing that Scenario 3 is optimal at time $\tau^*+\epsilon$. If Scenario 3 is played between time $\tau^*+\tau$ and time τ^*, then

$$\begin{aligned}
a_i(\tau^*+\tau) &= a_i(\tau^*) \\
b_i(\tau^*+\tau) &= a_i(\tau^*) - (a_i(\tau^*) - b_i(\tau^*) - c_i(\tau^*) + d_i(\tau^*))e^{-(\lambda_1+\lambda_2)\tau} + \\
&\quad + (d_i(\tau^*) - a_i(\tau^*)) + (a_i(\tau^*) - c_i(\tau^*)) \frac{\left(\lambda_1 e^{-\lambda_1\tau} - \lambda_2 e^{-\lambda_2\tau}\right)}{\lambda_1-\lambda_2} \\
c_i(\tau^*+\tau) &= a_i(\tau^*) + (c_i(\tau^*) - a_i(\tau^*))e^{-\lambda_1\tau} \\
d_i(\tau^*+\tau) &= a_i(\tau^*) - (a_i(\tau^*) - d_i(\tau^*))\, e^{-\lambda_2\tau} - (a_i(\tau^*) - d_i(\tau^*)) \frac{\lambda_2\left(e^{-\lambda_2\tau} - e^{-\lambda_1\tau}\right)}{\lambda_1-\lambda_2}
\end{aligned}$$

Recalling that $a_i = a_i(\tau^*)$, $d_i = d_i(\tau^*)$ and $d_2(\tau^*) = c_2(\tau^*)$, we have

$$a_1(\tau^*+\tau) - c_1(\tau^*+\tau) = (a_1 - c_1(\tau^*))\, e^{-\lambda_1\tau} > 0 \tag{23}$$

$$d_1(\tau^*+\tau) - b_1(\tau^*+\tau) = e^{-\lambda_1\tau}\left(c_1(\tau^*) - a_1 + (a_1 - c_1(\tau^*) + d_1 - b_1(\tau^*))\, e^{-\lambda_2\tau}\right) \tag{24}$$

$$a_2(\tau^*+\tau) - b_2(\tau^*+\tau) = e^{-(\lambda_1+\lambda_2)\tau}\left(a_2 - b_2(\tau^*) + \lambda_1(a_2-d_2)\frac{e^{\lambda_1\tau} - e^{\lambda_2\tau}}{\lambda_1-\lambda_2}\right) > 0 \tag{25}$$

$$d_2(\tau^*+\tau) - c_2(\tau^*+\tau) = \lambda_1(a_2-d_2)\frac{e^{-\lambda_1\tau} - e^{-\lambda_2\tau}}{\lambda_1-\lambda_2} < 0 \tag{26}$$

Inequalities (23), (25) and (26) follow from $a_2 > d_2$ and from expression (16)-(19). Note that as long as inequalities (23), (25) and (26) are satisfied the equilibrium play in the switching game is such that

A $keep; keep$	B $?; move$
C $move; keep$	D $?; move$

(27)

while the sign of (24) determines player 1's behavior in PAPs B and D. Note that the sign of (24) is equal to the sign of

$$c_1(\tau^*) - a_1 + (a_1 - c_1(\tau^*) + d_1 - b_1(\tau^*))\, e^{-\lambda_2 \tau}$$

that is positive for $\tau < \tau' := \frac{1}{\lambda_2} \ln \left(\frac{a_1 - c_1(\tau^*) + d_1 - b_1(\tau^*)}{a_1 - c_1(\tau^*)} \right) > 0$ and negative for $\tau > \tau'$. When (24) is positive, player 1 will "move" in PAP B and "keep" in PAP D: hence Scenario 3 represents the unique equilibrium play between time $\tau^* + \tau'$ and time τ^*. $\square$

