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Existence and Uniqueness of Equilibrium in Nonoptimal Unbounded Infinite Horizon Economics

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Abstract

In applied work in macroeconomics and finance, nonoptimal infinite horizon economies are often studied in which the state-space is unbounded. Important examples of such economies are single-sector growth models with production externalities, valued fiat money, monopolistic competition, and/or distortionary government taxation. Although sufficient conditions for existence and uniqueness of Markovian equilibrium are well known for the compact state space case, no similar sufficient conditions exist for unbounded growth. This paper provides such a set of sufficient conditions, and presents a computational algorithm that will prove asymptotically consistent when computing Markovian equilibrium.

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1 INTRODUCTION

This paper presents existence and uniqueness results of Markovian equilibrium for a broad class of dynamic non-optimal single-sector stochastic unbounded growth models used in applied macroeconomics and public finance. The primitive data describing the class of models under consideration include economies with a diverse set of potential equilibrium distortions such as distortionary taxes, valued fiat money, monopolistic competition, and various types of production externalities. The state space may be unbounded, and therefore not compact. While there is vast literature on unbounded endogenous growth, there are no general results establishing sufficient conditions under which there exist Markovian equilibria for such models, let alone sufficient conditions under which the equilibrium is unique.

The methodology in this paper is related to the Euler equation approach discussed in an important series of recent work on "monotone-map" methods pioneered in Coleman (1991) and Greenwood and Huffman (1995). In a seminal paper, Coleman (1991) proves existence of equilibrium for economies with an income tax under a standard condition concerning the boundedness of the production function, a condition that is sufficient to guarantee compactness of the state space. This compactness of the state space is critical in the work of Coleman (1991) and in all subsequent work (in particular, Greenwood and Huffman (1995) and Datta, et al (2002)), because it implies that the candidate set of equilibrium consumption (or investment) functions is compact in a uniform topology. The compactness property of the equilibrium set then allows to demonstrate that a nonlinear operator has the appropriate order continuity property, and that its domain (and range) is chain complete. Given these two properties, a version of Tarski's theorem presented in Dugundji and Granas (1982) leads to the existence argument. While this is precisely the existence proof in Coleman (1991), Greenwood and Huffman (1995) follow a related argument to demonstrate existence of non-optimal equilibrium in a version of Romer (1996) unbounded growth model that has a stationary representation. Such stationary representation exists in the case of log utility and Cobb-Douglass technology, so that the problem can be posed on a compact state space, but it is not clear how the argument can be generalized to other choices of promitives (for instance utility outside the CES class). Our work systematically explores this issue for the unbounded growth case, and provides an affirmative general result.

It is important to distinguish this paper from the recent work of Coleman (2000) and Datta, et.al (2002).[1] These papers, based on the important and innovative proof of uniqueness in Coleman (2000), demonstrate uniqueness of a continuous Markovian equilibrium consistent with a monotone investment function (and in Coleman (2000), additionally, a monotone consumption function) relative to a very large class of candidate continuous Markovian equilibrium.

[1]The Datta, et al (2002) paper concerns issues of existence and uniqueness in models of bounded growth with elastic labor supply, and the method of proof of uniqueness is somewhat different than Coleman (2000), but very related.

First, a careful reading reveals that the method of proof of existence of Markovian equilibrium in both of these papers does *not* apply when the state space is not compact. Without compactness, existence of Markovian equilibrium is not guaranteed, and thus knowledge of its uniqueness (by itself) is not particularly useful. Second, the proof of existence in both Coleman (2000) and Datta, et.al (2002) relies upon the contraction mapping theorem under bounded returns and state space, which cannot be directly applied in the context of unbounded growth. For example, the key assumptions in Coleman (2000) on the primitive data of the economy (i.e., Assumption 1 and 2) are not sufficient to demonstrate that a value function exists, and that an Euler equation can therefore be derived. Indeed, these assumptions allow for both production and utility to be unbounded, so that the return function may be unbounded as well; as discussed in Stokey et al. (1989) and Alvarez and Stokey (1997), the value function in such cases may not be defined, and studying an Euler equation to characterize solutions is then inappropriate. Uur work in the present paper resolves this issue.

