Efficiency and Collusion Neutrality of Solutions for Cooperative TU-Games

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Abstract

Three well-known solutions for cooperative TU-games are the *Shapley value*, the *Banzhaf value* and the *equal division solution*. In the literature various axiomatizations of these solutions can be found. Axiomatizations of the Shapley value often use *efficiency* which is not satisfied by the Banzhaf value. On the other hand, the Banzhaf value satisfies *collusion neutrality* which is not satisfied by the Shapley value. Both properties seem desirable. However, neither the Shapley value nor the Banzhaf value satisfy both. The equal division solution does satisfy both axioms and, moreover, together with symmetry these axioms characterize the equal division solution. Further, we show that there is no solution that satisfies efficiency, collusion neutrality and the null player property. Finally, we show that a solution satisfies efficiency, collusion neutrality and linearity if and only if there exist exogenous weights for the players such that in any game the worth of the 'grand coalition' is distributed proportional to these weights.

Keywords: Efficiency, Collusion neutrality, Shapley value, Banzhaf value, Equal division solution, Impossibility.
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1 Introduction

A situation in which a finite set $N = \{1, \ldots, n\}$ of n players can generate certain payoffs by cooperation can be described by a *cooperative game with transferable utility* (or simply a TU-game), being a pair (N, v) where $v \colon 2^N \to \mathbb{R}$ is a *characteristic function* on N satisfying $v(\emptyset) = 0$. For any *coalition* $S \subseteq N$, $v(S)$ is the *worth* of coalition S, i.e. the members of coalition S can obtain a total payoff of $v(S)$ by agreeing to cooperate.

A *payoff vector* $x \in \mathbb{R}^N$ of an n-player TU-game (N, v) is an n-dimensional vector giving a payoff $x_i \in \mathbb{R}$ to any player $i \in N$. A (single-valued) *solution* for TU-games is a function that assigns a payoff vector to every TU-game (N, v). Three well-known solutions for TU-games are the *Shapley value*, the *Banzhaf value* and the *equal division solution*. In the literature various axiomatizations of these solutions can be found. Most axiomatic characterizations of the Shapley value use efficiency. For example, the original characterization of the Shapley value characterizes it by efficiency, linearity, symmetry and the null player property, see Shapley (1953). Various characterizations of the Banzhaf value use some collusion neutrality axiom, see for example, Lehrer (1988), Haller (1994) and Malawski (2002) who characterize the Banzhaf value by linearity, symmetry, the null player property, the inessential game property and some collusion neutrality property. Collusion neutrality properties state that the sum of payoffs of two players does not change if these two players in some way 'collude' and act as one player. Both efficiency and collusion neutrality seem to be desirable properties. Clearly, by the above mentioned characterizations of the Shapley value and Banzhaf value, there does not exist a solution satisfying efficiency, collusion neutrality, linearity, symmetry and the null player property. A solution that does satisfy both efficiency and collusion neutrality is the *equal division solution*. In fact, we show that together with symmetry these axioms characterize the equal division solution if there are at least three players. Since the equal division solution does not satisfy the null player property, the next question is whether there is a solution that satisfies efficiency, collusion neutrality and the null player property. It turns out that such a solution does not exist for games with at least three players. Finally, we show that a solution satisfies efficiency, collusion neutrality and linearity if and only if there exist exogenous weights for the players such that in any game the worth of the 'grand coalition' is distributed proportional to these weights. Note that this implies that together with symmetry these axioms characterize the equal division solution but, as argued above, these axioms are not logically independent because we do not need linearity.

The paper is organized as follows. In Section 2 we discuss some preliminaries on TU-games. In Section 3 we show that there is a unique solution satisfying efficiency, collusion neutrality and symmetry, which is the equal division solution. In Section 4 we show that there is no

solution satisfying efficiency, collusion neutrality and the null player property. In Section 5 we characterize a class of proportional solutions by efficiency, collusion neutrality and linearity. Finally, in Section 6 we make some concluding remarks.

