TI 2006-001/1
Tinbergen Institute Discussion Paper

# A Theory of Procedurally Rational Choice:
# Optimization without Evaluation

*Stefano Ficco*[1]
*Vladimir Karamychev*[2]
*Peran Reeven*[2]

[1] *Erasmus University Rotterdam, and Tinbergen Institute;*
[2] *Erasmus University Rotterdam.*

**Tinbergen Institute**
The Tinbergen Institute is the institute for economic research of the Erasmus Universiteit Rotterdam, Universiteit van Amsterdam, and Vrije Universiteit Amsterdam.

**Tinbergen Institute Amsterdam**
Roetersstraat 31
1018 WB Amsterdam
The Netherlands
Tel.: +31(0)20 551 3500
Fax: +31(0)20 551 3555

**Tinbergen Institute Rotterdam**
Burg. Oudlaan 50
3062 PA Rotterdam
The Netherlands
Tel.: +31(0)10 408 8900
Fax: +31(0)10 408 9031

Please send questions and/or remarks of non-scientific nature to driessen@tinbergen.nl.
Most TI discussion papers can be downloaded at http://www.tinbergen.nl.

# A theory of procedurally rational choice:

# Optimization without evaluation

Stefano Ficco

Vladimir Karamychev

Peran van Reeven

This version: 11-2005

**Abstract:** This paper analyses the behavior of an individual who wants to maximize his utility function, but he is not able to evaluate it. There are many ways to choose a single alternative from a given set. We show that a unique utility maximizing procedure exists. Choices induced by this optimal procedure are always transitive but generally violate the Weak Axiom. In other words, utility maximizing individuals who are unable to evaluate their objective functions fail to exhibit rational revealed preferences.

**Key Words:** Bounded rationality, optimal selection procedure, procedural rationality.

**JEL Classification:** D01, D81

**Address for Correspondence:** Peran van Reeven, Department of Applied Economics, Erasmus University Rotterdam, H12-19, Burg. Oudlaan 50, 3062 PA Rotterdam, The Netherlands; tel.: +31-(10)-408.2113, E-mail: vanreeven@few.eur.nl.

We thank Maarten Janssen, Robert Dur and Han Bleichrodt for many discussions and very helpful comments.

## 1. Introduction

Misalignments between observed human behavior and predictions of rational choice theory have become a fertile ground for behavioral economics and new theories of bounded rationality. In accordance with these theories, a decision-maker follows a process of reasonable deliberation, often called a procedure, by which he chooses a single alternative from a given set. Economic literature offers many selection procedures, and in this paper we analyze which procedure is the best. Our goal is two-fold: to determine which procedure a utility maximizing decision-maker will choose, and to analyze the resulting choice function.

If the decision-maker is able to evaluate his utility function, then the optimal procedure is to evaluate all alternatives and to select the alternative with the highest level of utility. The behavior of such an individual conforms to rational choice theory and is called rational. One possible explanation why observed human behavior is in conflict with rational choices is that individuals do not have objective functions. However, this assumption would make individual choices completely unpredictable. Alternatively, it might be that individuals do have objective functions but they either cannot evaluate them or cannot compare the levels of utilities that each alternative yields. Since the evaluation of objective functions seems to be a much more complex task than the comparison of the resulting values, we assume in this paper that a decision-maker does have an objective function but, at the same time, cannot evaluate it.

Existing economic literature provides different arguments why individuals might not be able to evaluate objective functions. It could be the complexity of the environment that clarifies why individuals do not have a full understanding of the consequences of their decisions. Alternatively, individuals' cognitive limitations might prevent them from taking those consequences into account in a precise quantitative manner. For example, a choice of education has a large impact on future (career-)opportunities and well-being. However, the complexity of the environment makes it difficult to foresee all consequences of such a choice.

In addition, individuals' cognitive limitations make it difficult to take all these consequences simultaneously into account in a precise, quantitative, manner. In order to illustrate how environmental complexities and cognitive limitations affect individuals' ability to evaluate objective functions, let us consider the following casino game.

A croupier has three visibly identical coins on the table. Two coins are normal and show head and tail with equal probability. The other coin is a winning coin and shows head more often than tail. The croupier tosses the coins, and the player can select one or more coins for further play. The selected coins are tossed again, and the game repeats itself: the player selects some coins for the next round. The game ends when only a single coin remains, and the player wins if this coin is the winning coin.

Let us take a fully rational individual to play this game, and call him John. If the casino does not take any countermeasures, John can win the game almost surely. By continuously passing all coins for further play, he can count the number of heads that each coin shows. Then, by selecting the coin that showed the largest number of heads, he will select the coin with the highest likelihood of being a winning coin. As a preventive measure, whenever he faces John, the croupier tosses coins from a single tumble. This makes the game for John much more complex. Since coins are tossed from a single tumble, they loose their identities every time the croupier puts them back into the tumble. As a result, John cannot count the number of heads that each coin shows anymore. In other words, the complexity of the environment prevents him from evaluating his objective function for each alternative.

Now let us take a boundedly rational individual to play this game, and call him Bill. Bill has imperfect recall. He also wants to win the game, but when he plays he does not remember the results of previous tosses. Imperfect recall prevents Bill from counting the number of heads that each coin shows. Hence, Bill cannot evaluate his objective function, just like John.

As such, the casino game represents a decision problem in which the decision-maker cannot evaluate his objective functions. The complexity of the environment and cognitive limitations make it impossible to combine several pieces of information into a scalar performance measure for each alternative. This inability to evaluate objective functions urges the decision-maker to use another procedure in order to maximize his objective function. Let us focus on Bill. As he does not remember what happened before, Bill can only use the toss results of the current round as a basis for selecting one or more coins for the next round. More precisely, depending on how many coins show head and how many coins show tail in the current round, he decides how many of such heads and tails he selects for further play. All these selection decisions represent a selection procedure, and, in case of tree coins only, Bill has a total of $\prod_{n=1}^{n=3}\prod_{k=0}^{k=n}\left((k+1)(n-k+1)-1\right)=2700$ different selection procedures to choose from. Among those is Tversky's 'Elimination By Aspects', EBA hereafter, in which all coins that show tail are eliminated, and only coins that show head are selected. Another possible procedure is Simon's 'Satisficing', SAT hereafter, in which the very first coin that shows head is selected. If we now shift our focus to John, then we can observe that John has even more selection procedures to choose from than Bill. The number of heads and tails that John selects at a certain round not only depends on how many coins show head and how many coins show tail in that round, but also on the whole history of previous tosses and eliminations. Since so many procedures are feasible, our primary goal is to identify which procedure is optimal to follow.

In this paper, we study a decision-maker who chooses one alternative (a coin) from a set of $N$ available alternatives ($N$ coins). Each alternative comes in two types, high (winning coin) and low (normal coin). The decision-maker does not observe the true types of alternatives but views each alternative as a set of aspects. Each aspect represents one of his perceptions of how alternatives may affect his objective function. Hence, an aspect imprecisely reveals true types of the alternatives. If an alternative has a certain aspect (if a coin shows head) then it is good news about the quality of this alternative. If on the other hand, an alternative does not have this aspect (if a coin shows tail), then it is bad news. We assume that the decision-maker can use infinitely many different, but equally informative, aspects. Due to his inability to evaluate objective functions, the decision-maker cannot aggregate several aspects of an alternative into a single performance measure. Each aspect corresponds to a single selection stage (round of the game). In each stage, the decision-maker selects a non-empty sub-sample of alternatives that he will pass to the next stage. The selection procedure ends when only one alternative remains. The decision-maker maximizes the probability that this remaining alternative is of high type.

