TI 2006-105/4
Tinbergen Institute Discussion Paper

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Extracting business cycles using semi-parametric time-varying spectra with applications to U.S. macroeconomic time series

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November 2006

Abstract

A growing number of empirical studies provides evidence that dynamic properties of macroeconomic time series have been changing over time. Model-based procedures for the measurement of business cycles should therefore allow model parameters to adapt over time. In this paper the time dependencies of parameters are implied by a time dependent sample spectrum. Explicit model specifications for the parameters are therefore not required. Parameter estimation is carried out in the frequency domain by maximising the spectral likelihood function. The time dependent spectrum is specified as a semi-parametric smoothing spline ANOVA function that can be formulated in state space form. Since the resulting spectral likelihood function is time-varying, model parameter estimates become time-varying as well. This new and simple approach to business cycle extraction includes bootstrap procedures for the computation of confidence intervals and real-time procedures for the forecasting of the spectrum and the business cycle. We illustrate the methodology by presenting a complete business cycle analysis for two U.S. macroeconomic time series. The empirical results are promising and provide significant evidence for the great moderation of the U.S. business cycle.

Keywords: Frequency domain estimation, frequency domain bootstrap, time-varying parameters, unobserved components models.

JEL classification: C13, C14, C22, E32.

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1 Introduction

In this paper we consider the extraction of business cycles from macroeconomic time series. The business cycle is an important indicator of the state of the economy. Policy makers may adapt decisions when the economic situation changes. Since the business cycle is unobserved, we require methods to extract a business cycle indicator from macroeconomic time series. Business cycles are usually referred to as cyclical deviations from a long-term trend. Following Burns & Mitchell (1946), these fluctuations typically last between 1.5 and 8 years. For the purpose of cycle extraction, we can apply nonparametric filters or we may adopt parametric methods. Examples of nonparametric filters are those proposed by Hodrick & Prescott (1997), Baxter & King (1999) and Christiano & Fitzgerald (2003). The construction of these filters is based on the approximation to ideal band-pass and ideal high-pass filters in the frequency domain. The application of nonparametric filters has several drawbacks. Firstly, the aim of the filters is to extract signals with periods between a specified range. The filters do not take the dynamic properties of the time series into account and the extraction of spurious cycles may be the result. Secondly, it is not possible to obtain forecasts of the business cycle. This is usually solved by constructing leading indicators for the business cycle indicator. But the construction of a leading indicator that can forecast the business cycle well is difficult, since it may not cover all economic activity in a way the business cycle indicator does. Thirdly, we cannot obtain confidence intervals for the extracted cycle and the model parameters, as we do not consider a statistical model.

As an alternative to nonparametric filters, model-based approaches can be considered to extract cyclical components from time series. Examples of these are given by Beveridge & Nelson (1981), Clark (1987) and Harvey & Jaeger (1993). The Beveridge-Nelson decomposition is based on the modeling of a univariate time series by an autoregressive integrated moving average model. Clark (1987) and Harvey & Jaeger (1993) propose unobserved components time series models with trend and cycle factors, which need to be estimated from the univariate time series. Clark's model specifies the cycle component as an autoregressive process while the Harvey-Jaeger model has the cycle as a stationary and stochastic cyclical process based on cosine waves. The parameters of the models are estimated by the method of maximum likelihood for which state space methods can be adopted to evaluate the log-likelihood function, see Harvey (1989).

In recent years there is a growing interest in business cycle models with time-varying parameters, since it is unlikely that model parameters are constant over a period of many years. In Kim & Nelson (1999), Stock & Watson (2003), Sensier & van Dijk (2004), Kim, Nelson & Piger

(2004) and more recently Koopman, Lee & Wong (2006) and Korenok & Radchenko (2006) evidence has been found that the volatility in the business cycle has declined in the mid 1980s. The decline in volatility is often referred to as the *great moderation* of the business cycle. The moderation is mainly characterised by the decline in the volatility of the cycle and an increase in both the persistency and the duration of the cycle in the mid 1980s. These phenomena motivate us to consider a trend-cycle model with time-varying parameters. When we allow the model parameters to vary over time, we obtain cycle and trend estimates with properties that also vary over time. Consequently, the business cycle has different dynamic properties over time. This is of great importance when we wish to assess the current state of the economy. The most actual information on economic activity is obtained by extracting the business cycle with up-to-date estimates of the model parameters.

Several time-varying business cycle models are investigated in the literature. Stock & Watson (1996) apply time-varying autoregressive models to assess structural instabilities in macroeconomic time series. The autoregressive parameters are modeled as random walk processes. In Van Dijk, Terasvirta & Franses (2002), autoregressive models with smooth transition formulations for the parameters are considered, which are usually referred to as smooth transition autoregressive (STAR) models. In these models, parameters change smoothly or abruptly from one regime to another regime. Abrupt changes in model parameters are not plausible in large and stable economies. Another class of time-varying models are the Markov-Switching models, see Hamilton (1989) and Kim & Nelson (1999). These models allow the time series to be in any of a finite number of distinct regimes at any point in time. In a study to convergence in European business cycles, Luginbuhl & Koopman (2004) consider an unobserved components model with time-varying parameters by using a logit function to model the convergence parameters. A more flexible specification is considered in Koopman, Lee & Wong (2006), where cubic splines are used to model the time-varying model parameters. In contrary to the logit specification, spline functions are not monotone by construction and therefore vary more flexibly over time. We note that the estimation of the parameters in the aforementioned models is not straightforward, since these models are nonlinear and require an increase in the number of parameters. Estimation of the parameters in nonlinear models is involved and may require simulation-based and nonlinear estimation methods.

The business cycle extraction method proposed in this paper is mostly based on the frequency domain. The key idea in this paper is that we allow parameters to change in accordance with a smooth time-varying spectrum. A dynamic model specification is therefore not needed and an advanced estimation procedure is not required. The availability of a smooth estimate of a time-varying spectrum suffices. Since the spectral likelihood is a function of the model parameters and the spectrum, it is varying over time when the spectrum is time-varying. By maximising the time-varying spectral likelihood at all points in time, we obtain time-varying estimates of the parameters. The time-varying patterns of the parameters are therefore implied by the time-varying spectrum. Confidence intervals for the model parameters are obtained by adopting a frequency domain bootstrap, which is novel in this context. In the statistics literature, there exist numerous methods to estimate a time-varying spectrum. We adopt the Smoothing Spline ANOVA (SS–ANOVA) approach of Gu & Wahba (1993). This method estimates a spectrum smoothed over both time and frequency simultaneously. The smoothness property is important as sudden changes in the spectrum over time are unlikely in macroeconomic time series. Since it is possible to cast the SS–ANOVA model into state space form, we can apply the Kalman filter and smoother to obtain estimates of the time-varying spectrum. The state space approach also enables us to obtain forecasts of the spectrum. This implies that we are able to obtain forecasts for both the parameter estimates and the unobserved components such as trend and cycle. We apply the method in a business cycle analysis for two U.S. macroeconomic time series. The empirical results provide significant evidence for the *great moderation* of the U.S. business cycle.

This paper is organised as follows: section 2 presents the trend-cycle decomposition model. We discuss the estimation and the bootstrap procedure in the frequency domain. In section 3, we consider the estimation of the semi-parametric time-varying spectrum. Empirical illustrations of two U.S. macroeconomic time series are given in section 4. Section 5 concludes.

2 Unobserved Components Model and frequency domain estimation

2.1 Trend-cycle decomposition model

We consider an unobserved components time series model for the analysis of macroeconomic data. The decomposition model for a univariate time series is given by

$$y_t = \mu_t + \psi_t + \varepsilon_t, \quad t = 1, \ldots, T, \tag{1}$$

where y_t is the observation at time t, μ_t represents the trend, ψ_t is the cycle and ε_t is an idiosyncratic shock. The trend is modeled as a second order integrated random walk:

$$(1 - L)^2 \mu_t = \Delta^2 \mu_t = \eta_t, \quad \eta_t \sim \text{NID}(0, \sigma_\eta^2), \tag{2}$$

where L is the lag operator, i.e. $L^d \mu_t = \mu_{t-d}$ and $\Delta^d = (1-L)^d$ is the difference operator for $d = 1, 2, \ldots$. The trend specification (2) resembles a smooth trend process. We model the idiosyncratic shock ε_t as a Gaussian white noise process, i.e. $\varepsilon_t \sim \text{NID}(0, \sigma_\varepsilon^2)$. For the cycle component, we adopt Harvey's (1989) specification and formulate the cycle as a trigonometric process

$$\begin{pmatrix} \psi_{t+1} \\ \psi_{t+1}^+ \end{pmatrix} = \rho \begin{bmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{bmatrix} \begin{pmatrix} \psi_t \\ \psi_t^+ \end{pmatrix} + \begin{pmatrix} \kappa_t \\ \kappa_t^+ \end{pmatrix}, \begin{pmatrix} \kappa_t \\ \kappa_t^+ \end{pmatrix} \sim \text{NID}\left(0, \begin{bmatrix} \sigma_\kappa^2 & 0 \\ 0 & \sigma_\kappa^2 \end{bmatrix}\right), \quad (3)$$

where λ_c is the frequency of the cycle, ρ is the damping factor and σ_κ^2 is the variance of the disturbances in the cycle. The disturbances $\eta_t, \varepsilon_t, \kappa_t$ and κ_t^+ are mutually and serially independent. The period p_c of the cycle is related to the frequency, i.e. $p_c = 2\pi/\lambda_c$. The variance of the cycle is given by $\sigma_\psi^2 = \sigma_\kappa^2/(1-\rho^2)$. To maintain stationarity we restrict the damping factor ρ to the range $[0, 1)$. Values close to unity correspond to a more persistent cycle. We can also formulate the cycle ψ_t as an autoregressive moving average (ARMA) process. The autocorrelation function $r(\cdot)$ of the cycle at lag k is given by

$$r(k) = \rho^k \cos(\lambda_c k), \quad k = 1, 2, \ldots. \quad (4)$$

For this trend-cycle model we need to estimate five parameters: $\theta = (\sigma_\eta^2, \sigma_\varepsilon^2, \sigma_\kappa^2, \lambda_c, \rho)'$. We can estimate the model parameters in the time domain by casting the model into state space form and by evaluating the likelihood function using the Kalman filter. Estimation of the mean and variance of the state is carried out by the Kalman filter and smoother. Nevertheless, in this paper we use techniques based on the frequency domain to estimate the model parameters.