2 Preliminaries

In this paper we take the set of players $N = \{1, \ldots, n\}$ to be fixed, and therefore denote a TU-game (N, v) just by its characteristic function v. We assume that the game has at least three players[2]. The collection of all characteristic functions (which we will thus refer to as games) on N is denoted by $\mathcal{G}^N$. The increase in worth when player $i \in N$ joins coalition $S \subseteq N \setminus \{i\}$ is called the *marginal contribution* of player i to coalition S in game $v \in \mathcal{G}^N$ and is denoted by

$$m_i^S(v) = v(S \cup \{i\}) - v(S).$$

Assuming that the 'grand coalition' N forms in a way such that the players enter the coalition one by one, the Shapley value assigns to every player its expected marginal contribution to the coalition of players that enter before him given that all orders of entrance have equal probability. Thus, the *Shapley value* (Shapley (1953)) is the solution $Sh: \mathcal{G}^N \to \mathbb{R}^N$ given by

$$Sh_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{(n - |S| - 1)!(|S|)!}{n!} \, m_i^S(v) \text{ for all } i \in N.$$

On the other hand, the Banzhaf value (introduced by Banzhaf (1965) to measure voting power in voting games and generalized by Owen (1975) and Dubey and Shapley (1979) to general TU-games) is the solution $Ba: \mathcal{G}^N \to \mathbb{R}^N$ that assigns to every player its expected marginal contribution given that every combination of the other players has equal probability of being the coalition that is already present when that player enters. Thus, it assigns to every player in a game its average marginal contribution, i.e.

$$Ba_i(v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} m_i^S(v) \text{ for all } i \in N.$$

Players $i, j \in N$ are *symmetric* in game v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. Player $i \in N$ is a *null player* in game v if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$. For $v, w \in \mathcal{G}^N$, the game $(v + w) \in \mathcal{G}^N$ is defined by $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$. For $v \in \mathcal{G}^N$ and $\alpha \in \mathbb{R}$, the game $\alpha v \in \mathcal{G}^N$ is defined by $(\alpha v)(S) = \alpha v(S)$ for all $S \subseteq N$. Haller (1994) introduces some collusion neutrality properties which state that the sum

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[2]We make some remarks on two-player games in the final section.

of payoffs of two players does not change if they 'collude'. He used these properties to axiomatize the Banzhaf value. Later, Malawski (2002) showed that several other collusion neutrality properties can be used. In this paper we consider collusion between two players where they agree to 'act as one' in the sense that they contribute to a coalition only when they both are present. So, when players $i, j \in N$, $i \neq j$, collude in game $v \in \mathcal{G}^N$, then instead of game v we consider the game $v_{ij} \in \mathcal{G}^N$ given by

$$v_{ij}(S) = \begin{cases} v(S \setminus \{i, j\}) & \text{if } \{i, j\} \nsubseteq S \\ v(S) & \text{if } \{i, j\} \subseteq S. \end{cases} \tag{2.1}$$

Finally, a game v is called *inessential* if $v(S) = \sum_{i \in S} v(\{i\})$ for all $S \subseteq N$, i.e. for every player its marginal contribution to any coalition is the same. Various axiomatizations of the Shapley value and the Banzhaf value have been given in the literature. Some axioms that are used in these axiomatic characterizations are the following. A solution $f : \mathcal{G}^N \to \mathbb{R}^N$ satisfies

- *efficiency* if $\sum_{i \in N} f_i(v) = v(N)$ for all $v \in \mathcal{G}^N$;

- *linearity* if $f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)$ for all $v, w \in \mathcal{G}^N$ and $\alpha, \beta \in \mathbb{R}$;

- *symmetry* if $f_i(v) = f_j(v)$ whenever i and j are symmetric players in $v \in \mathcal{G}^N$;

- the *null player property* if $f_i(v) = 0$ whenever i is a null player in $v \in \mathcal{G}^N$;

- the *inessential game property* if $f_i(v) = v(\{i\})$ for all $i \in N$ and inessential games v;

- *collusion neutrality* if $f_i(v_{ij}) + f_j(v_{ij}) = f_i(v) + f_j(v)$ for all $i, j \in N$ and $v \in \mathcal{G}^N$, with v_{ij} given by (2.1).

Most axiomatic characterizations of the Shapley value use efficiency. For example, the original characterization in Shapley (1953) characterizes it by efficiency, linearity, symmetry and the null player property[3]. The Banzhaf value satisfies linearity, symmetry and the null player property, but it is not efficient. Malawski (2002) characterized the Banzhaf value by linearity, symmetry, the null player property, the inessential game property and collusion neutrality[4].

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[3]This axiomatization is more often presented in this way although Shapley (1953) combines efficiency and the null player property into a *carrier* axiom.