Our main finding is that a unique selection procedure exists that dominates all other selection procedures. This dominance holds irrespective of the source of the inability to evaluate objective function (environmental complexity or a cognitive limitation), and for all generic values of the parameters of the model (the size of the initial sample of alternatives, the number of high-type alternatives within this sample, and the degree in which aspects are informative about the true types). We call this procedure 'Single Worst Elimination', SWE hereafter, because an alternative must be eliminated at a particular stage if, and only if, it is a unique alternative at that stage that does not have the corresponding aspect. Applying this result to John and Bill implies that both players must select all coins for further play until they observe a single tail, in which case they eliminate the corresponding coin.

The optimal selection procedure SWE dominates both SAT and EBA if aspects do not fully reveal true types of alternatives. However, if an aspect does fully reveal that an alternative is of high type, then SAT is the optimal procedure. Similarly, if the absence of an aspect fully reveals that an alternative is of low type, then EBA is the optimal selection procedure. Therefore, SAT and EBA are only optimal in special limiting cases of the model.

Our secondary goal is to analyze the choice function of an individual who follows SWE. Our main finding here is that choices induced by SWE are always transitive but generally violate the Weak Axiom. Consequently, the optimal procedure may result in seemingly irrational behavior. In accordance with the existing literature on bounded rationality, it is not surprising that Bill behaves non-rationally due to his cognitive limitation. More interesting is the fact that John, who is a fully rational individual, exhibits the same behavior as Bill. On the basis of revealed preferences it is impossible to distinguish a fully rational individual from a boundedly rational individual. Moreover, even if we ask them how they arrived at their choices, they will give us the very same answer - SWE.

The rest of the paper is organized as follows. Section 2 provides motivation for our modeling assumptions and reviews the related literature. Section 3 states the model which is analyzed in section 4. Section 5 discusses SAT and EBA as special limiting cases of the optimal selection procedure SWE. Section 6 derives the properties of choice induced by SWE and section 7 concludes the paper. An appendix contains some proofs.

## 2. Relation to existing literature

Simon (1955) suggested that economists should take into account human's cognitive limitations as well as the complexity of the environment in analyzing individual decision-making. The paper follows this suggestion. Due to inability to evaluate objective functions, humans need a procedure that results in the selection of a single alternative. Any such a procedure can be characterized by the elimination of all available alternatives, except one. That is why we model decision-making as a 'Choice By Elimination' (Tversky, 1972), hereafter referred to as CBE-model. Following Tversky, each alternative is represented by a set of aspects, and the choice is made by successive elimination of alternatives. A decision-maker in the original CBE-model is unable to integrate multiple aspects into a single performance measure for each alternative, which makes him unable to establish and evaluate objective functions.

Although the set-up of our model is similar to the CBE-model, there is a significant difference. In the CBE-model the decision-maker does not optimize but follows a prescribed selection procedure, namely EBA. Consequently, there is no need to explicitly relate an alternative to the level of the objective function that this alternative yields. In the original CBE-model, aspects can simply be considered as good-characteristics (Lancaster, 1966) that a decision-maker likes. In our model, aspects provide information on how alternatives translate into utility. In the spirit of Milgrom (1981), each aspect can be seen as news about the type of an alternative. Another difference with the original CBE-model is that, due to the ignorance of the decision-maker about the impact of a particular aspect on his objective function, all aspects are assumed to be equally important.

Similar to the CBE-model, we make the following assumptions. First, no costs are involved in using aspects during the selection procedure. Humans accumulate an infinite number of aspects through experience and learning so that these aspects can be assumed readily available. Second, the implementation of the selection procedure is costless. We will see that the optimal selection procedure is rather simple and does not require any computations. Finally, there is no discounting. The objective function only depends on the selected alternative and does not depend on how it has been selected.

This paper can be positioned in literature on bounded rationality. This literature offers several approaches to explain seemingly irrational human behavior. One approach is to assume that individuals do not have a state-independent utility function. Economists propose several state-dependent functions that individuals maximize. A well-known example is the prospect theory of Kahneman and Tversky (1979). In Kalai *et al.* (2002) humans do not have a single preference relation, but several incomplete ones. Another example is Easley and Rustichini (2005) who consider an individual with preferences that evolve by learning from past experience.

Another approach is to assume that individuals do have an objective function but do not evaluate it, because either they cannot do that or it is prohibitively costly. Based on psychological evidences and experimental studies, literature offers many different selection procedures. Rubinstein (1988) analyzes behavior of an individual who simplifies alternatives by eliminating aspects that are similar for all alternatives. De Palma, Myers and Papageorgiou (1994) analyze an individual who myopically adjusts his current behavior in order to increase his utility. Gilboa and Schmeidler (1995) study an individual who maximizes the weighted average of the utility levels that resulted from using the same alternative in the past. The same authors (1997) offer another procedure in which an individual calculates the so-called cumulative index of satisfaction and selects an alternative that maximizes this index. These are just a few examples, and Bettman *et al.* (1998) provide an extensive overview. The focus of this literature is on the resulting choices and on the circumstances under which these revealed preferences are rational.

A third approach is to assume that individuals are able to evaluate objective functions, but comparisons of its values for different alternatives are costly. An example of this approach is Radner (1993) who assumes that individuals can only make pair-wise comparisons.

In this paper, we follow the second approach and assume that decision-makers cannot evaluate their objective function. However, rather than proposing a specific selection procedure and analyze resulting choices, we allow the decision-maker to use any selection procedure. The question that we address, and that, according to our knowledge, has not been addressed before, is what selection procedure a decision-maker will follow if he chooses it optimally. In order to answer this question, we introduce the optimality criterion in a procedural approach to decision-making, which allows us to compare different selection procedures.

## 3. The Model

A decision-maker has to choose a single alternative from a set of $N$ alternatives $\{x_i\}$. Each alternative is either a high type $\theta^H$ or a low type $\theta^L$. The decision-maker does not observe the true types of alternatives but views each alternative as a set of aspects. Whether an alternative has a certain aspect or not provides imprecise information about its true type. In particular, the decision-maker believes that a high type alternative has an aspect with probability $q^H$ and that a low type alternative has an aspect with probability $q^L$, where $q^L < q^H$.

The number of alternatives at the beginning of stage $t$ is denoted by $N_t$. Having observed $H_t$ alternatives with the corresponding aspect at stage $t$, called good alternatives hereafter, and $L_t = N_t - H_t$ alternatives without the aspect, called bad alternatives hereafter, the decision maker selects a sub-sample of alternatives to be passed to the next stage ($t + 1$). This sub-sample will consist of $h_t \in [0, H_t]$ good alternatives and $l_t \in [0, L_t]$ bad alternatives. At least one alternative must be selected, i.e., $h_t + l_t \geq 1$. The number of alternatives at the beginning of stage ($t + 1$) then becomes $N_{t+1} = h_t + l_t$. The selection procedure ends when $N_t = 1$.

We model the decision-maker as Bill and assume that he is not able to evaluate his objective function due to imperfect recall. As we will see in section 4, the problem of John can be solved in the very same way as that of Bill, although it is formally different. Due to his imperfect recall, the decision-maker does not remember what happened before. He only observes whether alternatives have an aspect for one aspect at a time. Therefore, in every stage $t$, the decision-maker has to choose the numbers of good and bad alternatives $h_t$ and $l_t$ based only on $H_t$ and $L_t$, and any selection procedure $S$ can be written as a mapping $S:(H,L) \rightarrow (h,l)$.

