TI 2005-025/1
Tinbergen Institute Discussion Paper

Auctions, Market Prices and the Risk Attitude Effect

*Maarten C.W. Janssen**
Vladimir A. Karamychev

Department of Economics, Erasmus University Rotterdam.

** Tinbergen Institute.*

Tinbergen Institute
The Tinbergen Institute is the institute for economic research of the Erasmus Universiteit Rotterdam, Universiteit van Amsterdam, and Vrije Universiteit Amsterdam.

Tinbergen Institute Amsterdam
Roetersstraat 31
1018 WB Amsterdam
The Netherlands
Tel.: +31(0)20 551 3500
Fax: +31(0)20 551 3555

Tinbergen Institute Rotterdam
Burg. Oudlaan 50
3062 PA Rotterdam
The Netherlands
Tel.: +31(0)10 408 8900
Fax: +31(0)10 408 9031

Please send questions and/or remarks of non-scientific nature to driessen@tinbergen.nl.
Most TI discussion papers can be downloaded at http://www.tinbergen.nl.

Auctions, Market Prices and the Risk Attitude Effect

Maarten C.W. Janssen and Vladimir A. Karamychev*

This Draft: February 2005

Abstract

This paper develops one possible argument why auctioning licenses to operate in an aftermarket may lead to higher prices in the aftermarket compared to a more random allocation mechanism. Key ingredients in the argument are differences in firms' risk attitudes and the fact that future market profits are uncertain so that winning an auction is like winning a lottery ticket. If one license is auctioned, auctions select the firm that is least risk averse. This is what we call the risk attitude effect. Firms that are less risk averse tend to set higher prices (or higher quantities in case quantity is the decision variable) in the marketplace than an average firm. When multiple licenses are auctioned, this conclusion gets strengthened when there is a differentiated Bertrand oligopoly in the marketplace. In case of Cournot competition, a strategic effect works against the risk attitude effect so that under certain conditions the more risk averse firms will be selected leading (again) to higher market prices.

Key Words: Auctions, Risk Attitude, Aftermarkets
JEL-codes: D43, D44, D82

*Correspondence Address: Prof. Maarten Janssen, Department of Economics, Erasmus University Rotterdam, P.O. Box 1738, 3000 DR Rotterdam, The Netherlands, Phone: 31 - 10 - 408.2341, Fax : 31 - 10 - 408.9149, e-mail: janssen@few.eur.nl. We thank audiences at ESEM 2004, ASSET 2004, Stockholm School of Economics and Tinbergen Institute for valuable comments.

1 Introduction

In recent years, governments throughout the world have extensively used auction formats to allocate to private enterprises licenses to operate in a market. This use of the auction format is to a large extent due to the success of the FCC auctions held in the first part of the 1990s and the attention that was drawn to these auctions by the large sums of money that were generated. After the FCC auctions, auctions have been used or have been considered to allocate, for example, 2G and 3G mobile telephony frequencies around the world, commercial radio frequencies and petrol stations. One element that all these examples have in common is that after the auction is held, firms operate in a market to sell their services to consumers (or to advertisers, in the case, of commercial radio). Moreover, while bidding in the auctions, there is considerable uncertainty about future demand. Winning such auctions is, therefore, very much like winning a lottery ticket: the costs are clear, but the revenues are highly uncertain.

Another element that most of these auctions have in common is the heated debates at the time the auction is prepared and held about the question whether it is in the interest of consumers that licenses are auctioned. Private enterprises consistently argue that the fee that has to be paid during an auction to obtain the license, is reflected in the market prices that these same enterprises later charge to consumers.[1] The more is money that is paid during an auction, the higher are future consumer prices. Economic theorists, on the other hand, have consistently argued that the fees paid to obtain a license are sunk costs and should have no effect on the prices that are later charged in the market (cf., Binmore and Klemperer 2002, Van Damme 2002). In the same vein, they argue that the views expressed by these enterprises may simply stem from the firms' own interests not to have to bid (and pay) in auctions.[2] Nevertheless, there is now experimental evidence that auctions may have a positive impact on market prices (Offerman and Potters, 2000).

In this paper, we look at the relation between auctions and prices in the aftermarket more closely. We will argue by means of a theoretical model that the sunk cost argument need not hold if bidders are risk averse and, more importantly, it does not hold if firms differ in their risk attitude. Of course, the usual assumption made in the literature is that firms are risk neutral, but relatively recent empirical studies in finance indicate that firms may indeed be risk-averse, or at least that their behavior is such that it is as if they were risk averse. Nance *et al.* (1993) and Geczy (1997), among others, argue that firms hedge against different types of exogenous uncertainties such as the volatility of exchange rates. They show that this is because

[1] Governments have also been worried about the possible effect on consumer prices; see e.g. European Commission (1994).

[2] There are some, see, e.g., McMillan (1994), arguing that if firms have to pay large sums of money, they may face an increase in the cost of obtaining capital, which may slow down innovation.

firms face non-linear tax systems or because they are liquidity constraint. In a study on the behavior of the gold mining industry, Tufano (1996) argues that delegation of control to a risk-averse manager, whose behavior is linked to the firm's performance, may cause the firm to take actions in a risk-averse manner even if owners themselves are not risk-averse. As delegation of control differs between firms and as the payment structure of managers differs from firm to firm, firms may very well act as if their attitude towards risk differs significantly from one to the other.

We will consider three different types of aftermarkets: monopoly, differentiated Bertrand oligopoly and Cournot oligopoly. Depending on whether the government issues one or multiple licenses, one of these market forms may apply. In the monopoly set-up, *i.e.*, if only one license is auctioned and future demand (and profit) is uncertain, auctions tend to select the bidder that is least risk averse and this firm chooses higher levels of its decision variable than an average firm as it concentrates more on the good states of demand. This is what we will call the *risk attitude effect*. Whether market prices are higher or lower because of auctions depend on whether the monopoly firm in the market is a price or a quantity setter. Price setting firms that win auctions, tend to set *higher* prices than a randomly selected firm. Quantity setting firms that win auctions, tend to set higher quantities than a randomly selected firm, resulting in *lower* expected market prices. Thus, the main idea incorporated in the paper is a selection argument: auctions select firms that do not set average market prices.

It is well-known that under general forms of risk aversion, the sunk cost argument may not hold. In particular, in a risky environment individual players with the same utility function may make different choices depending on how wealthy they are. The sunk cost argument continues to hold, however, for risk-averse players when this risk attitude is characterized by constant absolute risk aversion (CARA). It has been argued that as auctions force firms to pay considerable amounts of money for their licenses, auctions may force firms to behave differently in the marketplace (see, *e.g.*, MacMillan (1994) and footnote 2). As we want to concentrate on the *selection aspect* of auctions and not on the possible consequence of the fact that paying money for the license makes a firm poorer, we consider firms having CARA utility functions.

One other substantive question that may arise is whether the selection effect analyzed here has only short-term or also long-term implications. The answer depends on the way the model is interpreted. If a firm's quantity choice is interpreted as a capacity choice in the sense of Kreps and Scheinkman (1983) and if capacities have to be chosen before the uncertainty is realized, then it is clear that the selection effect has long-term consequences: a less-risk averse firm wins the auction and chooses a higher level of capacity, implying lower prices over a long period of time. In case the firms are price setters, the answer depends on whether the uncertainty about demand is quickly resolved or not. If demand remains uncertain for a long time, possibly because new types of demand shocks keep on arising, there are also long-term effects in case of price setting.

Next, we investigate whether these ideas are robust to settings where more licenses are auctioned and strategic effects may interfere with the risk attitude effect. This leads to the analysis of differentiated Bertrand oligopoly and Cournot oligopoly. A first point to note here is that in the oligopoly case the price that is paid to obtain a license effects the choice of a firm's market strategy through its expectation about the risk attitudes of other players even if firms have CARA utility functions. This considerably complicates the analysis of the oligopoly cases and that is why we restrict the analysis of the oligopoly cases to situations where there is relatively little uncertainty concerning future profits. Under differentiated Bertrand oligopoly, the strategic effect strengthens the risk attitude effect. The main reason is that with strategic complements every firm in the aftermarket benefits from a rival firm setting higher prices, *i.e.*, from the presence of a more risk-neutral firm. Therefore, the least risk-averse firms make a higher expected profit in the aftermarket than any other combination of firms. Combined with the fact that for a given distribution of profits their certainty equivalent, and therefore their willingness to pay, is higher these least risk-averse firms will win any type of auction and they will set higher prices than a randomly drawn sample of firms will do.

