TI 2006-028/1
Tinbergen Institute Discussion Paper

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Perfect Equilibria in a Negotiation Model with Different Time Preferences*

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March 2006

Abstract

There has been a long debate on equilibrium characterization in the negotiation model when players have different time preferences. We show that players behave quite differently under different time preferences than under common time preferences. Conventional analysis in this literature relies on the key assumption that all continuation payoffs are bounded from above by the bargaining frontier. However, when players have different time preferences, intertemporal trade may lead to continuation payoffs above the bargaining frontier. We provide a thorough study of this problem without imposing the conventional assumption. Our results tie up all the previous findings, and also clarify the controversies that arose in the past.

JEL Classification: C72 Noncooperative Games, C73 Stochastic and Dynamic Games, C78 Bargaining Theory

Keywords: Bargaining, Negotiation, Time Preference, Endogenous Threats

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*This project is supported by Netherlands Organization for Scientific Research (NWO), grant B45-271. Quan Wen is grateful to Gerard van der Laan, the Tinbergen Institute and the Vrije Universiteit for their hospitality and generosity.

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1 Introduction

Endogenous threats are an essential constituent of bargaining problems, as emphasized in Nash (1953) at the dawn of modern bargaining theory. The bargaining literature in the 1990s successfully incorporates endogenous threats into the alternating-offer bargaining model of Rubinstein (1982). The early contributions in this area, such as Fernandez and Glazer (1991), Haller (1991) and Haller and Holden (1990), study the selection of industrial action by a union during its contract negotiation with a firm. In contrast to Rubinstein (1982), after a proposal is rejected, the union needs to decide what course of industrial action to take before the next bargaining round.[1] Later contributions, such as Busch and Wen (1995, 2001), Houba (1997), and Slantchev (2003), allow for more general forms of endogenous threats, modeled as a normal-form game, called the *disagreement game*, to be played between offers and counter offers.[2] Except Fernandez and Glazer (1991), Muthoo (1999) and Slantchev (2003), all the above mentioned work treats the case of common times preferences.

Despite our well-understanding on the negotiation model with common time preferences, there has been a controversy on the equilibrium characterization in this model when time preferences differ. This class of models generally admits multiple equilibria and the set of the equilibria is fully characterized by so-called *extremal* equilibria that yield the lowest or highest equilibrium payoffs to a player. Bolt (1995) demonstrates that the strategy profile supporting the firm's worst equilibrium provided by Fernandez and Glazer (1991) fails to be an equilibrium when the firm is less patient than the union. He then provides a no-concession strategy to the firm and shows that it can be sustained in equilibrium. Recently, one of the claims by Slantchev (2003) suggests that the firm's no-concession strategy always supports the firm's worst equilibrium when the firm is less patient than the union. This implication, however, contradicts another finding reported in Bolt (1993) that an always-

[1] Fernandez and Glazer (1991), Haller (1991) and Haller and Holden (1990) consider two industrial actions. Houba and Bolt (2000) consider the strategic substitutability among several forms of industrial actions by the union.

[2] This negotiation model is surveyed in Muthoo (1999) and Houba and Bolt (2002).

strike strategy sometimes yields a even lower payoff to the firm. Instead of invoking the technique of Shaked and Sutton (1984) to derive extremal equilibria in this model, Bolt (1995) and Slantchev (2003) simply verify whether a given strategy profile constitutes an equilibrium. Muthoo (1999) notices the necessity to apply the technique of Shaked and Sutton (1984) in his study on the negotiation model with different time preferences, but seems unaware of this controversy.

We treat this controversy seriously, not only to settle the open issue of the extremal equilibria, but more importantly, to reexamine the methodology used in previous studies. For more than twenty years, the backward induction technique of Shaked and Sutton (1984) has been proven to be a very powerful and effective tool in studying bargaining problems. Application of this technique relies on all possible continuation payoffs being bounded by the bargaining frontier as specified in the bargaining problem, which holds in most bargaining problems. However, if some continuation payoffs lie above the bargaining frontier, there will be no mutually acceptable agreement available. In deriving the bounds of equilibrium payoffs, it is often treated as a fact that players always reach some agreement (in every subgame), such as in the original study of Shaked and Sutton (1984). For the model of Rubinstein (1982), it is without any loss of generality to assume that only acceptable offers count, as is demonstrated by Fudenberg and Tirole (1991), who incorporate the possibility of making unacceptable offers into this technique when discussing this model.

For the negotiation model, all continuation payoffs are bounded by the bargaining frontier under *common* time preferences. However, this is definitely not the case when the players have different time preferences. As been realized in other dynamic problems, Pareto improvement is possible through intertemporal trade among agents with different time preferences, see e.g., Ramsey (1928), Bewley (1972) and, more recently, Lehrer and Pauzner (1999). In the context of repeated games, Lehrer and Pauzner (1999) demonstrate that there are many individually-rational feasible payoffs outside the conventional (convex) set of 'feasible payoffs'.[3] What matters is that, under different discount factors, an infinite sequence of two payoff vectors does not lead to a trivial convex combination of these vectors. Unlike repeated games, feasible outcome paths in bargaining models are less flexible, because stationary agreements, by default, cease any future payoff variation. This is not a serious issue in the alternating-offer model of Rubinstein (1982), because all possible outcome paths are dominated by some immediate agreement. However, in the negotiation model it is possible for both players to benefit from playing some endogenous disagreement outcomes for some periods prior to an agreement that rewards the more patient player. Those benefits can be so dramatic that the resulting continuation payoff vector is above the bargaining frontier. This fact is exactly what has been overlooked in the negotiation model and the root of the controversy in the current literature.

Our analysis of the negotiation model avoids making any assumption on continuation payoffs. In order to provide a clear-cut demonstration of the issues and its resolution, we consider common interest disagreement games where there is a Pareto dominant disagreement outcome.[4] This class of negotiation models includes the models in Fernandez and Glazer (1991) and Slantchev (2003) as special cases, which are at the center of this controversy. In doing so, we are able to clarify why there is a problem in the less patient player's worst equilibrium. After identifying the source of the controversy, we demonstrate how the backward induction technique of Shaked and Sutton (1984) works with the possibility of unacceptable offers. These considerations significantly affect the nature of the backward induction argument, in the sense that we may have to trace how players behave in extremal equilibria for *more* than two periods. We show that, except in the less patient players' worst equilibrium, every player will make acceptable offers along the course of equilibrium play. This partly validates those results obtained under the presumption that all possible continuation payoffs are bounded by the bargaining frontier.

[3] Lehrer and Pauzner (1999) show not all individually rational payoffs can be supported in equilibrium, because every subsequence also has to be individually rational.

[4] Common interest games have been studied in other dynamic settings, see, e.g., Farrel and Saloner (1985) and Takahashi (2005).

Complications arise in deriving the less patient player's worst equilibrium. In this case, continuation payoffs in many subgames can be above the bargaining frontier so that there are no mutually acceptable agreements available. In the less patient player's worst equilibrium, we show that if delay happens, it will involve only an even number of periods prior to agreement. In this way, the less patient player is unable to exploit his advantage to propose at the time of the equilibrium agreement. Due to the complicated nature of the problem, we no long can have a closed-form solution to the less patient player's lowest equilibrium payoff. Instead, we first show that the less patient player's equilibrium payoffs are bounded from below by the least fixed point of a well-defined minimax problem. We then provide an equilibrium strategy profile, which is novel to the literature, to support such a least fixed point. In other words, this equilibrium is indeed the worst equilibrium to the less patient player. Our results also shed light on why those complications do not arise in Rubinstein (1982) even when players have different time preferences.

This paper is organized as follows. In Section 2, we introduce the negotiation model, summarize some undisputed results, and illustrate why continuation payoffs can be above the bargaining frontier if and only if the two players have different time preferences. We then provide an example in Section 3 to demonstrate some of the unsettled issues in this model. The analysis is partitioned into two sections. In Section 4 we derive a set of necessary conditions to characterize extremal equilibria in this model without relying on the conventional assumption discussed. We also show that, except in the less patient player's worst equilibrium, players behave similarly as in the case of common time preferences. In Section 5, we resolve the complications involved with the worst equilibrium payoff for the less patient player: First we show that equilibrium payoffs to this player are bounded from below by the least fixed point of a well-defined minimax problem. Then we provide an equilibrium strategy profile supporting the least fixed point, which demonstrates that it is the worst equilibrium to the less patient player. Section 6 offers some concluding remarks.

2 The Model and Pareto Efficiency

Consider the negotiation model in which two players, named 1 and 2, negotiate how to split an infinite stream of surpluses of certain value, all normalized to be 1 per period. In any period before reaching an agreement, one player makes a proposal on how to split this normalized value in all future periods and the other player either accepts or rejects the proposal. If the proposal is accepted, then it will be implemented immediately, which ceases any future strategic interaction between the players. If the proposal is rejected, then both players will have to play a disagreement game before the negotiations proceed to the following period.

