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# Simultaneous Pooled Auctions with Multiple Bids and Preference Lists

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**Abstract.** A simultaneous pooled auction with multiple bids and preference lists is a way to auction multiple objects, in which bidders simultaneously express a bid for each object and a preference ordering over which object they would like to get in case they have the highest bid on more than one object. This type of auction has been used in the Netherlands and in Ireland to auction available spectrum. We show that this type of auction does not satisfy elementary desirable properties such as the existence of an efficient equilibrium.

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## 1. Introduction

In the past few decades, it became more and more common that governments used auctions to allocate scarce resources such as spectrum for mobile communication or radio broadcasting, petrol station locations, telephone numbers, *etc.* Given the official goals of various allocation procedures, governments not always had a “lucky hand” in choosing the right auction design (see, *e.g.*, Klemperer, 2002, for a review). This paper adds to the list of unfortunate auction designs by analyzing the theoretical properties of an auction which properties were not yet known.

The allocation mechanism we study has been used in practice at least twice. The first time, it was used for allocating licenses for commercial radio stations in The Netherlands in 2003 (see Ministry of Education, Culture and Science, 2003, for the precise rules of the allocation mechanism used in The Netherlands). In 2005, it was used in Ireland to allocate licenses for wideband digital mobile data services (the auction documents of the Commission for Communications Regulation, ComReg, are confidential; the media release of the outcome of the auction, reference number PR211205, can be found on the website of ComReg, <http://www.comreg.ie>).

In the two auctions, multiple (possibly heterogeneous) licenses were allocated. If licenses differed, they differed in terms of their coverage (*i.e.*, the number of consumers reached) and as all bidders preferred a larger coverage, all bidders had the same ranking of licenses. Each firm was allowed to acquire at most one license. The auction format was sealed-bid, and firms could express different bids for different licenses. The firms also had to submit a list specifying their respective preferences over the licenses at stake. These preference lists played a role when a firm had submitted highest bids for several licenses. Each winning firm paid its own bid for the license it acquired. This allocation mechanism can be best described as a simultaneous pooled auction with multiple bids and preference lists.

In this paper, we show that this auction format fails to produce one of the most basic and desirable properties of an allocation mechanism, namely that it has an efficient equilibrium. In other words, the licenses do not always end up in the hands of those who value them the most. The reason for this result is as follows. Allocation efficiency requires that all bidders follow the same (symmetric) monotonically increasing (pure) bidding strategy. This implies that if an efficient equilibrium exists, the bidder with the highest possible valuation must submit the highest bids for all

objects, and he takes the most preferred one. However, this bidder can potentially increase his expected profit by changing his most preferred object and, at the same time, significantly reducing the bid for that object. In this deviation, the bidder's equilibrium (high) bid for his equilibrium (old) most preferred object remains the highest and, therefore, guarantees him his equilibrium pay-off. The bidder will obtain his equilibrium pay-off if the reduced bid for the 'new' most preferred object is not the highest. However, if the reduced bid turns out to be the highest bid, the bidder obtains his 'new' most preferred object for a very low price.

There are some indications that the outcome of the Dutch allocation mechanism was inefficient. A first indication is that not long after the auction was held, quite a few licenses were resold to third parties. Had the licenses ended up in the hands of those parties that valued them the most, reselling (not long after the auction) should not have taken place.<sup>1</sup> A second indication is that one of the licenses with a specific format requirement (these licenses were auctioned separately from the licenses for unrestricted programming at the same moment in time), was sold for a higher amount than the cheapest license for unrestricted programming (presumably, a more valuable license).<sup>2</sup>

This paper relates to a number of areas in the economic literature. First, it relates to the literature on simultaneous pooled auctions (see, *e.g.*, Menezes and Monteiro, 1998) in which, in contrast to the present paper, bidders are only allowed to submit a non-earmarked single bid for one of the objects in the pool. As bidders are uncertain about which object from the pool of objects they are going to win and are only allowed to submit a single bid, bidders may fall prey to some sort of "winner's curse". Salmon and Iachini (2007) experimentally show that bidders often overbid and incur losses because they are forced to buy objects that are not their most preferred objects. In the mechanism analyzed in the present paper, bidders do not suffer from this unexpected loss because they are allowed to submit as many bids as objects. Menezes and Monteiro (1998) show that in the homogeneous private-value case with risk-neutral bidders, simultaneous pooling auctions are revenue-equivalent to a first-price sealed-bid sequential auction.

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<sup>1</sup> In particular, Noordzee FM (Talpa Radio International) was sold to De Persgroep on 31 May 2005, Radio 538 (Advent International Corporation) to Talpa Radio International on 31 May 2005, Yorin FM (RTL Nederland) to SBS Broadcasting on 4 January 2006 and Sky Radio (News Corporation) to TMG (Telegraaf Media Groep) on 1 February 2006.

<sup>2</sup> Of course, it is difficult to be sure that the auction was indeed inefficient because the presence of economic inefficiency is difficult to test statistically given the data available.