2.2 Spectral likelihood

The frequency domain approach to estimate the model parameters is based on the maximisation of the spectral likelihood. To define the spectral likelihood we need to introduce the sample spectrum of the time series and the spectral generating function (sgf) of the model. The spectrum is an important concept in the frequency domain. It contains the same information as the autocovariance function but the information is presented in a different way. We refer to Brockwell & Davis (1991, chapter 10) and Priestley (1981, chapter 6) for more details on the spectrum. The sample spectrum defined at the frequency points $\lambda_k = \pi(k-1)/(M-1)$, $k = 1, \ldots, M$, is given by

$$I(\lambda_k) = \frac{1}{2\pi} \sum_{\tau=-T-1}^{T-1} \hat{\gamma}(\tau) e^{-i\lambda_k \tau}, \quad (5)$$

where $\hat{\gamma}(\tau)$ is the sample autocovariance function at lag τ and $i = \sqrt{-1}$. The spectral generating function is imposed by the model we wish to estimate. The sgf defined at frequency point λ_k, is given by:

$$g(\lambda_k) = \sum_{\tau=-\infty}^{\infty} \gamma(\tau) e^{-i\lambda_k \tau}, \quad k = 1, \dots, M, \tag{6}$$

where $\gamma(\tau)$ is the theoretical autocovariance function of the process at lag τ. The sgf is a function of the unknown vector of parameters.

The spectral likelihood is a function of the sample spectrum $I(\lambda_k)$ and the sgf $g(\lambda_k)$ and it is given by

$$\log L(\theta) = -\frac{T}{2} \log 2\pi - \frac{1}{2} \sum_{k=1}^{M} \log g(\lambda_k) - \pi \sum_{k=1}^{M} \frac{I(\lambda_k)}{g(\lambda_k)}, \tag{7}$$

where T is the number of observations and M is the number of frequency points. The spectral likelihood is maximised with respect to the unknown parameters θ, which are in $g(\lambda_k)$. Parameter estimation in the frequency domain can be considered for several models including ARMA models and regression models. Derivations of the sgf of several structural time series models can be found in Harvey (1989, chapter 2). Below, we give the derivation of the sgf of the trend-cycle model (1)–(3).

Derivation of the spectral generating function of the trend-cycle model

We derive the spectral generating function of the trend-cycle model in (1)–(3) by summing the sgf's corresponding to the trend, cycle and irregular components. This is possible since the unobserved components of the model are independent of each other. As the sgf is only defined for stationary processes, we transform the model such that the dependent variable is stationary. The main source of nonstationarity in the model comes from the trend component and since it is a second order integrated random walk process, we consider a model for $\Delta^2 y_t$:

$$\Delta^2 y_t = \Delta^2 \mu_t + \Delta^2 \psi_t + \Delta^2 \varepsilon_t. \tag{8}$$

The spectral generating functions of $\Delta^2 \mu_t$ and $\Delta^2 \varepsilon_t$ at λ_k are given by

$$g_\mu(\lambda_k) = \sigma_\eta^2, \quad g_\varepsilon(\lambda_k) = \sigma_\varepsilon^2 (2 - 2\cos\lambda_k)^2. \tag{9}$$

To derive the spectral generating function of the cycle component we can use the property that the cycle is an ARMA(2,1) process and derive its spectral density. However, it is easier to write ψ_t in a single equation form as a sum of two uncorrelated components, see Harvey (1989, section 2.4) for details. The sgf of $\Delta^2 \psi_t$ is equal to

$$g_\psi(\lambda_k) = \frac{\sigma_\kappa^2 \big(2 - 2\cos\lambda_k\big)^2 (1 + \rho^2 - 2\rho\cos\lambda_c \cos\lambda_k)}{1 + \rho^4 + 4\rho^2 \cos^2\lambda_c - 4\rho(1+\rho^2)\cos\lambda_c \cos\lambda_k + 2\rho^2 \cos 2\lambda_k}. \tag{10}$$

The spectral generating function of the model in (8) is then given by

$$g_\theta(\lambda_k) = g_\mu(\lambda_k) + g_\psi(\lambda_k) + g_\varepsilon(\lambda_k), \tag{11}$$

where we emphasise that the sgf is a function of the parameter vector θ. By maximising the spectral likelihood, we estimate the parameters in the frequency domain. However, the extraction of the trend and cycle components must be done in the time domain.

2.3 Bootstrap intervals for parameters

In this section we present the frequency domain bootstrap procedure to obtain confidence intervals for the estimated parameters. We adopt the method of bootstrapping, since in empirical work numerical difficulties may arise when we compute asymptotic standard errors for the parameter estimates. The use of bootstrap in the frequency domain is novel for this purpose and in the context of section 3. Asymptotic properties of the bootstrap in the frequency domain are studied by Dahlhaus & Janas (1996). In applied work, Wright (1999) uses the frequency domain bootstrap for the analysis of impulse response functions and Gerolimetto (2006) adopts the bootstrap approach to compute empirical distributions in a study into fractional cointegration regressions.

Bootstrapping in the time domain requires the simulation of time series. However, in the frequency domain we simulate the spectrum instead of the time series. The spectrum $I(\lambda_k)$ at frequency λ_k is mutually independent of the spectrum $I(\lambda_{j\neq k})$ at all other frequencies and it has the following chi-squared distribution, see Harvey (1989):

$$\begin{aligned} 4\pi I(\lambda_k)/g_\theta(\lambda_k) &\sim \chi^2_2 \quad \text{for } \lambda_k \neq 0, \pi, \\ 2\pi I(\lambda_k)/g_\theta(\lambda_k) &\sim \chi^2_1 \quad \text{for } \lambda_k = 0, \pi. \end{aligned} \tag{12}$$

Denote the estimated parameter vector by $\hat{\theta}$. In each bootstrap, we generate the bootstrap spectrum given the vector $\hat{\theta}$. The simulated spectrum and estimated bootstrap parameter vector in the ith bootstrap are respectively denoted by $I(\lambda)^{(i)}$ and $\hat{\theta}^{(i)}$, $i = 1, \ldots, B$, where B is the number of bootstraps. The bootstrap procedure consists of the following steps:

1. Draw a random series $\nu_k^{(i)}$, $k = 1, \ldots, M$ of independent chi-squared distributed variables with two degrees of freedom for $k = 2, \ldots, M-1$ and with one degree of freedom for $k = 1, M$.

2. Let $I(\lambda_k)^{(i)} = g_{\hat{\theta}}(\lambda_k)\nu_k^{(i)}/4\pi$ for $k = 2, \ldots, M-1$ and let $I(\lambda_k)^{(i)} = g(\lambda_k)\nu_k^{(i)}/2\pi$ when $k = 1, M$.

3. Maximise the spectral likelihood to obtain $\hat{\theta}^{(i)}$.

4. Repeat the steps 1–3 B times. Calculate the desired quantiles of the estimated $\hat{\theta}^{(i)}$. The quantiles are the limits of the confidence interval of the estimated parameter vector $\hat{\theta}$.

3 Time-varying spectrum and parameters

3.1 Time-varying spectral likelihood

In this section we discuss how to obtain implicitly a trend-cycle model with time-varying parameters. In the spectral likelihood (7) we substitute the spectrum for a time-varying spectrum. The spectral likelihood is now time-dependent:

$$\log L_t(\theta_t) = constant - \frac{1}{2}\sum_{k=1}^{M} \log g_{\theta_t}(\lambda_k) - \pi \sum_{k=1}^{M} \frac{I_t(\lambda_k)}{g_{\theta_t}(\lambda_k)}, \quad t = 1, \ldots, T, \tag{13}$$

where $I_t(\lambda)$ is the estimated spectrum at time t and θ_t is the vector of parameters at time t, that we do not specify further. We obtain time-varying estimates of the parameters $\hat{\theta}_t$ by maximising the time-dependent spectral likelihood at any point in time. The confidence intervals for $\hat{\theta}_t$ is obtained by the frequency domain bootstrap described in section 2.3. This implies that we carry out the bootstrap procedure T times to obtain confidence intervals for the time-varying patterns of the parameter estimates. We stress that our approach is not restricted to estimate the time-varying parameters of the trend-cycle model in (1)–(3). A large class of models can be treated in this way by specifying the spectral generating function.

Our approach can be classified as semi-parametric. The time-varying pattern of the model parameters is completely determined by the time-varying spectrum. We do not specify nor model the time-varying patterns of the parameters. In this sense, the approach is nonparametric. However, our approach is parametric since the model parameters are estimated by maximising the spectral likelihood and a statistical model is imposed for the time-varying spectrum, see section 3.2. Summarising, the method proposed in this paper is a novel and natural mix of both parametric and nonparametric approaches to estimation.

3.2 Smoothing Spline ANOVA model

In the statistics literature, methods to estimate a time-varying spectrum can be classified as nonparametric or parametric. Nonparametric methods include the approaches of Cohen (1989) and Adak (1998), who divide the times series into blocks and compute the sample spectrum at each block of observations. Examples of parametric methods are given by Jansen, Hasman &

Lenten (1981), Davis, Lee & Rodriguez-Yam (2006) and Kitagawa & Gersch (1996). Estimated time-varying parameters of autoregressive models are plugged in the spectral density of the corresponding models. The auto Smooth Localised complex Exponential (auto–SLEX) method is considered by Omboa, Raz, von Sachs & Malow (2001) and Guo, Dai, Omboa & von Sachs (2003) to estimate the time-varying spectrum.