More specifically, the model consists of infinitely many periods where player 1 proposes in all odd periods and player 2 proposes in all even periods. A proposal is a stationary contract and is denoted as

$$x = (x_1, x_2) \in \Delta = \left\{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 = 1\right\}$$

from which player i's payoff is x_i in every period after both players agree on $x \in \Delta$ (the unit simplex). We call Δ the *bargaining* frontier. The disagreement game is given in normal form:

$$G = \{A_1, A_2, d_1(\cdot), d_2(\cdot)\},$$

where A_i is the set of player i's disagreement actions that is assumed to be non-empty and compact, and $d_i(\cdot) : A \to \mathbb{R}$ is player i's disagreement payoff function that is assumed to be continuous, where $A = A_1 \times A_2$ is the set of disagreement outcomes. We also assume $d_1(a) + d_2(a) \leq 1$ meaning every disagreement outcome is weakly dominated by some agreement. To ease exposition, we denote player i's highest disagreement payoff when he deviates from $a \in A$ unilaterally by

$$g_i(a) = \max_{a_i' \in A_i} d_i(a_i', a_j).$$

Without loss of generality, every player's minimax value in G is normalized to be zero, i.e.,

$$\min_{a_j \in A_j} \max_{a_i \in A_i} d_i(a) = \min_{a \in A} g_i(a) = 0.$$

We consider *common interest* disagreement games with a unique dominant outcome. This model includes the class of negotiation models studied in Fernandez and Glazer (1991), Haller and Holden (1990), Bolt (1995), and Slantchev (2003). Formally, there is an $a^* \in A$ such that $d(a^*) \geq d(a)$ for all $a \in A$. Obviously, $a^* \in A$ is a Nash equilibrium in G. Without loss of generality, we assume that $d(a^*)$ is on the bargaining frontier Δ, i.e. $d_1(a^*) + d_2(a^*) = 1$. This last assumption does not qualitatively change our analysis, as we will argue in the concluding remarks.

A generic outcome path, denoted by $\pi = \left(a^1, a^2, \cdots, a^T, x\right)$ for $T \geq 0$, consists of all disagreement outcomes ($a^t \in A$ in period t for $t \leq T$) before the agreement $x \in \Delta$ is reached in period $T + 1$.[5] By convention, $T = \infty$ is an outcome path with perpetual disagreement. From such an outcome path π, player i's intertemporal time preferences are represented by the sum of his discounted payoffs from the disagreement game before the agreement and the agreement itself afterward:

$$(1 - \delta_i) \sum_{t=1}^{T} \delta_i^{t-1} d_i\left(a^t\right) + \delta_i^T x_i, \tag{1}$$

where $\delta_i \in (0, 1)$ represents player i's discount factor per period.

The negotiation model described so-far is a well-defined noncooperative game of complete information. A history is a complete description of how the game has been played up to a period. A player's strategy specifies one appropriate action for every finite history. Every strategy profile induces a unique outcome path and players evaluate their strategies based on their discounted payoffs from the induced outcome path. The equilibrium concept applied throughout this paper is subgame perfect equilibrium (SPE).

Next, we summarize some undisputed results for this model from previous studies in the form of a proposition for later reference. We state these results without proof and in terms of player $i \in \{1, 2\}$, while refer to his opponent as player $j \neq i$.[6]

[5] When $T = 0$, the outcome path specifies immediate agreement.

[6] Although $d_i(a^*) = 1 - d_j(a^*)$, one should not interchange them since most results we state in their current forms hold even if $d_i(a^*) + d_j(a^*) < 1$.

Proposition 1 *In the negotiation model, we have:*

(i) for all $(\delta_i, \delta_j) \in (0,1)^2$*, there is a stationary SPE where player* i *receives* $d_i(a^*)$ *in every period;*

(ii) for sufficiently large $(\delta_i, \delta_j) \in (0,1)^2$ *and* $\delta_i \geq \delta_j$*, there is a SPE where player* i *receives*

$$\frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j(a^*)\right] \text{ when player } i \text{ proposes and}$$
$$\frac{\delta_i(1-\delta_j)}{1-\delta_i\delta_j}\left[1-d_j(a^*)\right] \text{ when player } j \text{ proposes;}$$

(iii) for sufficiently large $(\delta_i, \delta_j) \in (0,1)^2$ *and* $\delta_i < \delta_j$*, there is a SPE where player* i *receives*

$$\frac{1}{1+\delta_i}d_i(a^*) \text{ when player } i \text{ proposes and}$$
$$\frac{\delta_i}{1+\delta_i}d_i(a^*) \text{ when player } j \text{ proposes.}$$

Slantchev (2003) claims that the non-stationary SPE of this proposition are the extremal SPE where player i receives his lowest SPE payoff.

Proposition 1 asserts that the model always has multiple equilibria when the players are sufficiently patient. In particular, there is a positive gap between a player's highest and lowest SPE payoff. As discussed in the introduction, what is less clear is a full characterization of the set of SPE payoffs when the discount factors are sufficiently large.

The Pareto frontier has not yet received enough attention in this type of model. In the alternating offer model of Rubinstein (1982), this issue is trivial, because the Pareto frontier coincides with the bargaining frontier. In the negotiation model, however, this is no longer the case. In the context of repeated games, Lehrer and Pauzner (1999) show that a sequence of two payoff vectors may not lead to a trivial convex combination of the two payoff vectors when the players have different time preferences. Pareto improvement can be realized if the less patient player trades his long-run payoffs for short-run payoffs.[7] In

[7] The results in e.g. Ramsey (1928) and Bewley (1972) indicate that similar insights already have a long history in economics.

fact, they demonstrate that when two players have different time preferences, many subgame perfect equilibrium payoffs in a repeated game are not in the set of feasible and individually rational payoffs, as traditionally defined. In the negotiation model, such Pareto improvement is also present, but it cannot be realized within stationary contracts. Consider an outcome path consisting of $a \in A$ for $T-1$ periods followed agreement on $(x_i, 1-x_i)$ in period T. The sum of their payoffs is equal to

$$d_i(a) + d_j(a) + \delta_i^T (x_i - d_i(a)) + \delta_j^T (1 - d_j(a) - x_i), \tag{2}$$

which can be greater than 1, such as for $x_i < d_i(a)$ and δ_j sufficiently large.[8] Applying the ideas of Lehrer and Pauzner (1999) in the negotiation model, we must first maximize (2) in order to obtain the Pareto frontier. This requires that $a = a^*$, so that (2) becomes

$$1 + \left(\delta_j^T - \delta_i^T\right)(d_i(a^*) - x_i), \tag{3}$$

and $x_i = 0$ whenever $\delta_i < \delta_j$, or $x_i = 1$ whenever $\delta_i > \delta_j$. Next, we convexify the payoff vectors for two consecutive values of T: play a^* for $T \geq 0$ periods with probability $1-p \in [0,1]$ and for $T+1$ periods with probability p. As a result, we obtain a piecewise linear curve with infinitely many segments. It becomes a concave and smooth curve as both δ_i and δ_j go to 1, see Lehrer and Pauzner (1999) for details. Note that the nonlinear curve only represents the part of the Pareto frontier that lies above the bargaining frontier, while the remaining part coincides with the bargaining frontier, as illustrated in Figure 1. The set of feasible payoffs is not convex and the entire Pareto frontier cannot be described by a concave function, which is quite different from the standard assumptions in the bargaining literature. If and only if $\delta_i = \delta_j$, the Pareto frontier is the same as the bargaining frontier.

As Fudenberg and Tirole (1991) show for the alternating offer model, it is without loss of generality to assume that all continuation payoffs are bounded by the bargaining frontier when applying the method of Shaked and Sutton (1984) to derive the bounds on equilibrium payoffs. This is not the case in the negotiation model since continuation payoffs can lie

[8] (2) is also greater than 1 for $1 - x_i < d_j(a)$ and δ_i sufficiently large.

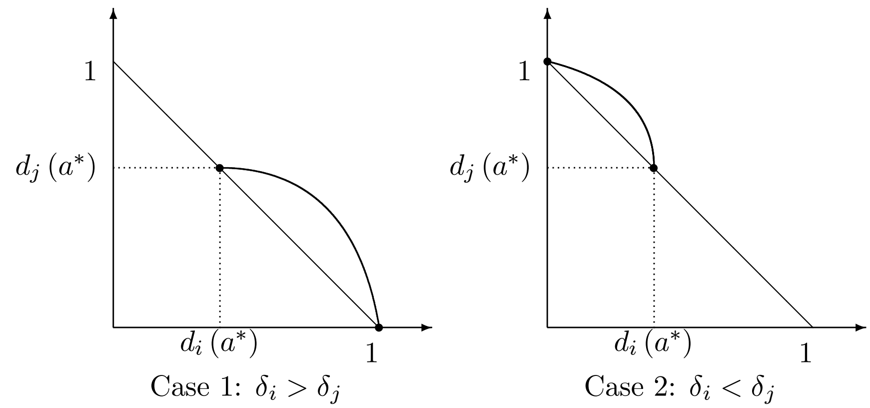

Figure 1: The Pareto frontier for $\delta_i > \delta_j$ and $\delta_i < \delta_j$.

above the bargaining frontier when players have different time preferences, which is why the conventional analysis breaks down. Furthermore, as we will make clear later, neither $x_i = 0$ or $x_i = 1$ corresponds to a SPE agreement. This hints at that Pareto efficient outcome paths may not be achievable in equilibrium. Instead, in equilibrium we need to consider continuations with x_i corresponding to extremal equilibrium agreements.