[4]As mentioned, other collusion neutrality properties that are used to axiomatize the Banzhaf value are stated in Haller (1994). The results in this paper also can be stated in terms of those neutrality properties. An axiomatization of the Banzhaf value with collusion properties in terms of inequalities can be found in Lehrer (1988).

In van den Brink (2007) it is shown that in several axiomatizations of the Shapley value, replacing an axiom about null players (such as the null player property) by a similar axiom about nullifying players (being players whose presence in a coalition implies that the worth of the coalition is zero) yields axiomatic characterizations of the *equal division solution* $ED: \mathcal{G}^N \to \mathbb{R}^N$ which is given by

$$ED_i(v) = \frac{v(N)}{n} \text{ for all } i \in N.$$

In this paper we find another axiomatization of the equal division solution by combining axioms that characterize the Shapley value and the Banzhaf value. The proof of this axiomatization uses the unanimity basis for TU-games. The *unanimity game* of coalition $T \subseteq N$, $T \neq \emptyset$, is the game $u_T \in \mathcal{G}^N$ given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. It is well-known that the set of unanimity games form a basis of $\mathcal{G}^N$: every game $v \in \mathcal{G}^N$ can be written as a linear combination of unanimity games $v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) u_T$ with $\Delta_v(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S)$ being the Harsanyi *dividends*, see Harsanyi (1959).

3 Dropping the null player property: an axiomatization of the equal division solution

Both efficiency and collusion neutrality seem to be desirable properties. Clearly, by axiomatizations of the Shapley value and the Banzhaf value mentioned before, there does not exist a solution satisfying efficiency, collusion neutrality, linearity, symmetry and the null player property. It turns out that for games with at least three players, dropping the null player property yields a characterization of the equal division solution. We can even state a stronger characterization result without linearity.

Theorem 1 *A solution $f: \mathcal{G}^N \to \mathbb{R}^N$ satisfies efficiency, collusion neutrality and symmetry if and only if it is the equal division solution.*

PROOF
It is easy to verify that the equal division solution satisfies the three properties. To show uniqueness, we proceed by induction on the smallest cardinality of the coalitions with non-zero dividend in a game. Before starting to show uniqueness, we introduce the following notation. For any game $v \in \mathcal{G}^N$, define $H(v) = \{T \subseteq N \mid \Delta_v(T) \neq 0\}$ as the set of coalitions with non-zero dividend, $d(v) = \min_{T \in H(v)} |T|$ as the smallest cardinality of coalitions with non-zero dividend, and $h(v) = |\{T \in H(v) \mid d(v) = |T|\}|$ as the number of coalitions of smallest cardinality with non-zero dividend.

Next we start the proof of uniqueness. Consider game $v \in \mathcal{G}^N$. First, if $d(v) = |N|$ then v is a scaled unanimity game of the 'grand coalition', i.e. $v = \Delta_v(N)u_N$ with $\Delta_v(N) \neq 0$. (Note that in this case $\Delta_v(N) = v(N)$.) Symmetry implies that all players earn the same payoff. Efficiency then determines that $f_i(v) = \frac{\Delta_v(N)}{n} = \frac{v(N)}{n} = ED_i(v)$ for all $i \in N$.

Proceeding by induction, assume that $f(v')$ is uniquely determined for all $v' \in \mathcal{G}^N$ with $d(v') > d(v)$.

In order to use collusion neutrality, we use the following result from Malawski (2002) concerning collusion between two players in unanimity games. For every coalition $R \subseteq N$ and two players $i, j \in N$, $i \neq j$, it holds that

$$(u_R)_{ij} = \begin{cases} u_R & \text{if either } [i, j \in R] \text{ or } [i, j \in N \setminus R] \\ u_{R \cup \{i,j\}} & \text{otherwise.} \end{cases} \tag{3.2}$$

We now start induction on $h(v)$. First, assume that $h(v) = 1$. Then there is a unique $T \in H(v)$ with $|T| = d(v)$, i.e. T is the (unique) smallest cardinality coalition with non-zero dividend. Take a specific $j \in T$ and $h \in N \setminus T$. Collusion neutrality implies that

$$f_i(v) + f_j(v) = f_i(v_{ij}) + f_j(v_{ij}) \text{ for all } i \in N \setminus T, \tag{3.3}$$

and

$$f_h(v) + f_g(v) = f_h(v_{hg}) + f_g(v_{hg}) \text{ for all } g \in T \setminus \{j\}, \tag{3.4}$$

while efficiency requires that

$$\sum_{i \in N} f_i(v) = v(N). \tag{3.5}$$

By (3.2) it follows that

(i) $(u_T)_{ij} = u_{T \cup \{i\}}$ for all $i \in N \setminus T$,

(ii) $(u_R)_{ij}$ is either equal to u_R or $u_{R \cup \{i,j\}}$ for every $R \subseteq N$ and $i \in N \setminus T$,