In this paper, we adopt the Smoothing Spline ANOVA method of Gu & Wahba (1993) and Qin (2004, chapter 3) to estimate the time-varying spectrum. The sample spectrum is approximately equal to the true spectrum times a standard exponential random variable. By applying the logarithmic transformation, we model the sample spectrum in an additive noise model. Let $x_k = \log I(\lambda_k) + c_{\lambda_k}$ be the sample log spectrum at frequency λ_k plus a constant[1], then the additive noise model is given by

$$x_k = f(\lambda_k) + \zeta_k, \quad k = 1, \dots, M, \tag{14}$$

where $f(\lambda_k)$ is the true log spectrum and ζ_k is an error term. In Qin (2004, chapter 3), the sample log spectrum is allowed to vary over time. The initial log sample spectra are estimated at N points on the rescaled time axis $t_j \in [0, 1], j = 1, \dots, N,\ N < T$. This time grid is not necessarily equally spaced. The transformed sample spectra is then related to the unknown two-dimensional function $G(t, \lambda)$ as follows

$$x_{jk} = G(t_j, \lambda_k) + \zeta_{jk}, \quad \zeta_{jk} \sim N(0, \sigma_\zeta^2), \quad j = 1, \dots, N,\ k = 1, \dots, M. \tag{15}$$

Following the SS–ANOVA approach of Gu & Wahba (1993) and Gu (2002), the two-dimensional smooth function $G(t, \lambda)$ is modeled as a tensor product of two cubic spline functions. In Qin (2004, chapter 3) two SS–ANOVA representations are considered for $G(t, \lambda)$: one representation includes the interaction between t and λ, whereas the other ignores this interaction. The former needs a comprehensive state space model with a large state vector of size $4M$, while the latter requires a more parsimonious state space model with a state vector of size $3M$. As we can see below and in appendix A, the time-varying system matrices in the state space model of the former representation are more complicated than those of the latter. It is obvious that the estimation based on the latter representation is faster than the estimation based on the former. However in our empirical study, we obtain similar estimates of the time-varying spectrum for the two models. For computational reasons, we adopt the representation that omits the interaction.

[1]For $\lambda_k \neq 0, \frac{1}{2}$, the constant c_{λ_k} is equal to Euler's constant: $c_{\lambda_k} = \gamma = 0.57721\dots$. For $\lambda_k = 0, \frac{1}{2}$ we have $c_{\lambda_k} = (\ln 2 + \gamma)/\pi$.

When we omit the interaction between t and λ, the function $G(t,\lambda)$ has the following SS–ANOVA representation:

$$G(t,\lambda) = G_1(t) + G_2(\lambda), \tag{16}$$

where $G_1(t)$ and $G_2(\lambda)$ are the smooth main effect in t and λ respectively. For completeness, the model including the interaction between t and λ is given in appendix A. By deriving the Bayesian stochastic model and the recursive Bayesian model for (16), we can formulate the state space model to estimate the smooth function $G(t,\lambda)$. The Bayesian stochastic model and the recursive Bayesian model of (16) can be found in appendix B.

Denote the estimate of $G(t_j,\lambda_k)$ by $g(t_j,\lambda_k)$ and define the state vector $\boldsymbol{\alpha}(t_j,\boldsymbol{\lambda})$ as

$$\underset{(3M\times 1)}{\boldsymbol{\alpha}(t_j,\boldsymbol{\lambda})} = \begin{pmatrix} \alpha(t_j,\lambda_1) \\ \alpha(t_j,\lambda_2) \\ \vdots \\ \alpha(t_j,\lambda_M) \end{pmatrix}, \quad \alpha(t_j,\lambda_k) = \left(g(t_j,\lambda_k),\ \frac{\partial g(t_j,\lambda_k)}{\partial t},\ \frac{\partial g(t_j,\lambda_k)}{\partial \lambda} \right)'.$$

With the definition of the state vector we have the following multivariate state space model:

$$\begin{aligned} \boldsymbol{x}_j &= Z\boldsymbol{\alpha}(t_j,\boldsymbol{\lambda}) + \boldsymbol{\zeta}_j, \quad \boldsymbol{\zeta}_j \sim N(0,G), \qquad j=1,\ldots,N, \\ \boldsymbol{\alpha}(t_j,\boldsymbol{\lambda}) &= T_j\boldsymbol{\alpha}(t_{j-1},\boldsymbol{\lambda}) + \boldsymbol{\xi}_j, \quad \boldsymbol{\xi}_j \sim N(0,H_j), \end{aligned} \tag{17}$$

where the variables, with the sizes in parentheses, are defined as

$$\underset{(M\times 1)}{\boldsymbol{x}_j} = \begin{pmatrix} x_{j1} \\ x_{j2} \\ \vdots \\ x_{jM} \end{pmatrix}, \quad \underset{(M\times 1)}{\boldsymbol{\zeta}_j} = \begin{pmatrix} \zeta_{j1} \\ \zeta_{j2} \\ \vdots \\ \zeta_{jM} \end{pmatrix}, \quad \underset{(3M\times 1)}{\boldsymbol{\xi}_j} = \begin{pmatrix} \xi_{j1} \\ \xi_{j2} \\ \vdots \\ \xi_{jM} \end{pmatrix}, \quad \underset{(3\times 1)}{\xi_{jk}} = \begin{pmatrix} e_{jk,1} \\ e_{jk,2} \\ e_{jk,3} \end{pmatrix}.$$

The vector $\boldsymbol{x}_j$ is the logarithm of the sample spectrum of the time series at time t_j of the initial time grid. For explicit expressions of $e_{jk,i},\ i=1,2,3$, we refer to appendix B. We need these expressions to derive the variance matrix H_j of the error vector in the state equation. The time-invariant system matrices Z and G and the time-varying system matrices T_j and H_j are given by:

$$Z = \mathbf{I}_M \otimes \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad G = \sigma_\zeta^2 \mathbf{I}_M,$$

$$T_j = \text{diag}\left[T_{j1}, \dots, T_{jM}\right] \quad \text{with } T_{jk} = \begin{pmatrix} 1 & \delta t_j & \delta\lambda_k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$H_j = \begin{pmatrix} H_j^{1,1} & H_j^{1,2} & \cdots & H_j^{1,M} \\ H_j^{2,1} & H_j^{2,2} & \cdots & H_j^{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ H_j^{M,1} & H_j^{M,2} & \cdots & H_j^{M,M} \end{pmatrix},$$

where the operator $\otimes$ denotes the Kronecker product, $\mathbf{I}_M$ is the identity matrix of size M, diag$[\cdot\ \cdot]$ refers to a block diagonal matrix, $\delta t_j = t_j - t_{j-1}$ is the gap between two successive points at the time grid and $\delta\lambda_k = \lambda_k - \lambda_{k-1}$ is the difference between two successive points at the frequency grid. The block $H_j^{a,b}$ with $a, b = 1, \dots, M$ in the variance matrix H_j is given by:

$$H_j^{a,b} = Cov(\xi_{ja}, \xi_{jb}) = \begin{cases} \begin{bmatrix} \omega_1 \delta t_j^3/3 + \omega_2 \delta\lambda_a^3/3 & \omega_1 \delta t_j^2/2 & \omega_2 \delta\lambda_a^2/2 \\ \omega_1 \delta t_j^2/2 & \omega_1 \delta t_j & 0 \\ \omega_2 \delta\lambda_a^2/2 & 0 & \omega_2 \delta\lambda_a \end{bmatrix} & \text{if } a = b \\[2em] \begin{bmatrix} \omega_1 \delta t_j^3/3 & \omega_1 \delta t_j^2/2 & 0 \\ \omega_1 \delta t_j^2/2 & \omega_1 \delta t_j & 0 \\ 0 & 0 & 0 \end{bmatrix} & \text{if } a \neq b \end{cases}$$

By applying the Kalman filter and smoother we obtain estimates of the mean and variance of the state vector. Maximum likelihood estimates of the parameters $\omega_1, \omega_2, \sigma_\zeta^2$ are obtained by maximising the Gaussian likelihood function. At the initial time grid, the spectrum is estimated at $N < T$ points. To estimate the spectrum at all T points, we set missing values at the other $T - N$ points of the time grid. The benefit of the state space form is that missing values can be handled easily, see Durbin & Koopman (2001, section 4.8). As a result we obtain estimates of the spectrum at any point in time. Following the same argument, we can fragment the frequency grid into a finer grid by adding missing observations at the new frequency points.

4 Empirical illustrations

4.1 Data description and computations

In the empirical study we apply our method to two macroeconomic time series from the U.S. economy: Gross Domestic Product (GDP) and Industrial Product Index (IPI). The GDP series is quarterly data, while IPI is monthly data. All series are collected from the Federal Reserve Bank of St. Louis. The GDP series represents the logarithm of the real gross domestic product in the period from 1947 to 2006. The IPI series is the logarithm of the industrial production index with 2002 as base year. The IPI series covers the period from the first month of 1947 to the fourth month of 2006. All series are seasonally adjusted. The description of the data is summarised in Table 1. The computations for this paper are done by using the object-oriented matrix language `Ox` of Doornik (2001) together with the state space functions of `SsfPack` version 3, developed by Koopman, Shephard & Doornik (1999).

[insert Table 1]

4.2 Time-invariant unobserved components model

We first estimate the parameters of the time-invariant unobserved components model (1)–(3) in the frequency domain. The estimated parameters with their 90% bootstrap confidence intervals are reported in Table 2. For the two macroeconomic time series, we obtain the smallest values for the variance of the irregular term. This result is typical for aggregated macroeconomic time series of large and stable economies. For both series, we obtain the largest variance for the cycle, implying that most of the variation in the data is in the cycle component. The estimated damping factor of the GDP cycle is 0.904, while for the IPI cycle it is equal to 0.973. These values are satisfactory, since they imply that the forecast function of the cycle contains sufficiently persistent cyclical variations. The period of the cycle of both series is estimated as approximately 3.5 years. This value is in the range of business cycle periods of Burns & Mitchell (1946). The trend and the cycle component of the GDP and IPI series are graphically presented in Figure 1. Summarising, for the GDP series and the IPI series we obtain similar estimation results, although the former is observed every quarter and the latter every month. The similarity of our initial results for the GDP and IPI series is not surprising, since the IPI series is often regarded as a proxy for the GDP series.

[insert Table 2]

[insert Figure 1]

Moreover, we split the sample in two subsamples and we estimate the parameters of the time-invariant model for these subsamples. The first subsample is for 1947-1981 (pre-1982 period), while the other subsample is for 1982-2006 (post-1982 period). The same partition of the sample is considered by Stock & Watson (2003). The estimation results for both subsamples are presented in Table 3.