This paper also relates to the literature on the so-called ‘right-to-choose’ auctions. A right-to-choose auction, which is also referred to as a sequential pooled auction or “condo” auction (as it is being used in selling condominiums in the United States), consists of a sequence of regular auctions in which bidders bid for the right to choose any object among the objects not yet sold. Burguet (2005) shows that ascending right-to-choose auctions, *i.e.*, right-to-choose auctions that consist of a sequence of regular English auctions, for two *ex-ante* symmetric objects are efficient. Gale and Hausch (1994) derive the same conclusion for a two-bidder model with more general preferences than in Burguet (2005). Goeree *et al.* (2004) introduce bidders’ risk-aversion into Burguet’s (2005) model. They show that ascending right-to-choose auctions raise more revenue than standard simultaneous ascending auctions. Eliaz *et al.* (2008) examine second-price sealed-bid right-to-choose auctions. They show that in thin markets where there is little interest per object both theoretically and experimentally the second-price sealed-bid right-to-choose auction raises more revenue than sequential auctions for the individual objects. They also provide experimental evidence that a right-to-choose auction can generate even more revenue than a theoretically optimal auction. Moreover, in contrast to the optimal auction, the right-to-choose auction is ‘approximately’ efficient in the sense that the surplus it generates is close to the maximal one.

Finally, the paper is related to the literature on the efficiency properties of auctions. Moldovanu and Sela (2003) show that standard auction mechanisms may lead to inefficient allocations if values are strongly interdependent. Janssen and Karamychev (2007) show that even if the externality (interdependence) is weak, efficient equilibria may fail to exist if the bidders’ types are strongly *ex-ante* correlated (affiliated). The present paper, in contrast, shows that simultaneous pooling auctions with multiple bids and preference lists can be inefficient even in the *independent private valuation* setting. The paper can also be related to the literature on price dispersion. In Subsection 3.2 we show that even if objects are perfect substitutes, firms bid for them differently.

The rest of the paper is organized as follows. In Section 2, we set up the model, which contains the key features of the design of auctions held in The Netherlands and in Ireland. In Section 3, we look at efficient Nash equilibria of the model and analyze their existence conditions. In particular, Subsection 3.1 analyzes the model with heterogeneous objects, and Subsection 3.2 analyzes the model with homogeneous objects. Section 4 concludes.

## 2. The model

There are two objects<sup>3</sup> for sale and  $N > 2$  bidders. Bidders are allowed to win at most one object. Bidder  $i$  assigns a value  $v_i$  to object 1 and a value  $\alpha v_i$  to object 2, where  $\alpha \in (0,1]$  is common for all bidders.<sup>4</sup> Valuations  $v_i$  are independently and identically distributed over the unit interval  $[0, 1]$  according to the distribution function  $F$ . The value of  $v_i$  is private information to bidder  $i$ . The values of  $\alpha$  and  $N$ , and the distribution function  $F$  are common knowledge.

If  $\alpha < 1$ , the goods are heterogeneous and the first object is preferred by all bidders to the second. The ratio of valuations for the two objects is identical for all bidders. This seems to capture the essence of the Dutch radio frequency auction quite well where the value of a license is directly related to the demographic coverage of a license. The licenses in the Dutch radio frequency auction differed in their demographic coverage. If the coverage of the license increases and a firm attracts a certain percentage of the population, then the total number of listeners (hence, firm's valuations for licenses) is proportional to that coverage. If licenses are *ex-ante* identical in terms of their demographic coverage, they can be analyzed by the model with homogeneous objects, where  $\alpha = 1$ . In what follows, we do not consider asymmetric auctions where different bidders are characterized by different values of  $\alpha$  as in case of asymmetries, a general argument can be easily invoked to establish the inefficiency of the auction.<sup>5</sup>

Every bidder  $i$  submits two bids,  $b_i^1$  and  $b_i^2$  in a sealed envelope, one for every object, and states his preference over the two objects in the event that both his bids turn out to be the highest. The preference is expressed in terms of a probability distribution over the two objects and is represented by  $p_i$ , the probability of taking object 1.

The auctioneer collects all triples  $(b_i^1, b_i^2, p_i)$  from all bidders and determines the highest bids for every object. If these highest bids belong to different bidders, these

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<sup>3</sup> The analysis for more than two objects is very similar and for simplicity in notation, we therefore concentrate on the two-object case.

<sup>4</sup> We assume linearity for simplicity; a common monotonically increasing scalar function would yield similar conclusions only adding to the notational complexity.