(iii) $(u_T)_{hg} = u_{T \cup \{h\}}$ for all $g \in T \setminus \{j\}$, and

(iv) $(u_R)_{hg}$ is either equal to u_R or $u_{R \cup \{h,g\}}$ for every $R \subseteq N$ and $g \in T \setminus \{j\}$.

Therefore, $d(v_{ij}) > d(v)$ for all $i \in N \setminus T$, and $d(v_{hg}) > d(v)$ for all $g \in T \setminus \{j\}$, and thus by the induction hypothesis, the equations (3.3) and (3.4) become

$$f_i(v) + f_j(v) = ED_i(v_{ij}) + ED_j(v_{ij}) = \frac{2v_{ij}(N)}{n} = \frac{2v(N)}{n} \text{ for all } i \in N \setminus T, \tag{3.6}$$

and

$$f_h(v) + f_g(v) = ED_h(v_{hg}) + ED_g(v_{hg}) = \frac{2v_{hg}(N)}{n} = \frac{2v(N)}{n} \text{ for all } g \in T \setminus \{j\}. \quad (3.7)$$

Since the $1 + (n - |T|) + (|T| - 1) = n$ equations given by (3.5), (3.6) and (3.7) are linearly independent[5], the values $f_i(v)$, $i \in N$, are uniquely determined and given by $f_i(v) = \frac{v(N)}{n} = ED_i(v)$.

Next, proceeding by induction on $h(v)$, assume that the result holds for every $v' \in \mathcal{G}^N$ with $d(v') \geq d(v)$ and $h(v') < h(v)$. Similar as above, take a $T \in H(v)$ with $|T| = d(v)$. (Note that now there are $h(v) > 1$ of such coalitions.) Collusion neutrality, the induction hypothesis and efficiency yield the same equations (3.5), (3.6) and (3.7), and it can be similarly shown that $f(v)$ is uniquely determined by $f(v) = ED(v)$ whenever $H(v) \neq \emptyset$.

Finally, we have to consider game $v \in \mathcal{G}^N$ with $H(v) = \emptyset$. Then, v is the *null game*, i.e. $v(S) = 0$ for all $S \subseteq N$. Symmetry implies that all players earn the same payoff. With efficiency it then follows that $f_i(v) = 0 = ED_i(v)$ for all $i \in N$. $\qquad \square$

Logical independence of the axioms of Theorem 1 is shown by the following solutions.

1. The Shapley value satisfies efficiency and symmetry. It does not satisfy collusion neutrality.

2. The Banzhaf value satisfies collusion neutrality and symmetry. It does not satisfy efficiency.

3. The solution $\overline{f}: \mathcal{G}^N \to \mathbb{R}^N$ that assigns all payoff to player 1, i.e. $\overline{f}_1(v) = v(N)$ and $\overline{f}_i(v) = 0$ for all $i \in N \setminus \{1\}$, satisfies efficiency and collusion neutrality. It does not satisfy symmetry.

4 Dropping symmetry: an impossibility

In the previous section we saw that dropping the null player property from our set of desirable properties yields an axiomatization of the equal division solution. The next question is what happens if we drop symmetry instead of the null player property. It turns out that there is no solution satisfying efficiency, collusion neutrality and the null player property. Note that for this impossibility result we do not need linearity.

Theorem 2 *There is no solution on $\mathcal{G}^N$ satisfying efficiency, collusion neutrality and the null player property.*

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[5]This follows from some easy but tedious computations.

By contradiction, suppose that solution $f: \mathcal{G}^N \to \mathbb{R}^N$ satisfies efficiency, collusion neutrality and the null player property.