The monotone method developed in this paper is not topological, but rather, is built upon the monotonicity properties of a particular nonlinear operator and the lattice completeness properties of the underlying domain of this operator. This operator is a self-map on a complete lattice of candidate equilibrium functions, and its monotonicity implies that it has a fixed point, through Tarski's fixed-point theorem.[2] Since we rely on an order-based construction as opposed to a purely topological one, the concerns of boundedness, compactness and continuity do not enter into the formulation of the existence and uniqueness problem, and we are able to prove existence within a very sharp set of monotone Markov processes, and uniqueness within a large class of Markovian equilibrium (not necessarily monotone). In addition, the monotone method employed is constructive, and allows the discussion of issues associated with computation as well as characterization of Markovian equilibrium.[3] In particular, we provide some comparative statics results in some key parameters of the underlying economy. The technology is required to have constant returns to scale in private inputs and to obey some standard first- order and second-order conditions. However, unlike in Coleman (1991), we do not make any assumption guaranteeing compactness of the domain of the endogenous state variable, and we introduce externalities in the production process. Utility obeys some standard assumptions, except for a boundedness requirement which can be relaxed for economies with homogeneous preferences and constant returns to scale.

The paper is organized as follows. Section 2 presents the class of environments studied. Section 3 proves existence of Markovian equilibrium as the fixed

[2]The methodology is closely related to the lattice theoretic constructions in recent work reported in Mirman et al. (2002) within the context of bounded state spaces.

[3]Coleman (1991) discusses the advantages of order theoretic fixed-point theory versus purely topological fixed point theory (e.g., Brouwer, Schauder, Fan-Glicksberg types of fixed-point theorems). The latter class of fixed-point theorems are existential, and therefore not useful for discussing the computation of infinite dimensional fixed-point problems. The former class, as they are constructive, can often be used to show that extremal fixed points can be computed by successive approximation.

point of a monotone operator A mapping a complete lattice into itself. The characterization of the set of equilibria is refined by constructing an algorithm converging to the maximal fixed point, and by deriving some simple comparative statics results in the strong set order. Section 4 shows that all the fixed points of the mapping A are fixed points of another mapping, denoted $\hat{A}$, and that $\hat{A}$ has at most one interior fixed point. This generalization of a method used in Coleman (2000) and Datta et al. (2002) forms the basis of our uniqueness argument. Section 5 provides some examples and concludes.

2 THE MODEL

Time is discrete and indexed by $t \in T = \{0, 1, 2, ...\}$, and there is a continuum of infinitely-lived and identical household/firm agents. The aggregate state variables for this economy consist of endogenous and exogenous variables and are denoted by the vector S. Uncertainty comes in the form of a finite state first-order Markov process denoted by $z_t \in \mathbb{Z}$ with stationary transition probabilities $\pi(z, z')$.[4] Let the set $\mathbb{K} \subset \mathbb{R}_+$ contain all the feasible values for the aggregate endogenous state variable K, and define the product space $\mathbb{S} : \mathbb{K} \times \mathbb{Z}$. Since the household also enters each period with an individual level of the endogenous state variable k, we denote the state of a household by the vector $s = (k, S)$ with $s \in \mathbb{K} \times \mathbb{S}$. We assume that the class of equilibrium distortions are consistent with the representative agent facing a set of feasible constraints summarized by a correspondence $\Omega(k, k', S) \subset \mathbb{K} \times \mathbb{K} \times \mathbb{S}$ in which k' is the next period value of the variable k. While more specific details will be provided below, for now we can think of Ω as simply the graph of the non-empty, continuous, convex and compact valued feasible correspondence for the household $\Gamma(s) : \mathbb{K} \times \mathbb{S} \to \mathbb{K}$.

We formulate the economy as in Coleman (1991), although Greenwood and Huffman (1995) show that our problem can be posed as an existence of Markov equilibrium problem for a broad class of models used in the macroeconomic literature (e.g., models with nonconvex production sets, monetary economies like many cash-in-advance models and shopping time models, monopolistic competition models, etc.). Each household assumes that the aggregate endogenous state variable evolves according to a continuous function $K' = h(K, z)$, and owns an identical production technology which exhibits constant returns to scale in private inputs for producing the output good. Production may also depend on the equilibrium level of inputs, and by allowing the technologies to be altered by per capita aggregates, the case of production externalities is included. Production takes place in the context of perfectly competitive markets for both the output good and the factors of production.

[4]To simplify the exposition, we use a finite state space for the exogenous shocks. See Hopenhayn and Prescott (1992) for a discussion of how to handle shock processes with more general state spaces, and what additional restrictions this case places on the transition processes for the shocks.