Now let us introduce $\tau'' := \tau' + \tau^*$. From Lemma (9) we know that at time τ'':

$$a_1(\tau'') - c_1(\tau'') > 0, \tag{28}$$

$$d_1(\tau'') - b_1(\tau'') = 0, \tag{29}$$

$$a_2(\tau'') - b_2(\tau'') > 0 \tag{30}$$

$$c_2(\tau'') - d_2(\tau'') > 0 \tag{31}$$

Consider then the equilibrium play at $\tau'' + \epsilon$ from the opening. For $\epsilon > 0$ and arbitrarily small, the play has to be consistent with Figure (27). This happens if the play reflects either Scenario 3 or one of the following three scenarios:

Scenario 4:

A *keep*; *keep*	B *keep*; *move*
C *move*; *keep*	D *keep*; *move*

A $\circlearrowright$	B $\leftharpoonup$
C $\uparrow$	D $\leftharpoonup$

(32)

Scenario 5:

A *keep*; *keep*	B *move*; *move*
C *move*; *keep*	D *move*; *move*

$S_t = A$ $\circlearrowright$	$S_t = B$ $\leftharpoonup\!\downharpoonleft$
$S_t = C$ $\uparrow$	$S_t = D$ $\leftharpoonup\!\upharpoonright$

(33)

Scenario 6:

A $keep; keep$	B $keep; move$
C $move; keep$	D $move; move$

A $\circlearrowright$	B $\leftharpoonup$
C $\uparrow$	D $\leftharpoonup\upharpoonright$

(34)

Lemma 10 *: For any $\tau > \tau''$, the unique equilibrium play at time $\tau > \tau''$ is given by Scenario 6.*

Proof: From the previous discussion we know that at time $(\tau'' + \epsilon)$ either Scenario 3, 4, 5, or 6 is played. Suppose Scenario 3 is played at time $\tau'' + \epsilon$ with $\epsilon > 0$ arbitrarily small. In this case $d_1(\tau'' + \epsilon) - b_1(\tau'' + \epsilon) = d_1(\tau^* + \tau' + \epsilon) - b_1(\tau^* + \tau' + \epsilon)$ where the right hand side is given by equation (24). However, from the definition of τ' it follows that $d_1(\tau'' + \epsilon) - b_1(\tau'' + \epsilon) < 0$ implying that it is not optimal for player 1 to keep his action at $\tau^* + \epsilon$ if $x(\tau^* + \epsilon) = D$. Hence Scenario 3 cannot be an equilibrium at time $\tau'' + \epsilon$.

Suppose then that Scenario 4 is played. Then

$$\begin{aligned} \dot{b}_i(\tau) &= \lambda_2(a_i(\tau) - b_i(\tau)) \\ \dot{d}_i(\tau) &= \lambda_2(c_i(\tau) - d_i(\tau)) \end{aligned}$$

Considering that at τ'', $b_1(\tau'') = d_1(\tau'')$, $a_1(\tau'') = a_1 > c_1(\tau'')$, we have $\dot{b}_1(\tau'') = \lambda_2(a_1 - d_1(\tau'')) > \lambda_2(c_1(\tau'') - d_1(\tau'')) = \dot{d}_1(\tau)$. Hence $b_1(\tau'' + \epsilon) > d_1(\tau'' + \epsilon)$, implying that if $x(\tau'' + \epsilon) = D$, player 1 prefers to "move". This contradicts the claim that Scenario 4 is played at time $\tau'' + \epsilon$. Suppose now that Scenario 5 is played. Then

$$\begin{aligned} \dot{b}_i(\tau) &= \lambda_2(a_i(\tau) - b_i(\tau)) + \lambda_1(d_i(\tau) - b_i(\tau)) \\ \dot{d}_i(\tau) &= \lambda_1(d_i(\tau) - b_i(\tau)) + \lambda_2(c_i(\tau) - d_i(\tau)) \end{aligned}$$