The goal of the decision-maker is to maximize his (expected) utility and his pay-off is

$$Utility = \Pr(\hat{x} = \theta^H)u(\theta^H) + \Pr(\hat{x} = \theta^L)u(\theta^L) = \Pr(\hat{x} = \theta^H)(u(\theta^H) - u(\theta^L)) + u(\theta^L),$$

where $\hat{x}$ is the finally selected alternative and $\Pr(\hat{x} = \theta^H)$ is the probability that it is a high type alternative $\theta^H$. The utility levels $u(\theta^H)$ and $u(\theta^L)$ are exogenously given, and the decision-maker knows that $u(\theta^H) > u(\theta^L)$. Therefore, maximization of the expected utility $U$ is equivalent to the maximization of the likelihood $\Pr(\hat{x} = \theta^H)$. This probability of selecting $\theta^H$

depends on the implemented selection procedure $S$, on the total number of alternatives $N$, and on the number of high type alternatives $N^H$ in the sample. Hence, it can be written as

$$\Pr\left(\hat{x} = \theta^H\right) = U\left(S, N, N^H\right),$$

where the function $U\left(S, N, N^H\right)$ becomes the objective function that the decision-maker must maximize over all feasible selection procedures.

## 4. Analysis of the model

Before choosing an alternative, the decision-maker needs to choose a selection procedure. Since the decision-maker does not know the number of high type alternatives $N^H$ in the sample, he is unable to evaluate $U\left(S, N, N^H\right)$ for all feasible selection procedures. In the following proposition, some properties of $U\left(S, N, N^H\right)$ are used in order to derive conditions that any optimal selection procedure must satisfy.

**Proposition 1.** For any values of $q^H$ and $q^L$ ($0 < q^L < q^H < 1$), for any number $N$ of alternatives in the initial sample, and for any nontrivial distribution of the number $N^H$ of high type alternatives $\theta^H$ in the initial sample, any optimal selection procedure $S^* : (H, L) \to \left(h^*, l^*\right)$ is such that:

a) $h^*\left(H_t, L_t\right) = H_t$, i.e., all good alternatives must be selected in each stage.

b) Either $l^*\left(H_t, L_t\right) = 0$ or $l^*\left(H_t, L_t\right) = L_t$, i.e., either all bad alternatives or none of them must be selected.

c) $l^*\left(H_t, L_t\right) = L_t$ for $L_t \geq 2$, i.e., if there are at least two bad alternative in stage $t$, then all bad alternatives must be selected.

d) If a selection procedure $S^*$ satisfies conditions (a)-(c) of the proposition, then $U\left(S^*, N, N^H\right)$ strictly increases in $N^H$.

The proof of Proposition 1 is in the appendix and is made by induction. If the decision-maker only had one alternative to choose from, i.e., $N = 1$, then the only feasible selection procedure $(h_t,l_t) = (H_t,L_t)$ is trivially optimal and satisfies the required properties. In addition, it is easy to see that for $N = 1$ the probability of selecting a high type alternative $U\left(S^*,1,N^H\right)$ coincides with $N^H$, i.e., $U\left(S^*,1,N^H\right) = N^H$. Thus, $U\left(S^*,1,N^H\right)$ strictly increases in $N^H$.

Now, suppose that we have already proven (induction assumption) that if the number of alternatives does not exceed $N$ then the properties (a)-(d) from Proposition 1 hold. The proof then shows that in the case of $N + 1$ alternatives, properties (a)-(d) continue to hold.

This result is proven as follows. Let the decision-maker face $N_t = N + 1$ alternatives in a stage $t$. Then, depending on the number of high types $N^H$ among them, each selection procedure $S$ induces a corresponding posterior distribution of the number of high type alternatives $N_{t+1}^H$ in the resulting set of $N_{t+1}$ selected alternatives. The probability of selecting a high type alternative $U\left(S, N_t, N_t^H\right)$ is, by definition, the expectation of $U\left(S^*, N_{t+1}, N_{t+1}^H\right)$ with respect to this posterior distribution of $N_{t+1}^H$. By the induction assumptions, $U\left(S^*, N_{t+1}, N_{t+1}^H\right)$ monotonically increases with $N_{t+1}^H$ for all $N_{t+1} \le N$. Hence, if a selection procedure $S$ induces a posterior distribution that stochastically (first-order) dominates the posterior distribution induced by another selection procedure $\hat{S}$, then $S$ yields a strictly higher pay-off than $\hat{S}$. This first-order stochastic dominance criterion makes it possible to compare different selection procedures without evaluations of the function $U\left(S, N, N^H\right)$ for different $S$ and to derive the properties that each optimal selection procedure must satisfy.

The following two peculiarities are worth mentioning here. First, it is not true that $U\left(S, N, N^H\right)$ strictly increases with $N^H$ for all selection procedures $S$, as the following example demonstrates.

**Example 1.** Let us take $N = 3$, $q^L = 0.5$ and the selection procedure $\overline{S}$ such that $(h_t,l_t) = (1,0)$ if $(H_t,L_t) = (1,2)$, $(h_t,l_t) = (0,1)$ if $(H_t,L_t) = (2,1)$, and $(h_t,l_t) = (H_t,L_t)$ otherwise. In other words, the decision-maker selects either a unique good alternative or a unique bad alternative in each selection stage and selects all alternatives otherwise. It is easy to get the following expressions for the objective function $U(\overline{S},3,N^H)$:

$$U(\overline{S},3,0)=0,\ U(\overline{S},3,1)=\frac{1}{3},\ U(\overline{S},3,2)=\frac{4q^H(1-q^H)}{2-(q^H)^2-(1-q^H)^2},\ U(\overline{S},3,3)=1\,.$$

It follows that $U(\overline{S},3,2)<U(\overline{S},3,1)$ if $q^H(1-q^H)<0.1$, so that the decision-maker is better-off when the initial sample has one high type alternative rather than two high type alternatives. //

This example stresses the fact that it is not the objective function $U(S,N,N^H)$ itself, but its maximum across all possible selected procedures, i.e., $U(S^*,N,N^H)$, that monotonically increases with the number of high type alternatives $N^H$. This observation leads us to the second peculiarity that we want to point out. It is not generally true that the decision-maker must select all good alternatives at a stage. It is the monotonicity of $U(S^*,N,N^H)$ that gives him incentives to do so. On the other hand, if the decision-maker always selects all good alternatives later on, the proof shows that $U(S,N,N^H)$ is monotonically increasing. Therefore, both properties are interconnected, non-trivial and have to be proven simultaneously.

There are two selection procedures that satisfy properties (a)-(c) from Proposition 1. The first procedure is the 'Always Pass' procedure with $S^*(H,L)=(H,L)$, i.e., all alternatives, whether they are good or bad, are passed to the next selection stage. Obviously, this procedure never ends and, therefore, is not feasible. The other optimal selection procedure is such that in each stage all alternatives must be selected unless only one alternative in the sample is bad. In this case, all good alternatives must be selected, and the single bad alternative must be eliminated. Formally:

$$S^*(H,L) = (h^*, l^*) = \begin{cases} (H,0), \text{if } H \geq 1 \text{ and } L = 1 \\ (H,L), \text{otherwise} \end{cases}$$

In accordance with its properties, we call this selection procedure 'Single Worst Elimination', abbreviated as SWE and denoted as $S^{\mathrm{SWE}}$. SWE prescribes only to eliminate an alternative if it is the single worst (without the aspect) alternative at a certain selection stage. Thus, we have proven the following proposition.

**Proposition 2.** For any values of $q^H$ and $q^L$ ($0 < q^L < q^H < 1$), for any number $N$ of alternatives in the initial sample, and for any nontrivial distribution of the number $N^H$ of high type alternatives in the initial sample, SWE is the unique optimal selection procedure.

In accordance with Proposition 2, it is optimal for any decision-maker who is subject to imperfect recall to follow SWE. In accordance with Proposition 1, the pay-off that he gets from implementing SWE strictly increases with the number of high type alternatives within the initial sample.