Consider unanimity games u_T, $\emptyset \neq T \subseteq N$. The null player property implies that

$$f_k(u_T) = 0 \text{ for all } k \in N \setminus T, \tag{4.8}$$

and with efficiency it then follows that

$$\sum_{k \in T} f_k(u_T) = 1. \tag{4.9}$$

Thus, if $|T| = 1$, i.e. $T = \{i\}$ for some $i \in N$, then the null player property and efficiency determine the payoffs $f_j(u_T) = 0$ for $j \in N \setminus \{i\}$ and $f_i(u_T) = 1$.

If $|T| = 2$, i.e. $T = \{i, j\}$ for some $i, j \in N$ with $i \neq j$, then the null player property and efficieny imply that $f_h(u_T) = 0$ for $h \in N \setminus \{i, j\}$, and $f_i(u_T) + f_j(u_T) = 1$. (Note that the same is implied by collusion neutrality and the case $|T| = 1$ considered above.)

Next consider the case $|T| = 3$, i.e. $T = \{i, j, h\}$ with i, j and h being three different players. (Recall that we only consider classes of games with at least three players.) By (4.9) we have that $f_i(u_T) + f_j(u_T) + f_h(u_T) = 1$. In order to apply collusion neutrality, notice that $\left(u_{\{i,j\}}\right)_{hi} = \left(u_{\{i,j\}}\right)_{hj} = \left(u_{\{i,h\}}\right)_{ji} = \left(u_{\{i,h\}}\right)_{jh} = \left(u_{\{j,h\}}\right)_{ij} = \left(u_{\{j,h\}}\right)_{ih} = u_T$. Then, collusion neutrality implies that

$$f_i(u_T) + f_j(u_T) = f_i(u_{\{i,h\}}) + f_j(u_{\{i,h\}}) = f_i(u_{\{j,h\}}) + f_j(u_{\{j,h\}}) \tag{4.10}$$

$$f_i(u_T) + f_h(u_T) = f_i(u_{\{i,j\}}) + f_h(u_{\{i,j\}}) = f_i(u_{\{j,h\}}) + f_h(u_{\{j,h\}}) \tag{4.11}$$

and

$$f_j(u_T) + f_h(u_T) = f_j(u_{\{i,j\}}) + f_h(u_{\{i,j\}}) = f_j(u_{\{i,h\}}) + f_h(u_{\{i,h\}}). \tag{4.12}$$

By (4.8) we have that this system of equations can be reduced to

$$f_i(u_T) + f_j(u_T) = f_i(u_{\{i,h\}}) = f_j(u_{\{j,h\}}) \tag{4.13}$$

$$f_i(u_T) + f_h(u_T) = f_i(u_{\{i,j\}}) = f_h(u_{\{j,h\}}) \tag{4.14}$$

and

$$f_j(u_T) + f_h(u_T) = f_j(u_{\{i,j\}}) = f_h(u_{\{i,h\}}). \tag{4.15}$$

It follows that $f_i(u_T) + f_j(u_T) = f_i(u_{\{i,h\}}) = 1 - f_h(u_{\{i,h\}}) = 1 - f_j(u_T) - f_h(u_T)$, where the first equality follows from (4.13), the second equality follows from (4.9) and the third equality follows from (4.15). By (4.9) we then have that $f_j(u_T) = 0$.

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Similar we can derive from the other equations that $f_i(u_T) = f_h(u_T) = 0$, which contradicts efficiency. So, there does not exist a solution satisfying efficiency, collusion neutrality and the null player property[6]. □

5 Dropping the null player property and symmetry: a characterization of a class of proportional solutions

From the original axiomatization of the Shapley value given by Shapley (1953) combined with collusion neutrality, we might still consider what solutions are left if we drop both the null player property and symmetry. It turns out that in that case we are left with a class of *proportional solutions* for which there exist exogenous weights for the players such that in any game the worth of the 'grand coalition' is distributed proportional to these weights[7]. Define $X^N := \{\lambda \in \mathbb{R}^N \mid \sum_{i \in N} \lambda_i = 1\}$. For $\lambda \in X^N$ we define

$$f_i^\lambda(v) = \lambda_i v(N) \text{ for all } i \in N. \tag{5.16}$$

Theorem 3 *A solution* $f: \mathcal{G}^N \to \mathbb{R}^N$ *satisfies efficiency, collusion neutrality and linearity if and only if there exists a vector of weights* $\lambda \in X^N$ *such that* $f = f^\lambda$.