The analysis is more complicated in case of Cournot oligopoly. In this case, the risk attitude effect and the strategic effect work in opposite directions. The main reason is that with strategic substitutes every firm in the aftermarket suffers from a rival firm choosing higher quantities, *i.e.*, from the presence of a less risk-averse firm. It remains true that for a given set of other players in the aftermarket, a firm that is relatively less risk averse makes more profits than a relatively more risk-averse firm. However, it may well be that two or more less risk-averse firms make less profits and as a consequence have a smaller willingness to pay for the licenses when they know they compete with each other in the aftermarket than more risk-averse firms. Depending on who will win the licenses, expected market prices will be lower or higher than the prices when firms are selected randomly. We show by appealing to special cases that both the risk attitude effect and the strategic effect may dominate. For example, in case of inelastic demand, a relatively large number of licenses being auctioned and a slight positive correlation between players' types, there exists a unique equilibrium in which the bids firms make increase in the degree of risk-aversion and the most risk-averse firms secure the licenses leading to lower quantity choices and higher market prices.

The paper borrows from the early literature on price and quantity setting behavior of a risk-averse monopolist (cf., Baron, 1971 and Leland, 1972). One important result of these papers is that a price setting risk-averse monopolist behaves differently from a quantity setting risk-averse monopoly. Moreover, Baron (1971) shows that the more risk averse a price setting monopolist the lower the price it sets. On the other hand, the more risk averse a quantity setting monopolist the lower the quantity it sets. This in turn implies that for each state of demand, market prices tend to be higher! The intuition behind these two results is that the more risk averse a firm the more it pays attention to the outcomes if demand is low. If demand is low, a firm sets relatively low prices or low quantities, whatever is its choice variable. These results have recently been generalized to the case of market competition (see, Asplund, 2001).

The paper is, of course, also related to the rapidly growing literature on auctions. There is a literature on the way risk aversion effects bidders' behavior in auctions (see, Krishna 2002, for an overview). Eso and White (2004) analyze the bidding behavior of risk-averse bidders in an affiliated valuation model where the influence of exogenous uncertainty on a player's valuation is independent of the private signals received. Our model may be considered in this light if one interprets risk attitudes as private signals. An important difference[3] between our paper and Eso and White (2004) is, however, that the influence of exogenous uncertainty about demand on a player's valuation is *not independent* of the private signal (the firm's risk attitude). There is also a growing literature studying the strategic interaction between bidding in auctions and firms' behavior in the aftermarket (see, *e.g.*, Binmore and Klemperer, 2002, Goeree, 2003, Janssen 2005, different papers in Janssen 2004, Jehiel and Moldovanu (1996a, 1996b, 2001) and Klemperer 2002a, 2002b). This paper is the first to consider the importance of differences in risk attitude for the interaction between auctions and aftermarkets. It turns out that these differences imply externalities between the players so that, for example, in the case of multiple licenses, a player's willingness-to-pay depends on his expectation about who else will win a license. These externalities may imply that there exist equilibria where players with higher willingness-to-pay do not win a license.

The rest of the paper is organized as follows. Section 2 deals with the case where monopoly rights are allocated. Section 3 first analyzes the case where multiple licenses are auctioned in a uniform price multi-unite auction in general terms. In two subsections, we then discuss price setting behavior in a differentiated Bertrand model and quantity setting behavior in Cournot competition. Section 4 analyses a multi-unit pay-your-bid auction. Section 5 concludes.

2 The Monopoly Setting

Consider a monopolistic market with uncertain demand, where the monopoly profit $\pi(s, u)$ depends on the monopolist's choice of the strategic variable s and the uncertainty u. The strategic variable s can be interpreted in many different ways, but

[3] This difference is further detailed at the end of Section 2.

two common interpretations are that of price strategy and/or of quantity or capacity strategy. If the firm chooses price, then $s = p$ and the firm fulfills a random demand $q(p, u)$. If the firm chooses quantity, then $s = q$ and the price consumers pay for this quantity is $p(q, u)$ and the firm accepts to sell its pre-determined quantity q at this random price. The uncertainty is represented by a random variable u which is distributed according to the distribution function F_u with support σ.[4] As usual, we assume that $\pi(s, u)$ is twice differentiable and strictly concave in s such that the profit-maximizing output always exists and is unique. With respect to the uncertainty, we follow Leland (1972) and assume that the Principle of Increased Uncertainty (PIU) holds, *i.e.*, marginal revenue is increasing in u, *i.e.*, $\pi_{s,u} > 0$. Moreover, we assume that $\pi(0, u) = 0$, *i.e.*, the profit function is continuous in the sense that the pay-off of not winning the auction, which is equal to 0, equals the pay-off of winning the auction and setting the strategic variable equal to 0. It follows that $\pi_u > 0$. All these conditions are satisfied in many instances that are commonly considered. For example, if demand is linear and given by $p = \alpha - \eta q$ and we look at a quantity setting monopolist, then the assumptions are satisfied when either α is an arbitrary increasing function of u, $\alpha(u)$, or η is an arbitrary decreasing function of u, $\eta(u)$. For an appropriate change in parameters, the conditions also hold true for a price-setting monopolist.

Access to the market is limited to the firm that has obtained the single license to operate in the market. The government considers two allocation mechanisms: a lottery where the licenses are randomly given to a firm and an auction where the highest bidding firm wins the license. To fix attention, we think of the auction as being an English auction, but the main ingredient that is important is that firms with higher valuations win the auction, *i.e.*, bidding in the auction can be characterized by an increasing bid function $\beta(v)$, where v represents firm's willingness to pay (specified in more detail below). A firm's actual bid is denoted by b. Even if a lottery is chosen as allocation mechanism, the winners have to pay a certain sum of money for the license.

Firms differ in their attitude towards risk. We assume that all firms are to a certain degree risk-averse, but some firms are more so than others. To make this more precise, we assume that a firm i has a strictly increasing and concave utility function, denoted by $U_i(\pi)$, $U_i' > 0$ and $U_i'' < 0$, and $U_i(0)$ is normalized to 0. A firm's attitude towards risk is represented in the standard way by the Arrow-Pratt measure of absolute risk aversion $-U_i''/U_i'$. To make comparisons between firms' risk attitudes feasible, we require that a firm's attitude towards risk can be captured by a single parameter. For easy reference we will use the symbol r_i to denote the

[4] Due to the fact that we can work with arbitrary profit functions, we can safely assume without loss of generality that u is uniformly distributed over the range $[-1, 1]$. In the oligopoly section, we will use this to make the expressions easier to handle.

parameter measuring the risk attitude of firm i and we assume that the individual signals (risk attitudes) are drawn from a common distribution F with support $[\underline{r}, \overline{r}]$, $\underline{r} > 0$. A player's risk attitude is private information to the player. Moreover, it is well-known that the sunk cost argument does not hold under general forms of risk aversion. As we want to concentrate in this paper on a pure selection argument, we concentrate on the case where the amount of money paid during the auction does not affect aftermarket behavior. This is the case when firms have a constant absolute risk aversion (CARA). Under CARA, the more risk averse a firm, the higher $r_i = -U_i''/U_i'$ is.

If a firm i with a risk attitude x secures the license at a price w and sets a level s of the choice variable in the aftermarket, its expected utility is given by

$$W(s, w, x) \equiv E_u U_i(\pi(s, u) - w) = \int_\sigma U_i(\pi(s, u) - w)\, dF_u.$$

As a firm that has not been successful in obtaining a license will make zero profit in the aftermarket, the firm i's maximum willingness to pay $v(x)$ for the license is implicitly determined by the following equation:

$$W(s^*, v(x), x) \equiv \max_s W(s, v(x), x) = \max_s E_u U_i(\pi(s, u) - v(x)) = 0, \tag{1}$$

i.e., $v(x)$ is the certainty equivalent of the random profit $\pi(s^*, u)$, where s^* is the optimal choice of the firm's decision variable s.

An interesting observation about this definition of a player's valuation is in place. Under a general form of risk aversion, the optimal value s^* a player chooses in the aftermarket and, therefore, the profit it makes, depends on the amount it has paid in the auction, $s^* = s^*(w, x)$. Thus, a player's expected utility from winning a license and paying a price w equals to

$$W(s^*(w, x), w, x) = E_u U_i(\pi(s^*(w, x), u) - w). \tag{2}$$

and we arrive at the unusual situation that the expected value of the license depends on the price a player paid! Taking the derivative of (2) with respect to w gives

$$\frac{d}{dw} W(s^*(w, x), w, x) = W_s(s^*, w, x)\frac{ds^*}{dw} + W_w(s^*, w, x).$$

The first term here equals to zero due to the first order condition in maximization of (1), whereas the second term is strictly negative as

$$W_w(s^*, w, x) = -E_u U_i'(\pi(s^*, u) - w) < 0.$$

This implies that even for non-CARA utility functions the solution of equation (1) for $v(x)$ is unique and thus, that the valuation $v(x)$ is properly defined.

Using the above notation and assumptions, one is able to arrive at the first set of results. We will prove the claim that auctions lead to either higher market prices (in case of price setting) or to lower market prices (in case of quantity setting) in two steps. First, we show that in the market environment, a less risk-averse firm will set a higher value of s. Second, we show that less risk-averse firms have a higher willingness to pay so that an auction selects the firm that is least risk averse among the firms that participate.