2.1 Assumptions

In each period, households are endowed with a unit of time which they supply inelastically to competitive firms. With the capital-labor ratio denoted by k, and the per-capita counterpart of this measurement by K, we assume that the production possibilities are represented by a function $f(k, K, z)$. A household's income before taxes and transfers is exactly:[5]

$$f(K, K, z) + (k - K)f_1(K, K, z),$$

where the equilibrium condition $K = k$ has been imposed on the firm's problem. The government taxes all income at the rate $t_1(K, z)$ and transfers the lump sum amount $t_2(K, z)$ to each household. In period t, a household must decide on an amount c to consume, and the capital-labor ratio carried over to the next period is thus:

$$k' = (1 - t_1(K, z))[f(K, K, z) + (k - K)f_1(K, K, z)] + t_2(K, z) - c.$$

We make the following assumptions on the primitives data:

Assumption 1. The production function $f(k, K, z)$ and the function governing taxes and transfers $t(K, z)$ are such that:

(i). $f : \mathbb{K} \times \mathbb{K} \times \mathbb{Z} \to \mathbb{K}$ is continuous and strictly increasing. Further, it is continuously differentiable in its first two arguments, and strictly concave in its first argument.

(ii). $f(0, K, z) = 0$ and $lim_{k \to 0} f_1(0, K, z) = \infty$ for all $(K, z) \in \mathbb{K} \times \mathbb{Z}$.

(iii). t_1 and t_2 are continuous and increasing in both their arguments.

(iv). The quantity $(1 - t_1(K, z))f_1(K, K, z)$ is weakly decreasing (i.e., non-increasing) in K.

Aside from the lack of any boundedness condition on the production function f, these restrictions are standard (e.g., Coleman (1991)). Given the nonexistence of continuous Markovian equilibrium results presented recently in the work of Santos (2000) and Mirman et al. (2001), it seems that assumptions *(iii)* and *(iv)* are necessary for the existence of continuous Markovian equilibrium.[6]

For each period and state, the preferences are represented by a period utility index $u(c_i)$, where $c_i \in \mathbb{K} \subset \mathbb{R}_+$ is period i consumption. Letting $z^i = (z_1, ..., z_i)$ denote the history of the shocks until period i, a household's lifetime preferences are defined over infinite sequences indexed by dates and histories $c = (c_{z^i})$ and are given by:

$$U(\mathbf{c}) = E\left\{\sum_{i=0}^{\infty} \beta^i u(c_i)\right\}, \tag{1}$$

[5]Many economies with production externalities, monetary distortions, and monopolistic competition are equivalent to economies with taxes. Therefore, although we write the equilibrium distortions in the form of taxes, we cover a much broader set of environments used in applied work.

[6]Santos (2000) presents a counterexample which highlights the need for monotonicity of distorted returns in models models such as ours. In particular, the non-existence of continuous Markovian equilibrium relies on the non-monotonicity of the distorted return on capital.

where the summation in (1) is with respect to the probability structure of future shocks given the history of shocks, the transition matrix π, and the optimal plans up to a given date i.

Assumption 2[7]. *The period utility function* $u : \mathbb{R}_+ \to \mathbb{R}$ *is bounded, continuously differentiable, strictly increasing, strictly concave,* $0 < \beta < 1$, *and* $u^{'}(0) = \infty$.

2.2 Value Function and Equilibrium

The value function V associated with the household's problem of choosing an optimal consumption level satisfies the Bellman's equation:

$$\begin{aligned} V(k, K, z) &= \sup_{c \in \Gamma(k,K,z)} \{u(c) + \beta E_z[V((1 - t_1(K, z))[f(K, K, z) \\ &\quad +(k - K)f_1(K, K, z)] + t_2(K, z) - c, h(K, z), z')]\} \end{aligned} \quad (1)$$

where the constraint set for the household's choice of consumption is the compact interval:

$$\Gamma(k, K, z) = [0, (1 - t_1(K, z))[f(K, K, z) + (k - K)f_1(K, K, z)] + t_2(K, z)].$$

Consider the complete metric space of bounded, continuous, real-valued functions $v : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{R}$ equipped with the sup norm, and W the subset of functions that are weakly increasing and concave in their first argument. The following is a standard result in the literature (see, for instance, Stokey et al., 1989).