Let us compare SWE with SAT and EBA. In accordance with SAT, a decision-maker tries to identify the alternative that yields the *highest* perceived level of utility and *chooses* this alternative as his final choice. SAT is a mirror image of SWE. In accordance with SWE, a decision-maker tries to identify the alternative that yields the *lowest* perceived level of utility and *eliminates* this alternative to make sure that it will never be his final choice. It is well-known that a risk-averse individual behaves in a similar fashion by putting higher weights on lower levels of utility. In this way, SWE might provide us with an explanation why individuals are often risk averse. Similar to SWE, EBA selects all good alternatives. The difference is that EBA eliminates *all* bad alternatives whereas SWE only eliminates a bad alternative if it is a *unique* bad alternative at a stage. Hence, SWE uses more selection stages than EBA, and, consequently, results in more accurate choices than EBA.

The number of selection stages at which alternatives are eliminated allows for another characterization of selection procedures. Each selection procedure guarantees that the finally selected alternative has at least a certain number of aspects. SWE maximizes the minimum number of aspects that the finally selected alternative has. SWE selects an alternative with at least ($N$ - 1) aspects. All other ($N$ - 1) alternatives were eliminate in ($N$ - 1) selection stages in which the finally selected alternative had the corresponding aspects. In contrast, SAT and EBA guarantee that the selected alternative has far fewer aspects. SAT selects an alternative that has at least one aspect, because this alternative is the very first alternative with an aspect. EBA might select an alternative that does not have an aspect at all, because it requires that one alternative must be selected even if all alternatives in the first stage do not have the corresponding aspect.

However, SWE is not the only procedure that satisfies this 'max-min'-property. For example, a procedure which specifies that in each stage a single randomly chosen alternative must be eliminated if, and only if, all alternatives have the aspect, also guarantees that the finally selected alternative has at least ($N$ - 1) aspects. Since such a procedure is equivalent to random choice, it is clearly not optimal. Hence, the 'max-min'-property is a necessary but not a sufficient condition for the optimality of a selection procedure.

So far we have analyzed the optimal play of Bill. His optimal strategy is to select all coins for further play until he observes a single coin that shows tail. When this happens, he eliminates this coin. Contrary to Bill, John has perfect memory so that he remembers the numbers of heads and tails that he observed in all previous rounds of the game. Due to the single tumble, and this is how we model the complexity of the environment, he is not able to keep track of tossing results for each coin. Therefore, the results of previous tosses only provide him aggregated information about the overall quality of the sample. This information can be used for updating his prior beliefs about the number of winning coins in the current sample. However, since SWE strictly dominates all the other selection procedures for *any* prior beliefs, SWE remains the unique optimal selection procedure. Hence, John, who is fully rational, will follow the same playing strategy as boundedly rational Bill. Consequently, Proposition 2 holds for any individual who is not able to evaluate his objective function.

## 5. Optimality of SAT and EBA

In the previous section it has been shown that SWE is the unique optimal selection procedure for $0 < q^L < q^H < 1$. In this section we show that if low type alternatives never have aspects, i.e., $q^L = 0$, then SAT is optimal. Alternatively, if high type alternatives always have aspects, i.e., $q^H = 1$, then EBA is optimal. In order to show this, we first need to modify the model. If $q^L = 0$ and the initial sample of alternatives has only low types, then no alternative will ever show an aspect. Since no alternative is satisfactory, it is optimal to pass all of them to the next stage and screen them again. Hence, if the number of aspects is infinite, an optimal selection procedure never ends. The same problem occurs for $q^H = 1$ if there are multiple high type alternatives in the sample. In this case, there will be multiple alternatives with an aspect in every selection stage. Since it is optimal to pass all these good alternatives to the next stage, the optimal selection procedure never ends. In order to avoid this problem, we assume that the number of aspects is arbitrary large but finite and equals to $T$. Then, if the decision-maker selects more than one alternative in the last stage $T$, one of these alternatives will be selected at random.

If $q^L = 0$, a low type alternative never has an aspect. Therefore, if an alternative has an aspect, it is a high type alternative. Selecting any number of such alternatives is optimal. If no alternative in the sample has an aspect, it is optimal to select all alternatives. Hence, if $q^L = 0$, there are multiple optimal selection procedures:

$$S^{\text{SAT}}(H,L) = (h^*, l^*) = \begin{cases} (h,0), \text{if } H \geq 1 \\ (0,L), \text{if } H = 0 \end{cases} \text{ for an arbitrary } h \in [1,H].$$

All these selection procedures are essentially SAT because, if feasible, they select only good alternatives.

If $q^H = 1$, a high type alternative always has an aspect. Therefore, if an alternative does not have an aspect, it is a low type alternative. In the presence of one or more alternatives with an aspect, removing all alternatives without the aspect is optimal. However, if no alternative in the sample has an aspect, then all alternatives are of low type, and all selection procedures are pay-off equivalent. Hence, if $q^H = 1$, there are multiple optimal selection procedures:

$$S^{\text{EBA}}(H,L) = (h^*, l^*) = \begin{cases} (H,0), \text{if } H \geq 1 \\ (0,l), \text{if } H = 0 \end{cases} \text{ for an arbitrary } l \in [1,L].$$

All these selection procedures are essentially EBA because, if feasible, they eliminate in every stage all bad alternatives.

Applying these results to our casino game implies that if normal coins have tails on both sides, both John and Bill play SAT. On the other hand, if the winning coin has head on both sides, they play EBA.

## 6. Properties of the choice induced by SWE

One of the pivotal questions in the literature on bounded rationality and behavioral economics is whether a particular procedure induces a rational choice of alternatives. In this section, we analyze the choice induced by SWE. Surprisingly, despite it is the unique optimal selection procedure, SWE induces choice that violates the rationality criterion.

**Proposition 3.** Choices induced by the SWE selection procedure

a) are always transitive;

b) generally violate the Weak Axiom.

**Proof.**

a) Let us take any three alternatives $x_1$, $x_2$ and $x_3$. Let the first stage where exactly one or exactly two alternatives have the corresponding aspect be denoted by $t$. Given infinitely many aspects and $q^L, q^H \in (0,1)$, there will be such a stage with probability one. At this stage $t$ either a single alternative, let's say $x_1$, has the aspect and $x_2$ and $x_3$ do not, or two alternatives, let's say $x_1$ and $x_2$, have the aspect and $x_3$ does not. In the former case, it is easy to see that $C(\{x_1, x_2\}) = \{x_1\}$. SWE passes $x_1$ and $x_2$ up to stage $t$ because they have identical aspects in all previous stages. Then, in stage $t$, SWE eliminates $x_2$ as the single worst alternative. Similarly, $C(\{x_1, x_3\}) = \{x_1\}$. Hence, intransitivity does not arise irrespective of the value of $C(\{x_2, x_3\})$. In the latter case, SWE results in $C(\{x_1, x_3\}) = \{x_1\}$ and $C(\{x_2, x_3\}) = \{x_2\}$. Also here, intransitivity does not arise irrespective of the choice $C(\{x_1, x_2\})$. Therefore, for any three alternatives $x_1$, $x_2$ and $x_3$, choices $C(\{x_1, x_2\}) = \{x_1\}$ and $C(\{x_2, x_3\}) = \{x_2\}$ imply $C(\{x_1, x_3\}) = \{x_1\}$. Consequently, SWE always leads to transitive choices.

b) Suppose that a decision-maker faces three alternatives with the following first three aspects.

| | Aspect 1 | Aspect 2 | Aspect 3 | … |
|---|---|---|---|---|
| $x_1$ | 1 | 0 | 0 | … |
| $x_2$ | 0 | 1 | 1 | … |
| $x_3$ | 0 | 1 | 0 | … |

Here '1' denotes the presence of an aspect and '0' denotes its absence. Applying SWE to the subsets $\{x_1, x_2\}$ and $\{x_1, x_2, x_3\}$ leads to $C(\{x_1, x_2\}) = \{x_1\}$ and $C(\{x_1, x_2, x_3\}) = \{x_2\}$. In the former case, SWE eliminates $x_2$ as the single worst alternative in stage one. In the latter case, SWE selects all three alternatives in stage one, eliminates $x_1$ in stage two, and finally eliminates $x_3$ in stage three. This choice structure clearly violates the weak axiom of revealed preferences. ■

Proposition 3 shows that the optimal choice of a selection procedure guarantees the transitivity of the resulting choices of alternatives, but not the fulfillment of the Weak Axiom. Although SWE is the optimal procedure, it leads to a non-rational choice of alternatives. As an illustration, let us go back to the casino game once more. Fully rational John plays SWE. SWE, in turn, may result in choices that violate the Weak Axiom. Hence, the revealed preferences of John might be non-rational. This seemingly irrational behavior arises because John does not directly choose a coin, but a particular selection procedure. Consequently, if a decision-maker, who is unable to evaluate objective functions, optimally chooses a selection procedure, then the rationality of his revealed preferences cannot be used as criterion for individual rationality.