Proposition 1 *Under CARA, the less risk averse the monopolist is, the higher the value of s^* it will choose,* i.e., $\partial s^*/\partial x < 0$. *Moreover, a price that a winning firm paid in an auction does not affect s^*,* i.e., $\partial s^*/\partial w = 0$.

Proof. Maximizing $W(s,w,x)$ w.r.t. s yields the necessary first-order condition

$$0 = W_s(s^*, w, x) = E_u(\pi_s(s^*, u)\, U_i'(\pi(s^*, u) - w)). \tag{3}$$

Differentiating this equation w.r.t. w yields the following expression for $\partial s^*/\partial w$:

$$\frac{\partial s^*}{\partial w} = \left(\frac{ds^*}{dw}\right)_x = -\frac{W_{s,w}(s^*, w, x)}{W_{s,s}(s^*, w, x)} = \frac{-xW_s(s^*, w, x)}{W_{s,s}(s^*, w, x)} = 0,$$

as because of CARA the numerator equals to zero while the denominator is strictly negative.

In order to show that $\partial s^*/\partial x < 0$ we first evaluate $W_{s,x}(s^*, w, x)$:

$$W_{s,x} = E_u\left(\pi_s \frac{\partial U_i'(\pi - w)}{\partial x}\right) = -E_u(\pi_s \pi U_i'(\pi - w)) = E_u(\pi_u J(u, x)) < 0,$$

where $J(t, x) = \int_{-1}^{t} \pi_s(s^*, u)\, U_i'(\pi(s^*, u) - w)\, du$. The first equality follows from CARA, the second equality is obtained by integrating in parts. It follows from the PIU that $J(-1, x) = J(1, x) = 0$ and $J(t, x) \leq 0$ for all $t \in \sigma$, which together with $\pi_u > 0$ implies the last inequality. Differentiating (3) but now w.r.t. x yields the desired inequality:

$$\frac{\partial s^*}{\partial x} = \left(\frac{ds^*}{dx}\right)_w = -\frac{W_{s,x}}{W_{s,s}} < 0.$$

■

In accordance with Proposition 1 the optimal level of the strategic variable can be written as $s^*(x)$. If the strategic variable is price, the Proposition says that less risk-averse firms will set higher prices. If the strategic variable is quantity or capacity, the Proposition says that less risk-averse firms will set higher quantities, leading to lower market prices. We next analyze which of the players an auction mechanism selects.

Proposition 2 *A firm's valuation is a strictly decreasing function of its risk attitude,* i.e.*,* $dv/dx < 0$*.*

Proof. $v(x)$ is defined by (1). It is well-known (see, *e.g.*, Mas-Colell *et al.* (1995, p. 191)) that this implies that a strictly less risk-averse player strictly prefers the same lottery, therefore $W_x < 0$. Differentiating $W(s^*(x), v(x), x) = 0$ w.r.t. x and taking into account (3) yields

$$\frac{dv}{dx} = -\frac{W_x}{W_w} < 0,$$

as $W_w = -E_u(U_i'(\pi(s^*, u) - v)) < 0$. ■

Proposition 2 tells us that in auctions that select the player with the highest willingness to pay, the least risk-averse player among all those who participate in the auction is selected.

One question that arises is whether commonly held auctions such as the English auction will select the player with the highest willingness-to-pay also in the present context. The next proposition answers this question. The proposition tells us that the standard result from auction theory with independent valuations, namely the equivalence between a second-price sealed-bid auction and an English auction, also holds in the present situation. Hence, in an English auction, the player with the highest valuation will win the auction and has to pay the valuation of the player who is least risk-averse of the remaining players. Moreover, Proposition 3 argues that the winning bidder is better off in a situation of uncertain market conditions than in a situation of certain demand.

Proposition 3 *The English auction and the second-price sealed-bid auction are strategically equivalent and the dominant bidding strategy in the latter (and the stopping rule in the former) is simply* $\beta(v) = v$*. The winning bidder receives a lower surplus in case of certain demand than in case of uncertain demand where the uncertainty takes the form of a mean-preserving spread of the certain demand case. The seller is worse off when auctioning at a moment demand is uncertain.*

Proof. The proof that a second-price sealed-bid auction in the present context has a dominant strategy equilibrium with $\beta(v) = v$ exactly follows the standard argument and is therefore omitted (see, *e.g.*, Krishna, 2002). The fact that the English auction is strategically equivalent simply follows from the fact that even if the risk attitudes of individual players are correlated, this is a private value auction where a firm i's valuation depends only on its own risk type. Thus, the players have nothing to learn from each other's bidding behavior (or from the moment a player stops bidding) in an English auction.

When demand is certain, *i.e.*, when $u = u_0$ is commonly known, all players have in fact the same valuation, which is then given by

$$\max_{s} U_i\left(\pi\left(s, u_0\right) - v_i\right) = 0.$$

Hence, a player's valuation in this case is $v_i^{\mathrm{c}} = \max_{s} \ \pi\left(s, u_0\right)$, which is independent of i. This implies that the winning player has to pay an amount equal to his own valuation so that his surplus equals to 0.

We next show that if demand is uncertain, all players' valuations are lower. This can be shown using a similar argument as used in Proposition 2. If we denote by v_i^{u} the valuation of player i in case of uncertain demand, then it follows $0 = EU_i(\pi\left(s^*, u\right) - v_i^{\mathrm{u}})$. If the player chooses the same s^* under certain demand as the optimal s^* under uncertain demand, it is well-known that a risk-averse player prefers the certain profits. As the player can choose a different optimal level of s under certain demand, it follows that $v_i^{\mathrm{c}} > v_i^{\mathrm{u}}$.

As Proposition 2 says that in case of uncertain demand if j is strictly less risk-averse than i, then $v_j > v_i$, it follows that the player with the highest valuation wins the auction and gets a positive surplus as it pays the price of the second highest valuation, which is strictly less. The seller gets lower revenue for two reasons: firstly, all valuations are lower and, secondly, the revenue it gets is lower than the valuation of the winning bidder. ■

Proposition 3 sheds some interesting light on the type of auction that is considered in this paper. The valuation of a player v depends on the player's risk attitude r and on the nature of market uncertainty, which we summarize by a symbol Ω, and can be expressed as follows: $v\left(r, \Omega\right)$. Regarding r as a signal the individual receives about his valuation of the object, this dependence makes clear that the model does not fit the general interdependent value model (see, Milgrom and Weber, 1982) that is standard in auction theory. The main difference is that in our model a player's valuation does not depend on the risk attitude of the other players. This also explains (unlike the interdependent value model) the strategic equivalence of the second-price sealed-bid auction and the English auction in the present context. The structure of $v\left(r, \Omega\right)$ also makes clear the difference with the model by Eso and White (2004). In their model, the influence of exogenous uncertainty on a player's valuation is independent of the influence of the private signals received. The combined influence of players' risk attitudes and the nature of market uncertainty makes that on one extreme end (when there is no market uncertainty) the present model just represents a simple common valuation auction, while at another extreme (when there is a lot of market uncertainty), the model is very much like a typical auction model with independent private valuations.

Taken together, the propositions prove one of the central claims of this paper, namely that typical auctions such as the English auction, select the player that is least risk-averse and that in case the decision variable is price, this player chooses a higher prices than a randomly selected firm.

3 The Oligopoly Setting: Uniform Price Auction

In many real-world cases, governments do not allocate a single license, but instead also rely on competitive forces in the marketplace by allocating as many licenses as is technically feasible. In case of the European UMTS-auctions, for example, governments have allocated between 4 and 6 licenses. In this section we will analyze whether the results of the previous section also hold true when the government decides to allocate $n > 1$ licenses. To this end, we first analyze a general model of oligopoly competition when goods are horizontally differentiated, and then we look at the specific features of price and quality competition in the aftermarket.

We will retain all the assumptions made in the previous section, if possible. The only element that needs to be changed is the profit function π. Assuming that apart from their risk attitudes, all firms are identical and denoting by s_i a level of the strategic variable chosen by a firm $i = 1, \ldots, n$ allows us to write its profit as $\pi(s_i, s_{-i}, u)$, where s_{-i} represents level of s chosen by all the other firms and the function π is symmetric in all s_j for $j \neq i$. For a short notation, we denote the partials of π as follows: $\pi_i \equiv \partial\pi/\partial s_i$ and $\pi_j \equiv \partial\pi/\partial s_j, j \neq i$. By indices i, j, k we will denote firms that won the auction whereas by index l we will refer to the firms that lost the auction, *i.e.*, that did not obtain a license.

In order to ensure the existence, uniqueness and stability of a Nash equilibrium in the aftermarket we assume that $\pi_{i,i} < 0$ and $\pi_{i,j} \in \left(\pi_{i,i}, -\frac{1}{n-1}\pi_{i,i}\right)$, see Dixit (1986). When $0 < \pi_{i,j} < -\frac{1}{n-1}\pi_{i,i}$ strategic variables s_i are strategic complements (Bertrand competition), whereas $\pi_{i,i} < \pi_{i,j} < 0$ corresponds to strategic substitutes (Cournot competition), see Bulow *et al.* (1985).