Proposition 1. Under Assumptions 1 and 2, given any continuous aggregate investment function h *and any transfer policy function* t_2*, there exists a unique* v *in* W *that satisfies Bellman's equation (2). Moreover, this* v *is strictly increasing and strictly concave in its first argument. The optimal policy* $c(k, K, z)$ *is single valued and continuous in its first argument.*

Note that v is generally not defined on a compact space, and that the proof of this proposition relies on applying the contraction mapping theorem for an operator defined on the Banach space $C(X)$ with the topology of the uniform convergence, without requiring X to be compact. However, the proof of existence and uniqueness of the value function in Stokey et al. (1989) relies crucially on the assumption of boundedness of utility, so that the contraction mapping theorem can be directly applied. Although there is no general theory for the case of unbounded utility and unbounded growth, some progress has been made by Alvarez and Stokey (1997) who demonstrate existence and uniqueness at

[7] The possibility of relaxing the boundedness assumption on u is discussed below.

least with CES utility and constant returns to scale in private inputs to production.[8] In Section 5 below, we provide an illustration of our method for a Romer type of model where utility is unbounded.

It is important to also note that the optimal policy $c(k, K, z)$ is strictly positive, that is $c(k, K, z) > 0$ when $k > 0$ and $K > 0$. Suppose that this is not the case, i.e., that there exists $(k, K) > 0$ such that $c(k, K, z) = 0$. Consider increasing consumption and decreasing investment by some amount $\varepsilon > 0$. The per unit increase in current utility is $[u(\varepsilon) - u(0)]/\varepsilon$, while the per unit decrease in expected future utility is $\beta E_t[v(k', K', z') - v(k' - \varepsilon, K', z')]/\varepsilon$. However:

$$\lim_{\varepsilon \to 0}[u(\varepsilon) - u(0)]/\varepsilon = u'(0) = \infty,$$

and the utility gain can therefore be made arbitrarily large by choosing ε small enough, while the utility loss are bounded since v is strictly increasing and concave in its first argument (and $k' > 0$ when $c(k, K, z) = 0$ and $K' = h(K, z) > 0$ as well). As a consequence, the policy of consuming nothing is not optimal.

We define an equilibrium as follows:

Definition: A stationary equilibrium consists of continuous functions h and t_2 mapping $\mathbb{R}_{++} * \mathbb{Z}$ into $\mathbb{R}_{++}$ such that:

(i). All tax revenues are lump-sum redistributed according to the transfer function $t_2 = t_1 f$.

(ii). The aggregate investment function h is such that households choose to invest according to the same rule:

$$h(K, z) = f(K, K, z) - c(K, K, z).$$

Proposition 2. Under Assumptions 1 and 2, if (h, t_2) is an equilibrium with the associated policy function c and value function v, then $c(K, K, z)$ always lies in the nonempty interior of $\Gamma(K, K, z)$, and v is continuously differentiable in its first argument k when $k = K$ for all (K, z).

Note again that the standard proof (See, for instance Stokey et al., 1989) relies on boundedness of utility, although the results have been extended to some cases where utility functions can be bounded above by a linear transformation of a CES utility.

Consequently, denoting $c(K, z) = c(K, K, z)$, $H(K, z) = (1 - t_1(K, z)) f_1(K, K, z)$ and $f(K, K, z) = F(K, z)$ for convenience, the optimal policy function necessarily satisfies the Euler equation:

$$u'(c(K, z)) = \beta E_z\{u'[c(F(K, z) - c(K, z), z')] * H(F(K, z) - c(K, z), z')\}, \quad (3)$$

and an equilibrium consumption is a strictly positive solution $c(K, z) > 0$ to this Euler equation.

[8] These results are extended in Miao (2001) to a larger class of primitives.

3 EXISTENCE AND CHARACTERIZATION OF EQUILIBRIUM

Since the seminal work of Arrow and Debreu, fixed point theorems have been at the core of general equilibrium analysis, especially for establishing existence of competitive equilibrium. Early work on existence appealed to topological constructions such as Brouwer's fixed point theorem, a theorem asserting that a single-valued continuous mapping from a compact convex subset of a vector space into itself has a fixed point.[9] In the context of a recursive dynamic monetary economy, Lucas and Stokey (1987) apply Schauder's fixed point theorem to establish that a nonlinear operator that maps a non-empty, closed, bounded and convex subset of continuous functions $C(X)$ defined on a compact subset X into itself has a fixed point if it is continuous and if the underlying subset is an equicontinuous set of functions. In the work of Jovanovic (1988), Bernhardt and Bergin (1992), and most recently Chakrabarti (2001), generalizations of Schauder's theorem for correspondences, the so-called Fan-Glicksberg class of fixed point theorems, are used to establish the existence of equilibrium for a class of large anonymous games and heterogeneous agent economies.