The two previous differential equations, together with the fact that $b_1(\tau'') = d_1(\tau'')$ imply that $\dot{b}_1(\tau'') = \lambda_2(a_1 - d_1(\tau'')) > \lambda_2(c_1(\tau'') - d_1(\tau'')) = \dot{d}_i(\tau)$. Hence $b_1(\tau'' + \epsilon) > d_1(\tau'' + \epsilon)$, implying that if $x(\tau'' + \epsilon) = B$, player 1 prefers to keep. This contradicts the claim that Scenario 5 is played at time $\tau'' + \epsilon$. Finally consider Scenario 6. Then:

$$\begin{aligned}
\dot{a}_i(\tau) &= 0 \\
\dot{b}_i(\tau) &= \lambda_2(a_i(\tau) - b_i(\tau)) \\
\dot{c}_i(\tau) &= \lambda_1(a_i(\tau) - c_i(\tau)) \\
\dot{d}_i(\tau) &= \lambda_2(c_i(\tau) - d_i(\tau)) + \lambda_1(b_i(\tau) - d_i(\tau))
\end{aligned}$$

Remark that $\dot{b}_i(\tau'') - \dot{d}_i(\tau'') = \lambda_2(a_1(\tau'') - c_1(\tau'')) > 0$. Hence from expressions (28)-(31), it results that for $\epsilon > 0$ and sufficiently small

$$\begin{aligned}
a_1(\tau'' + \epsilon) - c_1(\tau'' + \epsilon) &> 0, \\
d_1(\tau^* + \epsilon) - b_1(\tau'' + \epsilon) &< 0, \\
a_2(\tau'' + \epsilon) - b_2(\tau'' + \epsilon) &> 0 \\
d_2(\tau'' + \epsilon) - c_2(\tau'' + \epsilon) &< 0
\end{aligned}$$

guaranteeing that Scenario 6 is optimal at time $\tau'' + \epsilon$. If Scenario 6 is played between time $\tau'' + \tau$ and time τ'', then the dynamics of the continuation payoffs are given by

$$\begin{aligned}
a_i(\tau'' + \tau) &= a_i(\tau'') = a_i \\
b_i(\tau'' + \tau) &= a_i(\tau'') + (b_i(\tau'') - a_i(\tau''))e^{-\lambda_2 \tau} \\
c_i(\tau'' + \tau) &= a_i(\tau'') + (c_i(\tau'') - a_i(\tau''))e^{-\lambda_1 \tau} \\
d_i(\tau'' + \tau) &= a_i(\tau'') + (b_i(\tau'') - a_i(\tau''))e^{-\lambda_2 \tau} + (c_i(\tau'') - a_i(\tau''))e^{-\lambda_1 \tau} \\
&\quad + (a_i(\tau'') - b_i(\tau'') - c_i(\tau'') + d_i(\tau''))e^{-(\lambda_1 + \lambda_2)\tau}
\end{aligned}$$

Note that $a_i = a_1(\tau'')$, $d_1(\tau'') = b_1(\tau'')$ from the definition of τ''. Hence we have

$$a_1(\tau'' + \tau) - c_1(\tau'' + \tau) = (a_1 - c_1(\tau''))\, e^{-\lambda_1 \tau} > 0 \quad (35)$$

$$a_2(\tau'' + \tau) - b_2(\tau'' + \tau) = (a_2 - b_2(\tau''))\, e^{-\lambda_2 \tau} > 0 \quad (36)$$

$$d_1(\tau'' + \tau) - b_1(\tau'' + \tau) = (c_1(\tau'') - a_1) e^{-\lambda_1 \tau}(1 - e^{-\lambda_2 \tau}) < 0 \quad (37)$$

$$\begin{aligned} d_2(\tau'' + \tau) - c_2(\tau'' + \tau) &= e^{-\lambda_2 \tau} \left(b_2(\tau'') - a_2 + (a_2 - b_2(\tau'') - c_2(\tau'') + d_2(\tau''))e^{-\lambda_1 \tau} \right) \\ &< 0 \end{aligned} \quad (38)$$

where inequalities (35)-(37) follow from (28)-(31) while (38) follows from the fact that the sign of $d_2(\tau'' + \tau) - c_2(\tau'' + \tau)$ is equal to the sign of

$$b_2(\tau'') - a_2 + (a_2 - b_2(\tau'') - c_2(\tau'') + d_2(\tau''))e^{-\lambda_1 \tau}$$

which is negative far all $\tau \geq 0$ because,first, it is negative for for $\tau = 0$ (because of (31)) and second, it is either a decreasing function of τ or a sum of negative terms. Inequalities (35)-(38) hold for all $\tau > 0$ and imply that Scenario 6 must represent the only equilibrium play at any time $\tau > \tau''$. □