In Section 2 we have shown that under CARA the auction price does not affect the monopoly behavior in the aftermarket. In oligopoly settings this is not generally true any longer. Indeed, despite the fact that an amount w_i, which a CARA firm i has paid for the license, does not directly affect an optimal strategic choice s_i for *given values* of competitors' s_{-i}, it affects the *distribution* of their risk attitudes, and hence, their choices of s_{-i}, and, therefore, affects the optimal value of s_i indirectly. Hence, a proper definition of valuations is not guaranteed even for CARA utility functions. In order to keep the analysis tractable, we will restrict ourselves to the case of a small amount of uncertainty. Thus, we put $u = \alpha\varepsilon$, where $\alpha > 0$ and ε is uniformly distributed over the range $[-1, 1]$, and consider a limit case when $\alpha \to 0$

(see also footnote 4).

Suppose that a firm i with risk type x wins the auction and gets a license at price w. If risk types of all winning firms and the amounts they paid for the licenses had been revealed before the firms make their choices of s_j, that is under *full information* about risk types and prices, then the second stage Nash equilibrium strategic variable s_i^* would have been a function of all r_j and w_j, *i.e.*, $s_i^* = s^*(x, w, r_{-i}, w_{-i})$. Just before this information were available to firm i, its expected utility would have been given by

$$W^{FI}(w, x) = E\left(E_u U_i\left(\pi\left(s_i^*, s_{-i}^*, u\right) - w\right) | I_i\right),$$

where I_i is the information firm i has about risk types and auction prices of all the other winning firms. If, on the other hand, neither risk types, nor auction bids, nor the amounts others paid for the licenses are revealed before the firms make their choices of s_j, that is when they have *no information* about risk types and prices, then the second stage Nash equilibrium strategic variable s_i^* would be a function of x and w only, *i.e.*, $s_i^* = s^*(x, w)$. The expected utility of firm i in this case is given by the same expression

$$W^{NI}(w, x) = E\left(E_u U_i\left(\pi\left(s_i^*, s_{-i}^*, u\right) - w\right) | I_i\right),$$

with the important difference being the arguments the Nash equilibrium choices s^* depend on.

In the sequel we assume that the only information that is available to any winning firm i is its risk type x and the auction price w it has paid for the license,[5] that is, the no information case. However, as we will see, in the second-order approximation both functions $s^*(x, w, r_{-i}, w_{-i})$ and $s^*(x, w)$ are linear and the expected utilities $W^{FI}(w, x)$ and $W^{NI}(w, x)$ coincide! Hence, firms' bidding behavior is not affected by the informational assumption, although their strategic market behavior is affected.[6]

Suppose that a firm i with risk type x wins an auction and gets a license at price w. In a symmetric Nash equilibrium, each of the other winning firms j chooses $s_j^* = s^*(r_j, w_j)$. Thus, for any given values (r_j, w_j) of all competitors, if firm i chooses s_i, its conditional expected utility is

$$\widehat{W}(s_i, r_{-i}, w_{-i}, w, x) \equiv E_u U_i\left(\pi\left(s_i, s^*\left(r_{-i}, w_{-i}\right), u\right) - w\right), \tag{4}$$

[5] The reason a firm's bid is not included in I_i is that it provides no extra information to firm i as a firm i's bid is fully determined by its type x.

[6] There is an intermediate case where the bids rather than risk types are observed after the auction. In this case, bidding behavior were affected as firms would be willing to *signal* by submitting higher bids. This signalling behaviour is difficult to analyze in details. In a monotone bidding equilibrium, however, firms' strategic market behavior would not have been affected as they could have inverted bids back into risk types. Thus, in this intermediate case bidding behaviour is affected, but market behavior is not.

where $s^* (r_{-i}, w_{-i})$ represents choices of the other $(n-1)$ firms. The unconditional expected utility W of firm i is an expectation of $\widehat{W}$:

$$W (s_i, w, x) \equiv E \left(\widehat{W} (s_i, r_{-i}, w_{-i}, w, x) | I_i \right).$$

In other words, based on w and x, firm i estimates a joint distribution of (r_j, w_j) of all its competitors. In the rest of this paper we consider two commonly considered auction formats: a multi-unit uniform $(n+1)^{st}$ price auction (*i.e.*, a multi-unit generalization of the second price auction) and a pay-your-bid auction (*i.e.*, a multi-unit generalization of the first price auction). In both cases, the joint distribution of (r_j, w_j) conditional on (x, w) degenerates to a one-dimensional distribution. In a uniform $(n+1)^{st}$ price auction, where all n highest bids win the auction and pay the same price which is equal to the $(n+1)^{st}$ highest bid, the marginal distribution of w_j is degenerate as $\Pr (w_j = w) = 1$. On the other hand, in a pay-your-bid auction, where the n highest bids win the auction and each winning firm pays a price that is equal to the amount it has bid, the conditional distribution $(r_j | w_j)$ is degenerate as $\Pr (r_j = y | w_j = b (y)) = 1$, where b is a monotone equilibrium bidding function.

In the rest of this section we analyze the multi-unit uniform $(n+1)^{st}$ price auction, and we use a superscript (II) to indicate that this is a generalization of the second-price auction. In this auction format,[7] all winning firms pay the same price w and the conditional expected utility of firm i is

$$W^{(II)} (s_i, w, x) = E \left(\widehat{W} (s_i, r_{-i}, w, w, x) | I_i^{(II)} \right),$$

where information $I_i^{(II)}$ that is available to firm i consists of the following: $r_i = x$, $b^{(II)} (r_j) > w$, $b^{(II)} (r_l) \leq w$, where $b^{(II)}$ is a monotone equilibrium bidding function. Maximizing $W^{(II)}$ w.r.t. s_i yields the following first-order condition that $s^* (x, w)$ function has to satisfy:

$$W_{s_i}^{(II)} (s^* (x, w), w, x) = 0. \tag{5}$$

One may see that the main difficulty in using (5) for investigating the properties of the function $s^* (x, w)$ is that $W_{s_i}^{(II)}$ depends on (x, w) not only directly through $\widehat{W}$, but also indirectly through $I_i^{(II)}$. Even if the signals, *i.e.*, risk attitudes, are statistically independent, $I_i^{(II)}$ still depends on w. This is the main reason why we consider a limit case $\alpha \to 0$. Without uncertainty, *i.e.*, if $\alpha = 0$, the aftermarket game has a unique and stable Nash equilibrium $s^{(0)}$ that satisfies $\pi_i \left(s^{(0)}, s_{-i}^{(0)}, 0 \right) = 0$. In equilibrium all firms get the same profit $\pi \left(s^{(0)}, s_{-i}^{(0)}, 0 \right)$ and, therefore, bid this amount in an auction and get zero utility.

[7] In case of statistically independent signals, the analysis below also holds true for a multi-unit version of the English ascending auction.

In the following proposition we analyze the market stage Nash equilibrium function $s^*(x,w)$ for small values of α. From this moment on we will drop the arguments that the profit function π and all its derivatives depend on, and we implicitly assume that they are evaluated at the point $\left(s^{(0)}, s^{(0)}_{-i}, 0\right)$.

Proposition 4 *In the second-order approximation $s^*(x,w)$ can be written as*

$$s^*(x,w) = s^{(0)} + \left(A^{(0)} + A^{(1)}x + A^{(2)}E\left(r_j | I_i^{II}\right)\right)\alpha^2,$$

where

$$\begin{aligned} A^{(0)} &= -\frac{\pi_{i,u,u}}{6\left(\pi_{i,i} + (n-1)\pi_{i,j}\right)},\ A^{(1)} = \frac{\left(\pi_{i,i} + (n-2)\pi_{i,j}\right)\pi_{i,u}\pi_u}{3\left(\pi_{i,i} - \pi_{i,j}\right)\left(\pi_{i,i} + (n-1)\pi_{i,j}\right)} < 0, \\ A^{(2)} &= -\frac{(n-1)\pi_{i,j}\pi_{i,u}\pi_u}{3\left(\pi_{i,i} - \pi_{i,j}\right)\left(\pi_{i,i} + (n-1)\pi_{i,j}\right)}. \end{aligned}$$

Proof. We begin with the first-order approximation $s^*(x,w) = s^{(0)} + s^{(1)}(x,w)\alpha$ and show that $s^{(1)}(x,w) = 0$. To this end we write the first order approximation of $\pi_i(s^*, s^*(r_{-i},w), u)$ as $\pi_i(s^*, s^*(r_{-i},w), u) = \pi_i + \pi_i^{(1)}\alpha$, where

$$\pi_i^{(1)} = \pi_{i,i}s^{(1)}(x,w) + \pi_{i,j}\sum_{j\neq i} s^{(1)}(r_j,w) + \pi_{i,u}\varepsilon.$$