There are, however, some major impediments and limitations when attempting to apply these topological constructions to the class of unbounded growth models considered in this paper. First is the standard problem of trivial fixed points: The theorems of Schauder and Fan-Glicksberg are existential, but, unfortunately, the operators we study often contain trivial fixed points that cannot be decentralized under a price system with strongly concave households. Ruling out trivial fixed points would therefore require constructing domains of functions that exclude from consideration such trivial elements, an often intractable problem which would therefore make these theorems difficult to employ. Second, to apply this collection of theorems, the state space X has to be compact, which is obviously not the case for unbounded growth models. Third, proving topological continuity of a nonlinear operator in a particular topology when the state space is not compact often proves to be difficult task.

This leads one to consider fixed point arguments that are not topological and more specific to the problem under consideration, i.e., that exploit some additional structure of the particular problem being studied. An interesting application of a non topological fixed point theorem is the case of bounded growth models with equilibrium distortions: Coleman (1991) pioneered an application of a version of Tarski's fixed point theorem to demonstrate existence of equilibrium in an infinite horizon stochastic framework with an income tax. Tarksi's theorem establishes that a monotone operator from a complete lattice into itself has a fixed point.

Tarski's fixed point theorem (Tarski, 1955). If f is an increasing mapping

[9] In more general situations, Kakutani's fixed point theorem, Schauder's fixed point theorem, or the Fan-Glicksberg fixed point theorem are often required. All of these theorems are topological in nature, and essentially extend the result of Brouwer to the case of correspondences and infinite dimensional spaces.

of a complete lattice X into itself, then the set of fixed points is a nonempty complete lattice.

Coleman uses a version of this theorem for order continuous operator on countably chain complete partially ordered sets, in a situation where there exist two elements , l and u, that are mapped "up" and "down" , i.e. $A(l) \geq l$ and $A(u) \leq u$. With these additional hypothesis, the version of Tarski's theorem due to Dugundji and Granas (1982) can be applied, and the minimal and maximal fixed points can be constructed by successive approximations. As emphasized by Coleman, the advantage of using a version of Tarski's fixed point theorem with an explicit algorithm is to be able to rule out the zero consumption as the maximal fixed point.[10]

This method thus seems promising for our problem.[11] However, for the class of models studied in Coleman (1991), a standard restriction on the production function insures that the state space X is compact (in addition to being convex and closed), which implies, through the Arzela-Ascoli theorem, that a set of equicontinuous functions (endowed with the sup norm) defined on X is a countably chain complete lattice because it is a compact subset of a Banach lattice of continuous functions. Coleman then constructs a monotone and continuous operator from this compact subset of a Banach lattice of continuous functions into itself, and an application of a topological version of Tarski's fixed point theorem generates an algorithm that converges to the fixed point, shown to be unique and strictly positive. Unfortunately, in distorted unbounded growth models where X is not compact the strategy in Coleman (1991) cannot be directly applied because Coleman's set of equicontinuous functions cannot easily be shown to be a compact subset of a Banach lattice of continuous functions. Similarly, the absence of compactness renders the proof of continuity of the operator very difficult.

We show below that neither compactness of the state space nor continuity of the operator are needed to prove existence of equilibrium. We demonstrate that the set of functions considered in Coleman is in fact a complete lattice, whether or not the state space is compact. The key insight is that continuity of the candidate equilibrium policies need not be assumed because it is implied by a double monotonicity assumption. This key insight enables us to use purely order-based methods . We rely on a version of Tarski's theorem due to Veinott (1992) , which is a slightly stronger version of the main theorem in Abian and Brown (1962) to produce additional characterizations of the set of fixed points.

[10] Greenwood and Huffman (1995) provide a weaker set of conditions than Coleman under which the maximal fixed point is strictly positive.

[11] An alternative strategy could be to apply a fixed point theorem that combines continuity of an operator mapping an ordered Banach space into itself with order preserving monotonicity to deliver, under some conditions, a minimal and maximal fixed point (see Amann, 1976), as discussed in Datta, Mirman, Morand and Reffett (2002) However, interesting Banach spaces of functions are hard to come by for our environments, because they are vector spaces, which, by definition, rule out any explicit bounds imposed on the functions (such as resource or budget constraints).