3 An Example

In this section, we present an example to demonstrate some of the unsolved issues and problems in the negotiation model when players have different time preferences. Consider the model described in Section 2 with the following 2×2 disagreement game for $\varepsilon \geq 0$:

Player 1 \ Player 2	L	R
U	0.5, 0.5	$-\varepsilon$, 0.5
D	0.5, 0	0, -1

where $a^* = (U, L)$. For simplicity, we consider pure actions only.

To support the non-stationary SPE stated in Proposition 1 for $i = 1$, two players would play (U, L) in any odd period and (D, R) in any even period. When $\delta_1 \geq \delta_2$, both players behave as if in the alternating offer model with disagreement point $(0, 0.5)$, from which player 1 receives $\frac{1-\delta_2}{1-\delta_1\delta_2} \cdot 0.5$ in any odd period. By the one-stage deviation principle, see e.g., Fudenberg and Tirole (1991), player 1 prefers to make such an offer if and only if

$$\frac{1-\delta_2}{1-\delta_1\delta_2}\cdot 0.5 \geq (1-\delta_1)\cdot 0.5 + \delta_1 \cdot \frac{1-\delta_2}{1-\delta_1\delta_2}\cdot 0.5 \quad \Leftrightarrow \quad \delta_1 \geq \delta_2. \tag{4}$$

For $\delta_1 < \delta_2$, the non-stationary SPE of Proposition 1 requires that only player 2 makes the least acceptable offer. Consequently, player 1 receives $\frac{1}{1+\delta_1}\cdot 0.5$ in any odd period, which is equal to player 1's present value from the infinite sequence of alternating disagreement outcomes.

The open issue we concern in this paper is the worst SPE to the less patient player. Incorporating the possibility of unacceptable offers, we will show that the non-stationary SPE of Proposition 1 is indeed player 1's worst SPE when $\delta_1 \geq \delta_2$ (Proposition 7). When $\delta_1 < \delta_2$, contrary to common belief, player 1's worst SPE has not been established yet in the literature. First, player 1's worst SPE when $\delta_1 \geq \delta_2$ is no longer an equilibrium when $\delta_1 < \delta_2$, because of (4). Next, we demonstrate that the non-stationary SPE of Proposition 1 when $\delta_1 < \delta_2$ is not player 1's worst SPE in general.

Consider the following strategy profile:

- In an odd period, player 1 demands

$$x^* = \frac{1-\delta_2}{1-\delta_1\delta_2}\left[\frac{1}{2} + \frac{\delta_2}{\delta_1}\frac{1-\delta_1}{1-\delta_2}\varepsilon\right] \tag{5}$$

  and player 2 will reject if and only if player 1 demands more than x^*.

  If player 1 demands more than x^* and player 2 rejects, then (U, R) will be played.

  Since player 1 may want to deviate from U, continuation must depend on whether player 1 deviates or not.

- In an even period, if (U, R) is played in the last period, player 2 will offer $\delta_1 x^* + \frac{1-\delta_1}{\delta_1}\varepsilon$ and player 1 will accept any offer no less than $\delta_1 x^* + \frac{1-\delta_1}{\delta_1}\varepsilon$. Otherwise, player 2 will offer $\delta_1 x^*$ and player 1 will accept any offer no less than $\delta_1 x^*$.

  If player 1 rejects an offer that should be accepted, then (D, R) will be played.

- If player 2 deviates from the strategies described above, such as offers less than what should be offered, or rejects what should be accepted, or deviates in the disagreement game, then the continuation involves an immediate switch to the stationary SPE of Proposition 1.

It is easy to verify that this strategy profile constitutes a SPE when δ_2 is sufficiently large. First, player 2 is indifferent between accepting and rejecting player 1's demand x^*:

$$1 - x^* = (1 - \delta_2) \cdot 0.5 + \delta_2 \left[1 - \left(\delta_1 x^* + \frac{1 - \delta_1}{\delta_1}\varepsilon\right)\right],$$

which yields x^* as given by (5). Rewarding player 1 with additional $\frac{1-\delta_1}{\delta_1}\varepsilon$ is just enough to induce player 1 to play U. Unlike in the non-stationary SPE of Proposition 1, player 1 is better off to demand x^* than to make an acceptable proposal since

$$x^* > \delta_1 \left(\delta_1 x^* + \frac{1 - \delta_1}{\delta_1}\varepsilon\right) \Leftrightarrow x^* > \frac{\varepsilon}{1 + \delta_1},$$

which is true when ε is not too large. Player 2's incentive to comply is enforced by the stationary equilibrium as punishment if player 2 ever deviates, which is credible when δ_2 is sufficiently large.

Figure 2 shows that for $\delta_1 = 0.8$ and $\varepsilon = 0.15$, x^* is lower than $\frac{0.5}{1+\delta_1}$ for all $\delta_2 \in (0.925, 1)$, which contradicts Slantchev's (2003) claim about player 1's extremal SPE. This observation is quite robust, in the sense that it holds for a wide range of values for ε and δ_1. If $\varepsilon = 0$, $x^* = 0.5 \cdot \frac{1-\delta_2}{1-\delta_1\delta_2} < 0.5 \cdot \frac{1}{1+\delta_1}$ for all $\delta_2 > \delta_1$. As we will show in Section 5.4, the SPE we have just studied is not player 1's worst SPE when $\delta_1 < \delta_2$. Player 1's worst SPE is somewhat more complicated, so we postpone discussion. The neglected issue in supporting a player's worst SPE is that the players may not reach an immediate agreement. As it will become clear later, delay in reaching an agreement can happen to support the less patient player's worst SPE. When it happens, it dominates reaching agreement immediately. As we have argued, the players may trade-off their differences in time preferences with a delayed agreement and this may Pareto dominate reaching an agreement immediately.

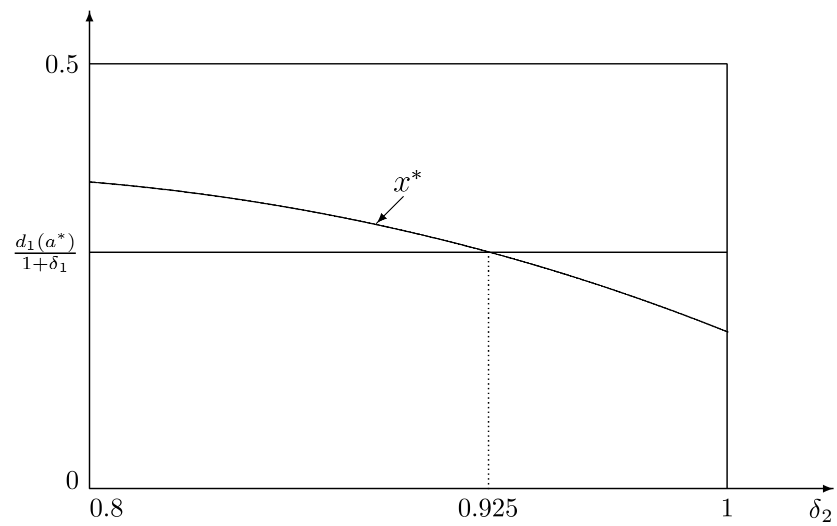

Figure 2: Plot of x^* with respect to $\delta_2 \in (\delta_1, 1)$ for $\delta_1 = 0.8$ and $\varepsilon = 0.15$.

4 Extreme Bounds of SPE Payoffs

The key to characterize the set of SPE payoffs is to derive each player's lowest and highest SPE payoff. With the possibility of unacceptable offers, we first provide a set of necessary conditions for the extreme bounds of SPE payoffs by generalizing the backward induction technique of Shaked and Sutton (1984). We then solve these extreme bounds and support them by SPEs for sufficiently large discount factors.

Let E^i be the set of SPE payoffs in any period in which player i proposes and player j responds for $j \neq i$. For simplicity, we suppress all the other parameters that E^i may depend on, such as the discount factors. Given the existence of SPE (Proposition 1) and the model setup, the set E^i is a non-empty and bounded subset of $[0, 1]^2$. For $l = i, j$ we define

$$m_l^i = \inf_{v \in E^i} v_l \quad \text{and} \quad M_l^i = \sup_{v \in E^i} v_l, \tag{6}$$

the infimum and the supremum of player l's SPE payoffs in any period where player i proposes. Proposition 1 implies that $m_l^i \leq d_l(a^*) \leq M_l^i$ for all $(\delta_i, \delta_j) \in (0, 1)^2$, with strict inequalities for sufficiently large $(\delta_i, \delta_j) \in (0, 1)^2$.

In any period in which player i proposes, if player j rejects player i's offer then both players will have to play a disagreement outcome $a \in A$, followed by a continuation SPE with payoff vector $v = (v_i, v_j) \in E^j$ in the following period in which player j proposes. The continuation payoff vector v generally depends on the disagreement outcome $a \in A$ so that if a player deviates from a then the continuation payoff vector will change accordingly. By definition, we have $m_l^j \leq v_l \leq M_l^j$ for $l = i, j$. Given the continuation payoff vector $v \in E^j$, playing $a \in A$ in G is sequential rational for player l if and only if

$$(1-\delta_l)d_l(a) + \delta_l v_l \geq (1-\delta_l)g_l(a) + \delta_l m_l^j. \tag{7}$$

Inequality (7) states that player l's payoff from complying is at least what he could obtain by deviating from $a \in A$. Obviously, any Nash equilibrium of G, including $a^* \in A$, satisfies the credibility constraint (7) for all discount factors and all continuation payoff vectors.