In order to conclude the proof of the Theorem note that the above Lemma shows that Scenario 6 is played when the time left until the opening is more than τ'', i.e. from date 0 until date $T - \tau''$. In Scenario 6 player 1 (resp. player 2) moves his action whenever his posted action is not "up" (resp. "left"). Hence both players keep their posted action only if the PAP is A. Thus, for any $\varepsilon > 0$ there exists a sufficiently large T such that if Scenario 6 is played from time T to time τ'', then $\Pr(x(\tau'') = A) > 1 - \varepsilon$. Once PAP A is reached players will not move away from it neither in Scenario 3 (which, by Lemma 9 follows Scenario 6), nor in the equilibrating scenario that by Proposition 4 and Lemma 9 follows scenario 3 until the opening. Hence the opening will be at A. ■

Proof of Proposition 6: See supplementary material.

Proof of Proposition 7: We denote with $H(t, x) \subset H(t)$ the set of histories of length t ending with a PAP equal to x. For any given history $h(t) \in H(t)$, let $h(t, x) \in H(t, x)$ be the history identical to $h(t)$ for $t' < t$ and such that the PAP in t is x. Consider a Nash equilibrium of the preopening game, and let $h(t) \in H(t, (., x_{-i}))$ be a history of length t where player $-i$ posted action at date t is x_{-i}. Let $h(t, (y_i, x_{-i}))$ be a history identical to $h(t)$ for all $t' < t$ and such that the posted action of player i at t is equal to y_i. Let $y_i^*(h(t))$ be the y_i that maximizes player i's expected continuation payoff after history $h(t, (y_i, x_{-i}))$:

$$y_i^*(h(t)) := \arg\max_{y_i \in X_i} E\left[u_i(x(T)) | h(t, (y_i, x_{-i}))\right].$$

The following lemma shows that if for all $h(t) \in H(t, (., x_{-i}))$, $y_i^*(h(t))$ is unique and it is equal to some $y_i^*(x_{-i})$, then it must be that at date t in equilibrium player i submits $y_i^*(x_{-i})$ whenever $h(t) \in H(t, (., x_{-i}))$. Formally,

Lemma 11 *: Under Assumption 3, if at date t for some $x \in X$, there is a unique $y_i^*(x_{-i})$ such that $y_i^*(h(t)) = y_i^*(x_{-i})$ for all $h(t) \in H(t, (., x_{-i}))$, then in any Nash equilibria of the preopening game player i submits $y_i^*(x_{-i})$ after history $h(t) \in H(t, (., x_{-i}))$.*

We provide here a short intuition of the proof of Lemma 11. The formal proof is provided in supplementary material. Consider player i at instant t after observing the history $h(t) \in H(t, (., x_{-i}))$. If he submits action y_i and his action is not posted, then his choice of y_i does not affect his continuation payoff. If his action is posted, then by Assumption 3 the other player's submission is not posted and i's continuation payoff is equal to $E\left[u_i(x(T)) | h(t, (y_i, x_{-i}))\right]$, which is maximized at $y^i = y_i^*(x_{-i})$ by hypothesis.