Proof
It is easy to verify that f^λ, $\lambda \in X^N$, satisfies the three properties. To show uniqueness, assume that solution $f: \mathcal{G}^N \to \mathbb{R}^N$ satisfies the three properties. We first prove uniqueness for unanimity games u_T, $\emptyset \neq T \subseteq N$. We do this by induction on $|T|$. If $|T| = n$, i.e. $T = N$, then efficiency implies that there exists a vector $\lambda \in X^N$ such that $f(u_N) = \lambda = f^\lambda(u_N)$.
Proceeding by induction, suppose that $f(u_{T'}) = \lambda$ for all $T' \subseteq N$ with $|T'| > |T|$. Take a specific $j \in T$ and $h \in N \setminus T$. Since $(u_T)_{ij} = u_{T \cup \{i\}}$ when $i \in N \setminus T$, collusion neutrality implies that

$$f_i(u_T) + f_j(u_T) = f_i(u_{T \cup \{i\}}) + f_j(u_{T \cup \{i\}}) \text{ for all } i \in N \setminus T \tag{5.17}$$

and since $(u_T)_{hg} = u_{T \cup \{h\}}$ when $g \in T \setminus \{j\}$, collusion neutrality also implies that

$$f_h(u_T) + f_g(u_T) = f_h(u_{T \cup \{h\}}) + f_g(u_{T \cup \{h\}}) \text{ for all } g \in T \setminus \{j\}. \tag{5.18}$$

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[6]One also obtains the impossibility by showing directly that the system of six equations (4.10), (4.11) and (4.12) together with the four efficiency equalities $f_i(u_T) + f_j(u_T) + f_h(u_T) = f_i(u_{\{i,j\}}) + f_j(u_{\{i,j\}}) = f_i(u_{\{i,h\}}) + f_h(u_{\{i,h\}}) = f_j(u_{\{j,h\}}) + f_h(u_{\{j,h\}}) = 1$ has no solution.

[7]Although we use linearity, this theorem also can be stated by using the weaker additivity axiom.

With the induction hypothesis the above two equations yield

$$f_i(u_T) + f_j(u_T) = \lambda_i + \lambda_j \text{ for all } i \in N \setminus T \tag{5.19}$$

and

$$f_h(u_T) + f_g(u_T) = \lambda_h + \lambda_g \text{ for all } g \in T \setminus \{j\}. \tag{5.20}$$

Then

$$
\begin{aligned}
\sum_{i \in N} f_i(u_T) &= \sum_{g \in T \setminus \{j\}} f_g(u_T) + f_j(u_T) + \sum_{i \in N \setminus (T \cup \{h\})} f_i(u_T) + f_h(u_T) \\
&= \sum_{g \in T \setminus \{j\}} (\lambda_h + \lambda_g - f_h(u_T)) + f_j(u_T) + \sum_{i \in N \setminus (T \cup \{h\})} (\lambda_i + \lambda_j - f_j(u_T)) + f_h(u_T) \\
&= \sum_{g \in T \setminus \{j\}} \lambda_g + (|T| - 1)(\lambda_h - f_h(u_T)) + f_j(u_T) \\
&\quad + \sum_{i \in N \setminus (T \cup \{h\})} \lambda_i + (n - |T| - 1)(\lambda_j - f_j(u_T)) + f_h(u_T) \\
&= \sum_{i \in N} \lambda_i + (|T| - 2)(\lambda_h - f_h(u_T)) + (n - |T| - 2)(\lambda_j - f_j(u_T)),
\end{aligned}
$$

where the second equality follows from (5.19) and (5.20). With efficiency it follows that $\sum_{i \in N} f_i(u_T) = 1 = \sum_{i \in N} \lambda_i$, and thus

$$(|T| - 2)(\lambda_h - f_h(u_T)) + (n - |T| - 2)(\lambda_j - f_j(u_T)) = 0. \tag{5.21}$$

For $j \in T$ and $h \in N \setminus T$, (5.19) yields that

$$f_j(u_T) + f_h(u_T) = \lambda_j + \lambda_h. \tag{5.22}$$

Solving (5.21) and (5.22) yields $f_j(u_T) = \lambda_j$, $f_h(u_T) = \lambda_h$, and with (5.19) and (5.20) this yields that $f_i(u_T) = \lambda_i$ for all $i \in N$.