As we have argued, the controversy in the current literature is due to the negligent of unacceptable offers when applying the backward induction technique of Shaked and Sutton (1984). By incorporating the possibility of unacceptable offers explicitly, we first obtain the following proposition:

Proposition 2 *For all $(\delta_i, \delta_j) \in (0,1)^2$ and $l = i$ and j,*

$$m_l^i \geq \inf_{a \in A, v \in E^j} U_l^i(a, v), \quad s.t.\ (7), \tag{8}$$

$$M_l^i \leq \sup_{a \in A, v \in E^j} U_l^i(a, v), \quad s.t.\ (7), \tag{9}$$

where

$$U_i^i(a, v) = \max \begin{cases} (1-\delta_i)d_i(a) + \delta_i v_i, \\ 1 - (1-\delta_j)d_j(a) - \delta_j v_j, \end{cases} \tag{10}$$

$$U_j^i(a, v) = (1-\delta_j)d_j(a) + \delta_j v_j. \tag{11}$$

Proof. Given the necessary structure of any SPE in a period in which player i proposes, if player i makes an unacceptable offer, such as offering player j less than $(1-\delta_j)d_j(a)+\delta_j v_j$, then player i will receive $(1-\delta_i)d_i(a) + \delta_i v_i$. On the other hand, in order to induce player j to accept, player i will have to offer at least $(1-\delta_j)d_j(a) + \delta_j v_j$ to player j, from which player i will receive at most $1-(1-\delta_j)d_j(a)-\delta_j v_j$. By sequential rationality, player i will choose either an unacceptable offer or the least acceptable offer to player j, whichever yields player i a higher payoff. Therefore, player i's payoff when proposing is equal to $U_i^i(a,v)$ as given by (10). Whether player i makes an unacceptable offer or the least acceptable offer, player j will always receive $U_j^i(a,v)$, as given by (11).

For $l=i$ and j, player l's SPE payoffs in any period in which player i proposes must be between the infimum and supremum of $U_l^i(a,v)$ for all $a \in A$ and $v \in E^j$ under the credibility constraint (7). ■

Although the objective functions in (8) and (9) are continuous and A is compact by assumption, we know nothing about E^j at this stage. Therefore, the extremum of (8) and (9) may not be achievable. In other words, infimum and supremum cannot be replaced by minimum and maximum in these optimization problems at this stage of the analysis.

From Proposition 2, we are able to solve m_j^i, M_j^i and M_i^i in terms of m_i^i and m_j^j, which we will do in the remainder of this section. In Section 5, we will derive m_i^i, which requires us to understand the set E^j when both players have different time preferences.

Given Proposition 2, the following conditions on the extreme bounds of player j's SPE payoffs (as the responding player), m_j^i and M_j^i, are immediate:

Proposition 3 *For all $(\delta_i,\delta_j)\in(0,1)^2$, we have*

$$m_j^i \geq \delta_j m_j^j, \tag{12}$$

$$M_j^i \leq (1-\delta_j)d_j(a^*)+\delta_j M_j^j. \tag{13}$$

Proof. Substituting (11) into (8), we have

$$\begin{aligned}
m_j^i &\geq \inf_{a\in A, v\in E^j}\left[(1-\delta_j)\,d_j(a)+\delta_j v_j\right] \quad \text{s.t. (7)} \\
&\geq \min_{a\in A}\left[(1-\delta_j)g_j(a)+\delta_j m_j^j\right] \qquad \text{due to (7)} \\
&= (1-\delta_j)\min_{a\in A} g_j(a)+\delta_j m_j^j \\
&= \delta_j m_j^j,
\end{aligned}$$

which is (12). Notice that player j's minimax value $\min_{a\in A} g_j(a) = 0$ by assumption. Substituting (11) into to (9), we have

$$\begin{aligned} M_j^i &\leq \sup_{a\in A, v\in E^j} [(1-\delta_j)d_j(a) + \delta_j v_j] && \text{s.t. } (7) \\ &\leq (1-\delta_j)\max_{a\in A} d_j(a) + \delta_j \sup_{v\in E^j} v_j && \text{due to } (7) \\ &= (1-\delta_j)d_j(a^*) + \delta_j M_j^j, && \text{due to } (6), \end{aligned}$$

which is (13). ■

For sufficiently large $(\delta_i, \delta_j) \in (0,1)^2$, Proposition 3 implicitly specifies how the players behave in the responder's worst and best SPE. In player j's worst SPE, if player j rejects any offer, he will receive his minimax value of 0 in the current period followed by his lowest SPE payoff m_j^j in the following period. In player j's best SPE, on the other hand, if player j rejects any offer, he will receive his highest disagreement payoff $d_j(a^*)$ in the current period followed by his highest SPE payoff M_j^j in the following period. In fact, when the players are sufficiently patient, (12) and (13) hold with equalities for the responding player's lowest and highest SPE payoffs. These results are similar to those of Busch and Wen (1995), where both players have the same discount factor.

We now turn to M_i^i, the supremum of the proposing player's SPE payoffs.

Proposition 4 *For sufficiently large* $(\delta_i, \delta_j) \in (0,1)^2$,

$$M_i^i \leq 1 - m_j^i. \tag{14}$$

Proof. Substituting (10) into (9), we have

$$\begin{aligned} M_i^i &\leq \sup_{a\in A, v\in E^j} \max \begin{cases} (1-\delta_i)d_i(a) + \delta_i v_i, \\ 1-(1-\delta_j)d_j(a) - \delta_j v_j, \end{cases} && \text{s.t. } (7) \\ &= \max \begin{cases} \sup_{a,v} [(1-\delta_i)d_i(a) + \delta_i v_i], & \text{s.t. } (7), \\ \sup_{a,v} [1-(1-\delta_j)d_j(a) - \delta_j v_j], & \text{s.t. } (7), \end{cases} \\ &\leq \max \begin{cases} (1-\delta_i)d_i(a^*) + \delta_i M_i^j, & \text{due to } d_i(a) \leq d_i(a^*) \text{ and } (6), \\ 1 - m_j^i, & \text{due to } (8) \text{ and } (11). \end{cases} \end{aligned}$$

For sufficiently large $(\delta_i, \delta_j) \in (0,1)^2$, however, it cannot be the case that $1 - m_j^i \leq (1 - \delta_i)d_i(a^*) + \delta_i M_i^j$. Suppose not, then $M_i^i \leq (1-\delta_i)d_i(a^*) + \delta_i M_i^j$ and (13) imply $M_i^i \leq d_i(a^*)$, which contradicts $M_i^i > d_i(a^*)$ as we established for sufficiently large discount factors. Hence, (14) must hold. ■

In obtaining player i's highest SPE payoff when this player proposes, Proposition 4 implies player i offers m_j^i to player j and player j will accept. For sufficiently large discount factors, (12), (13) and (14) yield that

$$\begin{aligned} m_j^i &\geq \delta_j m_j^j, \\ M_j^i &\leq (1-\delta_j)d_j(a^*) + \delta_j \left(1 - \delta_i m_i^i\right), \\ M_i^i &\leq 1 - \delta_j m_j^j. \end{aligned}$$

Therefore, m_i^i and m_j^j are the key to determine the other extremal bounds of players' equilibrium payoffs.

5 The Infimum of the Proposer's SPE Payoffs

From Proposition 2 and $m_i^j \geq \delta_i m_i^i$, we rewrite the condition for m_i^i as

$$m_i^i \geq \inf_{a \in A, v \in E^j} \max \left\{ \begin{array}{l} (1-\delta_i)d_i(a) + \delta_i v_i, \\ 1 - (1-\delta_j)d_j(a) - \delta_j v_j, \end{array} \right. \tag{15}$$

$$\text{s.t.} \quad v_i \geq \frac{1-\delta_i}{\delta_i}\left[g_i(a) - d_i(a)\right] + \delta_i m_i^i. \tag{16}$$

In solving the minimax problem (15), we proceed in two steps: First, it is necessary to establish some properties of the set E^j, in particular, the Pareto frontier of E^j that contains all effective continuation payoffs to solve (15). Next, we use these properties to characterize the infimum for each of the two cases identified in Section 2, namely $\delta_i \geq \delta_j$ and $\delta_i < \delta_j$. Obviously, any SPE where player i receives m_i^i would be player i's worst SPE. It turns out that the non-stationary SPE of Proposition 1 is in fact player i's worst SPE when $\delta_i \geq \delta_j$. For $\delta_i < \delta_j$, we provide an equilibrium strategy profile to support m_i^i. In order to illustrate the characterization and the novel insights we obtained, especially for the second case, we reconsider the example of Section 3. We conclude this section with a technical discussion on robustness.