<sup>5</sup> If bidders are asymmetric, their types are drawn from different distributions or the ratios of their valuations for the two objects are different. Efficient equilibria do not exist in either one of these cases as efficiency requires that bidding functions for different bidders must be identical (players with higher valuations must bid higher), while asymmetry requires different bidders to use different bidding functions (as the distribution of valuations of a bidder's competitors has an impact on the bidder's equilibrium bidding functions).

bidders win the objects for which they are the highest bidders, and they pay their winning bid as a price. If, however, it is one and the same bidder  $j$  who has submitted the highest bids for both objects, then this bidder gets object 1 with probability  $p_j$  and object 2 with probability  $(1 - p_j)$ . Bidder  $j$  pays his bid for the object that he gets. The other object goes to the bidder who has submitted the second highest bid for that object. This bidder also pays his bid as a price.

### 3. Analysis

We will search for efficient Nash equilibria of this game, *i.e.*, equilibria in which the two bidders with the two highest valuations win the objects, and, furthermore, in case  $\alpha < 1$ , the bidder with the highest value wins object 1 (the most valuable object) and the bidder with the second highest value wins object 2. As equilibrium efficiency requires that bidders follow a symmetric monotonically increasing bidding strategy, we focus only on such equilibria. We distinguish two cases.

In Subsection 3.1, we assume that  $\alpha < 1$  so that the objects are heterogeneous. In an efficient monotone symmetric equilibrium, if it exists, each bidder  $i$  with valuation  $v_i$  submits a bid  $b_i^1 = b^1(v_i)$  for object 1, a bid  $b_i^2 = b^2(v_i)$  for object 2, and sets his preferences for object 1, *i.e.*,  $p_i = 1$ .

In Subsection 3.2, we assume that the objects are homogeneous. If bidders are not able to coordinate their bids  $b_i^1$  on one object and their bids  $b_i^2$  on the other object, a different type of equilibrium may emerge. In a symmetric monotone bidding equilibrium, each bidder  $i$  places both his bids on both objects with equal probability, in the spirit of the strategic uncertainty assumption of Crawford and Haller (1990). If an efficient equilibrium exists, each bidder  $i$  sets his preferences for the object on which he submits the lowest bid. As all  $N$  bidders place their bids on both objects independently of each other, each of  $2^N$  possible distributions of  $2N$  bids across two objects occurs with equal probability. This equilibrium can alternatively be viewed as a symmetric mixed-strategy bidding equilibrium.



### 3.1. Heterogeneous objects

We first show that for any  $\alpha < 1$  there is no efficient Nash equilibrium if the number of bidders  $N$  is sufficiently large. The intuition is as follows. In an efficient equilibrium, a bidder  $i$  with the highest possible valuation  $v_i = 1$  submits the highest bid on both objects with certainty and wins object 1, because the equilibrium efficiency requires that  $p_i = 1$ . By significantly reducing his bid on object 2 and making this object his most preferred choice by submitting  $p_i = 0$ , he can increase his expected pay-off (due to a higher surplus in the event he is still the highest bidder on object 2), which constitutes a profitable deviation. As the realized profit from such a deviation is strictly positive and independent of the number of bidders  $N$  whereas the profit in the proposed efficient equilibrium asymptotically decreases to zero, a larger number of bidders  $N$  makes the deviation relatively more profitable.

**Proposition 1.** For any  $\alpha \in (0,1)$ , there exist a number  $\hat{N}(\alpha)$  such that for all  $N > \hat{N}$ , no pure strategy monotone symmetric equilibrium exists.

**Proof.** First of all, in any existing equilibrium the surplus of every type must converge to zero when  $N \rightarrow \infty$ . This can be seen as follows. In a symmetric equilibrium, a bidder has the highest bid on both objects or on neither object. If a bidder  $j$  has a valuation  $v_j < 1$ , then the probability that he wins any of the objects converges to zero when  $N \rightarrow \infty$ . Consequently, the *ex-ante* expected surplus of bidder  $j$  also converges to zero with  $N$ . If, however,  $v_j = 1$  then the winning probability is equal to 1 for any number of bidders, and bidder  $j$ 's expected surplus is equal to the surplus of his most preferred object revealed by  $p_i$ , *i.e.*, the largest of the two surpluses,  $1 - b_j^1$  or  $\alpha - b_j^2$ . If it were that  $1 - b_j^1 > \varepsilon > 0$  for all  $N$ , then the bidder  $k$  with value  $v_k = 1 - \varepsilon/3$  (who is receiving asymptotically zero expected surplus in equilibrium as we explained above) would have got, by bidding  $b_k^1 = b_j^1 + \varepsilon/3$ , a strictly positive surplus of  $v_k - b_k^1 = (1 - \varepsilon/3) - (b_j^1 + \varepsilon/3) > \varepsilon/3 > 0$ . If, on the other hand, it were that  $\alpha - b_j^2 > \varepsilon > 0$  for all  $N$ , then the bidder  $k$  with value  $v_k = 1 - \varepsilon/(3\alpha)$  would have got, by bidding  $b_k^2 = b_j^2 + \frac{1}{3}\varepsilon$ , a strictly positive surplus of

$\alpha v_k - b_k^2 = \alpha(1 - \varepsilon/(3\alpha)) - (b_j^2 + \varepsilon/3) > \varepsilon/3 > 0$ . In both cases, there is a bidder  $k$  who can profitably deviate. Hence, for any  $\varepsilon > 0$ ,  $1 - b_j^1 < \varepsilon$  and  $\alpha - b_j^2 < \varepsilon$  if  $N$  is taken to be large enough.