In the previous section we have shown that the decision-maker follows SAT in case $q^L = 0$ and that he follows EBA in case $q^H = 1$. In both cases, the resulting CBE-model is a random utility model (see Tversky, 1972), and the revealed preferences are always rational. Thus, individual rationality only coincides with a rational choice of alternatives for extreme values of the revealing probabilities. In contrast, had the decision-maker been able to evaluate his objective function for each alternative, his revealed preferences would have been always rational.

## 7. Conclusion

In this paper we took a procedural approach to decision-making by assuming that humans cannot evaluate objective functions and have to base their decisions on partial information provided by aspects. In the resulting CBE-model, we found a unique selection procedure that strictly dominates all the other selection procedures. This dominance is independent of sample sizes, prior distributions of the number of high type alternatives within the initial sample, and the degree in which aspects are informative about the true types. We call this optimal selection procedure 'Single Worst Elimination', SWE in short, because it eliminates an alternative if, and only if, it is the single worst alternative in the sample. Interestingly, when aspects become fully informative about either high type or low type alternatives, either SAT or EBA become optimal.

Choices induced by SWE are always transitive, but generally violate the Weak Axiom. Usually utility maximizing behavior is considered to be rational. However, a utility maximizing decision-maker who is unable to evaluate his utility function fails to exhibit rational revealed preferences. This seemingly irrational behavior arises because the decision-maker does not select an alternative, but a procedure. In our model, the choice of a selection procedure is rational, whereas the choice of alternatives is not a conscious choice but just an outcome of a selection procedure. Consequently, whether choice conforms to the Weak Axiom or not is irrelevant for assessing the rationality of the decision-maker who must choose a specific selection procedure due to his inability to evaluate his objective function.

We have shown that the procedure that maximizes the chances of selecting the best alternative can be found without the necessity of evaluating this objective function. In this way, the rationality of choice of a selection procedure appears to be a natural ground for defining procedural rationality. As a result, the distinction between substantive rationality and procedural rationality can be consistently redefined as a distinction between the ability and the inability of a decision-maker to evaluate objective functions.

The optimality of SWE has been derived under the following three assumptions. First, alternatives and aspects come in binary types; second, all aspects are equally informative; and lastly, individuals have infinitely many aspects. Relaxing these assumptions will become next steps in analyzing optimal selection procedures and, consequently, procedural rationality.

Finally, the properties of SWE make this procedure very attractive in explaining individuals' risk aversion from a procedural point of view.

## References


Bettman, J., M. Luce, and J. Payne, 1998, Constructive consumer choice processes, *Journal of Consumer Research* 25, 187-217.

De Palma, A., G. Myers and Y. Papageorgiou, 1994, Rational choice under an imperfect ability to choose, *American Economic Review* 84, 419-440.

Easley, D. and A. Rustichini, 2005, Optimal guessing: Choice in complex environments, *Journal of Economic Theory* 124, 1-21.

Gilboa, I. and D. Schmeidler, 1995, Case-based decision theory, *Quarterly Journal of Economics* 110, 605-639.

Gilboa, I. and D. Schmeidler, 1997, Cumulative utility consumer theory, *International Economic Review* 38, 737-761.

Kahneman, D. and A. Tversky, 1979, Prospect theory: An analysis of decision under risk, *Econometrica* 47, 263-292.

Kalai, G., A. Rubinstein and R. Spiegler, 2002, Rationalizing choice functions by multiple rationales, *Econometrica* 70, 2481-2488.

Lancaster, K., 1966, A new approach to consumer theory, *Journal of Political Economy* 74, 132-157.

Milgrom, P., 1981, Good news and bad news: Representation theorems and applications, *Bell Journal of Economics* 12, 380-391.

Radner, R., 1993, The organization of decentralized information processing, *Econometrica* 61, 1109-1146.

Rubinstein, A., 1988, Similarity and decision-making under risk (is there a utility theory resolution to the Allais paradox?), *Journal of Economic Theory* 46, 145-153.

Simon, H., 1955, A behavioral model of rational choice, *Quarterly Journal of Economics* 69, 99-118.

Tversky, A., 1972, Choice by elimination, *Journal of Mathematical Psychology* 9, 341-367.

## Appendix

**Proof of Proposition 1.** The proof is done by induction in six steps.

In step one, we assume that in stage *t* the decision-maker faces a set of $N_t \geq 2$ alternatives and that by applying a selection procedure ($h_t$,$l_t$), he selects $h_t + l_t \leq N_t - 1$ alternatives. It is also assumed that for all $N \leq N_t - 1$ the properties (a)-(d) of ($h^*$,$l^*$) stated in the proposition hold. The (unknown) prior distribution of the number of high type alternatives $N_t^H$ at this stage *t* is denoted by $p_t(z) \equiv \Pr(N_t^H = z)$. Under these assumptions, implementation of a selection procedure ($h_t$,$l_t$) at stage *t* induces the following posterior distribution of the number of high type alternatives $N_{t+1}^H$ at stage (*t* + 1):

$$p_{t+1}\left(N_{t+1}^H = z \middle| N_t, N_t^H, H_t, h_t, l_t\right) \equiv \sum_{N_t^H=0}^{N} p_t \Pr\left(N_{t+1}^H = z \middle| N_t, N_t^H, H_t, h_t, l_t\right).$$

Let us denote by $\hat{S}(h_t, l_t)$ the selection procedure ($h_t$,$l_t$) at stage *t* followed by an optimal selection procedure $S^*$ thereafter. The pay-off from $\hat{S}(h_t, l_t)$ for any given $N_t^H$ reads as

$$\begin{aligned} U\left(\hat{S}(h_t, l_t), N_t, N_t^H\right) &= \sum_{z=0}^{z=h+l} p_{t+1}\left(N_{t+1}^H = z \middle| N_t, N_t^H, H_t, h_t, l_t\right) U\left(S^*, h_t + l_t, z\right) \\ &= 1 - \sum_{z=0}^{z=h_t+l_t-1} F_{t+1}\left(z \middle| N_t, N_t^H, H_t, h_t, l_t\right)\left(U\left(S^*, h_t + l_t, z+1\right) - U\left(S^*, h_t + l_t, z\right)\right), \end{aligned}$$

where $F_{t+1}\left(w \middle| N_t, N_t^H, H_t, h_t, l_t\right) \equiv \sum_{z \leq w} p_{t+1}\left(N_{t+1}^H = z \middle| N_t, N_t^H, H_t, h_t, l_t\right)$ is the corresponding cumulative distribution function. By the induction assumption, $U\left(S^*, h_t + l_t, z+1\right) - U\left(S^*, h_t + l_t, z\right) > 0$. Therefore, if a distribution $F_{t+1}\left(w \middle| N_t, N_t^H, H_t, h_t, l_t\right)$

induced by $(h_t, l_t)$ first-order stochastically dominates a distribution $F_{t+1}\left(w \middle| N_t, N_t^H, H_t, h_t', l_t'\right)$ induced by $(h_t', l_t')$, then the selection procedure $(h_t, l_t)$ yields a strictly higher pay-off than the pay-off from the selection procedure $(h_t', l_t')$.