5.1 Effective Continuation Payoffs

In order to fully understand the issues involved, we have to discuss what continuation payoffs are most effective in solving (15). For every $a \in A$, credibility constraint (16) requires that player i receives at least $\frac{1-\delta_i}{\delta_i}[g_i(a) - d_i(a)]$ more than $\delta_i m_i^i$ in the following period. For $(v_i, v_j) \in E^j$ and $(v_i, v_j') \in E^j$, if $v_j \geq v_j'$ then

$$\max \begin{cases} (1-\delta_i)d_i(a) + \delta_i v_i, \\ 1-(1-\delta_j)d_j(a) - \delta_j v_{j,} \end{cases} \leq \max \begin{cases} (1-\delta_i)d_i(a) + \delta_i v_i, \\ 1-(1-\delta_j)d_j(a) - \delta_j v_j'. \end{cases}$$

Therefore, at any solution of (15), player j must receive his highest continuation payoff while player i's continuation payoff satisfies the credibility constraint (16). Given the setup of our model, define

$$K = \max_{a \in A} [g_i(a) - d_i(d)] < \infty.$$

For sufficiently large (δ_i, δ_j) such that $m_i^j \leq \frac{\delta_i}{1+\delta_i} d_i(a^*)$ from Proposition 1, we have that $\left(1-\delta_i^2\right) K < \delta_i d_i(a^*)$ is a sufficient condition for

$$\frac{1-\delta_i}{\delta_i}[g_i(a) - d_i(a)] + m_i^j \leq \frac{1-\delta_i}{\delta_i} K + \frac{\delta_i}{1+\delta_i} d_i(a^*) < d_i(a^*).$$

This means that for sufficiently large discount factors, we only have to consider those continuation payoffs where player i receives strictly less than $d_i(a^*)$.

As argued in Section 2, the literature on the negotiation model has taken for granted that all continuation payoffs lie on or below the bargaining frontier. Since this is not the case here, we must explicitly take into account the possibility of continuation payoffs that lie above the bargaining frontier. For $\delta_i > \delta_j$, those payoffs all lie on the irrelevant side of $d(a^*)$, i.e., $[d_i(a^*), 1]$ while $m_i^i < d_i(a^*)$. Therefore, it is without of loss of generality in this case to assume the bargaining frontier is the Pareto frontier in solving (15).

In the remainder of this subsection, we deal with the more complicated case: $\delta_i < \delta_j$. In Section 2, we show that the Pareto frontier corresponds to $x_i = 0$. However, $x_i = 0$ cannot be supported in equilibrium, as Lemma 5 shows. This implies that Pareto frontier cannot be part of any SPE, including player i's worst SPE. The following lemma provides a lower bound for m_i^i, which may not be attainable in equilibrium either. The proofs of the lemmas in this subsection are deferred to the appendix.

Lemma 5 *For all* $(\delta_i, \delta_j) \in (0,1)^2$*, we have*

$$m_i^i \geq \frac{1-\delta_j}{1-\delta_i\delta_j}\,[1-d_j(a^*)] > 0.$$

The most effective way in supporting player i's worst SPE corresponds to the smallest $\tilde{x}_i$ in any SPE. This requires that $\tilde{x}_i = m_i^j$ for even T and $\tilde{x}_i = m_i^i$ for odd T. What is clear from Section 2 is that in order to solve (15), it is sufficient to consider continuation paths where the players play a^* for $T \geq 0$ periods followed by player i's worst SPE in period $T+1$. Accordingly, we define

$$v_i(T) \;=\; (1-\delta_i^T)d_i(a^*) + \delta_i^T \tilde{x}_i, \tag{17}$$

$$v_j(T) \;=\; (1-\delta_j^T)d_j(a^*) + \delta_j^T (1-\tilde{x}_i), \tag{18}$$

where we suppress $\tilde{x}_i$ in our notation. Note that the effective continuation payoff vector $(v_i(T), v_j(T)) \in E^j$ is above the bargaining frontier:

$$v_i(T) + v_j(T) = 1 + (\delta_j^T - \delta_i^T)[d_i(a^*) - \tilde{m}_i] > 1 \quad \Leftrightarrow \quad \delta_i < \delta_j. \tag{19}$$

It is clear that such continuations may occur in supporting player i's worst SPE, which is what has been overlooked in previous studies.

We now turn our attention to the set of effective continuation payoffs, i.e., the Pareto frontier of E^j. This frontier is fully characterized by $\{(v_i(T), v_j(T))\}_{T \in 2\mathbb{N}}$, where $2\mathbb{N}$ denotes the set of all even number, implied by the following lemma:

Lemma 6 *For any even* $T \geq 0$ *and all* $\delta_i < \delta_j$,

$$v_i(T) \;<\; v_i(T+1) \leq v_i(T+2) < d_i(a^*) \tag{20}$$

$$v_j(T) \;>\; v_j(T+2) < v_j(T+1) < d_j(a^*) \tag{21}$$

Lemma 6 is best illustrated by Figure 3, where $(v_i(T), v_j(T))$ is represented by solid dots for even $T \leq 8$ and open dots for odd $T < 8$. It implies that for any even $T \geq 0$, $(v_i(T+1), v_j(T+1))$ is dominated by a segment of convex combinations of $(v_i(T), v_j(T))$ and $(v_i(T+2), v_j(T+2))$. Intuitively, if the continuation path were associated with an odd T, then player i would propose an offer along such a continuation, from which player i could exploit his first-mover advantage. Consequently, such a continuation cannot be effective in solving (15).

Any convex combination of $(v_i(T), v_j(T))$ and $(v_i(T+2), v_j(T+2))$ can be achieved by the following path: First play a^* for T periods, then for $p \in [0,1]$, agree on m_i^i in period $T+1$ with probability $1-p$, and with probability p, continue to play a^* for two more periods followed by agreement m_i^i . Such a continuation path yields $(1-p)v_l(T)+pv_l(T+2)$ to player $l = i, j$. Similar as for the Pareto frontier, effective continuation payoffs consist of the piecewise linear part above the bargaining frontier. We incorporate these effective continuation payoffs into a single function. Formally, given m_i^i, we define the function $\varphi(\cdot\,, \delta_i m_i^i) : [0,1] \to [0,1]$, as

$$\varphi(v_i, \delta_i m_i^i) \equiv \max\left\{1 - v_i, \min_{T \in 2\mathbb{N}} \left\{v_j(T) + \frac{v_j(T+2) - v_j(T)}{v_i(T+2) - v_i(T)}\,[v_i - v_i(T)]\right\}\right\}. \tag{22}$$

The graph of $\varphi(\cdot, \delta_i m_i^i)$ describes all possibly effective continuation payoff vectors. In contrast to the current literature, this function corrects the problem of assuming only the first part of (22) as the set of effective continuation payoffs.

The function $\varphi(v_i\,, \delta_i m_i^i)$ is continuous and monotonically decreasing in v_i and m_i^i (also through the v's). Given m_i^i, $\varphi(v_i, \delta_i m_i^i) > 1 - v_i$ for all $v_i \in (\delta_i m_i^i, d_i(a^*))$, and $\varphi(v_i, \delta_i m_i^i) = 1 - v_i$ otherwise. In particular, $\varphi(\delta_i m_i^i, \delta_i m_i^i) = 1 - \delta_i m_i^i$, just as Figure 3 illustrates. Note that (16) and $(v_i, v_j) \in E^j$ imply that

$$v_{j \leq} \varphi(v_i, \delta_i m_i^i) \leq \varphi\left(\frac{1-\delta_i}{\delta_i}\,[g_i(a) - d_i(a)] + m_i^j, \delta_i m_i^i\right).$$

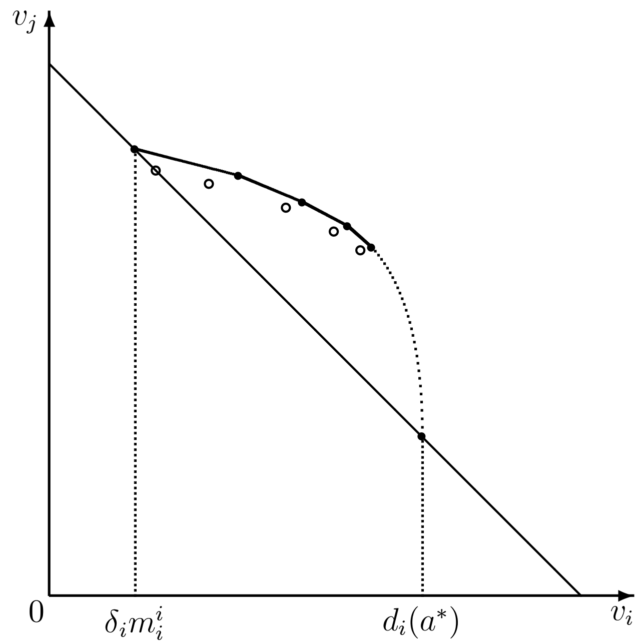

Figure 3: Effective continuation payoffs for $T \leq 8$.