Then, $\widehat{W}_i(s^*, r_{-i}, w, w, x)$ can be written as

$$\begin{aligned} \widehat{W}_i &= E_u\left(\pi_i\left(s^*, s^*(r_{-i},w), u\right) U_i'\left(\pi\left(s^*, s^*(r_{-i},w), u\right) - w\right)\right) \\ &= E_u\left(\left(\pi_{i,i}s^{(1)}(x,w) + \pi_{i,j}\sum_{j\neq i} s^{(1)}(r_j,w) + \pi_{i,u}\varepsilon\right) U_i'(\pi - w)\right)\alpha \\ &= \left(\pi_{i,i}s^{(1)}(x,w) + \pi_{i,j}\sum_{j\neq i} s^{(1)}(r_j,w)\right) U_i'(\pi - w)\,\alpha. \end{aligned}$$

Therefore, in the first-order approximation equation (5) reads as

$$\pi_{i,i}s^{(1)}(x,w) + \pi_{i,j}\sum_{j\neq i} E\left(s^{(1)}(r_j,w) | I_i^{(II)}\right) = 0. \tag{6}$$

For another firm j equation (6) becomes

$$\pi_{i,i}s^{(1)}(r_j,w) + \pi_{i,j}\sum_{k\neq j} E\left(s^{(1)}(r_k,w) | I_j^{(II)}\right) = 0.$$

Taking an expectation of this equation conditional on $I_i^{(II)}$ and using the law of iterative expectation yields

$$\left(\pi_{i,i} + (n-2)\,\pi_{i,j}\right) E\left(s^{(1)}\left(r_j, w\right) | I_i^{(II)}\right) + \pi_{i,j} s^{(1)}\left(x, w\right) = 0. \tag{7}$$

Plugging $E\left(s^{(1)}\left(r_j, w\right) | I_i^{(II)}\right)$ from (7) into (6) leads to

$$\frac{\left(\pi_{i,i} - \pi_{i,j}\right)\left(\pi_{i,i} + (n-1)\,\pi_{i,j}\right)}{\left(\pi_{i,i} + (n-2)\,\pi_{i,j}\right)} s^{(1)}\left(x, w\right) = 0.$$

Hence, $s^{(1)}\left(x, w\right) = 0$.

As $s^{(1)}\left(x, w\right) = 0$, in the second-order approximation $s^*\left(x, w\right) = s^{(0)} + s^{(2)}\left(x, w\right)\alpha^2$ and $\pi_i\left(s^*, s^*\left(r_{-i}, w\right), u\right)$ can be written as $\pi_i\left(s^*, s^*\left(r_{-i}, w\right), u\right) = \pi_{i,u}\varepsilon\alpha + \pi_i^{(2)}\alpha^2$ where

$$\pi_i^{(2)} = \pi_{i,i} s^{(2)}\left(x, w\right) + \pi_{i,j}\sum_{j \neq i} s^{(2)}\left(r_j, w\right) + \frac{1}{2}\pi_{i,u,u}\varepsilon^2.$$

In a similar way one obtains the following approximations:

$$\begin{aligned} \pi\left(s^*, s^*\left(r_{-i}, w\right), u\right) &= \pi + \pi_u \varepsilon\alpha, \\ U_i'\left(\pi\left(s^*, s^*\left(r_{-i}, w_{-i}\right), u\right) - w\right) &= U_i'\left(\pi - w\right)\left(1 - \pi_u x \varepsilon\alpha\right). \end{aligned}$$

Plugging the above approximations for π and U_i' into (5) and dropping all terms that are $o\left(\alpha^2\right)$, we see that the first-order term vanishes, as it must do, and equating the second-order term to zero yields the following second order approximation of (5):

$$\pi_{i,u,u} - 2\pi_{i,u}\pi_u x + 6\pi_{i,i} s^{(2)}\left(x, w\right) + 6\pi_{i,j}\sum_{j \neq i} E\left(s^{(2)}\left(r_j, w\right) | I_i^{(II)}\right) = 0.$$

In a similar way, as in the first-order approximation, the above equation reduces to

$$s^{(2)}\left(x, w\right) = A^{(0)} + A^{(1)}x + A^{(2)}E\left(r_j | I_i^{(II)}\right),$$

where $A^{(0)}$, $A^{(1)}$ and $A^{(2)}$ are defined as in the proposition. It also follows that

$$E\left(s^{(2)}\left(r_j, w\right) | I_i^{(II)}\right) = A^{(0)} - \frac{\pi_{i,j} A^{(1)} x}{\left(\pi_{i,i} + \pi_{i,j}\left(n-2\right)\right)} - \frac{\pi_{i,i} A^{(2)} E\left(r_j | I_i^{(II)}\right)}{\left(n-1\right)\pi_{i,j}}, \tag{8}$$

which will be used in the next proposition. ■

There are two ways a risk type affects market behavior. First, it directly influences the strategic variable s through $A^{(1)}$, which is what we call the *risk attitude effect*. This effect, just like in the monopoly settings, manifests that a more risk-averse firm chooses a lower level of s. The other effect, which we call the *strategic effect*, influences s indirectly through the fact that a change in a player's risk attitude affects the player's expectations about other firms' risk types either because these types are not independently distributed or because of the indirect effect through the price paid for the license during the auction. This last, indirect, effect also holds true when types are statistically independently distributed. The sign of this strategic effect depends on the exact correlation structure and on the sign of $A^{(2)}$.

In the second-order approximation, Nash equilibrium profit of firm i is given by

$$\pi\left(s^*\left(x,w\right),s^*\left(r_{-i},w_{-i}\right),u\right)=\pi+\pi_u\varepsilon\alpha+\left(\pi_j\sum_{j\neq i}s^{(2)}\left(r_j,w\right)+\frac{1}{2}\pi_{u,u}\varepsilon^2\right)\alpha^2.$$

Now we are going to analyze the equilibrium bidding behavior. As there is no first-order term in $s^*\left(x,w\right)$, there is no first-order term in the bidding function $b^{(II)}\left(x\right)$ either. Hence, we write the bidding function as $b^{(II)}\left(x\right)=\pi+b^{(II),(2)}\left(x\right)\alpha^2$ and $w=b^{(II)}\left(z\right)=\pi+b^{(II),(2)}\left(z\right)\alpha^2$. Expected utility $\widehat{W}\left(s^*\left(x,b^{(II)}\left(z\right)\right),r_{-i},b^{(II)}\left(z\right),b^{(II)}\left(z\right),x\right)$ in the second order approximation then becomes

$$\widehat{W}=U_i'\left(0\right)\left(\pi_j\sum_{j\neq i}s^{(2)}\left(r_j,w\right)+\frac{1}{6}\left(\pi_{u,u}-x\left(\pi_u\right)^2\right)-b^{(II),(2)}\left(z\right)\right)\alpha^2.$$

Hence, the conditional expected utility, being written as a function of (x,z), is

$$\begin{aligned}\widehat{V^{(II)}}\left(x,z\right) &= W^{(II)}\left(s^*\left(x,b^{(II)}\left(z\right)\right),b^{(II)}\left(z\right),x\right) \qquad (9)\\ &= U_i'\left(0\right)\left(\left(n-1\right)\pi_jE\left(s^{(2)}|I_i^{(II)}\right)+\frac{\pi_{u,u}-x\left(\pi_u\right)^2}{6}-b^{(II),(2)}\left(z\right)\right)\alpha^2.\end{aligned}$$

The unconditional (*ex-ante*) expected utility of a firm i having a risk attitude x and bidding $b\left(y\right)$ is:

$$V^{(II)}\left(x,y\right)=\int_{b^{(II)}(z)<b^{(II)}(y)}\widehat{V^{(II)}}\left(x,z\right)dG,$$

where z is a risk attitude of a firm which submits the $\left(n+1\right)^{st}$-highest bid and $G\left(z\mid x\right)$ is the conditional distribution function of z. We explore the conditions under which a decreasing and/or an increasing equilibrium exist. In the first (second) case, firms bid higher (lower) the less risk-averse they are.