[insert Table 3]

For the GDP series, we obtain estimates for the variances in the post-1982 period which are significantly smaller than the variances in the pre-1982 period. This means that the volatility in the trend and the cycle of the GDP series has declined over the years. The ratio $\sigma_\eta^2/\sigma_\kappa^2$ is increasing, implying that the trend is relatively more volatile than the cycle in the first sample. We also observe that the GDP cycle is more persistent and lasts longer in the post-1982 period. For the IPI cycle, we only obtain significantly different estimates for the variances of the components. The estimates of the persistency parameter and the period of the cycle are not significantly different for the two subsamples. These results indicate that some of the model parameters are not constant over time. The approach to split the sample in two subsamples is ad-hoc and restrictive, since the sample may be partitioned in more subsamples. In the next section, we present the results of the trend-cycle model, for which the model parameters are estimated at any point in time.

4.3 Estimation results of the time-varying spectra

To obtain estimates of the time-varying patterns for the model parameters we first need to estimate the time-varying spectra for the two macroeconomic time series. The frequency grid $[0, \pi]$ is fragmented in $M = 100$ equidistant points. We have also considered other values for M, but larger values for M increase the length of the state vector and thus the computing time without changing our results. The initial time grid is fragmented such that about 40 observations are used to estimate the initial sample spectra. By setting other observations as missing values, we estimate the time-varying spectra of Δ^2 GDP and Δ^2 IPI, which are presented in Figures 2 and 3, respectively. The three-dimensional graphs show the considerable changes of the spectrum over time.

[insert Figure 2]

[insert Figure 3]

At the beginning of the sample, the spectra behave clearly differently than at the end of the sample. For both series it is reasonable that the time-varying spectrum does not have high values in the low frequencies since we estimate the spectrum of the twice differenced series. The estimated SS–ANOVA parameters and variance of the error are given in Table 4. The high values for ω_1 indicate that the model produces smooth estimates of the spectrum over time, whereas the low values for ω_2 imply that the estimated time-varying spectrum has a better fit of the data over the frequency axis.

[insert Table 4]

In the time domain, the time-varying spectra imply that the two twice differenced series exhibit different sample autocovariance/autocorrelation functions over time. To show this, we estimate the sample autocovariance function of the twice differenced series by taking the inverse Fourier transform of the spectrum at the beginning, in the middle and at the end of the sample. In Figure 4, we see that the sample autocorrelation functions are different for the different time points.

[insert Figure 4]

4.4 Time-varying parameter estimates for GDP series

Given the time-varying spectrum, we can now estimate the time-varying parameters of the trend-cycle model for the GDP series. We do not estimate the variance of the irregular term σ_ε^2, since these estimates are very small. Therefore, we set σ_ε^2 equal to zero. The estimated parameters and the 90% bootstrap confidence intervals are presented in Figure 5. In panel A and B, the time-varying patterns of the variances σ_η^2 and σ_κ^2 are shown respectively. The variance of the shocks in the trend shows a generally decreasing pattern. with a temporary increase at the end of the 1970s. A similar increase is found in the time-varying pattern of the variance in the cycle component. These increments in the variances are likely to be the result of large shocks in the economy due to the oil crises in the 1970s. We observe that the volatility of the cycle decreases rapidly after 1980. At the end of the sample, the variance of the cycle appears to increase slowly. Panel C shows that the trend-cycle ratio $\sigma_\eta^2/\sigma_\kappa^2$ is decreasing over time in a smooth way. The trend appears to be relatively less volatile over time than the cycle. Furthermore, it shows that the trend is more smooth in comparison to the cycle from 1980 onwards than in the years before 1980. The bootstrap confidence intervals show that the variances change significantly over time. The persistency parameter ρ, see panel D, decreases from the beginning of the sample until the 1980s and then increases until the end of the sample. We conclude that the cycle is more persistent in the last two decades. In panel E, we find evidence that the period of the cycle shows an increasing pattern, rising from three years in the beginning of the sample to more than ten years at the end of the sample. The lower limit of the bootstrap confidence interval of the damping factor shows a decreasing pattern, due to the increasing period and the small variance of the cycle. As these two factors cause that it is more difficult to identify the cycle and consequently, there is more uncertainty in the persistency parameter.

[insert Figure 5]

Summarising for the GDP series, we have found that the dynamic properties of the series are subject to changes over time. The variances of the trend and the cycle decline in the 1980s and the cycle is more persistent and lasts longer. These results provide further evidence for the existence of the *great moderation* in the business cycle. Following Stock & Watson (2003), we address the changes in the business cycle to the improved monetary policy in the 1980s, the developments of financial markets and the improved inventory management systems. In the mid 1980s, the monetary policy in the U.S. was mainly focused on the reduction of high inflation. One of the consequences of this policy is that the volatility in the GDP series has been reduced. The declined volatilities in both the trend and the cycle support this fact. Financial developments reduce the volatility in the business cycle due to improved access for firms to financial markets and new ways to hedge risks. Shocks in the investments of firms are smoothed by these developments. Since the GDP series includes investments of all firms, shocks in the aggregated series have been smoothed to some extent. The improved inventory management allows firms better usage of their inventories to smooth their production when the sales shows unexpected shifts. It is evident that the smoothed production also contributes to a decrease in volatility in the GDP series. These factors partly explain the decreasing pattern of the ratio $\sigma_\eta^2/\sigma_\kappa^2$. The developments of financial markets and the improved inventory management obviously smooth the shocks in the level of the GDP series directly. As a result, the decrease of the variance of the trend is larger than the decrease of the variance of the cycle. At the end of the sample, we see an increase in the volatility of the cycle. We may address this to the increased volatility in financial markets due to the collapse of the technology stock markets at the start of the new century.

Given the time-varying estimates of the model parameters, we extract the cycle based on the estimated parameters at the beginning, in the middle and at the end of the sample using the Kalman filter and smoother applied to the trend-cycle model (1)–(3). These three different cycles are shown in Figure 6, along with the implied theoretical autocorrelation function of the cycle, the implied weighting pattern to extract the cycle and the gain function corresponding to the filter used.

[insert Figure 6]

The cycle extracted with the parameters of 1947 occurs more frequently at the end of the sample than the cycle extracted with the 2006 parameters. The former has more peaks and troughs in the period between 1990 and 2006, while for the latter we just observe two global peaks in 1990 and 2000 and one global trough in the mid 1990s. This is not surprising, since the 2006-cycle is extracted given a longer period. Furthermore, the 1947-cycle indicates that there is a peak in 2004 and that the cycle is in the downturn in 2006. However, the 2006-cycle points out that there is negative growth in 2004 and that the cycle is in the upturn in 2006. This remarkable difference shows the importance of extracting the business cycle with the most actual estimates of the model parameters. We draw the same conclusions when we compare the 2006-cycle with the cycle of the time-invariant model. Moreover, we see that the theoretical autocorrelation functions and the gain functions of the cycles clearly differ at the three time points. The acf of the 1976-cycle shows a pattern which goes to zero rapidly implying that the cycle has short memory, i.e. shocks in the cycle die out quickly. The 2006-cycle has large values for both persistency and periodicity and this results in a more persistent pattern of the acf. Shocks in the cycle still have an effect on the cycle for a long time. For the analysis in the frequency domain we see that given the 2006-parameters, a gain function is obtained which approximates the ideal filter reasonably well. Nevertheless, at the beginning of the sample, the approximation to the ideal filter is poor. We draw these conclusions by considering the compression[2] effect, which is smallest for the parameters at the end of the sample and largest for those at the beginning of the sample.

4.5 Time-varying parameter estimates for IPI series

For the IPI series, we set the variance of the irregular term equal to zero as in the case of the GDP series. The estimation results for the IPI series are depicted in Figure 7. In panel A and B, we observe a generally decreasing pattern in the variances of the trend and the cycle. We can roughly identify three periods with different levels of the variances: 1947-1965, 1965-1985 and 1985-2006. The variances are highest in the first period and lowest in the last period. The change in the level of σ_κ^2 in the mid 1980s is in line with the GDP results. The bootstrap confidence intervals of the variances show that the variances vary over time significantly. These results also provide evidence of the great moderation in the business cycle. We address the decline of the volatilities to the three possible causes of Stock & Watson (2003). The policy

[2]The compression effect refers to the tendency for a filter to have less than a unit frequency response, i.e. the filter fails to retain the frequencies perfectly.

to reduce the inflation does not only affect the volatility in the GDP series, but affects the IPI series as well. It is obvious that shocks in the IPI series are smoothed by the financial developments and improved inventory management, since these developments affect the IPI series directly. In panel C, we see that the ratio $\sigma_\eta^2/\sigma_\kappa^2$ is decreasing over time, implying that the cycle is relatively more volatile than the trend. Noticing that the ratio is decreasing and that the cycle is less erratic at the end of the sample, we conclude that the decrease in the variance of the trend is substantial.

[insert Figure 7]

The persistency parameter ρ and the period of the cycle both display increasing patterns. The persistency parameter raises from 0.82 to 0.97, while the period of the IPI cycle varies between 1.5 and 7.5 years. However, from the bootstrap confidence interval of the persistency parameter we cannot conclude that it varies over time significantly. Here again, we see that the lower limit of the confidence interval decreases over time, due to the increasing period and the small variance of the cycle. We notice that the estimated period of the IPI cycle is shorter than the period of the GDP cycle. At the end of the sample, the IPI cycle lasts 7.5 years, while the GDP cycle lasts 15 years. The persistency parameter of the IPI cycle is increasing over the whole time grid, but the one of the GDP cycle is not. Although there are some differences in the time-varying patterns of the model parameters of the GDP series and the IPI series, we conclude that the estimation results of both series provide sufficient evidence for the great moderation in the business cycle.

In Figure 8, we compare the IPI cycles extracted with the time-varying parameters estimated at the beginning, the middle and the end of the sample. The cycle extracted at the end of the sample is less erratic than the cycle based on parameters at the two other points of the sample. This is obviously due to the longer period at the end of the sample. Furthermore, it is hard to distinguish the cycle at the beginning of the sample from a noise process.

[insert Figure 8]

The differences in the extracted cycles are also reflected in the theoretical autocorrelation function of the cycles. The parameters corresponding to the 2006-cycle imply a persistent acf, whereas the acf of the two other cycles goes to zero rapidly. The weighting patterns of the filter to extract the cycle component do not show remarkable differences. However, differences appear when we consider the gain functions. The time-varying model parameters correspond to filters with different frequency domain properties. We obtain the best approximation to the ideal filter at the end of the sample. The compression effect of the filter is largest at the beginning of the sample and smallest at the end of the sample.