5.2 Player i is at least as patient as player j

As we have argued, all effective continuation payoffs in solving (15) must be on the bargaining frontier when $\delta_i \geq \delta_j$. That is, two players always reach an agreement in the following period where player j proposes. Then, (16) becomes

$$v_j \leq 1 - v_i \leq 1 - \delta_i m_i^i - \frac{1-\delta_i}{\delta_i}[g_i(a) - d_i(a)].$$

Substituting the last inequality into (15) yields

$$\begin{aligned} m_i^i &\geq \min_a \max \begin{cases} (1-\delta_i) g_i(a) + \delta_i m_i^j, \\ (1-\delta_j)\left[1 - d_j(a) + \frac{\delta_j(1-\delta_i)}{\delta_i(1-\delta_j)}[g_i(a) - d_i(a)]\right] + \delta_j m_i^j, \end{cases} \\ &\geq \min_a \left\{(1-\delta_j)\left[1 - d_j(a) + \frac{\delta_j(1-\delta_i)}{\delta_i(1-\delta_j)}[g_i(a) - d_i(a)]\right] + \delta_i \delta_j m_i^i\right\} \\ &\geq (1-\delta_j)\left[\min_a [1 - d_j(a)] + \frac{\delta_j(1-\delta_i)}{\delta_i(1-\delta_j)} \min_a [g_i(a) - d_i(a)]\right] + \delta_i \delta_j m_i^i \\ &= (1-\delta_j)[1 - d_j(a^*)] + \delta_i \delta_j m_i^i. \end{aligned}$$

Solving for m_i^i, we obtain (again) the lower bound of Lemma 5.

Proposition 7 *For sufficiently large* $(\delta_i, \delta_j) \in (0,1)^2$ *and* $\delta_i \geq \delta_j$, *we have*

$$m_i^i \geq \frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j(a^*)\right]. \tag{23}$$

The non-stationary SPE of Proposition 1 yields player i exactly $\frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j(a^*)\right]$. In other words, this non-stationary SPE is indeed player i's worst SPE. It follows from Proposition 7 and 8 that it is also player j's best SPE. To summarize, for sufficiently large (δ_i, δ_j) and $\delta_i \geq \delta_j$, we have

$$\begin{array}{ll} m_i^i = \frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j(a^*)\right], & M_j^i = d_j(a^*) + \frac{\delta_j(1-\delta_i)}{1-\delta_i\delta_j}\left[1-d_j(a^*)\right], \\ m_i^j = \frac{\delta_i(1-\delta_j)}{1-\delta_i\delta_j}\left[1-d_j(a^*)\right], & M_j^j = d_j(a^*) + \frac{1-\delta_i}{1-\delta_i\delta_j}\left[1-d_j(a^*)\right]. \end{array}$$

When $\delta_i \geq \delta_j$, player i's worst SPE resembles the unique SPE in the alternating-offer model in which player i receives 0 and player j receives $d_j(a^*)$ in every disagreement period, which is identical to the situation when $\delta_i = \delta_j$. These findings validate previous results concerning the more patient player's worst SPE by taking explicitly into account the possibility of unacceptable offer.

5.3 Player i is less patient than player j

The complications that arise in solving (15) when $\delta_i < \delta_j$ are incorporated in the function φ. Since (16) and $(v_i, v_j) \in E^j$ imply that

$$v_j \leq \varphi(v_i, \delta_i m_i^i) \leq \varphi\left(\frac{1-\delta_i}{\delta_i}\left[g_i(a) - d_i(a)\right] + m_i^j, \delta_i m_i^i\right)$$

and $m_i^j \geq \delta_i m_i^i$, problem (15) can be rewritten as the fixed point problem $m_i^i \geq \Lambda(m_i^i)$, where

$$\Lambda(m_i^i) \equiv \min_{a\in A} \max \left\{ \begin{array}{l} (1-\delta_i) g_i(a) + \delta_i^2 m_i^i, \\ 1-(1-\delta_j) d_j(a) - \delta_j \varphi\left(\frac{1-\delta_i}{\delta_i}\left[g_i(a) - d_i(a)\right] + \delta_i m_i^i, \delta_i m_i^i\right). \end{array}\right. \tag{24}$$

This is a well-defined minimax problem where the two functions under the maximum in (24) are continuous and increasing in m_i^i. Our main result of this subsection is the following proposition that provides a lower bound of m_i^i for all $\delta_i < \delta_j$.

Proposition 8 *For all* $\delta_i < \delta_j$, *function* $\Lambda(\cdot)$ *has at least one fixed point in the relevant interval for* m_i^i, *i.e.,* $\left[\frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j\left(a^*\right)\right], \frac{1}{1+\delta_i} d_i\left(a^*\right)\right]$, *and* m_i^i *is bounded from below by the least fixed point of* $\Lambda(\cdot)$.

Proof. Recall that $\Lambda(\cdot)$ is well-defined, continuous and monotonically increasing. For all $m_i^i \in \left[\frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j\left(a^*\right)\right], \frac{1}{1+\delta_i}d_i\left(a^*\right)\right]$ and $v_i \in [\delta_i m_i^i, d_i\left(a^*\right)]$, we have

$$d_j\left(a^*\right) \leq \varphi\left(v_i, \delta_i m_i^i\right) \leq 1-\delta_i m_i^i. \tag{25}$$

Evaluating $\Lambda(\cdot)$ at the upper end of this relevant interval, we obtain

$$\begin{aligned}
\Lambda\left(\frac{1}{1+\delta_i}d_i\left(a^*\right)\right) &\leq \max\begin{cases} (1-\delta_i)d_i(a^*)+\delta_i^2\frac{1}{1+\delta_i}d_i\left(a^*\right) \\ 1-(1-\delta_j)d_j(a^*)-\delta_j\varphi\left(\delta_i\frac{1}{1+\delta_i}d_i\left(a^*\right), \delta_i\frac{1}{1+\delta_i}d_i\left(a^*\right)\right)\end{cases} \\
&= \max\begin{cases} \frac{1}{1+\delta_i}d_i\left(a^*\right) \\ 1-(1-\delta_j)d_j(a^*)-\delta_j\left(1-\delta_i\frac{1}{1+\delta_i}d_i\left(a^*\right)\right)\end{cases} \\
&= \max\begin{cases} \frac{1}{1+\delta_i}d_i\left(a^*\right) \\ \frac{1-(\delta_j-\delta_i)}{1+\delta_i}d_i\left(a^*\right)\end{cases} = \frac{1}{1+\delta_i}d_i\left(a^*\right),
\end{aligned}$$

where the first inequality holds because at $a=a^*$ we have $v_i=\delta_i m_i^i=\delta_i\frac{1}{1+\delta_i}d_i\left(a^*\right)$, and the second inequality is due to (25). Next, evaluating $\Lambda(\cdot)$ at the lower end of relevant interval, we have

$$\begin{aligned}
&\Lambda\left(\frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j\left(a^*\right)\right]\right) \\
&= \min_{a\in A}\max\begin{cases} (1-\delta_i)g_i(a)+\delta_i^2\frac{1-\delta_j}{1-\delta_i\delta_j}d_i\left(a^*\right) \\ 1-(1-\delta_j)d_j(a)-\delta_j\varphi\left(\frac{1-\delta_i}{\delta_i}\left[g_i\left(a\right)-d_i\left(a\right)\right]+\delta_i m_i^i, \delta_i m_i^i\right)\Big|_{m_i^i=\frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j\left(a^*\right)\right]}\end{cases} \\
&\geq \min_{a\in A}\max\begin{cases} (1-\delta_i)g_i(a)+\delta_i^2\frac{1-\delta_j}{1-\delta_i\delta_j}d_i\left(a^*\right) \\ 1-(1-\delta_j)d_j(a)-\delta_j\varphi\left(\delta_i\frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j\left(a^*\right)\right], \delta_i\frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j\left(a^*\right)\right]\right)\end{cases} \\
&= \min_{a\in A}\max\begin{cases} (1-\delta_i)g_i(a)+\delta_i^2\frac{1-\delta_j}{1-\delta_i\delta_j}d_i\left(a^*\right) \\ 1-(1-\delta_j)d_j(a)-\delta_j\left[1-\delta_i\frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j\left(a^*\right)\right]\right]\end{cases} \\
&\geq \min_{a\in A}\left\{(1-\delta_j)\left[1-d_j(a)\right]+\delta_i\delta_j\frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j\left(a^*\right)\right]\right\} \\
&= (1-\delta_j)\left[1-\max_{a\in A}d_j(a)\right]+\delta_i\delta_j\frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j\left(a^*\right)\right] \\
&= (1-\delta_j)\left[1-d_j\left(a^*\right)\right]+\delta_i\delta_j\frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j\left(a^*\right)\right] \\
&= \frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j\left(a^*\right)\right].
\end{aligned}$$

Once again, the first inequality is due to (25), and the other (in)equalities are trivial. Because of its monotonicity, function $\Lambda(\cdot)$ maps from $\left[\frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j\left(a^*\right)\right], \frac{1}{1+\delta_i}d_i\left(a^*\right)\right]$ into itself. By

Brouwer's fixed point theorem, $\Lambda(\cdot)$ has at least one fixed point in the relevant interval. Since $\Lambda(\cdot)$ is monotonically increasing, any value of m_i^i that is strictly less than the least fixed point of $\Lambda(\cdot)$ certainly violates inequality (24). This concludes the proof of the proposition.

■

Our next proposition asserts that when the discount factors are sufficiently large, the least fixed point $\Lambda(\cdot)$ can be supported as player i's SPE payoff. Therefore, m_i^i is indeed the least fixed point of $\Lambda(\cdot)$ whenever it can be supported as player i's SPE payoff. Since the proof is rather long, we defer it to the appendix.