Theorem 1.[12] *Let $(X, \geq)$ be a complete lattice $A : X \to X$ an increasing mapping. The set of fixed points of A is a non-empty complete lattice. Further, the sets of excessive and deficient point (resp. $s \geq A(s)$ and $s \leq A(s)$) are non-empty complete lattices, and the greatest (resp. least) fixed point is the greatest deficient point (resp. least excessive).*

3.1 Existence

Consider the space E of functions $h : \mathbb{X} = \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{R}$, endowed with the pointwise partial order $\leq$[13], and satisfying the following conditions (recall that $F(K, z) = f(K, K, z)$ is continuous and strictly increasing in its arguments):

(i). $0 \leq h(K, z) \leq F(K, z)$.

(ii). h is weakly increasing in K.

(iii). $F - h$ is weakly increasing in K.

Recall that a space $(E, \leq)$ is a lattice if it is endowed with two binary operations for any two points $h, g \in E$ referred to as the meet and the join (denoted respectively $h \wedge g$ and $h \vee g$). These two operations are given as follows:

$$(h \vee g)(x, y, z) = \max\{h(x, y, z), g(x, y, z)\}$$

and,

$$(h \wedge g)(x, y, , z) = \min\{h(x, y, z), g(x, y, z)\}.$$

It is important to notice that the combined double monotonicity of h and $F - h$ implies that $h \in E$ is a continuous function in K. This can be easily seen graphically, as any discontinuous jump up of h would necessarily violate the assumption that $F - h$ is weakly increasing, and is demonstrated rigorously in the proof of the following Lemma establishing an important property of E.[14]

Lemma 1. E is a complete lattice. Also, for all $h \in E$, h is continuous in K.

In the footsteps of Coleman (1991), we define the mapping A on E from the Euler equation (3) as follows:

$$\begin{aligned} u'((Ac)(K, z)) &= \beta E_z\{u'[c(F(K, z) - (Ac)(K, z), z')]4 \\ &\cdot H[F(K, z) - (Ac)(K, z), z']\} \end{aligned} \tag{2}$$

so that any fixed point of A is an equilibrium consumption function. Note that 0 is a fixed point of A (i.e., $A0 = 0$), and that $AF \leq F$.

Lemma 2. Under Assumption 1-2, for any c in E, a unique $A(c)$ in E exists. Furthermore, the operator A is monotone.

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[12] For a proof, see Veinott, 1992 Theorem 11.

[13] The pointwise partial order $\leq$ defined as $h \leq g$ if $g(x', y', z') \geq h(x, y, z)$ for all $(x', y', z') \geq (x, y, z)$ in $\mathbb{X}$, where the last inequality holds componentwise.

[14] Proofs are in the Appendix.

Then, by Lemma 1-3, the operator A maps a complete lattice into itself and is monotone. By Tarski's theorem, the set of fixed point is a nonempty complete lattice. Note that F is a deficient point ($AF \leq F$) and 0 is an excessive point ($A0 \geq 0$) so the hypothesis of Theorem 1 above are satisfied. The following proposition, which follows directly from Theorem 1 above, states the existence result of this paper.

Proposition 1. Under Assumption 1-2, the set of fixed points is nonempty, and there exist greatest and least fixed points.

3.2 Maximal Fixed Point

As noted above, order based fixed point theorems are generally more than existential and can provide additional characterization of the set of fixed point. In our problem, we exploit the order structure to establish a computational procedure that converges to the maximal fixed point of the operator A, as stated in the following proposition.

Proposition 2. The sequence $\{A^n F\}_{n=0}^{\infty}$ converges to the maximal fixed point.

Denote $\widetilde{c}$ the maximal fixed point, and consider the sequence of value functions $\{\widehat{v}_n\}_{n=0}^{\infty}$ generated from the following recursion:

$$\begin{aligned} \widehat{v}_n(k, K, z) &= \sup_{c \in \Gamma(k,K,z)} \{u(c) + \\ &\quad \beta E_z[\widehat{v}_{n-1}(f(k, K, z) - c, F(K, z) - A^{n-1}F(K, z), z')]\}, \end{aligned}$$

and with $\widehat{v}_0 \equiv 0$. Our strategy is to demonstrate that the sequence $\widehat{v}_n$ converges to the solution v of Bellman's equation associated with the household's maximization problem. If $\{\widehat{v}_n\}_{n=0}^{\infty}$ converges to v, since by construction the optimal policy function maximizing the right side of the previous equality, evaluated along the equilibrium path, is exactly $A^{n-1}F(K, z)$, then by Theorem 9.9 in Stokey et al. (1989), the sequence of functions $A^{n-1}F(K, z)$ converges pointwise to the optimal policy associated with v, which we have demonstrated must be strictly positive in Section 2. We now show that the above stated convergence is true.