Note that condition (1) holds under Assumption 3: this in turn implies that when the opening date is close enough the probability that the players succeed in changing their posted actions before the opening gets arbitrarily close to zero. Hence, for t sufficiently close to T, independently of their strategies, the players' equilibrium expected payoffs are arbitrarily close to the payoffs obtained when the opening action profile is the one posted at t. Formally, for any strategy profile σ and any history $h(t) \in H(t)$:

$$\lim_{t \to T} E\left[u_i(x(T)) | h(t), \sigma\right] = u_i(x(t)). \tag{39}$$

where $x(t)$ is the PAP at the end of history t.[13] This implies that for t sufficiently close to T and a generic payoff matrix u_i, Lemma 11 applies and in all Nash equilibria players adopt the equilibrating strategy. As a consequence, the equilibrium continuation payoff of each player at time t depends only on the PAP $x(t)$ and on the time remaining to the opening $T - t$. Hence players' equilibrium strategies as well must depend only on the current PAP and the time to the opening (i.e. Markov strategies). Moving backward the same argument applies so that the submission preferred by each player at time τ to the opening is unique and the backward dynamics of players' continuation payoffs is uniquely defined. Nevertheless, as it is illustrated in the proof of Theorem 5, there two points in time (namely τ^* of Proposition 3 and τ'' of Lemma 10) at which a player i is indifferent between submitting different actions. For example at τ^* Lemma 11 does not apply and the action submitted by player i might depend on the whole history of past PAPs. However, the proof of Theorem 5 also shows that generically only one of the submissions that are optimal at τ^* is also optimal slightly before, i.e. at time $\tau^* + \varepsilon$. Hence a Nash equilibrium of the preopening game can differ from the Markov SPE only for some actions submitted at two precise dates. Since the probability that any action is instantaneously posted is equal to zero, all Nash equilibria are observational equivalent to the Markov SPE we have analyzed in the case $r = 0$. ■

[13] Recall that Assumption 4 implies that for any $x \in X$ and $t > 0$, there is strictly positive probability that $h(t) \in H(t, x)$.

Supplementary Material to "Preopening and Equilibrium Selection"

Proof of Proposition 2

We prove the proposition for $r = 1$ and hence by continuity for r close to 1. W.l.o.g. fix $q^1_{11} = 1$ and $q^2_{11} = 1$ and let $\lambda_1 = \lambda_2 = \lambda$. In this case equality (12) in Lemma 1 implies that player i chooses

$$y_i(\tau) \in \arg\max_{y_i \in X_i} \pi_i(y_i, y_{-i}, \tau).$$

Fix a pure strategies Nash equilibrium of the underlying game $x^N = (x^N_1, x^N_2) \in N$. We show now that the strategy in which for all PAPs and all τ, player 1 submits $y_1(\tau) = x^N_1$ and player 2 submits $y_2(\tau) = x^N_2$ is a Markov equilibrium of the preopening game. In other words, at any time τ and for all PAPs $x(\tau) \in X$ we have that if $y_{-i}(\tau) = x^N_{-i}$, then player i's best reply is to submit $y_i(\tau) = x^N_i$. Consider player i. We have to show that

$$\forall \tau \geq 0, x_i \in X, \quad \pi_i(x^N_i, x^N_{-i}, \tau) - \pi_i(x_i, x^N_{-i}, \tau) \geq 0 \tag{40}$$

Note that (4) implies that (40) is satisfied for τ close enough to 0 since $x^N \in N$. That is to say that when the opening is close enough, it is an equilibrium for player i to submit x^N_i given that the other player is submitting x^N_{-i}. This strategy profile induces the following backward dynamics for players' expected continuation payoffs at τ from the opening:

$$\frac{\partial \pi_i(x_1, x_2, \tau)}{\partial \tau} := \dot{\pi}_i(x_1, x_2, \tau) = \lambda(\pi_i(x^N_1, x_2, \tau) - \pi_i(x_1, x_2, \tau) + \pi_i(x_1, x^N_2, \tau) - \pi_i(x_1, x_2, \tau))$$

for any given PAP $x(\tau) = (x_1, x_2)$. In particular, we have

$$\begin{aligned}
\dot{\pi}_i(x^N_1, x^N_2, \tau) &= 0 \\
\dot{\pi}_i(x_1, x^N_2, \tau) &= \lambda(\pi_i(x^N_1, x^N_2, \tau) - \pi_i(x_1, x^N_2, \tau)) \\
\dot{\pi}_i(x^N_1, x_2, \tau) &= \lambda(\pi_i(x^N_1, x^N_2, \tau) - \pi_i(x^N_1, x_2, \tau))
\end{aligned}$$