Proposition 9 *For all $\delta_i \in (0,1)$, there exists $\hat{\delta}_j \in (\delta_i, 1)$ such that for all $\delta_j \in \left(\hat{\delta}_j, 1\right)$, there is a SPE in which player i receives the least fixed point of $\Lambda(\cdot)$.*

Propositions 8 and 9 imply that when the discount factors are large enough, the least fix point of $\Lambda(\cdot)$ is indeed an SPE payoff and, of course, it is player i's lowest SPE payoff. All the other extreme bounds derived in Section 4 can also be supported as SPE payoffs. To support $m_i^j = \delta_i m_i^i$ as player i's SPE when player j proposes, consider the following strategy profile: Player j offers $m_i^j = \delta_i m_i^i$ to player i and player i accepts any offer higher than $m_i^j = \delta_i m_i^i$. If player i rejects $m_i^j = \delta_i m_i^i$, then player i will be minimaxed followed by his worst SPE from which player i receives m_i^i. Any of player j's deviation will be followed by the stationary SPE of Proposition 1, which is sufficient to enforce player j to comply. In this equilibrium, player j receives his highest SPE payoff $M_j^j = 1 - \delta_i m_i^i$ as the proposer and player i receives his lowest SPE payoff $m_i^j = \delta_i m_i^i$ as the responder. The strategy profile that supports the responder's highest equilibrium payoff will simply call for a^* in the first period, followed by his best SPE (as the proposer) in the following period. Consequently, we have completely characterized the set of equilibrium payoffs in this model when discount factors are sufficiently large.

5.4 The Example Revisited

We now derive player 1's worst SPE when $\delta_1 < \delta_2$ in the example studied in Section 3. Proposition 9 implies that delay will happen (with certain probability) after player 1's (non-equilibrium) demand is rejected. Consider the following strategy profile:

- In an odd period, player 1 demands

$$x^{**} = \frac{1-\delta_2}{1-\delta_1\delta_2}\left[\frac{1}{2} + \frac{\delta_2}{\delta_1}\frac{1+\delta_2}{1+\delta_1}\varepsilon\right] \tag{26}$$

  and player 2 will reject if and only if player 1 demands more than x^{**}.

  If player 1 demands more than x^{**} and player 2 rejects, then (U, R) will be played.

- In an even period, if player 1 deviates from U in the last (odd) period, player 2 will offer $\delta_1 x^{**}$ and player 1 will accept.

- Otherwise, with probability $1-p$, player 2 will offer $\delta_1 x^{**}$ in the current even period, and with probability p, (U, L) will be played for two periods, followed by player 2's offer $\delta_1 x^{**}$. Player 1 accepts in both cases. In this equilibrium,

$$p = \frac{1}{\delta_1(1-\delta_1)} \cdot \frac{\varepsilon}{0.5-\delta_1 x^{**}}. \tag{27}$$

- In an even period, if player 1 rejects $\delta_1 x^{**}$ (that should be accepted), then (D, R) will be played once followed by player 1's demand x^{**}.

- If player 2 deviates from the strategies described above, then continuation will switch immediately to the stationary SPE of Proposition 1.

To verify that the above strategy profile constitutes a SPE, first note that player 1 has no incentive to deviate from (U, R) if his payoff from deviation is the same as what player 1 receives if he does not:

$$\delta_1^2 x^{**} = (1-\delta_1)\cdot(-\varepsilon) + \delta_1\left[(1-p)\delta_1 x^{**} + p\left(0.5(1-\delta_1^2) + \delta_1^3 x^{**}\right)\right]. \tag{28}$$

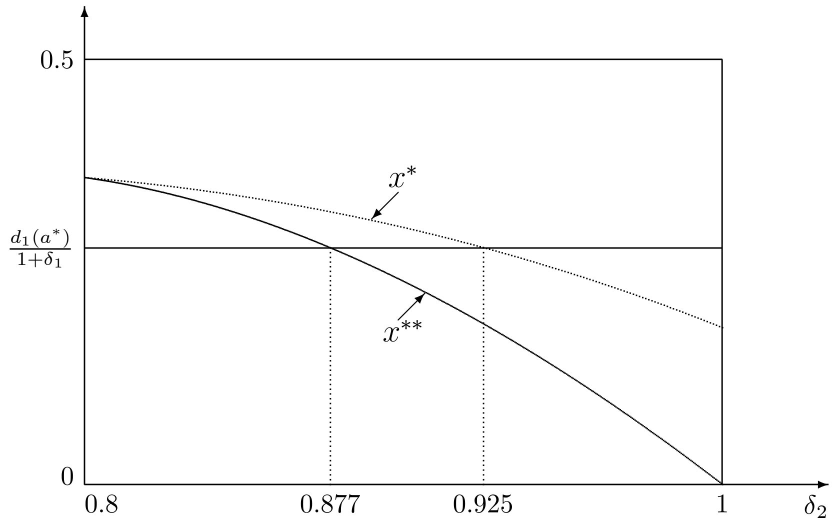

Figure 4: Plots of x^{**} with respect to $\delta_2 \in (\delta_1, 1)$ for $\delta_1 = 0.8$ and $\varepsilon = 0.15$.

One can show that (28) holds for p as given by (27). Next, player 1 should demand x^{**} rather than making an unacceptable proposal,

$$x^{**} \geq (1-\delta_1)\cdot(-\varepsilon) + \delta_1\left[(1-p)\delta_1 x^{**} + p\left(0.5(1-\delta_1^2) + \delta_1^3 x^{**}\right)\right] = \delta_1^2 x^{**},$$

which follows from (28). Lastly, player 1 cannot demand more than x^{**} since $1 - x^{**}$ is exactly equal to player 2's continuation payoff after rejecting any demand higher than x^{**}:

$$1 - x^{**} = 0.5(1-\delta_2) + \delta_2[(1-p)(1-\delta_1 x^{**}) + p\left[0.5(1-\delta_2^2) + \delta_2^2(1-\delta_1 x^{**})\right]]. \tag{29}$$

In fact, (28) and (29) yield x^{**} and p as given by (26) and (27), respectively.

For the same set of parameters, i.e., $\delta_1 = 0.8$ and $\varepsilon = 0.15$, Figure 4 shows x^{**} is less than x^* for all $\delta_2 > \delta_1$. For $\delta_2 \in (0.877, 1)$, we have $x^{**} \leq \frac{0.5}{1+\delta_1}$. When the difference between the players' time preferences is not significant enough such as $\delta_2 \in (0.8, 0.877)$, it would not be optimal to have delay in the continuation while compensating player 1. In such a case, the non-stationary SPE of Proposition 1 is player 1's worst SPE. However, such incidence diminishes as the value of ε decreases.

5.5 Discussion

In this section we further discuss our new results and relate them to the literature.

First, in the proof of Proposition 9, player j's behavior is enforced by the stationary SPE of Proposition 1 rather than player j's worst SPE. When $\delta_i < \delta_j$, player j's worst SPE is characterized by Proposition 7, where

$$m_j^j = \frac{1-\delta_i}{1-\delta_i\delta_j} d_j(a^*) < d_j(a^*).$$

If player j is punished by his worst SPE, the condition on δ_j yields a lower threshold for δ_j than the one used in the proof. Adopting the stationary equilibrium simplifies the proof since player j has a constant continuation payoff $d_j(a^*)$ whenever player j deviates.

Second, it is generally impossible to obtain a closed-form solution for m_i^i from (24). However, if there is no need to compensate player i, the continuation payoff after player i's (non-equilibrium) offers are rejected will be on the bargaining frontier, i.e., at $(\delta_i m_i^i, \varphi(\delta_i m_i^i)) = (\delta_i m_i^i, 1-\delta_i m_i^i)$ associated with $T=0$. In such cases, we have

Proposition 10 *Suppose $\hat{a} \in A$ solves (24) at the least fixed point of $\Lambda(\cdot)$ and $g_i(\hat{a}) = d_i(\hat{a})$. Then*

$$m_i^i = \max\left\{\frac{1}{1+\delta_i} d_i(\hat{a}), \frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j(\hat{a})\right]\right\}. \tag{30}$$

Proof. In (24), since $g_i(\hat{a}) = d_i(\hat{a})$, we have

$$\begin{aligned} m_i^i &= \max\left\{\begin{array}{l} (1-\delta_i)d_i(\hat{a}) + \delta_i^2 m_i^i \\ 1-(1-\delta_j)d_j(\hat{a}) - \delta_j\varphi\left(\delta_i m_i^i, m_i^i\right) \end{array}\right. \\ &= \max\left\{\begin{array}{l} (1-\delta_i)d_i(\hat{a}) + \delta_i^2 m_i^i \\ 1-(1-\delta_j)d_j(\hat{a}) - \delta_j\left[1-\delta_i m_i^i\right] \end{array}\right. \\ &= \max\left\{\begin{array}{l} (1-\delta_i)d_i(\hat{a}) + \delta_i^2 m_i^i \\ (1-\delta_j)\left[1-d_j(\hat{a})\right] - \delta_i\delta_j m_i^i, \end{array}\right. \end{aligned}$$

which yields (30). ■

Third, all SPE with immediate agreement $x_i \in (m_i^i, d_i(a^*))$ fail Pareto efficiency, when $\delta_i < \delta_j$. This fact is quite different from the results thus far obtained in the bargaining literature.