Next, let us consider a bidder  $j$  with valuation  $v_j = 1$ , who in an efficient equilibrium submits the highest bid on both objects and surely wins his most preferred object, *i.e.*, object 1 (as efficiency requires  $p_i = 1$ ). His surplus from this object converges to zero when  $N \rightarrow \infty$  (as shown above). Hence, there exists a number  $\hat{N}(\alpha)$  such that  $(1 - b_j^1) < \alpha/2$  for all  $N > \hat{N}(\alpha)$ . Bidder  $j$  can profitably deviate by bidding  $\gamma < \alpha/2$  for object 2, and submitting  $p_i = 0$  though. He is then still the bidder with the highest bid on object 1, which assures him his equilibrium profit. In addition, with a small probability, he has the highest bid on object 2 as well. In that case he wins object 2 at price  $\gamma$ . For all  $N > \hat{N}(\alpha)$  this deviation is profitable because  $\alpha - \gamma > \alpha/2 > (1 - b_j^1)$ . ■

In accordance with Proposition 1, if the number of bidders is sufficiently large, only inefficient equilibria may exist. This inefficiency result might not really be a problem because the game might still be efficient for small  $N$ . The following proposition, however, shows that the non-existence of efficient equilibria may even appear for  $N = 3$  if the objects are sufficiently equal in value.

**Proposition 2.** For any  $N \geq 3$ , there exists a number  $\hat{\alpha}(N) \in (0,1)$  such that for all  $\alpha \in (\hat{\alpha}(N), 1)$ , no pure strategy monotone symmetric equilibrium exists, so that all Nash equilibria of this game are inefficient.

**Proof.** Assume that a pure strategy monotone symmetric equilibrium exists, and let  $s(\alpha, N, v)$  denote the equilibrium surplus of the bidder of type  $v$ . Obviously,  $s(\alpha, N, v) < v$  for all values of  $v \in (0, 1]$ , and uniformly for all values of  $\alpha \in (0, 1)$ . Let us consider a bidder  $i$  of type  $v_i = 1$ . The deviation by bidding  $b_i^2 = \gamma$  for object 2 and submitting  $p_i = 0$  is profitable if  $\alpha - \gamma > s(\alpha, N, 1)$ . Hence, as long as  $\alpha > s(\alpha, N, 1)$ , there exists a sufficiently small  $\gamma$  so that the deviation is indeed profitable, and the

game has no pure strategy monotone symmetric equilibrium. Taking  $\hat{\alpha}(N) = \sup_{\alpha} s(\alpha, N, 1) < 1$  completes the proof. ■

The conditions in Proposition 1 and Proposition 2 can be made more precise if we make an extra assumption about the distribution function  $F$ . Proposition 3 below shows that when valuations are uniformly distributed, the game does not have efficient equilibria even if the objects are quite different, *i.e.*, even if  $\alpha$  is relatively small (but larger than  $1/(N-1)$ ).

**Proposition 3.** Let  $N \geq 3$  and the valuations be uniformly distributed over  $[0, 1]$ . If  $\alpha > 1/(N-1)$  then the game has no efficient Nash equilibria.

**Proof.** First, the efficiency criterion requires that every bidder puts his preference on object 1. Second, using a standard technique from auction theory we assume that all bidders  $j \neq i$  follow a symmetric bidding strategy and bid  $(b_j^1, b_j^2, p_j) = (b^1(v_j), b^2(v_j), 1)$ . If bidder  $i$  bids  $(b_i^1, b_i^2, p_i) = (b^1(x^1), b^2(x^2), 1)$ , where  $b^1(v)$  and  $b^2(v)$  are assumed to be strictly increasing and continuously differentiable bidding functions, he gets the following expected pay-off:

$$\pi_i = (v_i - b_i^1) \Pr(b_i^1 > b_{-i}^1) + (\alpha v_i - b_i^2) \Pr(b_i^2 > b_{-i}^2 | \text{NOT}(b_i^1 > b_{-i}^1)) (1 - \Pr(b_i^1 > b_{-i}^1)).$$