5 Discussion and conclusion

In this paper we consider a frequency domain based method to estimate time-varying parameters of a trend-cycle model. We adopt the semi-parametric SS–ANOVA approach to estimate the time-varying spectrum at all points in time. When the spectrum in the spectral likelihood is allowed to vary over time, the spectral likelihood is also time-varying. We maximise the spectral likelihood at all points in time to obtain time-varying estimates of the model parameters. Furthermore, we use a frequency domain bootstrap to compute confidence intervals for the estimated time-varying parameters.

In our empirical study, we estimate time-varying parameters for two U.S. macroeconomic time series: Gross Domestic Product and Industrial Product Index. We found strong evidence that some of the model parameters are varying over time. The estimation results show a generally decreasing trend for the variance of both the trend and cycle component. This confirms that the business cycle is less volatile in the last decades. In addition, we show that the business cycle lasts longer and is more persistent. The bootstrap shows that the variances are significantly changing over time. With the time-varying estimates of the model parameters, we extract cycles with different dynamic properties over time. The implied theoretical autocorrelation functions of the cycle and the gain functions are also clearly varying over time.

In future research we will consider the extraction of smooth cycles from economic time series. Economists may prefer more smooth cycles, since the state of the economy is reflected more clearly. The cycle process in the trend-cycle model must be extended to a higher order cycle process, see Harvey & Trimbur (2003). This implies a gain function of the filter which has similar properties as the filter proposed by Baxter & King (1999). However, estimation of generalised trend-cycle models in the frequency domain is not straightforward and further research is required.

Appendix

A General Smoothing Spline ANOVA model

The general SS–ANOVA model considers the interaction between t and λ. Following Qin (2004, chapter 3) the unknown two-dimensional function $G(t, \lambda)$ has the following decomposition:

$$G(t, \lambda) = b_0 + b_1 t + b_2 \lambda + b_3 t\lambda + G_1(t) + G_2(\lambda) + G_3(t, \lambda) + G_4(t, \lambda) + G_5(t, \lambda), \qquad (A.1)$$

where $b_0 + b_1 t + b_2 \lambda + b_3 t\lambda$ is the parametric trend, $G_1(t)$ is the smooth main effect in t, $G_2(\lambda)$ is the smooth main effect in λ, $G_3(t, \lambda)$ is linear in λ and smooth in t, $G_4(t, \lambda)$ is linear in t and smooth in λ and $G_5(t, \lambda)$ is the smooth interaction in t and λ. The Bayesian stochastic model for $G(t, \lambda)$ is given by:

$$\begin{aligned} g(t, \lambda) = {} & b_0 + b_1 t + b_2 \lambda + b_3 t\lambda + \sqrt{\omega_1} \int_0^t (t-u) dW_1(u) \\ & + \sqrt{\omega_2} \int_0^\lambda (\lambda - u) dW_2(u) + \sqrt{\omega_3} \lambda \int_0^t (t-u) dW_3(u) \\ & + \sqrt{\omega_4} t \int_0^\lambda (\lambda - u) dW_4(u) + \sqrt{\omega_5} \int_0^t (t-u) dW_5(u) \int_0^\lambda (\lambda - u) dW_6(u), \end{aligned} \qquad (A.2)$$

where $W_k(u)$, $k = 1, \ldots, 6$ are mutually independent Wiener processes and $\omega_1, \ldots, \omega_5$ are the smoothing parameters. In order to set up the state space form of the SS–ANOVA model we need to formulate the recursive Bayesian model:

$$\begin{aligned} g(t_j, \lambda_k) &= g(t_{j-1}, \lambda_{k-1}) + \frac{\partial g(t_{j-1}, \lambda_{k-1})}{\partial t} \delta t_j + \frac{\partial g(t_{j-1}, \lambda_{k-1})}{\partial \lambda} \delta \lambda_k + \\ & \quad \frac{\partial g^2(t_{j-1}, \lambda_{k-1})}{\partial t \partial \lambda} \delta t_j \delta \lambda_k + e_{jk,1}, \\ \frac{\partial g(t_j, \lambda_k)}{\partial t} &= \frac{\partial g(t_{j-1}, \lambda_{k-1})}{\partial t} + \frac{\partial g^2(t_{j-1}, \lambda_{k-1})}{\partial t \partial \lambda} \delta \lambda_k + e_{jk,2}, \\ \frac{\partial g(t_j, \lambda_k)}{\partial \lambda} &= \frac{\partial g(t_{j-1}, \lambda_{k-1})}{\partial \lambda} + \frac{\partial g^2(t_{j-1}, \lambda_{k-1})}{\partial t \partial \lambda} \delta t_j + e_{jk,3}, \\ \frac{\partial g^2(t_j, \lambda_k)}{\partial t \partial \lambda} &= \frac{\partial g^2(t_{j-1}, \lambda_{k-1})}{\partial t \partial \lambda} + e_{jk,4}, \end{aligned} \qquad (A.3)$$

where $\delta t_j = t_j - t_{j-1} \geq 0$ and $\delta \lambda_k = \lambda_k = \lambda_{k-1} \geq 0$.

The error terms $e_{jk,i}, i = 1, \ldots, 4$ in the recursive Bayesian model are given by:

$$\begin{aligned}
e_{jk,1} &= \sqrt{\omega_1}\int_{t_{j-1}}^{t_j}(t_j - u)dW_1(u) + \sqrt{\omega_2}\int_{\lambda_{k-1}}^{\lambda_k}(\lambda_k - u)dW_2(u) \\
&+ \sqrt{\omega_3}\lambda_k\int_{t_{j-1}}^{t_j}(t_j - u)dW_3(u) + \sqrt{\omega_4}t_j\int_{\lambda_{k-1}}^{\lambda_k}(\lambda_k - u)dW_4(u) \\
&+ \sqrt{\omega_5}\int_0^{t_j}(t_j - u)dW_5(u)\int_0^{\lambda_k}(\lambda_k - u)dW_6(u) \\
&- \sqrt{\omega_5}\int_0^{t_{j-1}}(t_j - u)dW_5(u)\int_0^{\lambda_{k-1}}(\lambda_k - u)dW_6(u), \\
e_{jk,2} &= \sqrt{\omega_1}\int_{t_{j-1}}^{t_j}dW_1(u) + \sqrt{\omega_3}\lambda_k\int_{t_{j-1}}^{t_j}dW_3(u) + \sqrt{\omega_4}\int_{\lambda_{k-1}}^{\lambda_k}(\lambda_k - u)dW_4(u) \\
&+ \sqrt{\omega_5}\left(\int_0^{t_j}dW_5(u)\int_0^{\lambda_k}(\lambda_k - u)dW_6(u) - \int_0^{t_{j-1}}dW_5(u)\int_0^{\lambda_{k-1}}(\lambda_k - u)dW_6(u)\right), \\
e_{jk,3} &= \sqrt{\omega_2}\int_{\lambda_{k-1}}^{\lambda_k}dW_2(u) + \sqrt{\omega_3}\int_{t_{j-1}}^{t_j}(t_j - u)dW_3(u) + \sqrt{\omega_4}t_j\int_{\lambda_{k-1}}^{\lambda_k}dW_4(u) \\
&+ \sqrt{\omega_5}\left(\int_0^{t_j}(t_j - u)dW_5(u)\int_0^{\lambda_k}dW_6(u) - \int_0^{t_{j-1}}(t_j - u)dW_5(u)\int_0^{\lambda_{k-1}}dW_6(u)\right), \\
e_{jk,4} &= \sqrt{\omega_3}\int_{t_{j-1}}^{t_j}dW_3(u) + \sqrt{\omega_4}\int_{\lambda_{k-1}}^{\lambda_k}dW_4(u) \\
&+ \sqrt{\omega_5}\left(\int_0^{t_j}dW_5(u)\int_0^{\lambda_k}dW_6(u) - \int_0^{t_{j-1}}dW_5(u)\int_0^{\lambda_{k-1}}dW_6(u)\right).
\end{aligned}$$

Define the state vector $\boldsymbol{\alpha}(t_j, \boldsymbol{\lambda})$ by

$$\underset{(4M\times 1)}{\boldsymbol{\alpha}(t_j, \boldsymbol{\lambda})} = \begin{pmatrix} \alpha(t_j, \lambda_1) \\ \alpha(t_j, \lambda_2) \\ \vdots \\ \alpha(t_j, \lambda_M) \end{pmatrix}, \quad \alpha(t_j, \lambda_k) = \left(g(t_j, \lambda_k), \; \frac{\partial g(t_j, \lambda_k)}{\partial t}, \; \frac{\partial g(t_j, \lambda_k)}{\partial \lambda}, \; \frac{\partial g^2(t_j, \lambda_k)}{\partial t \partial \lambda}\right)'.$$

The multivariate state space model has the following representation:

$$\begin{aligned}
\boldsymbol{x}_j &= Z\boldsymbol{\alpha}(t_j, \boldsymbol{\lambda}) + \boldsymbol{\zeta}_j, \quad \boldsymbol{\zeta}_j \sim N(0, G), \qquad j = 1, \ldots, N, \\
\boldsymbol{\alpha}(t_j, \boldsymbol{\lambda}) &= T_j\boldsymbol{\alpha}(t_{j-1}, \boldsymbol{\lambda}) + \boldsymbol{\xi}_j, \quad \boldsymbol{\xi}_j \sim N(0, H_j),
\end{aligned} \tag{A.4}$$

where the vector $\boldsymbol{x}_j$ is the log sample spectrum of the time series at time t_j in the initial time grid. The error vector in the state equation is given by

$$\begin{aligned}
\underset{(4M\times 1)}{\boldsymbol{\xi}_j} &= \left(\xi_{j1}, \; \xi_{j2} \; \cdots \; \xi_{jM}\right)', \quad j = 1, \ldots, N, \\
\xi_{jk} &= \left(e_{jk,1}, \; e_{jk,2}, \; e_{jk,3}, \; e_{jk,4}\right)', \quad k = 1, \ldots, M.
\end{aligned} \tag{A.5}$$

Note that the error terms $e_{jk,i}, i = 1, \ldots, 4$, are included in the vector $\boldsymbol{\xi}_j$. The time-invariant system matrices Z and G and the time-varying system matrices T_j and H_j are given by:

$$Z = \mathbf{I}_M \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \quad G = \sigma_\zeta^2 \mathbf{I}_M,$$

$$T_j = \text{diag}\,[T_{j1}, \ldots, T_{jM}] \quad \text{with } T_{jk} = \begin{pmatrix} 1 & \delta t_j & \delta\lambda_k & \delta t_j \delta\lambda_k \\ 0 & 1 & 0 & \delta\lambda_k \\ 0 & 0 & 1 & \delta t_j \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$H_j = \begin{pmatrix} H_j^{1,1} & H_j^{1,2} & \cdots & H_j^{1,M} \\ H_j^{2,1} & H_j^{2,2} & \cdots & H_j^{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ H_j^{M,1} & H_j^{M,2} & \cdots & H_j^{M,M} \end{pmatrix},$$

where the operator $\otimes$ denotes the Kronecker product, diag$[\cdot\ \cdot]$ refers to a block diagonal matrix, $\delta t_j = t_j - t_{j-1}$ is and $\delta\lambda_k = \lambda_k - \lambda_{k-1}$. To derive the blocks $H_j^{a,b}$, $a, b = 1, \ldots, M$, we have to derive the covariance matrix of ξ_{ja} and ξ_{jb} in (A.5). The expressions for these blocks can be found in Qin (2004).