In Proposition 5, we first present a general condition under which an increasing and a decreasing equilibrium exist. Subsequently, we will look at the case of Bertrand and Cournot competition to give the proposition below more economic content. Let

$$\begin{aligned}H^{(-)}\left(z,x\right) &\equiv E\left(r_j|r_i=x,r_j<z,r_l\geq z\right),\\ H^{(+)}\left(z,x\right) &\equiv E\left(r_j|r_i=x,r_j>z,r_l\leq z\right).\end{aligned}$$

In other words, $H^{(-)}$ ($H^{(+)}$) is an expectation of risk attitudes of $(n-1)$ winning firms j conditional on these risk attitudes being below (above) z, the risk attitudes of all the other $(N-n)$ loosing firms l are above (below) z and the risk attitude of firm i itself being x. In case the signals r_i are affiliated (including the case of statistically independent signals), both partials of $H^{(\pm)}(x,z)$ are non-negative (see, *e.g.*, Krishna, 2002, pp. 272)

Proposition 5 *Let*

$$\begin{aligned}
B^{(0)} &= \frac{1}{6}\left(\pi_{u,u} - \frac{(n-1)\,\pi_j \pi_{i,u,u}}{(\pi_{i,i} + \pi_{i,j}(n-1))}\right), \\
B^{(1)} &= \frac{\pi_u}{6}\left(\frac{2(n-1)\,\pi_j \pi_{i,j} \pi_{i,u}}{(\pi_{i,i} - \pi_{i,j})(\pi_{i,i} + \pi_{i,j}(n-1))} + \pi_u\right), \\
B^{(2)} &= \frac{(n-1)\,\pi_j \pi_{i,i} \pi_{i,u} \pi_u}{3(\pi_{i,i} - \pi_{i,j})(\pi_{i,i} + \pi_{i,j}(n-1))}, \text{ and} \\
v^{(\pm)}(z,x) &= B^{(0)} - B^{(1)}x + B^{(2)} H^{(\pm)}(z,x)
\end{aligned}$$

(i) A decreasing bidding equilibrium $b^{(II)}(x) = \pi + v^{(-)}(x,x)\,\alpha^2$ *exists, if, and only if*

$$\max\left\{v_x^{(-)}(x,x),\, v_x^{(-)}(x,x) + v_z^{(-)}(x,x)\right\} < 0. \tag{10}$$

There is at most one decreasing equilibrium.

(ii) An increasing bidding equilibrium $b^{(II)}(x) = \pi + v^{(+)}(x,x)\,\alpha^2$ *exists, if and only if*

$$\min\left\{v_x^{(-)}(x,x),\, v_x^{(-)}(x,x) + v_z^{(-)}(x,x)\right\} > 0. \tag{11}$$

There is at most one increasing equilibrium.

Proof. Let a firm i has a risk attitude x and submits a bid $b^{(II)}(y)$ where $b^{(II)}$ is a strictly decreasing bidding function. In equilibrium, it must be that

$$x = \arg\max_y V^{(II)}(x,y).$$

The first and the second order conditions for a decreasing equilibrium are:

$$\widehat{V^{(II)}}(x,x) = 0,\ \widehat{V^{(II)}}_z(x,x) > 0.$$

Using (9) and (8) we rewrite the first order condition as

$$b^{(II),(2)}(x) = (n-1)\,\pi_j E\left(s^{(2)}\left(r_j, b^{(II)}(x)\right)|I_i^{(II)}\right) + \frac{1}{6}\left(\pi_{u,u} - x(\pi_u)^2\right) = v^{(-)}(x,x).$$

Hence, an equilibrium bidding function must be

$$b^{(II)}(x) = \pi + v^{(-)}(x,x)\,\alpha^2,$$

such that in equilibrium:

$$\widehat{V^{(II)}}(x,z) = U_i'(0)\left(v^{(-)}(z,x) - v^{(-)}(z,z)\right)\alpha^2.$$

However, it determines a unique decreasing equilibrium if and only if it is a decreasing function and such that $\widehat{V^{(II)}}_z(x,x) > 0$. The first condition requires $v_x^{(-)}(x,x) + v_z^{(-)}(x,x) < 0$. The second condition can be transformed as follows:

$$0 < \widehat{V^{(II)}}_z = \frac{d\widehat{V^{(II)}}(x,x)}{dx} - \widehat{V^{(II)}}_x(x,x) = -\widehat{V^{(II)}}_x = -U_i'(0)\, v_x^{(-)}(z,x)\,\alpha^2,$$

that is $v_x^{(-)}(z,x) < 0$. Combining them together yields the necessary and sufficient condition for a decreasing equilibrium is

$$\max\left\{v_x^{(-)}(x,x), v_x^{(-)}(x,x) + v_z^{(-)}(x,x)\right\} < 0.$$

In a similar way, one gets the necessary and sufficient condition for an increasing equilibrium. ■

One may see that now the functions $v^{(\pm)}(z,x)$ represent a valuation function, *i.e.*, a certainty equivalent of the market-stage game, of a firm i, which has a risk type x and pays an auction price determined by a firm j of type z. When $x = z$, *i.e.*, both firms i and j have the same risk type and compete for only one remaining license, they bid their values $v^{(\pm)}(z,z)$, hence, the auction price w must be equal to $v^{(\pm)}(z,z)$. Then, the existence condition $v_x^{(-)}(x,x) < 0$ ($v_x^{(+)}(x,x) > 0$) requires that if a risk type of firm i marginally differs from z such that $x < z$ ($x > z$), then firm i must have a valuation $v^{(\pm)}(z,x)$ that is strictly higher than the valuation of firm j, which is $v^{(\pm)}(z,z)$, so that firm i bids higher than $v^{(\pm)}(z,z)$ and wins the license. The other existence condition $v_x^{(-)}(x,x)+v_z^{(-)}(x,x) < 0$ ($v_x^{(+)}(x,x)+v_z^{(+)}(x,x) > 0$) then guarantees that the actual bid of firm i $v^{(\pm)}(x,x)$ is indeed higher than the bid of firm j, $v^{(\pm)}(z,z)$. Finally, if one of the conditions (10) and (11) is violated, then the functions $v^{(\pm)}(z,x)$ do not represent a firm's maximum willingness to pay because there will be a risk type x such that if a firm of this type x bids an amount $v^{(\pm)}(x,x)$, it is strictly better off with an auction price $w > v^{(\pm)}(x,x)$ rather than $w < v^{(\pm)}(x,x)$. In this case the valuation is improperly defined.

It is interesting to observe that the ex-post valuations of firms depend not only on their own signals, but also on those of their rivals. Still, the model is different from a standard interdependent valuation model. The main reason is that in the standard interdependent valuation model, a player's valuation depends on the signals of *all* players *participating* in the auction, whereas in our model, a player's valuation depends only on the signals of *all winning* firms, who are endogenously determined.

We will now investigate the implications of the above general proposition in case of price and quality setting market behavior when the signals players receive about their risk attitudes are affiliated.

3.1 Bertrand Competition

Under Bertrand competition, it is clear that both $\pi_{i,j}$ and π_j are positive and, therefore, $B^{(1)}$ and $B^{(2)}$ as defined in Proposition 5 are such that $B^{(1)} > 0$ and $B^{(2)} < 0$. As both partials of $H^{(\pm)}(x,z)$ are non-negative if the signals are affiliated, the equilibrium existence conditions (10) and (11) reduce to

$$\frac{B^{(1)}}{B^{(2)}} < H_x^{(-)}(x,x),$$

for the decreasing equilibrium and to

$$\frac{B^{(1)}}{B^{(2)}} > H_x^{(+)}(x,x) + H_z^{(+)}(x,x),$$

for the increasing equilibrium.

As in the case of Bertrand competition, $B^{(1)}/B^{(2)} < 0$, it immediately follows that the decreasing equilibrium always exists, whereas the increasing equilibrium never exists. This result is summarized in the corollary below.

Corollary 6 *If n licenses are auctioned and aftermarket behavior is characterized by differentiated Bertrand competition and firms' risk attitudes are affiliated, then the least risk-averse players will win the auction.*

Thus, the result of the monopoly price setting case analyzed in Section 2 generalizes to the case of price setting behavior in an oligopoly context: the least risk-averse firms are selected and they set higher aftermarket prices.

3.2 Cournot Competition

Under Cournot competition both $\pi_{i,j}$ and π_j are negative and, therefore, $B^{(1)} > 0$ and $B^{(2)} > 0$. With affiliated signals, the equilibrium existence conditions (10) and (11) reduce to

$$\frac{B^{(1)}}{B^{(2)}} > H_x^{(-)}(x,x) + H_z^{(-)}(x,x),$$

for the decreasing equilibrium and to

$$\frac{B^{(1)}}{B^{(2)}} < H_x^{(+)}(x,x),$$

for the increasing equilibrium.

Unlike the case of Bertrand competition, these conditions do not lead to a situation that can be easily characterized. The main reason is that in this case the strategic effect and the risk attitude effect work in opposite directions. In general, four different cases are possible: (*i*) only a decreasing equilibrium exists, (*ii*) only an increasing equilibrium exists, (*iii*) no monotonic equilibrium exists and (*iv*) both equilibria exists. Below we consider some special cases and show that certainly the first three situations can occur in economically relevant environments.

(*i*) If

$$\frac{B^{(1)}}{B^{(2)}} > \max\left\{H_x^{(-)}(x,x) + H_z^{(-)}(x,x), H_x^{(+)}(x,x)\right\},$$

then only a decreasing equilibrium exists. This situation occurs, *e.g.*, if the strategic effect is very small and market demand of firm i is almost independent of the quantity set by firm j, *i.e.*, if π_j is very close to zero. This implies that the monopoly result for quantity setting generalizes to a "neighborhood of the monopoly case", namely where firms almost have "local" monopolies. Another case where this situation of the existence of only a decreasing equilibrium occurs, is if marginal profit π_i is almost independent of u, such that $\pi_{i,u}$ is very close to zero and uncertainty about market conditions does not effect the level of the strategic variable firms choose.