Let us consider the following two cases in which we denote the first and the second order statistics of  $(N-1)$  competitors' valuations  $v_{-i}$  by  $y$  and  $z$  respectively:

a) If  $x^1 \geq x^2$  then  $\Pr(b_i^1 > b_{-i}^1) = \Pr(x^1 > y) = (x^1)^{N-1}$  and

$$\begin{aligned} \Pr(b_i^2 > b_{-i}^2 | \text{NOT}(b_i^1 > b_{-i}^1)) (1 - \Pr(b_i^1 > b_{-i}^1)) &= \Pr(y > x^1 \geq x^2 > z) \\ &= (N-1)(1-x^1)(x^2)^{N-2}, \end{aligned}$$

so that

$$\pi_i = (v_i - b_i^1)(x^1)^{N-1} + (\alpha v_i - b_i^2)(N-1)(1-x^1)(x^2)^{N-2}.$$

b) If  $x^1 \leq x^2$  then, again,  $\Pr(b_i^1 > b_{-i}^1) = \Pr(x^1 \geq y) = (x^1)^{N-1}$  and

$$\begin{aligned} \Pr(b_i^2 > b_{-i}^2 | \text{NOT}(b_i^1 > b_{-i}^1)) (1 - \Pr(b_i^1 > b_{-i}^1)) &= \Pr(y > x^1, x^2 > z) \\ &= \Pr(x^2 > y > x^1) + \Pr(y > x^2 > z) \\ &= (x^2)^{N-1} - (x^1)^{N-1} + (N-1)(1-x^2)(x^2)^{N-2}, \end{aligned}$$

so that

$$\pi_i = (v_i - b_i^1)(x^1)^{N-1} + (\alpha v_i - b_i^2)((x^2)^{N-1} - (x^1)^{N-1}) + (N-1)(1-x^2)(x^2)^{N-2}.$$

Combining both cases, we can rewrite  $\pi_i$  as follows:

$$\begin{aligned} \pi_i(v_i, x^1, x^2) = & (v_i - b^1(x^1))(x^1)^{N-1} + (\alpha v_i - b^2(x^2))(N-1)(1-x^1)(x^2)^{N-2} + \\ & + (\alpha v_i - b^2(x^2))((x^2)^{N-1} - (x^1)^{N-1}) - (N-1)(x^2 - x^1)(x^2)^{N-2} \end{aligned}$$

where

$$\mathbf{1}(x) \equiv \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0 \end{cases}$$

Bidder  $i$  maximizes  $\pi_i$  with respect to  $x^1$  and  $x^2$ , and the maximum must be attained at  $(v_i = x^1 = x^2 = v)$ , which is the truth-telling condition for the mechanism. The first-order conditions are:

$$\begin{cases} 0 = \frac{\partial \pi_i}{\partial x^1}(v, v, v) = -\left( v \frac{db^1}{dv}(v) - (N-1)(v - b^1(v)) + (N-1)(\alpha v - b^2(v)) \right) v^{N-2} \\ 0 = \frac{\partial \pi_i}{\partial x^2}(v, v, v) = -(N-1) \left( v \frac{db^2}{dv}(v) - (N-2)(\alpha v - b^2(v)) \right) (1-v) v^{N-3} \end{cases}$$

Solving this system of differential equations yields the following unique candidate bidding functions:

$$\begin{cases} b^1(v) = \frac{N-(1+\alpha)}{N} v \\ b^2(v) = \frac{N-2}{N-1} \alpha v \end{cases}$$

Thus, if an efficient Nash equilibrium exists, it must be given by the above bidding functions. Let us consider a bidder  $j$  with valuation  $v_j = 1$ . His equilibrium pay-off is

$$\pi_j(1,1,1) = (1+\alpha)/N.$$

Deviating by bidding  $b_j^2 = b^2(\varepsilon) = \alpha\varepsilon(N-2)/(N-1)$ , where  $\varepsilon$  is arbitrarily small, bidder  $j$  has still the highest bid on object 1, but with a small probability he is also the highest bidder for object 2. Stating his preference as  $p_j = 0$  yields him in such rare occasions a pay-off of  $\alpha - \varepsilon$ . Thus, the pay-off  $\tilde{\pi}_j$  of bidder  $j$  from such a deviation is:

$$\begin{aligned}
\tilde{\pi}_j &\equiv (\alpha v_j - b_j^2) \Pr(b^2(\varepsilon) > b^2(v_{-j})) + \\
&\quad + (v_j - b_j^1) \Pr(b_j^1 > b_{-j}^1 | \text{NOT}(b^2(\varepsilon) > b^2(v_{-j}))) (1 - \Pr(b^2(\varepsilon) > b^2(v_{-j}))) \\
&= (\alpha v_j - b^2(\varepsilon)) \Pr(b^2(\varepsilon) > b^2(v_{-j})) + (v_j - b_j^1) (1 - \Pr(b^2(\varepsilon) > b^2(v_{-j}))) \\
&= (\alpha v_j - b^2(\varepsilon)) \Pr(v_{-j} < \varepsilon) + (v_j - b_j^1) (1 - \Pr(v_{-j} < \varepsilon)) \\
&= \left( \alpha - \frac{N-2}{N-1} \alpha \varepsilon \right) \varepsilon^{N-1} + \left( 1 - \frac{N-(1+\alpha)}{N} \right) (1 - \varepsilon^{N-1}) \\
&= \frac{(1+\alpha)}{N} + \left( \alpha - \frac{(1+\alpha)}{N} - \frac{N-2}{N-1} \alpha \varepsilon \right) \varepsilon^{N-1}.
\end{aligned}$$

This implies that if  $\alpha > (1+\alpha)/N$ , there exists an  $\varepsilon$ :

$$\varepsilon < \left( \alpha - \frac{(1+\alpha)}{N} \right) / \left( \frac{N-2}{N-1} \alpha \right) = \frac{(N-1)((N-1)\alpha - 1)}{(N-2)N\alpha}$$

such that  $\tilde{\pi}_j(v_j, b^1(v_j), b^2(\varepsilon)) > (1+\alpha)/N = \pi_j(1,1,1)$  so that the deviation is profitable.