Considering that equality (4) imposes the final condition $\pi_i(x_1, x_2, 0) = u_i(x_1, x_2)$, it results that for τ close to 0 we have

$$\begin{aligned}
\pi_i(x_1, x_2^N, \tau) &= u_i(x^N) + (u_i(x_1, x_2^N) - u_i(x^N))e^{-\lambda\tau} \\
\pi_i(x_1^N, x_2, \tau) &= u_i(x^N) + (u_i(x_1^N, x_2) - u_i(x^N))e^{-\lambda\tau} \\
\pi_i(x_1^N, x_2^N, \tau) &= u_i(x^N)
\end{aligned}$$

hence

$$\pi_i(x_1^N, x_2^N, \tau) - \pi_i(x_i, x_{-i}^N, \tau) = \lambda e^{-\lambda\tau}(u_i(x^N) - u_i(x_i, x_{-i}^N)) \geq 0 \tag{41}$$

where the last inequality follows from $x^N \in N$: in turn, (41) implies that condition (40) is satisfied for all τ. In other words, it is an equilibrium for players to continuously submit x^N during all the preopening phase. If this phase is long enough, then assumption (2) implies that the probability of having x^N posted at the opening can be made arbitrarily close to one irrespective of the starting action profile. In order to see that the same result applies for r sufficiently to 1, it is sufficient to note the following: first, equation (12) varies continuously with r, implying that when the opening is close and r is close to 1, submitting x^N remains an equilibrium for both players; second, the backward dynamics $\dot{\pi}_i(x_1, x_2, \tau)$ induced by such strategy is also continuous in r, implying that the resulting $\pi_i(x_1, x_2, \tau)$ can be made arbitrarily close to the one obtained in the case $r = 1$. These two last observations in turn guarantee that submitting x^N remains an equilibrium. ■

Proof of Proposition 6

We refer here to the underlying game of Figure 4 where payoffs are symmetric. The symmetry in the payoff structure and $\lambda_1 = \lambda_2$ imply that $\tau_1^* = \tau_2^* = \tau^*$. This in turn implies that, at time τ^* from the opening, $c_1(\tau^*) = a_1(\tau^*) = a_1 = 1$ and $c_2(\tau^*) = d_2(\tau^*) = d_2 = 1$ i.e. player 1 (resp. player 2) is indifferent between keeping or moving his action at PAP A and C (resp. D and C). However, $d_1(\tau^*) > b_1(\tau^*)$ and $a_2(\tau^*) > b_2(\tau^*)$: hence both players prefer to move at PAP B, while player 1 (resp. player 2) keeps his action at PAP D (resp. at A). In order to determine the equilibrium of the sub-game starting at time $\tau' = \tau^* + \epsilon$, with $\epsilon > 0$ arbitrarily small, we have to analyze four possible scenarios for each player (we will derive the best reply of player 2 fixing the behavior of player 1, and obtain the reverse by symmetry):

	PAP A	*PAP C*
	Player 1 action	*Player 1 action*
Scenario 1	*keep*	*move*
Scenario 2	*keep*	*keep*
Scenario 3	*move*	*keep*
Scenario 4	*move*	*move*

Across all the four scenarios, recall that player 1 at time τ' keeps at PAP D (and moves at PAP B), since $d_1(\tau^*) > b_1(\tau^*)$.