6 Concluding Remarks

In this paper, we pin down what has been overlooked in previous studies on the negotiation model when players have different time preferences. Players may trade the difference in time preferences from which both of them could be better off than from reaching an immediate agreement. Such a trade is possible only if they disagree for some periods. Therefore, simply disagree does not necessarily imply inefficiency when the players have different time preferences. This matters in a player's worst equilibrium outcome as we need to count for the least irresistible offer to the other player. In our study, we formally incorporate this line of argument in the equilibrium analysis. One should not put too much emphasis on what numerical value we get, but what is rather important is how players behave in those extreme situations. It is also important not to overlook the possibility of unacceptable offers in bargaining models.

The results in this paper are quite robust. First, since (2) is applicable to any disagreement game, it is generally the case that some continuation payoffs may lie above the bargaining frontier when the players have different time preferences. Second, all our results as stated, namely Proposition 6 to 10, will not be effected if $d(a^*)$ lies below the bargaining frontier. The reason is that the $\varphi(\cdot)$ function we define in (22) is independent of the assumption of $d(a^*)$ being on the bargaining frontier. However, if $d(a^*)$ is below the bargaining frontier, it will enlarge the set of discount factors for which delay does not occur in supporting the less patient players' worst SPE, and a closed-form solution becomes available. At the extreme case $d(a^*) = 0$, the negotiation model is equivalent to the alternating-offer model in Rubinstein (1982), and only then all continuation payoffs will be bounded by the bargaining frontier for all discount factors.

7 Appendix: Proofs

Proof of Lemma 5

Note that for all $a \in A$ and $v \in E^j$, $d_j(a) \leq d_j(a^*)$ and $M_j^j \leq 1 - \delta_i m_i^i$ imply that

$$\begin{aligned} 1-(1-\delta_j)d_j(a)-\delta_j v_j &\geq 1-(1-\delta_j)d_j(a^*)-\delta_j M_j^j \\ &\geq 1-(1-\delta_j)d_j(a^*)-\delta_j(1-\delta_i m_i^i) \\ &= (1-\delta_j)\left[1-d_j(a^*)\right]+\delta_j\delta_i m_i^i. \end{aligned}$$

Together with (15), we have $m_i^i \geq (1-\delta_j)\left[1-d_j(a^*)\right]+\delta_j\delta_i m_i^i$, which leads to the stated result. ■

Proof of Lemma 6

Recall the following inequalities:

$$\frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j(a^*)\right] \leq m_i^i \leq \frac{1}{1+\delta_i}d_i(a^*) < d_i(a^*).$$

For any even $T \geq 0$, we have

$$\begin{aligned} v_i(T) &= (1-\delta_i^T)d_i(a^*)+\delta_i^{T+1}m_i^i \\ &< (1-\delta_i^{T+1})d_i(a^*)+\delta_i^{T+1}m_i^i = v_i(T+1), \\ v_i(T+1) &= (1-\delta_i^{T+1})d_i(a^*)+\delta_i^{T+1}m_i^i \\ &\leq (1-\delta_i^{T+2})d_i(a^*)+\delta_i^{T+3}m_i^i = v_i(T+2), \end{aligned}$$

where the first inequality is trivial, and the second inequality is due to $m_i^i \leq \frac{1}{1+\delta_i}d_i(a^*)$. Comparing player j's payoffs, we have

$$\begin{aligned} v_j(T) &= (1-\delta_j^T)d_j(a^*)+\delta_j^T(1-\delta_i m_i^i) \\ &> (1-\delta_j^{T+2})d_j(a^*)+\delta_j^{T+2}(1-\delta_i m_i^i) = v_j(T+2), \\ v_j(T+2) &= (1-\delta_j^{T+2})d_j(a^*)+\delta_j^{T+2}(1-\delta_i m_i^i) \\ &\geq (1-\delta_j^{T+1})d_j(a^*)+\delta_j^{T+1}(1-m_i^i) = v_i(T+1), \end{aligned}$$

where the first inequality is trivial, and second inequality is due to $\frac{1-\delta_j}{1-\delta_i\delta_j}\left[1-d_j(a^*)\right] \leq m_i^i$.

■

Proof of Proposition 9

The proof of this proposition is constructive. Given $\delta_i \in (0,1)$, $\exists\ \hat{\delta}_j \in (\delta_i, 1)$ large enough so that for all $a \in A$ and $\delta_j \in \left(\hat{\delta}_j, 1\right)$,

$$(1-\delta_j)d_j(a) + \delta_j\left(1 - \frac{d_i(a^*)}{1+\delta_i}\right) > d_j(a^*). \tag{31}$$

Given $\delta_i \in (0,1)$ and $\delta_j \in \left(\hat{\delta}_j, 1\right)$, denote $\hat{m}$ as the least fix point of $\Lambda(\cdot)$. Accordingly, there exist $\hat{a} \in A$, $\hat{T} \in 2\mathbb{N}$, and $\hat{p} \in [0,1]$ such that (15) holds with equality at $\hat{m}$:

$$\begin{aligned} \hat{m} &= \max\left\{\begin{array}{l} (1-\delta_i)d_i(\hat{a}) + \delta_i\left[(1-\hat{p})v_i(\hat{T}) + \hat{p}v_i(\hat{T}+2)\right], \\ 1-(1-\delta_j)d_j(\hat{a}) - \delta_j\left[(1-\hat{p})v_j(\hat{T}) + \hat{p}v_j(\hat{T}+2)\right], \end{array}\right. \\ & \quad \text{s.t.} \quad (1-\hat{p})v_i(\hat{T}) + \hat{p}v_i(\hat{T}+2) \geq \frac{1-\delta_i}{\delta_i}\left[g_i(a) - d_i(a)\right] + \delta_i\hat{m}. \end{aligned}$$

We have the following two cases to examine:

Case 1: $\hat{m} = (1-\delta_i)d_i(\hat{a}) + \delta_i\left[(1-\hat{p})v_i(\hat{T}) + \hat{p}v_i(\hat{T}+2)\right]$.

Consider the following strategy profile: Player i makes an unacceptable offer (such as demands $\hat{m}$ or more). Player j rejects if and only if player i offers less than

$$(1-\delta_j)d_j(\hat{a}) + \delta_j[(1-\hat{p})v_j(\hat{T}) + \hat{p}v_j(\hat{T}+2)] \geq 1-\hat{m},$$

followed by $\hat{a}$ once. If player i deviates from $\hat{a}$, player j will offer $\delta_i\hat{m}$ and player i will accept in the following period. Otherwise, they play a^* until player j will offer $\delta_i\hat{m}$ either in period $\hat{T}$ with probability $1-\hat{p}$ or in period $\hat{T}+2$ with probability $\hat{p}$, while player i will accept. If player i rejects $\delta_i\hat{m}$ that should be accepted, player i will be minimaxed once, followed by what is described above, from which player i receives $\delta_i\hat{m}$ (player i's worst SPE when player j proposes). During this course, no one will make any acceptable offer. If player j deviates from what is described above, player j will be punished by the stationary equilibrium of Proposition 1, from which player j receives $d_j(a^*)$.

We now verify sequential rationality. It is clear from the construction that no one deviates in the proposing and responding stages. For example, player i has to offer at least

$$(1-\delta_j)d_j(\hat{a}) + \delta_j \left[(1-\hat{p})v_j(\hat{T}) + \hat{p}v_j(\hat{T}+2)\right]$$

in order to induce player j to accept, from which player i receives less than $\hat{m}$. In any period where a^* should be played, there is no mutually acceptable proposal because the continuation payoff vector lies strictly above the bargaining frontier. Player i has no incentive to deviate from either a^* or $\hat{a}$, due to the construction of the continuation payoffs. On the other hand, if player j deviates from $\hat{a}$, he will receive no more than $(1-\delta_j)g_j(\hat{a}) + \delta_j d_j(a^*) \leq d_j(a^*)$. If player j deviates elsewhere, he cannot receive more than $d_j(a^*)$ either. Inequality (31) implies that if player j ever deviates, his payoff will be less than his lowest continuation payoff. Hence, player j will not deviate.

Case 2: $\hat{m} = 1 - (1-\delta_j)d_j(\hat{a}) - \delta_j \left[(1-\hat{p})v_j(\hat{T}) + \hat{p}v_j(\hat{T}+2)\right]$.

Consider the following strategy profile: Player i demands $\hat{m}$. Player j rejects if and only if player i demands more than $\hat{m}$. If player i demands more and player j rejects (which should not occur), two players will play a^* until player j offers $\delta_i\hat{m}$ in either period $\hat{T}$ with probability $1-\hat{p}$ or in period $\hat{T}+2$ with probability $\hat{p}$, which will be accepted by player i.

Similar to Case 1, no one will deviate after player i demands more $\hat{m}$ and player j rejects. If player i demands more than $\hat{m}$ at the beginning, player j will reject, and player i will receive

$$(1-\delta_i)d_i(\hat{a}) + \delta_i \left[(1-\hat{p})v_i(\hat{T}) + \hat{p}v_i(\hat{T}+2)\right] \leq \hat{m}.$$

Therefore, player i will demand $\hat{m}$, which will be accepted by player j. In summary, no one has incentive to deviate when player i is supposed to demand $\hat{m}$.

We have shown that in either case, there is an equilibrium where player i receives $\hat{m}$, the least fixed point of $\Lambda(\cdot)$, when proposing an offer. ■

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