Hence, an efficient equilibrium does not exist for  $\alpha > 1/(N-1)$ . ■

In summary, Proposition 3 shows that the non-existence of efficient Nash equilibria is not only an asymptotic property of the game, as established in Proposition 1 and Proposition 2. Efficient equilibria may fail to exist also for small  $N$ , as long as the objects are sufficiently equal in value. For the uniform distribution, efficient Bayes-Nash equilibria fail to exist for any  $N \geq 3$  provided  $\alpha > 0.5$ . On the other hand, efficient equilibria may also fail to exist even for small values of  $\alpha$ , *i.e.*, when objects are very different in terms of valuations, as long as the number of bidders is large. For the uniform distribution, efficient Bayes-Nash equilibria fail to exist for any  $\alpha > 0$  provided  $N > 1 + 1/\alpha$ .

### 3.2. Homogeneous objects

In Section 3.1, Proposition 1, we have given a reason why efficient equilibria may not exist in case objects are heterogeneous. In short, when all bidders submit their high bids  $b^1(v_i)$  for object 1 and their low bids  $b^2(v_i)$  for object 2, some bidders, in particular a bidder  $i$  with the highest possible valuation  $v_i = 1$ , unilaterally have an incentive to switch their preference from object 1 to object 2, *i.e.*, to put  $p_i = 0$  and to reduce their bid for object 2. This deviation breaks an efficient equilibrium which requires  $p_i = 1$ .

However, when objects are homogeneous, *i.e.*, when  $\alpha = 1$ , the efficiency criterion does not require that  $p_i = 1$  anymore. This may lead to another type of equilibrium, where bidders do not coordinate on bidding  $b^1(v_i)$  for object 1 and  $b^2(v_i)$  for object 2. Bidders simply submit two bids for two objects, and they do not pay any attention whether the bid  $b^1(v_i)$  is placed on object 1 and the bid  $b^2(v_i)$  is placed on object 2, or the other way around. From the point of view of one bidder, any other bidder puts  $b^1(v_i)$  and  $b^2(v_i)$  on object 1 with equal probability.<sup>6</sup> In other words, bidders put their (deterministic) bids randomly on both objects. In order to get the highest possible expected surplus, every bidder will set his preferences for the object on which he submits the lowest bid.

In the following proposition, we show that in case objects are homogeneous no efficient Nash equilibria exist.

**Proposition 4.** The auction with homogeneous objects has no efficient symmetric Nash equilibria.

**Proof.** Suppose an efficient Nash equilibrium does exist, and let its monotone and symmetric bidding functions be  $b^1(v)$  and  $b^2(v)$ . Then, it cannot be that  $b^1(1) = b^2(1)$  due to the following reason. If a bidder  $j$  with the highest possible valuation  $v_j = 1$  bids the same amounts for both objects, he is the highest bidder for both of them, and he gets the pay-off  $1 - b^2(1)$ . By deviating and reducing his second bid  $b_j^2 < b^2(1)$  he gets a higher pay-off  $1 - b_j^2 > 1 - b^2(1)$  with a strictly positive probability. Thus, the bidding functions must not coincide so that there must be an open interval of valuations  $\Theta = (\underline{v}, \bar{v})$  on which  $b^1(v) > b^2(v)$ .

However, if  $b^1(v) > b^2(v)$  for all  $v \in (\underline{v}, \bar{v})$ , then there is a positive probability that all bidders' valuations will be drawn from  $\Theta$  in such a way that

- a)  $\bar{v} > v_1 > v_2 > v_3 > \dots > v_i > \dots > v_N > \underline{v}$  and  $b^1(v_3) > b^2(v_1)$ ; and

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<sup>6</sup> The superscripts of the bidding functions  $b^1(v_i)$  and  $b^2(v_i)$  do not anymore refer to the objects and are only used to make a distinction between the bids.

b)  $b^1(v_1)$ ,  $b^1(v_2)$ , and  $b^2(v_3)$  are placed on object 1, and  $b^2(v_1)$ ,  $b^2(v_2)$ , and  $b^1(v_3)$  are placed on object 2.