B SS–ANOVA model without interaction

The Bayesian stochastic model for the two-dimensional function $G(t, \lambda)$ in the SS–ANOVA model which omits the interaction between t and λ is given by

$$g(t, \lambda) = \sqrt{\omega_1} \int_0^t (t - u) dW_1(u) + \sqrt{\omega_2} \int_0^\lambda (\lambda - u) dW_2(u), \tag{B.1}$$

where $W_1(u)$ and $W_2(u)$ are mutually independent Wiener processes and ω_1, ω_2 are the smoothing parameters of the spline functions. Some straightforward derivations lead to the recursive Bayesian model

$$\begin{aligned} g(t_j, \lambda_k) &= g(t_{j-1}, \lambda_{k-1}) + \frac{\partial g(t_{j-1}, \lambda_{k-1})}{\partial t} \delta t_j + \frac{\partial g(t_{j-1}, \lambda_{k-1})}{\partial \lambda} \delta\lambda_k + e_{jk,1}, \\ \frac{\partial g(t_j, \lambda_k)}{\partial t} &= \frac{\partial g(t_{j-1}, \lambda_{k-1})}{\partial t} + e_{jk,2}, \\ \frac{\partial g(t_j, \lambda_k)}{\partial \lambda} &= \frac{\partial g(t_{j-1}, \lambda_{k-1})}{\partial \lambda} + e_{jk,3}, \end{aligned} \tag{B.2}$$

where

$$\begin{aligned} e_{jk,1} &= \sqrt{\omega_1} \int_{t_{j-1}}^{t_j} (t_j - u) dW_1(u) + \sqrt{\omega_2} \int_{\lambda_{k-1}}^{\lambda_k} (\lambda_k - u) dW_2(u) \\ e_{jk,2} &= \sqrt{\omega_1} \int_{t_{j-1}}^{t_j} dW_1(u), \quad e_{jk,3} = \sqrt{\omega_2} \int_{\lambda_{k-1}}^{\lambda_k} dW_2(u). \end{aligned} \tag{B.3}$$

Equations (B.2) and (B.3) are used to formulate the state equation of the state space model, i.e. the equations in (B.2) represents the Markovian structure of the state equation and the e_{jk}'s are the error terms of the state equation.

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Table 1: Description of the dataset. All series are obtained form the Federal Reserve Bank of St. Louis, see `http://research.stlouisfed.org`.

series	frequency	data range	description
GDP	quarterly	1947:1–2006:1	log of the U.S. Real Gross Domestic Product; seasonally adjusted.
IPI	monthly	1947:1–2006:4	log of the U.S. Industrial Production Index; Index 2002=100; seasonally adjusted.

Table 2: Estimation results of the time-invariant trend-cycle model for GDP and IPI. 90% bootstrap confidence intervals are given between brackets.

	GDP	IPI
σ_η^2	3.134×10^{-6} [1.448×10^{-6}; 5.252×10^{-6}]	1.702×10^{-6} [1.771×10^{-8}; 3.945×10^{-7}]
σ_κ^2	4.698×10^{-5} [2.758×10^{-5}; 5.470×10^{-5}]	6.754×10^{-5} [5.577×10^{-5}; 7.891×10^{-5}]
σ_ε^2	3.020×10^{-7} [1.990×10^{-14}; 8.723×10^{-6}]	1.607×10^{-13} [1.607×10^{-13}; 1.607×10^{-13}]
$\sigma_\eta^2/\sigma_\kappa^2$	0.067	0.0025
ρ	0.904 [0.863; 0.942]	0.973 [0.935; 0.986]
period in years	3.404 [2.969; 3.911]	3.709 [3.040; 4.813]

Table 3: Estimation results of the time-invariant model on two subsamples: 1947-1981 (pre-82) and 1982-2006 (post-82) for GDP and IPI. The 90% bootstrap confidence intervals are given between brackets.

	GDP		IPI	
	pre-82	post-82	pre-82	post-82
σ_η^2	4.291×10^{-6} $[3.541\times10^{-6};\ 5.761\times10^{-6}]$	2.286×10^{-6} $[2.023\times10^{-6};\ 3.256\times10^{-6}]$	5.995×10^{-7} $[1.616\times10^{-7};\ 1.006\times10^{-6}]$	1.756×10^{-7} $[4.677\times10^{-8};\ 2.767\times10^{-7}]$
σ_κ^2	3.747×10^{-5} $[2.178\times10^{-5};\ 3.691\times10^{-5}]$	3.058×10^{-6} $[7.498\times10^{-7};\ 4.099\times10^{-6}]$	5.128×10^{-5} $[3.888\times10^{-5};\ 6.353\times10^{-5}]$	5.800×10^{-6} $[3.447\times10^{-6};\ 8.468\times10^{-6}]$
σ_ε^2	1.238×10^{-6} $[7.295\times10^{-7};\ 5.834\times10^{-6}]$	8.521×10^{-7} $[6.594\times10^{-7};\ 1.596\times10^{-6}]$	2.80×10^{-13} $[2.797\times10^{-13};\ 2.797\times10^{-13}]$	1.644×10^{-6} $[1.183\times10^{-6};\ 2.430\times10^{-6}]$
$\sigma_\eta^2/\sigma_\kappa^2$	0.114	0.747	0.012	0.030
ρ	0.906 [0.892; 0.935]	0.965 [0.944; 0.990]	0.964 [0.960; 0.971]	0.964 [0.955; 0.981]
period in years	3.222 [2.807; 3.542]	5.294 [4.686; 5.695]	2.924 [2.638; 4.210]	3.218 [2.736; 4.879]

Table 4: Estimation results of the SS–ANOVA model with $M = 100$. The two smoothing spline parameters ω_1, ω_2 and the variance σ_ζ^2 are estimated.

	ω_1	ω_2	σ_ζ^2
GDP	922.594	1.563	0.365
IPI	119481.417	0.000046	0.431

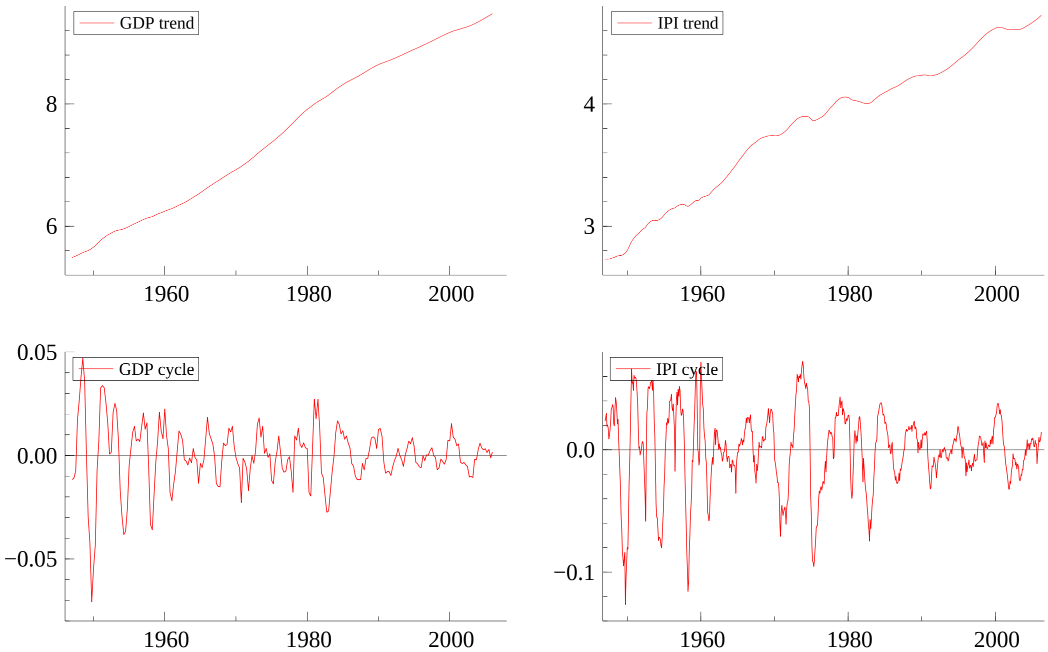

Figure 1: Estimated trend and cycle components of time-invariant model for GDP and IPI. Trend and cycle components of GDP and IPI are depicted on the left and the right, respectively.