Scenario 1:

Player 1 uses the equilibrating strategy at τ'. We know that the equilibrating strategy cannot be the best reply for player 2 since $\tau' > \tau^*$. It is easy to verify that the best action for player 2 is actually to keep at PAP C and to move at PAP D. Indeed,

$$\dot{c}_2(\tau^*) = \lambda(a_2(\tau^*) - c_2(\tau^*)) > 0 \quad \text{if player 2 keeps in PAP } C$$
$$\dot{c}_2(\tau^*) = \lambda(a_2(\tau^*) - c_2(\tau^*)) + \lambda(d_2(\tau^*) - c_2(\tau^*)) \quad \text{if player 2 moves in PAP } C$$

while

$$\dot{d}_2(\tau^*) = \lambda(c_2(\tau^*) - d_2(\tau^*)) = 0 \quad \text{if player 2 moves in PAP } D$$
$$\dot{d}_2(\tau^*) = 0 \quad \text{if player 2 keeps in PAP } D$$

Together with $c_2(\tau^*) = d_2(\tau^*)$ this ensures that $c_2(\tau') > d_2(\tau')$. Thus player 2 keeps his action at $x(\tau') = C$ and moves when $x(\tau') = D$.

Scenario 2:

At time τ' player 1 keeps his action in both $x(\tau') = A$ and $x(\tau') = C$. Then the payoffs dynamics is

$$\dot{c}_2(\tau^*) = 0 \quad \text{if player 2 keeps in PAP } C$$
$$\dot{c}_2(\tau^*) = \lambda(d_2(\tau^*) - c_2(\tau^*)) = 0 \quad \text{if player 2 moves in PAP } C$$

and $\dot{d}_2(\tau^*) = 0$ irrespective of the choice of player 2 at $x(\tau') = D$. Given that $c_2(\tau^*) = d_2(\tau^*)$ this in turn implies that $c_2(\tau') = d_2(\tau')$: hence player 2 is indifferent between keeping and moving at τ' when the game is in PAPs C or D.

Scenario 3:

Using the same reasoning as in the previous scenarios, we have:

$$\begin{aligned} \dot{c}_2(\tau^*) &= 0 && \text{if player 2 keeps in PAP } C \\ \dot{c}_2(\tau^*) &= \lambda(d_2(\tau^*) - c_2(\tau^*)) = 0 && \text{if player 2 moves in PAP } C \end{aligned}$$

and for PAP D :

$$\begin{aligned} \dot{d}_2(\tau^*) &= \lambda(c_2(\tau^*) - d_2(\tau^*)) = 0 && \text{if player 2 moves in PAP } D \\ \dot{d}_2(\tau^*) &= 0 && \text{if player 2 keeps in PAP } D \end{aligned}$$

which in turn shows that again player 2 is indifferent between keeping and moving at τ' when the game is in PAPs C or D.

Scenario 4:

Player 1 moves both in $x(\tau') = A$ and $x(\tau') = C$. In this case

$$\begin{aligned} \dot{c}_2(\tau^*) &= \lambda(a_2(\tau^*) - c_2(\tau^*)) > 0 && \text{if player 2 keeps in PAP } C \\ \dot{c}_2(\tau^*) &= \lambda(a_2(\tau^*) - c_2(\tau^*)) + \lambda(d_2(\tau^*) - c_2(\tau^*)) > 0 && \text{if player 2 moves in PAP } C \end{aligned}$$

$$\begin{aligned} \dot{d}_2(\tau^*) &= 0 && \text{if player 2 moves in PAP } D \\ \dot{d}_2(\tau^*) &= \lambda(c_2(\tau^*) - d_2(\tau^*)) = 0 && \text{if player 2 keeps in PAP } D \end{aligned}$$

so that $c_2(\tau') > d_2(\tau')$ and for player 2 the best reply is to keep in PAP C and to move in D.

We can repeat the construction of the four scenarios by symmetry fixing the action of player 2 at $x(\tau') = C$ and at $x(\tau') = D$ and determining the best replies for player 1 in A and C. Putting together the best replies of the two players, we find that the sub-game at τ' has three possible equilibria when the PAP at τ' is C: "keep, keep", "keep, move" and "move, keep". Each of the three equilibria then originates a possible scenario at $\tau' = \tau^* + \epsilon$:

Scenario (i) :

A	B
?; *keep*	?; ?
C	D
keep; *keep*	*keep*; ?