In such a case, bidder 1 takes object 1 (as  $b^1(v_1)$  is the highest overall bid and bidder 1's low bid on object 2 does not turn out to be the highest bid on object 2) and bidder 3 takes object 2 (as  $b^1(v_3) > b^2(v_2)$ ). Hence, the game has no efficient Nash equilibria. ■

It turns out that the analytical derivation of the equilibrium bidding functions is very complicated due to the random (binomial) distributions of bidders' high and low bids over the two objects. This randomness also complicates a numerical analysis for a general number of bidders. For the simplest case where  $N = 3$  and valuations being uniformly distributed over the  $[0,1]$  interval, we numerically obtain equilibrium bidding functions. Figure 1 presents the bidding functions themselves whereas Figure 2 presents their derivatives. The horizontal axis in both pictures denotes valuations, and the vertical axes denote the bidding functions, Figure 1, and their derivatives, Figure 2.

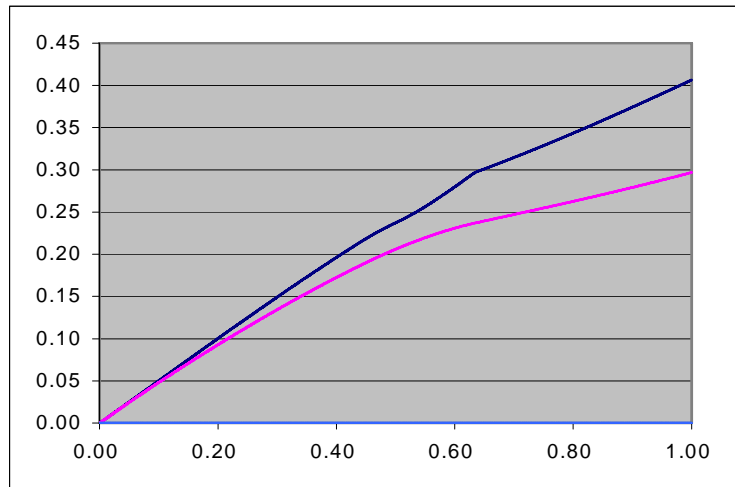


Figure 1. Equilibrium bidding functions  $b^1(v)$  and  $b^2(v)$  for  $N = 3$ ,  $\alpha = 1$  and uniform distributions of bidders' valuations.

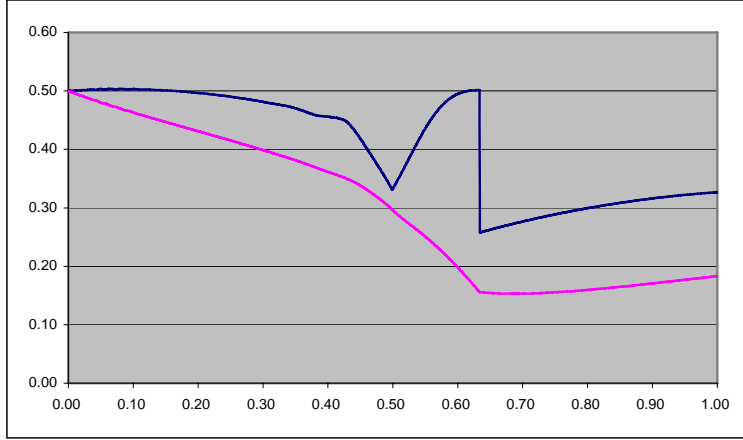


Figure 2. Derivatives of equilibrium bidding functions.

Bidding behavior of a bidder with a value close to zero can be described as follows. The probability that he outbids two competitors is negligibly small compared to outbidding only one. Therefore, his bidding strategy is based on out-competing only one bidder. Consequently the auction game for a low-valuation bidder is like a single-object first-price sealed-bid auction with one competitor, where bidding half of his value is an equilibrium strategy.

Observe that  $b^1(v_i)$  contains a kink at  $v \approx 0.633$ . The reason that there is a kink in  $b^1(v_i)$  is the following. Let us have a look at a bidder with  $v = 0.80$ . His high bid is always higher than the low bid of the highest value bidder ( $v = 1$ ). For a bidder with for example  $v = 0.40$  this is not the case. The kink is exactly at the level  $v$  for which  $b^1(v) = b^2(1)$ . This kink of  $b^1(v_i)$  causes a kink in the derivative of  $b^2(v_i)$  and, therefore, it causes a discontinuity in its second-order derivative. This discontinuity, in turn, causes a kink in the derivative of  $b^1(v_i)$ , and, therefore, it causes a kink in the second derivative of  $b^2(v_i)$ . This latter kink, in turn, causes a discontinuity in the third-order derivative of  $b^2(v_i)$ , and so on, *ad infinitum*. It makes numerical calculations of the bidding functions unstable and complex.

Further numerical calculations with the same parameters show that with probability of about 4.3% the outcome of the auction is not efficient, *i.e.*, the second highest valuation bidder does not get the second object.<sup>7</sup> This 4.3% is a lower bound (when valuations are uniformly distributed over the  $[0,1]$  interval): if the number of bidders

<sup>7</sup> The highest value bidder always wins an object in a monotone symmetric bidding equilibrium.



increases, then this probability increases too. The intuition is twofold. First, if more than three bidders compete, then more than one bidder can outbid the second highest valuation bidder on one of the objects, and, second, the expected difference between the low bid of the second highest valuation bidder and the high bid of the lower valuation bidders is smaller. On the other hand, the expected efficiency loss goes down when the number of bidders increases.