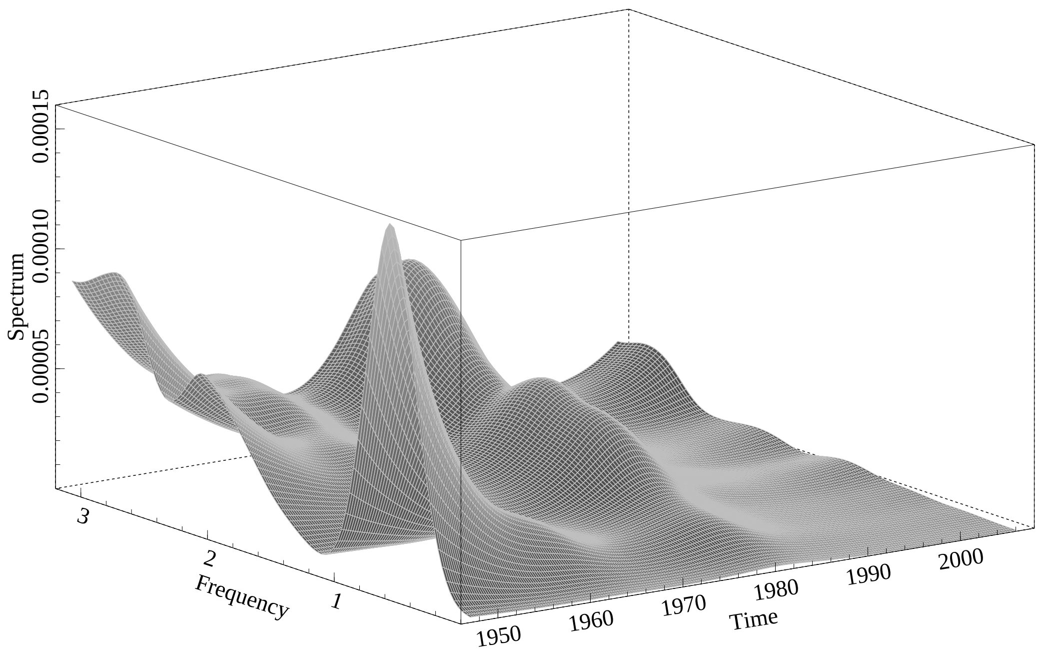

Figure 2: Estimated time-varying spectrum of the GDP series. Spectrum is estimated on a 100 × 237 grid.

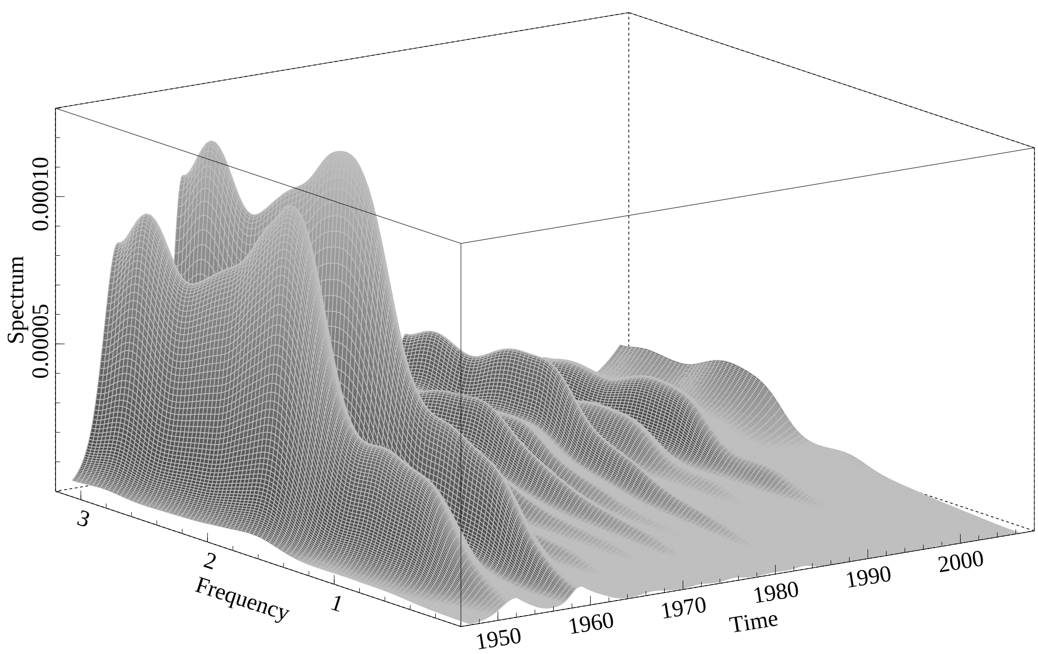

Figure 3: Estimated time-varying spectrum of the IPI series. Spectrum is estimated on a 100 × 712 grid.

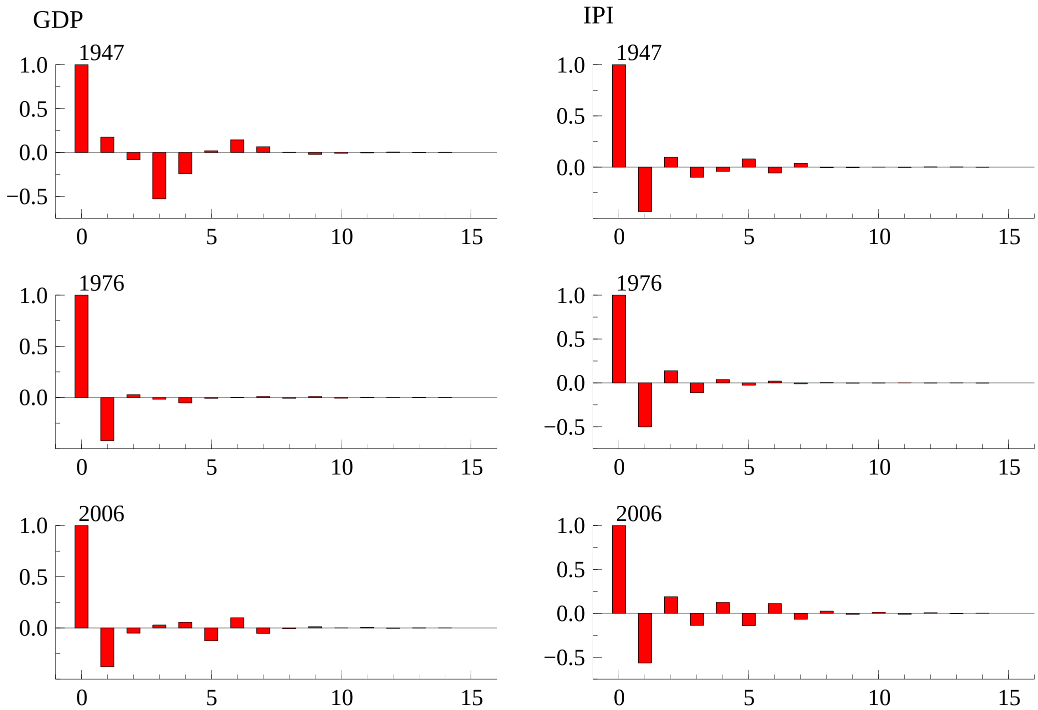

Figure 4: Estimated sample autocorrelation function of Δ^2 GDP (left) and Δ^2 IPI (right). The sample ACF's are implied by the estimated time-varying spectrum. The depicted sample ACF's correspond to the beginning, the middle and the end of the sample.

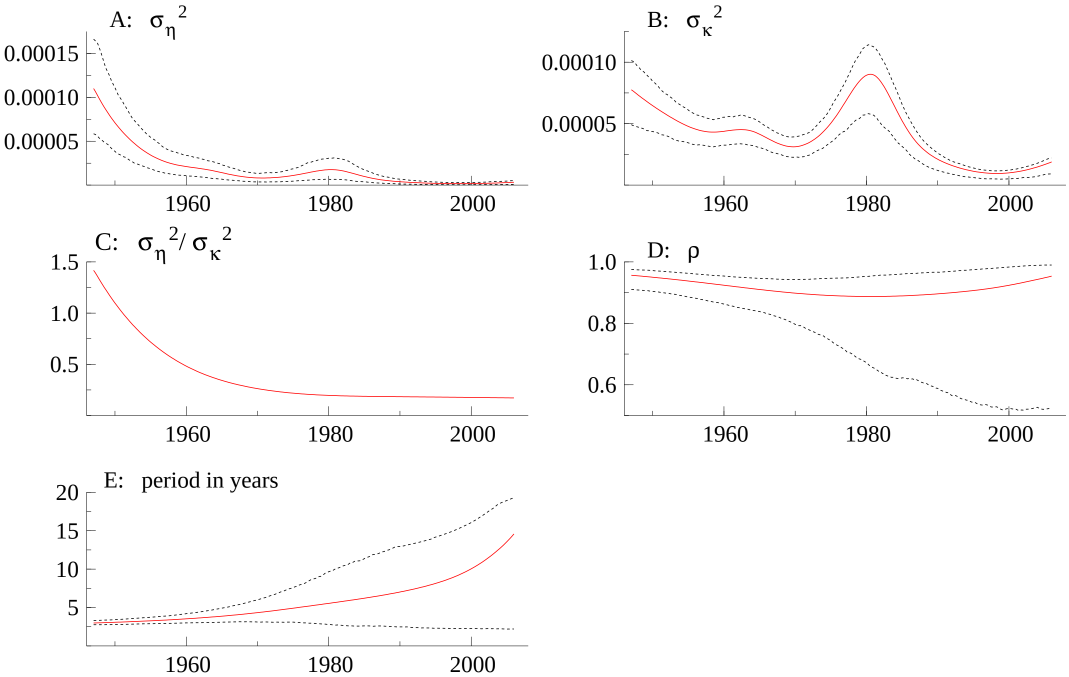

Figure 5: Estimated time-varying model parameters for the GDP series with the 90% bootstrap confidence intervals (dotted lines). The variances of the trend and cycle components are given in panel A, B. Panel C shows the ratio $\sigma_\eta^2/\sigma_\kappa^2$, while panel D and E present the damping factor of the cycle and the period in years of the cycle, respectively.