Scenario (ii) :

A	B
?; *keep*	?; ?
C	D
keep; *move*	*keep*; ?

Scenario (iii) :

A	B
?; *keep*	?; ?
C	D
move; *keep*	*keep*; ?

with question marks indicating that the best reply in the respective PAP still has to be characterized.

Starting from PAPs A and D, it is easy to verify that both players are indifferent between keeping and moving their actions when their opponent keeps his own. Indeed, in this case, given that player 2 keeps in PAP A:

$$\dot{a}_1(\tau^*) = 0 \qquad \text{if player 1 keeps in PAP } A$$
$$\dot{a}_1(\tau^*) = \lambda(c_1(\tau^*) - a_1(\tau^*)) = 0 \quad \text{if player 1 moves in PAP } A$$

while, given that player1 keeps in PAP D:

$$\dot{d}_2(\tau^*) = 0 \qquad \text{if player 2 keeps in PAP } D$$
$$\dot{d}_2(\tau^*) = \lambda(c_2(\tau^*) - d_2(\tau^*)) = 0 \quad \text{if player 2 moves in PAP } D$$

There exists then an equilibrium at which both players keep their action when $x(\tau') = A$ and $x(\tau') = D$. As long as this is true, the continuation payoffs $a_i(\tau')$ and $d_i(\tau')$ are constant. Moreover: $a_1(\tau') = 1$, $a_2(\tau') = \alpha$, $d_1(\tau') = \alpha$, $d_2(\tau') = 1$, where α is the maximum attainable payoff for both players. Since $\forall \tau \geq \tau^* : b_1(\tau) < d_1(\tau)$ and $b_2(\tau) < a_2(\tau)$ at equilibrium both players move their action in PAP B. This allows us to show that only one of the following three scenarios may arise at equilibrium at $\tau' > \tau^*$:

Equilibrium (i) :

A $keep; keep$	B $move; move$
C $keep; keep$	D $keep; keep$

Equilibrium (ii) :

A $keep; keep$	B $move; move$
C $keep; move$	D $keep; keep$

Equilibrium (iii) :

A $keep; keep$	B $move; move$
C $move; keep$	D $keep; keep$

The text of the Proposition refers to the equilibrium (i) since the PAP C corresponds to the action each player would play in their preferred UGE. ■

Proof of Lemma 11

Suppose that (i) at some date $t - \Delta$ the history $h(t - \Delta) \in H(t - \Delta, x)$, i.e. the PAP at date $t - \Delta$ is x, and (ii) from time $t - \Delta$ until t players consistently submit action profile y. Then the payoff for player i is

$$\int_{h(t) \in H(t)} \Pr\left(h(t) | h(t - \Delta), y(t') = y \text{ for } t' \in [t - \Delta, t]\right) E\left[u_i(x(T)) | h(t)\right]$$

Dividing this expression by Δ and considering the limit for $\Delta \to 0$, we obtain

$$\begin{aligned}\sum_{z\in X}(\lambda_1 q^1_{xyz} + \lambda_2 q^2_{xyz})\left[u_i(x(T))|h(t,z)\right] &= (\lambda_1 q^1_{xy(y_i,x_{-i})} + \lambda_2 q^2_{xy(y_i,x_{-i})})\left[u_i(x(T))|h(t,(y_i,x_{-i}))\right] \\ &+ \sum_{z\in X, z_i\neq y_i}(\lambda_1 q^1_{xyz} + \lambda_2 q^2_{xyz})\left[u_i(x(T))|h(t,z)\right]\end{aligned}$$

where the equality follows from Assumption 3. Note that the second term on the r.h.s. does not depend on action y_i: hence, after observing history $h(t)$ player i optimally submits the action that maximizes the first term. However, for the hypothesis of the lemma, $y_i^*(x_{-i})$ is the unique maximizer of $E\left[u_i(x(T))|h(t,(y_i,x_{-i}))\right]$ for all $h(t) \in H(t,(.,x_{-i}))$. $\square$