In steps two, three and four, using the first-order stochastic dominance criterion, we compare different selection procedures $(h_t,l_t)$ and derive properties (a)-(c) of the proposition for $N = N_t$. In particular, in step two it is established that $h^* = H_t$ for $l_t = 0$, whereas in step three it is shown that $h^* = H_t$ also for $l_t > 0$. Finally, in step four it is shown that it is either $l^* = 0$ or $l^* = L_t$, and also that $l^* = L_t$ for $L_t \geq 2$. Thus, steps two, three and four, under assumptions made in step one, prove that any optimal selection procedure $S^*$ must satisfy properties (a)-(c) also for $N = N_t$ as it maximizes $\sum_{z=0}^{z=N_t} p_t(z) U\left(\hat{S}(h_t, l_t), N_t, N_t^H\right)$ for all prior distributions $p_t(z)$.

In step five, part (d) of the proposition is proven for $N = N_t$, which ends the induction arguments. Finally, in step six, it is shown that the induction assumptions from step one are valid for $N_t = 1$, which ends the whole induction.

The following notations are used throughout the proof:

$$\alpha \equiv \frac{q^H\left(1-q^L\right)}{q^L\left(1-q^H\right)} = 1 + \frac{q^H - q^L}{q^L\left(1-q^H\right)} \in (1, \infty),\ Q_t \equiv \frac{C_{N_t}^{H_t}}{C_{N_t}^{N_t^H} \sum_y C_{N_t^H}^{y} C_{N_t - N_t^H}^{H_t - y} \alpha^y},$$

where, here and after, all summation indices implicitly take all integer values, and the binomial coefficients are assumed to be zero if they are not defined for given values of its entries.

**Step one.** In order to derive the distribution $p_{t+1}\left(N_{t+1}^H = z \middle| N_t, N_t^H, H_t, h_t, l_t\right)$, let assume that in stage $t$ exactly $y$ out of $H_t$ good alternatives are of high type, and the remaining $(H_t - y)$ good alternatives are of low type. Selecting a sample $(h_t,l_t)$ yields the following probability of having exactly $z$ high type alternatives among selected $(h_t + l_t)$ alternatives

$$\Pr\left(N_{t+1}^H = z \middle| N_t, N_t^H, H_t, h_t, l_t, y_t\right) = \frac{\sum_n C_y^n C_{N^H - y}^{z-n} C_{H_t - y}^{h_t - n} C_{N_t - N^H - H_t + y}^{l_t - z + n}}{C_{H_t}^{h_t} C_{N_t - H_t}^{l_t}}.$$

Taking expectations of the above probability with respect to $y$, which follows a distribution

$$\Pr\left(y_t = y \middle| N_t, N_t^H, H_t, h_t, l_t\right) \equiv \frac{C_{N_t^H}^{y}\left(q^H\right)^y\left(1-q^H\right)^{N_t^H-y} C_{N_t-N_t^H}^{H_t-y}\left(q^L\right)^{H_t-y}\left(1-q^L\right)^{N_t-N_t^H-H_t+y}}{\sum_y C_{N_t^H}^{y}\left(q^H\right)^y\left(1-q^H\right)^{N_t^H-y} C_{N_t-N_t^H}^{H_t-y}\left(q^L\right)^{H_t-y}\left(1-q^L\right)^{N_t-N_t^H-H_t+y}}$$

$$= \frac{C_{N_t^H}^{y} C_{N_t-N_t^H}^{H_t-y}\alpha^y}{\sum_y C_{N_t^H}^{y} C_{N_t-N_t^H}^{H_t-y}\alpha^y},$$

yields the following expression for $p_{t+1}\left(N_{t+1}^H = z \middle| N_t, N_t^H, H_t, h_t, l_t\right)$:

$$p_{t+1}\left(N_{t+1}^H = z \middle| N_t, N_t^H, H_t, h_t, l_t\right) \equiv \sum_y \Pr\left(N_{t+1}^H = z \middle| N_t, N_t^H, H_t, h_t, l_t, y_t\right) \cdot \Pr\left(y_t = y \middle| N_t, N_t^H, H_t, h_t, l_t\right)$$

$$= \frac{\sum_y \sum_n C_{N_t^H}^{y} C_{N_t-N_t^H}^{H_t-y} C_y^n C_{N_t^H-y}^{z-n} C_{H_t-y}^{h_t-n} C_{N_t-N_t^H-H_t+y}^{l_t-z+n}\alpha^y}{C_{H_t}^{h_t} C_{N_t-H_t}^{l_t} \sum_y C_{N_t^H}^{y} C_{N_t-N_t^H}^{H_t-y}\alpha^y}$$

$$= Q_t \sum_n C_{h_t}^{n} C_{l_t}^{z-n} \sum_y C_{H_t-h_t}^{y-n} C_{N_t-H_t-l_t}^{N_t^H-y-z+n}\alpha^y,$$

and for $F_{t+1}\left(w \middle| N_t, N_t^H, H_t, h_t, l_t\right)$:

$$F_{t+1}\left(w \middle| N_t, N_t^H, H_t, h_t, l_t\right) = Q_t \sum_n \sum_y \sum_{z \le w} C_{h_t}^{n} C_{l_t}^{z-n} C_{H_t-h_t}^{y-n} C_{N_t-H_t-l_t}^{N_t^H-y-z+n}\alpha^y .$$

**Step two** (derivation of $h^* = H_t$ for $l_t = 0$). Suppose that the decision-maker does not select bad alternatives in stage $t$, i.e., $l_t = 0$, and selects $h_t \le H_t - 1$, i.e., not all, good alternatives. The corresponding type's distribution in stage $(t + 1)$ is, by definition, $F_{t+1}\left(w \middle| N_t, N_t^H, H_t, h_t, 0\right)$. However, he could get a better distribution (here and after, in terms of the first-order stochastic dominance) if he uses the following modified selection procedure $\tilde{S}$. Let the decision-maker select $(h_t + 1)$ good alternatives, i.e., one good alternative more. This will induce the distribution $p_{t+1}\left(N_{t+1}^H = z \middle| N_t, N_t^H, H_t, h_t + 1, 0\right)$. Then, let him wait until at a certain stage $(t + m)$ there are exactly $h_t$ good alternatives and exactly one bad alternative. In that stage, let the decision-maker select only $h_t$ good alternatives. This modified selection procedure $\tilde{S}$, provided $N_{t+1}^H = N_{t+m}^H$, induces the distribution $p_{t+m+1}\left(N_{t+m+1}^H = z \middle| h_t + 1, N_{t+1}^H, h_t, h_t, 0\right)$. Taking expectations of the latter probability with respect to $N_{t+1}^H$, which is distributed in accordance with $p_{t+1}\left(N_{t+1}^H = z \middle| N_t, N_t^H, H_t, h_t + 1, 0\right)$, yields the following distribution $\widetilde{F}_{t+m+1}(w)$ of the number of high type alternatives $N_{t+m+1}^H$ among $h_t$ alternatives in stage ($t + m + 1$):

$$\widetilde{F}_{t+m+1}(w) \equiv \sum_{z \le w} \sum_{\mu} p_{t+m+1}\left(N_{t+m+1}^H = z \middle| h_t + 1, \mu, h_t, h_t, 0\right) p_{t+1}\left(N_{t+1}^H = \mu \middle| N_t, N_t^H, H_t, h_t + 1, 0\right),$$