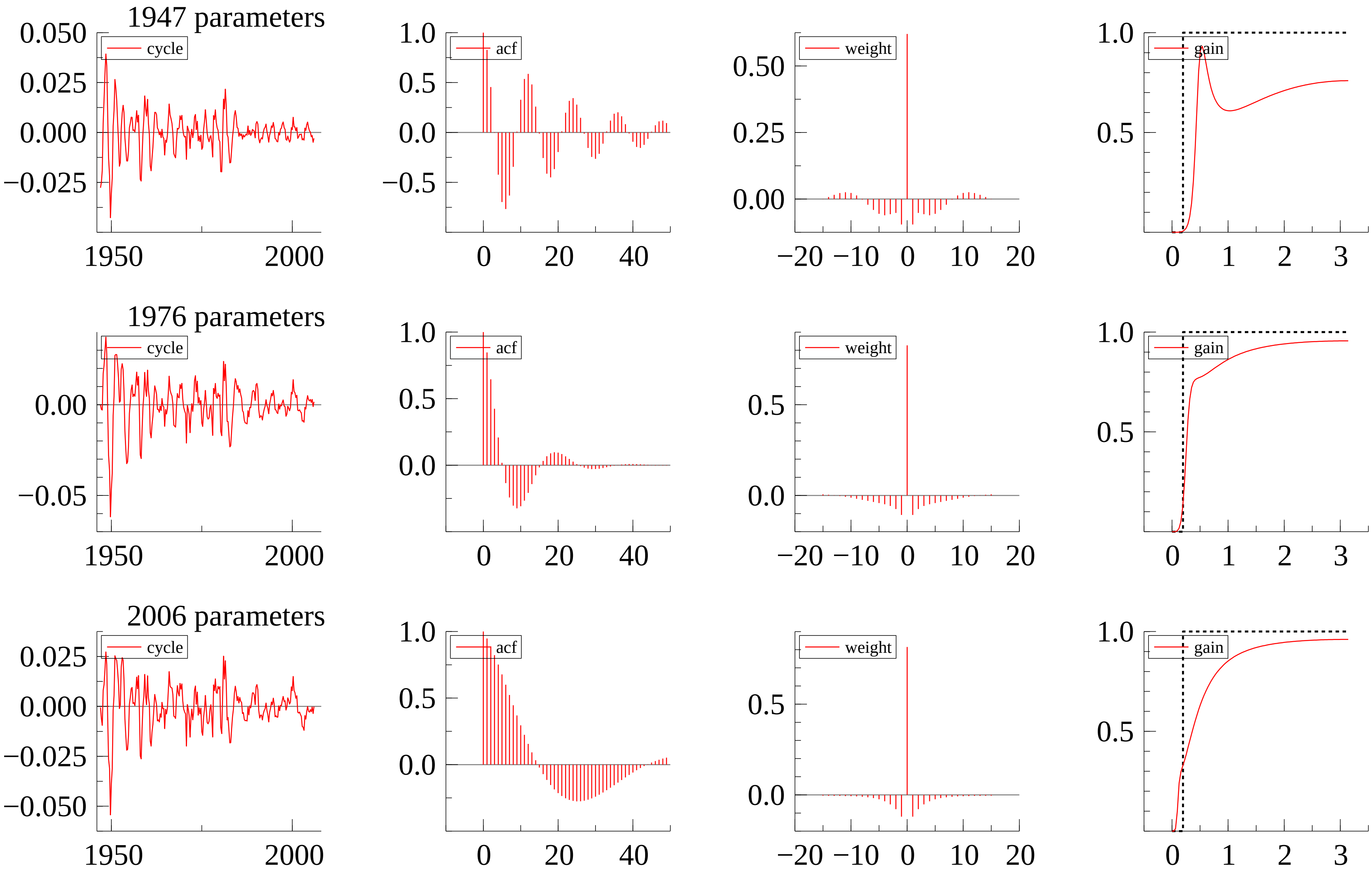

Figure 6: Extracted cycles of the GDP series from the time-varying model along with the theoretical autocorrelation function, weighting pattern and gain function. The first, second and third row of the figure correspond to the parameters estimated at the beginning, the middle and the end of the sample, respectively.

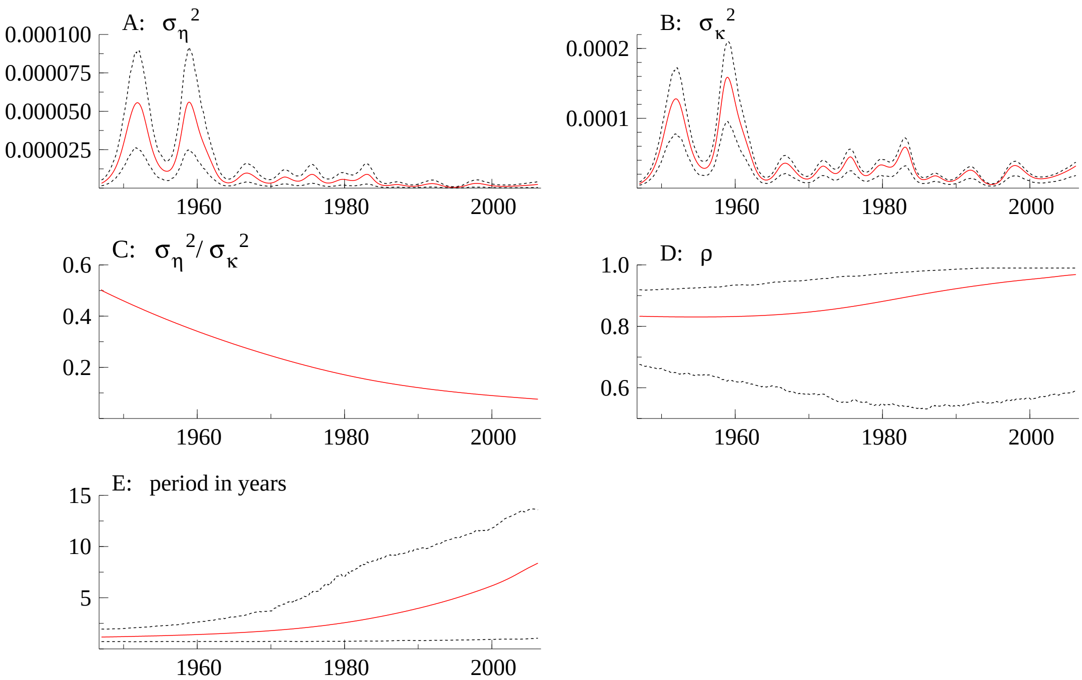

Figure 7: Estimated time-varying model parameters of the IPI series with the 90% bootstrap confidence intervals (dotted lines). The variances of the trend and cycle components are given in panel A, B. Panel C shows the ratio $\sigma_\eta^2/\sigma_\kappa^2$, while panel D and E present the damping factor of the cycle and the period in years of the cycle, respectively.

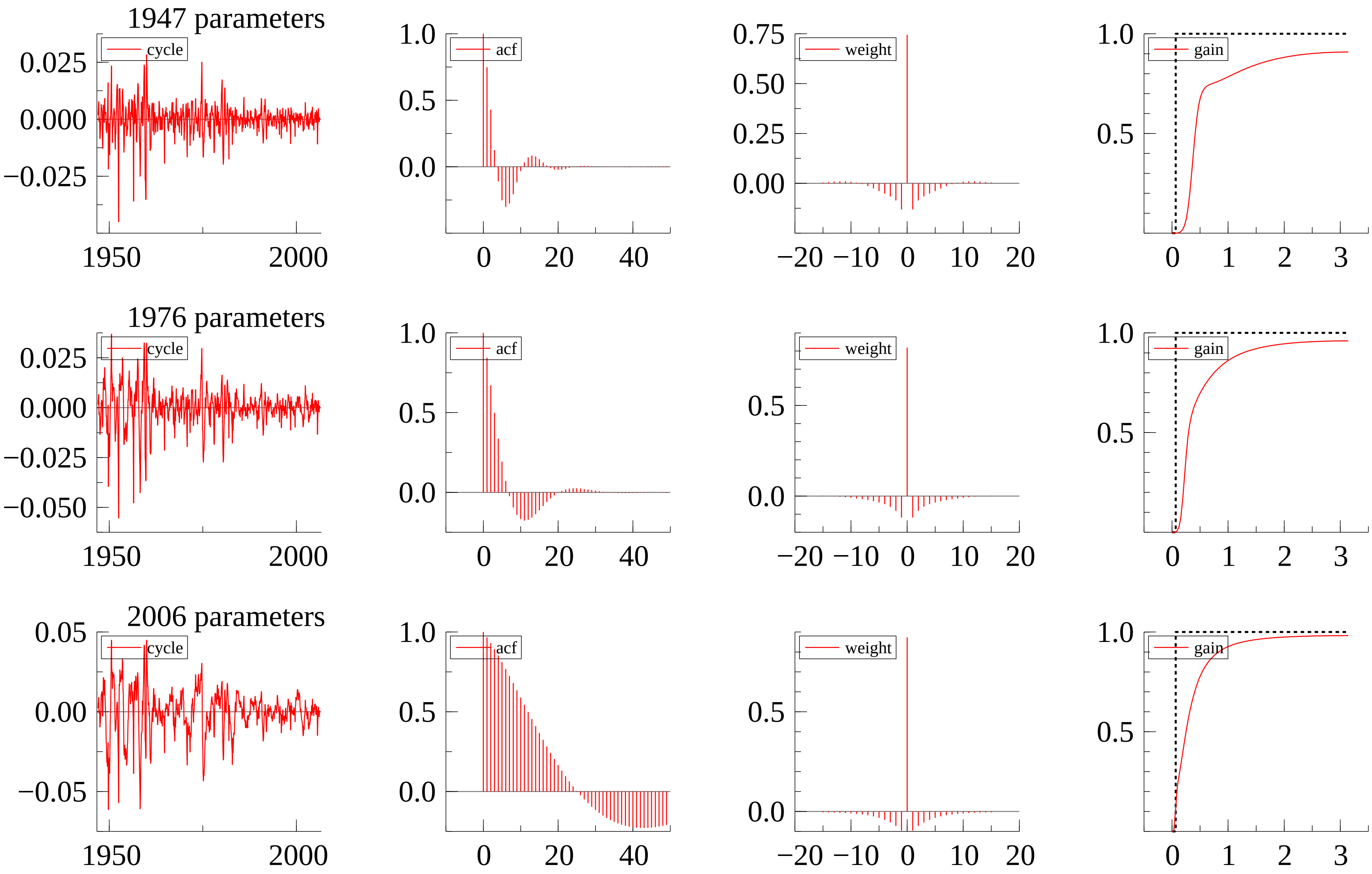

Figure 8: Extracted cycles of the IPI series from the time-varying model along with the theoretical autocorrelation function, weighting pattern and gain function. The first, second and third row of the figure correspond to the parameters estimated at the beginning, the middle and the end of the sample, respectively.