(*ii*) If

$$\frac{B^{(1)}}{B^{(2)}} < \min\left\{H_x^{(-)}(x,x) + H_z^{(-)}(x,x), H_x^{(+)}(x,x)\right\}, \tag{12}$$

then only an increasing equilibrium exists. The following example represents a case where this condition is satisfied. Let the inverse market demands be given by $p_i = \left(q_i + \sum_{j\neq i} q_j\right)^{-\frac{1}{e}}$, where e is the price elasticity of demand, and let firms' uncertain marginal costs be given by $c - u$. One can easily verify that with n licenses, the unique Cournot-Nash equilibrium in case of no uncertainty, *i.e.*, $\alpha = 0$, is given by

$$q^{(0)} = \frac{1}{n}\left(\frac{ne-1}{nec}\right)^e.$$

This equilibrium is stable if $ne > 1$ and quantities are strategic substitutes provided $ne - 1 > e$. Evaluation of the partial derivatives of $\pi = (p_i - c + u)\, q_i$ yields

$$\frac{B^{(1)}}{B^{(2)}} = \frac{(3n-1)(ne-1-e) + 1 + e}{2\left((2n-1)(ne-1-e) + n\right)}.$$

Now it is seen that when $e \to 1/(n-1)$, $B^{(1)}/B^{(2)}$ converges to $\frac{1}{2(n-1)}$. Hence, for a given structure of strictly positively affiliated signals, if the number of licenses is sufficiently large, condition (12) is satisfied and only an increasing equilibrium exists.

What happens in this example is that when demand becomes relatively inelastic, a small change in output has a relatively large impact on price. This means that a firm's market profit is highly sensitive to the output chosen by the competitors and thus, that a given firm strongly prefers to compete with the more risk-averse firms. This strategic effect gets stronger the more firms there are in the marketplace.

(iii) Finally, if

$$H_x^{(+)}(x,x) < \frac{B^{(1)}}{B^{(2)}} < H_x^{(-)}(x,x) + H_z^{(-)}(x,x),$$

then no monotonic equilibrium exists. This happens, for example, when signals are statistically independent, such that $H_x^{(\pm)}(x,x) = 0$, and the corresponding density function rapidly increases. For example, if we take the following distribution function

$$F(x) = \left(\frac{x-\underline{r}}{\overline{r}-\underline{r}}\right)^{\gamma},$$

with $\gamma > 0$, then $H^{(-)}(x,z) = E(r|\underline{r} < r < z)$ and $H_z^{(-)}(x,z) = \gamma/(1+\gamma)$. Thus, for any market structure that satisfies $B^{(1)}/B^{(2)} \in (0,1)$ (see, *e.g.*, the example analyzed under (ii)) there exists a value of γ such that $0 < B^{(1)}/B^{(2)} < \gamma/(1+\gamma)$ and no monotonic equilibrium exists.[8]

Summarizing, we conclude that if firms set market prices, the risk attitude effect is reinforced by the strategic effect and, like in the case of a single license (monopoly), the least risk-averse firms win the auction and, as a result, set higher prices than a randomly selected group of firms. If, however, firms set quantities, the strategic effect offsets the risk attitude effect and it may happen that the equilibrium allocation of licenses is reversed: the most risk-averse firms win the auction and set higher prices than a randomly selected group of firms.

4 The Oligopoly Setting: Pay-Your-Bid Auction

In a pay-your-bid auction, n highest bids win the auction and each winning firm pays a price that is equal to the amount it has bid, *i.e.*, $w_i = b_i^{(I)}$. Therefore, the conditional expected utility of firm i, provided it has a risk type x, submits a bid $b^{(I)}$ and wins the auction is given by

$$W^{(I)}(s_i,b,x) \equiv E\left(\widehat{W}\left(s_i,r_{-i},b^{(I)}(r_{-i}),b,x\right)|I_i^{(I)}\right),$$

where $b^{(I)}(x)$ is a monotone equilibrium bidding function and the information $I_i^{(I)}$ that is available to firm i consists of the following: $r_i = x$, $b^{(I)}(z) < b$, $b^{(I)}(r_j) > b^{(I)}(z)$, $b^{(I)}(r_l) \leq b^{(I)}(z)$, and z is a risk type that submitted the $(n+1)^{st}$-highest

[8] In the case of linear uncertain demand of the form $p = 1 + u - \sum q_j$, it is easily seen that $B^{(1)}/B^{(2)} = \frac{3n-1}{4(n-1)}$ so that $B^{(1)}/B^{(2)} \in (0,1)$ for any $n \geq 4$.

bid. The superscript (I) refers to the fact that we consider a multi-unit generalization of the first-price sealed-bid auction.

It is easily seen that behavior of firms in the aftermarket is characterized by the very same proposition 4, if we take into account differences in information that firms have in these two auction formats. In contrast to the uniform $(n+1)^{st}$ price auction, in a pay-your-bid auction, under CARA prices paid in an auction do not affect a market behavior. This is so because a price that a firm i paid for a license, does not provide any information about distributions of the other competitors' risk types. Hence, the second stage Nash equilibrium strategic variable depends, in fact, only on the risk type x, that is

$$s^*(x) = s^{(0)} + \left(A^{(0)} + A^{(1)}x + A^{(2)}E\left(r_j|I_i^{(I)}\right)\right)\alpha^2.$$

In order to analyze the equilibrium bidding behavior, we assume that a firm i with a risk type x bids an amount $b^{(I)}(y)$, *i.e.*, as if it were of a type y. In the second-order approximation $b^{(I)}(x) = \pi + b^{(I),(2)}(x)\alpha^2$. Plugging this together with the above expression for $s^*(x)$ into (4) yields the following second-order approximation for a firm i's expected utility:

$$\widehat{W^{(I)}} = U_i'(0)\left(\pi_j\sum_{j\neq i}s^{(2)}(r_j) + \frac{\pi_{u,u} - x(\pi_u)^2}{6} - b^{(I),(2)}(y)\right)\alpha^2.$$

If a firm i knew z, then for this given value of z firm i would get the following conditional expected utility

$$\begin{aligned}\widehat{V^{(I)}}(x,y,z) &= E\left(\widehat{W^{(I)}}\left(s_i, r_{-i}, b^{(I)}(r_{-i}), b^{(I)}(y), x\right)|I_i^{(I)}, z\right) \\ &= U_i'(0)\left(B^{(0)} - B^{(1)}x + B^{(2)}E\left(r_j|I_i^{(I)}, z\right) - b^{(I),(2)}(y)\right)\alpha^2 \\ &= U_i'(0)\left(v^{(\pm)}(z,x) - b^{(I),(2)}(y)\right)\alpha^2,\end{aligned}$$

where, as before, +/- corresponds to a decreasing/increasing equilibrium and $v^{(\pm)}(z,x)$ is the corresponding valuation function defined in Proposition 5. Without knowing z, firm i takes into account that z is a random variable that follows a distribution $G(z \mid x)$ with support $(x,\overline{r})$ for a decreasing bidding equilibrium and with support $(\underline{r},x)$ for an increasing bidding equilibrium. Then, the unconditional (*ex-ante*) expected utility of a firm i having a risk attitude x and bidding $b^{(I)}(y)$ is:

$$V^{(I)}(x,y) = \int_{b^{(I)}(z)<b^{(I)}(y)} \widehat{V^{(I)}}(x,y,z)\,dG(z \mid x).$$

It is important to note that if risk types are affiliated, then z and x are affiliated as well, such that $\partial G(z \mid x)/\partial x < 0$, $\partial \lambda(z \mid x)/\partial x < 0$, $\partial \sigma(z \mid x)/\partial x < 0$, where

$$\lambda(z \mid x) \equiv \frac{g(z \mid x)}{1 - G(z \mid x)}, \text{ and } \sigma(z \mid x) \equiv \frac{g(z \mid x)}{G(z \mid x)}$$

are the conditional hazard rate of z (the n^{th} lowest type amongst $N-1$ remaining firms) and the conditional reverse hazard rate of z (the n^{th} highest type amongst $N-1$ remaining firms), and $g(z \mid x)$ is the corresponding conditional distribution density function of z. In the following proposition we use the properties of the functions g, G, λ and σ in order to establish existence conditions for decreasing and increasing bidding equilibria.