$$\widetilde{F}_{t+m+1}(w) = Q_t \sum_{z \le w} \sum_{\mu = z}^{z+1} \frac{C_\mu^z C_{h_t+1-\mu}^{h_t - z} \alpha^{z+1-\mu}}{\mu + (h_t + 1 - \mu)\alpha} \sum_y C_{h_t+1}^{\mu} C_{H_t - h_t - 1}^{y-\mu} C_{N_t - H_t}^{N_t^H - y} \alpha^y$$

$$= Q_t \sum_y C_{N_t - H_t}^{N_t^H - y} \alpha^y \left( \sum_{z \le w} \frac{(h_t + 1 - z) C_{h_t+1}^z C_{H_t - h_t - 1}^{y-z} \alpha}{z + (h_t + 1 - z)\alpha} + \sum_{z \le w} \frac{(z+1) C_{h_t+1}^{z+1} C_{H_t - h_t - 1}^{y-z-1}}{z + 1 + (h_t - z)\alpha} \right)$$

$$= Q_t \sum_y C_{N_t - H_t}^{N_t^H - y} \alpha^y \left( \sum_{z \le w} C_{h_t+1}^z C_{H_t - h_t - 1}^{y-z} + \frac{(w+1) C_{h_t+1}^{w+1} C_{H_t - h_t - 1}^{y-w-1}}{w + 1 + (h_t - w)\alpha} \right).$$

It can be seen now that distribution $\widetilde{F}_{t+m+1}(w)$ dominates $F_{t+1}\left(w \middle| N_t, N_t^H, H_t, h_t, 0\right)$ as

$$D_0(w) \equiv \frac{1}{Q_t}\left( F_{t+1}\left(w \middle| N_t, N_t^H, H_t, h_t, 0\right) - \widetilde{F}_{t+m+1}(w) \right),$$

$$D_0(w) = \sum_y C_{N_t - H_t}^{N_t^H - y} \alpha^y \left( \sum_{z \le w} C_{h_t}^z C_{H_t - h_t}^{y-z} - \sum_{z \le w} C_{h_t+1}^z C_{H_t - h_t - 1}^{y-z} - \frac{(w+1) C_{h_t+1}^{w+1} C_{H_t - h_t - 1}^{y-w-1}}{w + 1 + (h_t - w)\alpha} \right)$$

$$= \sum_y C_{N_t - H_t}^{N_t^H - y} \alpha^y \left( C_{h_t}^w C_{H_t - h_t - 1}^{y-w-1} - \frac{(w+1) C_{h_t+1}^{w+1} C_{H_t - h_t - 1}^{y-w-1}}{w + 1 + (h_t - w)\alpha} \right)$$

$$= \frac{(h_t - w) C_{h_t}^w (\alpha - 1)}{w + 1 + (h_t - w)\alpha} \sum_y C_{H_t - h_t - 1}^{y-w-1} C_{N_t - H_t}^{N_t^H - y} \alpha^y > 0$$

for all $w \le h_t - 1$. Thus, if $l_t = 0$, an optimal selection procedure requires $h^* = H_t$.

**Step three** (derivation of $h^* = H_t$ for $l_t > 0$). Suppose next that the decision-maker does select bad alternatives in stage $t$, i.e., $l_t > 0$, and also selects $h_t \le H_t - 1$, i.e., not all, good alternatives. This induces the distribution $F_{t+1}\left(w \middle| N_t, N_t^H, H_t, h_t, l_t\right)$ in stage ($t + 1$). However, he could get a better distribution if he replaces one bad alternative with one good alternative. Defining

$$D_l(w) \equiv \frac{1}{Q_t}\left( F_{t+1}\left(w \middle| N_t, N_t^H, H_t, h_t, l_t\right) - F_{t+1}\left(w \middle| N_t, N_t^H, H_t, h_t + 1, l_t - 1\right) \right)$$

yields:

$$
\begin{aligned}
D_l(w) &= \sum_n \sum_y \alpha^y \sum_{z \le w} \left( C_{h_t}^n C_{l_t}^{z-n} C_{H_t-h_t}^{y-n} C_{N_t-H_t-l_t}^{N_t^H-y-z+n} - C_{h_t+1}^n C_{l_t-1}^{z-n} C_{H_t-h_t-1}^{y-n} C_{N_t-H_t-l_t+1}^{N_t^H-y-z+n} \right) \\
&= \sum_n \sum_y \alpha^{y+1} \sum_{z \le w} \left( C_{h_t}^n C_{l_t-1}^{z-n} C_{H_t-h_t-1}^{y-n} C_{N_t-H_t-l_t}^{N_t^H-y-z+n-1} - C_{h_t}^n C_{l_t-1}^{z-n-1} C_{H_t-h_t-1}^{y-n} C_{N_t-H_t-l_t}^{N_t^H-y-z+n} \right) - \\
&\quad - \sum_n \sum_y \alpha^y \sum_{z \le w} \left( C_{h_t}^n C_{l_t-1}^{z-n} C_{H_t-h_t-1}^{y-n} C_{N_t-H_t-l_t}^{N_t^H-y-z+n-1} - C_{h_t}^n C_{l_t-1}^{z-n-1} C_{H_t-h_t-1}^{y-n} C_{N_t-H_t-l_t}^{N_t^H-y-z+n} \right) \\
&= (\alpha - 1) \sum_n C_{h_t}^n \sum_y C_{H_t-h_t-1}^{y-n} \alpha^y \left( \sum_{z \le w} C_{l_t-1}^{z-n} C_{N_t-H_t-l_t}^{N_t^H-y-z+n-1} - \sum_{z \le w-1} C_{l_t-1}^{z-n} C_{N_t-H_t-l_t}^{N_t^H-y-z+n-1} \right) \\
&= (\alpha - 1) \sum_n C_{h_t}^n C_{l_t-1}^{w-n} \sum_y C_{H_t-h_t-1}^{y-n} C_{N_t-H_t-l_t}^{N_t^H-y-w+n-1} \alpha^y > 0.
\end{aligned}
$$

Thus, an optimal selection procedure requires $h^* = H_t$ for all $l_t \ge 0$.

**Step four** (derivation of $l^*$). Suppose that the decision-maker, in addition to all good alternatives, also selects $l_t \le N_t - H_t - 1$, i.e., not all, bad alternatives. This induces the distribution $F_{t+1}\left(w \middle| N_t, N_t^H, H_t, H_t, l_t\right)$ in stage $(t + 1)$. Let us consider the following deviation $\tilde{\tilde{S}}$ from this selection procedure. Let the decision-maker select $(l_t + 1)$ bad alternatives, i.e., one bad alternative more. This will induce the distribution $p_{t+1}\left(N_{t+1}^H = z \middle| N_t, N_t^H, H_t, H_t, l_t + 1\right)$. Then, let him wait until at a certain stage $(t + m)$ there are exactly $(H_t + l_t)$ good alternatives and exactly one bad alternative. In that stage, let the decision-maker select only $(H_t + l_t)$ good alternatives. This modified selection procedure $\tilde{\tilde{S}}$, provided on $N_{t+1}^H = N_{t+m}^H$, induces the distribution $p_{t+m+1}\left(N_{t+m}^H = z \middle| H_t + l_t + 1, N_{t+1}^H, H_t + l_t, H_t + l_t, 0\right)$ at stage $(t + m + 1)$. Taking expectations of the latter probability with respect to $N_{t+1}^H$ distributed in accordance with $p_{t+m+1}\left(N_{t+m+1}^H = z \middle| H_t + l_t + 1, N_{t+1}^H, H_t + l_t, H_t + l_t, 0\right)$ yields the following distribution $\tilde{\tilde{F}}_{t+m+1}(w)$ of the number of high type alternatives $N_{t+m+1}^H$ among $(H_t + l_t)$ alternatives in stage $(t + m + 1)$:

$$
\tilde{\tilde{F}}_{t+m+1}(w) \equiv \sum_{z \le w} \sum_{\mu} p_{t+m+1}\left(N_{t+m+1}^H = z \middle| H_t + l_t + 1, \mu, H_t + l_t, H_t + l_t, 0\right) p_{t+1}\left(N_{t+1}^H = \mu \middle| N_t, N_t^H, H_t, H_t, l_t + 1\right),
$$

$$\tilde{\tilde{F}}_{t+m+1}(w) = Q_t \sum_{z \le w} \sum_{\mu=z}^{z+1} \frac{C_\mu^z C_{H_t+l_t+1-\mu}^{H_t+l_t-z} \alpha^{z+1-\mu}}{\mu + (H_t + l_t + 1 - \mu)\alpha} \sum_y C_{H_t}^y C_{l_t+1}^{\mu-y} C_{N_t-H_t-l_t-1}^{N_t^H-\mu} \alpha^y$$
$$= Q_t \sum_y C_{H_t}^y \alpha^y \left( \sum_{z \le w} \frac{(H_t + l_t + 1 - z) C_{l_t+1}^{z-y} C_{N_t-H_t-l_t-1}^{N_t^H-z} \alpha}{z + (H_t + l_t + 1 - z)\alpha} + \sum_{z \le w} \frac{(z+1) C_{l_t+1}^{z+1-y} C_{N_t-H_t-l_t-1}^{N_t^H-z-1}}{z + 1 + (H_t + l_t - z)\alpha} \right)$$
$$= Q_t \sum_y C_{H_t}^y \alpha^y \left( \sum_{z \le w} C_{l_t+1}^{z-y} C_{N_t-H_t-l_t-1}^{N_t^H-z} + \frac{(w+1) C_{l_t+1}^{w+1-y} C_{N_t-H_t-l_t-1}^{N_t^H-w-1}}{w + 1 + (H_t + l_t - w)\alpha} \right).$$

In order to compare distributions $F_{t+1}\left(w \middle| N_t, N_t^H, H_t, H_t, l_t\right)$ and $\tilde{\tilde{F}}_{t+m+1}(w)$, we define

$$D_h(w, H_t, l_t) \equiv \frac{w + 1 + (H_t + l_t - w)\alpha}{Q_t C_{N_t-H_t-l_t-1}^{N_t^H-w-1}} \left( F_{t+1}\left(w \middle| N_t, N_t^H, H_t, H_t, l_t\right) p - \tilde{\tilde{F}}_{t+m+1}(w) \right).$$

Then

$$D_h(w, H_t, l_t) = \sum_y C_{H_t}^y \alpha^y \left( \frac{w + 1 + (H_t + l_t - w)\alpha}{C_{N_t-H_t-l_t-1}^{N_t^H-w-1}} \sum_{z \le w} \left( C_{l_t}^{z-y} C_{N_t-H_t-l_t}^{N_t^H-z} - C_{l_t+1}^{z-y} C_{N_t-H_t-l_t-1}^{N_t^H-z} \right) - (w+1) C_{l_t+1}^{w+1-y} \right)$$
$$= \sum_y C_{H_t}^y \alpha^y \left( (w + 1 + (H_t + l_t - w)\alpha) C_{l_t}^{w-y} - (w+1) C_{l_t+1}^{w+1-y} \right)$$
$$= \sum_y (H_t + l_t - w) C_{H_t}^y C_{l_t}^{w-y} \alpha^{y+1} - \sum_y (w+1) C_{H_t}^y C_{l_t}^{w+1-y} \alpha^y$$
$$= \sum_y \left( (H_t + l_t - w) C_{H_t}^{y-1} - (w+1) C_{H_t}^y \right) C_{l_t}^{w-y+1} \alpha^y.$$

It is easily seen that $D_h(w, 0, l_t) = (l_t - w) C_{l_t}^w \alpha - (w+1) C_{l_t}^{w+1} = l_t C_{l_t-1}^w (\alpha - 1) > 0$ for all $l_t \ge w + 1 \ge 1$, and that $D_h(w, 0, 0) = 0$. In addition, $D_h(w, H_t, l_t)$ can be written recursively as

$$D_h(w, H_t + 1, l_t) = D_h(w, H_t, l_t) + \alpha D_h(w - 1, H_t, l_t).$$

Therefore, $D_h(w, H_t, 0) = 0$ and $D_h(w, H_t, l_t) > 0$ for all $l_t \ge 1$ by induction.

This implies that in stage *t*, any selection procedure $(H_t, l_t)$ with $l_t \in [1, N_t - H_t - 1]$ is strictly dominated by the selection procedure $\tilde{\tilde{S}}$. Hence, if some bad alternatives are eliminated, then all the other bad alternatives must also be eliminated, or, stated differently, if some bad alternatives are selected, then all the other bad alternatives must also be selected. In other words, all bad alternatives must be treated equivalently, i.e., either $l^*(H_t, L_t) = 0$ or $l^*(H_t, L_t) = L_t$.

Suppose now, that $H_t \leq N_t - 2$, i.e., at least two bad alternatives are available. Then, selecting ($H_t$,0) is not better than selecting ($H_t$,1), whereas selecting ($H_t$,1) is strictly dominated by selecting ($H_t$,2). Hence, selecting only good alternatives is also strictly dominated by passing all alternatives provided $H_t \leq N_t - 2$, i.e., $l^*(H_t, L_t) = L_t$ if $L_t \geq 2$.

**Step five** (monotonicity of $U(S^*, N, N^H)$). From steps 2, 3 and 4 it follows that it is only optimal to eliminate an alternative if it is a unique bad alternative at a stage. Suppose that the decision-maker eliminates such a unique bad alternative, i.e., $(H_t, L_t) = (N_t - 1, 1)$ and $(h_t, l_t) = (N_t - 1, 0)$. Evaluating $F_{t+1}(w | N_t, N_t^H, N_t - 1, N_t - 1, 0)$ yields:

$$F_{t+1}(w | N_t, N_t^H, N_t - 1, N_t - 1, 0) = \frac{N_t}{C_{N_t}^{N_t^H}\left(N_t^H + \left(N_t - N_t^H\right)\alpha\right)} \sum_{z \leq w} C_{N_t - 1}^{z} C_1^{N_t^H - z} \alpha^{z + 1 - N_t^H}$$

$$= \begin{cases} 0, \text{ if } w < N_t^H - 1 \\ \dfrac{N_t^H}{N_t^H + \left(N_t - N_t^H\right)\alpha}, \text{ if } w = N_t^H - 1 \\ 1, \text{ if } w > N_t^H - 1 \end{cases}$$

It is easy to see that $F_{t+1}(w | N_t, N_t^H + 1, N_t - 1, N_t - 1, 0) \leq F_{t+1}(w | N_t, N_t^H, N_t - 1, N_t - 1, 0)$ so that the distribution $F_{t+1}(w | N_t, N_t^H + 1, N_t - 1, N_t - 1, 0)$ stochastically dominates the distribution $F_{t+1}(w | N_t, N_t^H, N_t - 1, N_t - 1, 0)$ and, therefore, $U(S^*, N, N^H + 1) > U(S^*, N, N^H)$ for all $N \leq N_t$. This ends the induction arguments.

**Step six** (proof of the assumptions from step one for $N_t = 1$). In case $N_t = 1$, it is either $(H_t, L_t) = (0,1)$ or $(H_t, L_t) = (1,0)$ such that the only feasible selection procedure $(h_t, l_t) = (H_t, L_t)$ is trivially optimal and satisfies the required properties. In this case, the pay-off function is $U(S, 1, N^H) = N^H$ which strictly increases in $N^H$. Thus, the assumptions from step one are confirmed. This ends the proof of Proposition 1. ■