Since every game $v \in \mathcal{G}^N$ can be written as a linear combination of unanimity games $v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) u_T$, with $\Delta_v(T) = \sum_{S \subseteq T} (-1)^{|T| - |S|} v(S)$ being the Harsanyi dividends, uniqueness for arbitrary $v \in \mathcal{G}^N$ then follows since linearity of f implies that $f_i(v) = f_i(\sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) u_T) = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) f_i(u_T) = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \lambda_i \Delta_v(T) = \lambda_i \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) = \lambda_i v(N)$ for all $i \in N$. $\qquad \square$

Note that as a corollary it follows that adding symmetry to the axioms of Theorem 3 yields that all weights λ_i should be equal, and thus characterizes the equal division solution.

However, these axioms are not logically independent. In Section 3 we showed that for this characterization linearity is not necessary since it is sufficient to require efficiency, collusion neutrality and symmetry.

Also note the difference between the solutions characterized in this section for TU-games and the (unique) proportional solution for bankruptcy problems. In that solution the total estate is distributed proportionally to the claims of the agents, while in the proportional solutions defined by (5.16) for TU-games, the weights that are used are exogenous and need not be related to the game.

Logical independence of the axioms of Theorem 3 is shown by the following solutions.

1. The Shapley value satisfies efficiency and linearity. It does not satisfy collusion neutrality.

2. The Banzhaf value satisfies collusion neutrality and linearity. It does not satisfy efficiency.

3. The solution $\widehat{f} \colon \mathcal{G}^N \to \mathbb{R}^N$ given by $\widehat{f}(v) = \overline{f}(v)$ if $v(N) \leq 10$ and $\widehat{f}(v) = ED(v)$ if $v(N) > 10$, with $\overline{f}$ as given at the end of Section 3, satisfies efficiency and collusion neutrality. It does not satisfy linearity.

A final question we consider is what solutions satisfy efficiency and collusion neutrality. The answer is that these are kind of proportional solutions, but the weights λ that determine what share in the worth of the grand coalition the players get depends on the worth of the grand coalition.

Theorem 4 *A solution f satisfies efficiency and collusion neutrality if and only if there is a function $L \colon \mathbb{R} \to X^N$ such that $f = f^{L(v(N))}$.*

So, in two games $v, w \in \mathcal{G}^N$ with $v(N) = w(N)$ the payoff distributions are the same. The proof of this theorem goes along similar lines as the proof of Theorem 1 and is therefore omitted.

6 Concluding remarks

In this paper we have studied the possibilities of having solutions for TU-games that satisfy efficiency and collusion neutrality. We have seen that for games with at least three players, additionally requiring the symmetry property characterizes the equal division solution. So, the equal division solution is the unique symmetric solution that satisfies these two properties (that distinguish the Shapley value and Banzhaf value from each

other). Additionally requiring the null player property instead of symmetry yields an impossibility. In both results linearity of the solution is not necessary. Other relations between the Shapley value and equal division solution are given in van den Brink (2007). Finally, we showed that a solution satisfies efficiency, collusion neutrality and linearity if and only if there exist exogenous weights for the players such that in any game the worth of the 'grand coalition' is distributed proportional to these weights. We summarize these results in Table 1.

Properties/Solutions	Sh	Ba	ED	$f^{\lambda}, \lambda \in X^{N}$	Impossibility
Efficiency	x		x	x	x
Collusion neutrality		x	x	x	x
Symmetry	x	x	x		
Null player property	x	x			x
Linearity	x	x		x	

Table 1: Characterizing properties of solutions

We remark that collusion neutrality can be replaced by other (but similar) axioms that reflect collusion between two players going to 'act' as one, see also Footnote 4.

Note that the multiplicative normalization of the Banzhaf value (i.e. dividing the worth of the 'grand coalition' proportional to the Banzhaf values of the players) as axiomatized in van den Brink and van der Laan (1998) does not satisfy collusion neutrality nor linearity, while the additive normalization of the Banzhaf value (i.e. adding or subtracting from the Banzhaf value of every player the same amount to obtain an efficient payoff vector) as considered in Ruiz, Valenciano and Zarzuelo (1998) does not satisfy collusion neutrality nor the null player property.

In the proofs of Theorems 1 and 2 we needed the player set to contain at least three players. For two-player games there are more solutions satisfying efficiency, collusion neutrality and symmetry (and linearity). An example is the Shapley value which on the class of one- and two-player games is equal to the Banzhaf value. Moreover, for two player games there exist solutions satisfying efficiency, collusion neutrality and the null player property (and linearity), which is again illustrated by the Shapley value.

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