#### 4. Concluding remarks

This paper shows that simultaneous pooled auctions with multiple bids and preference lists, where single-object demand bidders are allowed to make separate bids for each object and submit a preference list to rank these objects, never have efficient equilibria unless objects are sufficiently heterogeneous. In so far, as efficiency of auctions' outcome is an important consideration for governments – and which government would ever want to openly deny that this is the case? – the paper shows that this type of auction format, *i.e.*, a multi-object sealed-bid auction with right-to-choose ingredients, should not be used (anymore). Other mechanisms exist that exhibit these efficiency properties (under fairly general conditions), like the Vickrey-Clarke-Groves mechanism,<sup>8</sup> the simultaneous ascending auction,<sup>9</sup> and the right-to-choose auction. In laboratory experiments, Goeree *et al.* (2006) show that with respect to efficiency, the simultaneous ascending auction performs better than auctions with a first-price element like the simultaneous first-price auction, the sequential first-price auction and the simultaneous descending auction. As other auction formats perform better, we do not see good economic arguments why the auctions analyzed in this paper should be used in future allocation processes.

#### References

Burguet, R., 2005. The condominium problem: Auctions for substitutes. *Review of Economic Design* 9(2), 73-90.

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<sup>8</sup> This mechanism is developed by Clarke (1971) and Groves (1973) and generalizes the (multi-unit) Vickrey auction.

<sup>9</sup> Leonard (1983) and Demange *et al.* (1986) show that in a simultaneous ascending auction, an efficient equilibrium is established when bidders bid 'straightforwardly', *i.e.*, in each round, each bidder who currently does not have a standing high bid, bids for the object which currently offers him the highest surplus; and they drop out once the highest available surplus becomes negative.

- Clarke, E.H., 1971. Multipart pricing of public goods. *Public Choice* 11(1), 17-33.
- Commission for Communications Regulation, 2005. Wideband Digital Mobile Data Services Licence Awarded by ComReg. Media Release - 21 December 2005, ComReg PR211205. Available at [http://www.comreg.ie/\\_fileupload/publications/PR211205.pdf](http://www.comreg.ie/_fileupload/publications/PR211205.pdf).
- Crawford, V.P., Haller, H., 1990. Learning how to cooperate: Optimal play in repeated coordination games. *Econometrica* 58(3), 571-595.
- Demange, G., Gale, D., Sotomayor, M., 1986. Multi-item auctions. *Journal of Political Economy* 94(4), 863-872.
- Eliasz, K., Offerman, T., Schotter, A., 2008. Creating competition out of thin air: An experimental study of right-to-choose auctions. *Games and Economic Behavior* 62(2), 383-416.
- Gale, I.L., Hausch, D.B., 1994. Bottom-fishing and declining prices in sequential auctions. *Games and Economic Behavior* 7(3), 318-331.
- Goeree, J.K., Offerman, T., Schram, A., 2006. Using first-price auctions to sell heterogeneous licenses. *International Journal of Industrial Organization* 24(3), 555-581.
- Goeree, J.K., Plott, C.R., Wooders, J., 2004. Bidders' choice auctions: Raising revenues through the right to choose. *Journal of the European Economic Association* 2(2-3), 504-515.
- Groves, T., 1973. Incentives in teams. *Econometrica* 41(4), 617-631.
- Janssen, M.C.W., Karamychev, V.A., 2007. Do auctions select efficient firms? Discussion Paper TI 2007-001/1. Tinbergen Institute, Erasmus University Rotterdam. Available at <http://www.tinbergen.nl/discussionpapers/07001.pdf>.
- Klemperer, P., 2002. How (not) to run auctions: The European 3G telecom auctions. *European Economic Review* 46(4-5), 829-845.
- Leonard, H.B., 1983. Elicitation of honest preferences for the assignment of individuals to positions. *Journal of Political Economy* 91(3), 461-479.
- Menezes, F.M., Monteiro, P.K., 1998. Simultaneous pooled auctions. *Journal of Real Estate Finance and Economics* 17(3), 219-232.

Ministry of Education, Culture and Science, 2003. Regeling aanvraag en vergelijkende toets vergunningen commerciële radio-omroep 2003. Staatscourant 40, 19, 26 February 2003. Available at <http://www.overheid.nl/op/>.

Moldovanu, B, Sela, A., 2003. Patent licensing to Bertrand competitors. *International Journal of Industrial Organization* 21(1), 1-13.

Salmon, T.C., Iachini, M., 2007. Continuous ascending vs. pooled multiple unit auctions. *Games and Economic Behavior* 61(1), 67-85.