Proposition 7 *Let*

$$L^{(-)}(z,x) \equiv \exp\left(-\int_x^z \lambda(t \mid t)\,dt\right) \text{ and } L^{(+)}(z,x) \equiv \exp\left(-\int_z^x \sigma(t \mid t)\,dt\right).$$

(i) If condition (10) is satisfied, then there exists a unique decreasing bidding equilibrium $b^{(I)}(x) = \pi + b^{(I),(2)}(x)\,\alpha^2$, *where*

$$b^{(I),(2)}(x) = v^{(-)}(x,x) + \int_x^{\overline{r}} \left(v_x^{(-)}(z,z) + v_z^{(-)}(z,z)\right) L^{(-)}(z,x)\,dz.$$

(ii) If condition (11) is satisfied, then there exists a unique increasing bidding equilibrium $b^{(I)}(x) = \pi + b^{(I),(2)}(x)\,\alpha^2$, *where*

$$b^{(I),(2)}(x) = v^{(+)}(x,x) - \int_{\underline{r}}^{x} \left(v_x^{(+)}(z,z) + v_z^{(+)}(z,z)\right) L^{(+)}(z,x)\,dz.$$

Proof. We begin with a decreasing equilibrium. In equilibrium the unconditional expected utility $V^{(I)}(x,y)$ is maximized at $y = x$, hence, the first-order condition is $V_y^{(I)}(x,x) = 0$, that is

$$0 = -U_i'(0)\,\alpha^2\,(1 - G(x \mid x))\left(\left(v^{(-)}(x,x) - b^{(I),(2)}(x)\right)\lambda(x \mid x) + \frac{db^{(I),(2)}}{dx}\right).$$

The solution to this differential equation that satisfies the initial condition $b^{(I),(2)}(\overline{r}) = v^{(-)}(\overline{r},\overline{r})$ is

$$b^{(I),(2)}(x) = v^{(-)}(x,x) + \int_x^{\overline{r}} \left(v_x^{(-)}(z,z) + v_z^{(-)}(z,z)\right) L^{(-)}(z,x)\,dz.$$

The derivative of $b^{(I),(2)}(x)$ can then be written as

$$\frac{db^{(I),(2)}}{dx} = \lambda(x \mid x) \int_x^{\overline{r}} \left(v_x^{(-)}(z,z) + v_z^{(-)}(z,z)\right) L^{(-)}(z,x)\, dz < 0,$$

such that $b^{(I),(2)}(x)$ is a decreasing function indeed. The second-order condition requires that

$$\begin{aligned}
0 &> V_{y,y}^{(I)}(x,x) = -V_{y,x}^{(I)}(x,x) \\
&= \frac{\partial}{\partial x}\left(\widehat{V^{(I)}}(x,y,y)\, g(y \mid x) - \int_y^{\overline{r}} \widehat{V_y^{(I)}}(x,y,z)\, dG(z \mid x)\right) \\
&= U_i'(0)\,\alpha^2 (1 - G(y \mid x)) \left(\frac{\partial \left(v^{(-)}(y,x) - b^{(I),(2)}(y)\right) \lambda(z \mid x)}{\partial x}\right)_{y=x}, \\
0 &> \lambda(x \mid x)\, v_x^{(-)}(y,x) + \left(v^{(-)}(x,x) - b^{(I),(2)}(x)\right) \frac{\partial \lambda(y \mid x)}{\partial x}.
\end{aligned}$$

Hence, the second-order condition is always satisfied under the conditions of the proposition and $b^{(I)}(x) = \pi + b^{(I),(2)}(x)\,\alpha^2$ defines a unique decreasing bidding equilibrium.

In a similar way, one proves the sufficiency of condition (11) for an increasing equilibrium. ■

It is important to note that the necessary and sufficient conditions for an increasing or decreasing equilibrium to exist in a uniform price auction are sufficient but not necessary for the existence of similar equilibria in a pay-your-bid auction. In fact, the necessary existence conditions for a pay-your-bid auction are weaker than (10) or (11). Therefore, it might happen that there is an equilibrium in a pay-your-bid auction whereas a uniform price auction does not have monotone bidding equilibria.

Once again, we conclude that the selection arguments that are valid in the monopoly settings may fail if firms compete in quantities and if the strategic effect dominates the risk attitude effect. In the latter case, the most risk-averse firms win the auction and they set higher prices than a randomly selected group of firms.

5 Conclusion

In this paper, we have considered a selection argument to connect the auctioning of licenses with the choice of market prices in the aftermarket. Crucial to the argument is that firms may differ in their attitude towards risk and that firms with different risk attitudes behave differently in the marketplace. To this end, we have considered three prototypes of market structures: monopoly, differentiated Bertrand competition and Cournot competition. In the monopoly situation, we have argued that auctions select the least risk-averse player (the risk attitude effect) and that this player chooses higher prices (quantities) than a randomly chosen player will do.

This general argument cannot easily be fully generalized to the case where multiple licenses are auctioned as the price that is paid to obtain a license effects the choice of a firm's market strategy through its expectation about the risk attitudes of other players. We, therefore, restrict the analysis of the two oligopoly cases to situations where uncertainty is relatively small. In this case, the monopoly argument about price setting behavior is robust to allowing for competition in the market place. In particular, in case of differentiated Bertrand competition, the least risk-averse firms will win the auction and set higher prices than a randomly selected group of firms. The monopoly argument about quantities is, however, not robust to allowing for competition in the marketplace. In particular, as firms have a preference for competing with the most risk-averse players, the strategic effect counteracts the risk attitude effect and in certain cases may even dominate the risk attitude effect. One such a case is when demand is relatively inelastic and small changes in output have large effects on price. In this case, more risk-averse bidders will bid more than less risk-averse bidders in case the risk attitudes of firms are strictly positively affiliated and the number of licenses is relatively large. This leads to higher market prices than when firms were selected in a more random fashion.

6 References

Asplund M. 2001. Risk-averse firms in oligopoly. *International Journal of Industrial Organization* 20, p 995-1012.

Baron, D.P. 1970. Price Uncertainty, Utility and Industry Equilibrium in Pure Competition. *International Economic Review* 11, 463-480.

Baron, D.P. 1971. Demand Uncertainty and Imperfect Competition. *International Economic Review* 12, 196-208.

Binmore, K. and P. Klemperer. 2002. The Biggest Auction Ever: The Sale of the British 3G Telecom Licenses. *Economic Journal 112:* C-*74-96.*

Bulow, J., J. Geanokoplos, and P. Klemperer. 1985. Multi-market Oligopoly: strategic substitutes and complements. *Journal of Political Economy* 93, pp. 488-511.

Dixit, A. 1986. Comparative Statics for Oligopoly. *International Economic Review* 27, pp. 107-22.

Eso, P. and L. White. 2004. Precautionary Bidding in Auctions. *Econometrica* 72, pp. 77-92.

European Commission. 1994. Green paper on a common approach in the field of mobile and personal communication in the European Union. Office for official publications of the European Communities, Luxembourg.

Geczy, C., B.A. Minton and C. Schrand. 1997. Why Firms use Currency Derivatives? *Journal of Finance* 52, pp. 1323-54.

Goeree, J. 2003. Bidding for the Future. *Journal of Economic Theory* 108 (2), pp. 345-364.

Janssen, M.C.W. (ed.). 2004. *Auctioning Public Assets: Analysis and Alternatives*. Cambridge University Press.

Janssen, M.C.W. 2005. Auctions as Coordination Devices. *European Economic Review* (forthcoming).

Jehiel, P. and B. Moldovanu. 1996a. Strategic Nonparticipation. *Rand Journal of Economics* 27, 84-98.

Jehiel, P. and B. Moldovanu. 1996b. How (not) to sell Nuclear Weapons. *American Economic Review* 86, 814-29.

Jehiel, P. and B. Moldovanu. 2001. Auctions with Downstream Interaction among Buyers. *Rand Journal of Economics* 31, 768-91.

Klemperer 2002 (a). What really matters in auction design. *Journal of Economic Perspectives* 16, p. 169-189.

Klemperer 2002 (b). How not to run auctions: the European 3G mobile telecom auctions. *European Economic Review* 46, p. 829-45.

Kreps, D. and J. Scheinkman. 1983. Quantity Pre-commitment and Bertrand Competition yield Cournot Outcomes. *Bell Journal of Economics* 14, pp. 326-37.

Krishna, V. 2002. *Auction Theory.* Academic Press.

Leland, H. 1972. Theory of the firm facing uncertain demand. *American Economic Review* 62, 278-91.

Mas-Colell, A. M. Whinston, J, Green. 1995. *Microeconomic Theory.* New York: Oxford University Press.

Milgrom, P. and R. Weber. 1982. A Theory of Auctions and Competitive Bidding, *Econometrica* 50, pp. 1089-1122.

McMillan, J. 1994. Selling Spectrum rights. *Journal of Economic Perspectives* 8, pp. 145-162.

Nance, D.R. Smith C.W. and Smithson, C.W. 1993. On the determinants of corporate hedging. *Journal of Finance* 48, pp. 267-84.

Offerman, Th. and J. Potters. 2000. Does Auctioning of Entry Licenses affect Consumer Prices? Tinbergen Institute Discussion Paper 00-046/1.

Tufano, P. 1996. Who manages risk? An Empirical Examination of Risk Management Practices in the Gold Mining Industry. *Journal of Finance* 51, pp. 1097-1137.

Van Damme, E. 2002. The European UMTS-auctions. *European Economic Review* 46, pp. 846-858.