First, notice that the sequence $\{\widehat{v}_n\}_{n=0}^{\infty}$ is convergent. To demonstrate this property, define the operator T_n as follows:

$$\begin{aligned} (T_{n-1}\widehat{v}_{n-1})(k, K, z) &= \sup_{c \in \Gamma(k,K,z)} \{u(c) + \\ &\quad \beta E_z[\widehat{v}_{n-1}(f(k, K, z) - c, F(K, z) - A^{n-1}F(K, z), z')]\}, \end{aligned}$$

for $n \geq 1$, and $\widehat{v}_n = T_{n-1}\widehat{v}_{n-1}$ Obviously, each T_j is a contraction of modulus $\beta < 1$ so that the sequence $T_n \circ T_{n-1} \circ ... \circ T_0(v_0)$ is a Cauchy sequence, and therefore converges to a unique limit.

Second, applying the same argument as in Greenwood and Huffman (1995) establishes that the sequence $\{\widehat{v}_n\}_{n=0}^{\infty}$ converges to v on any compact subset of the state space. Together these two results imply that $\lim_{n\to\infty}\{\widehat{v}_n\}_{n=0}^{\infty} = v$.

3.3 Comparative Statics Results

The monotonicity of the mapping A can be exploited to derive comparative statics results in some of the deep parameters of the problem (i.e., parameters of the preferences and of technology). Recall that the set of equilibria is a non-empty complete lattice, so, in the absence of uniqueness result, comparative statics analysis requires defining orders on both the set of parameters considered and on the set of equilibria.

We define the following two set orders. Consider a set Y, and two subsets $A, B \in P(Y)$. The strong set order $\geq_a$ is defined on $P(Y)$ as follows:

$$A \geq_a B \text{ iff for any } a \in A \text{ and } b \in B,\ a \wedge b \in B \text{ and } a \vee b \in B.$$

Then, in Veinott's (1992) terminology, we show that the set of equilibria is ascending in the strong set order in a parameter $t \in T$, and consequently, that the minimal and maximal fixed points also increase in this parameter.

Theorem 2. Suppose that the assumptions of Theorem 1 are satisfied for each mapping A_t belonging to the set $\{A_t : X \to X,\ t \in T\}$, where $(T, \geq_T)$ is a poset. If A_t is increasing in t, that is if $t' \geq_T t$ implies that, for all x in X, $A_{t'}x \geq A_t x$, then the minimal and maximal fixed points of A_t are increasing in t.

For an application of this result, consider a perturbation in the discount rate β. Since the right side of equation (4), which defines implicitly the mapping A, is increasing in β, as a consequence, $A_{t=\beta}(c)$ is increasing in $\beta \in]-1, 0[= T$, where T is endowed with the dual order $\geq_T$ on the real line (i.e., $\beta^{'} \geq_T \beta$ if $\beta^{'} \leq \beta$). By Theorem 2, the maximal and minimal fixed points increase with t (i.e., decrease with β). For another application, consider the tax rate $t \in T$, where T is the set of continuous functions $t(K, z) \in [0, 1]$ that are monotone in K.[15] Endow T with the standard pointwise Euclidean order for a space of functions, i.e., $t' \geq_T t$ if $t'(K, z) \geq t(K, z)$ for all (K, z). Then $A_{t'}c \geq A_t c$ in the order defined on E and the equilibrium set (the set of fixed points of the operator A_t) is monotone in t the strong set order.

4 UNIQUENESS OF EQUILIBRIUM

This section establishes uniqueness of equilibrium under fairly standard assumptions by following a method similar to the one in Coleman (2000) and Datta et al. (2002). As discussed in Coleman (1991), in the presence of uncertainty the concavity of an operator is not sufficient for proving uniqueness of the fixed point

[15] And also consistent with Assumption 1.

of the operator. However, pseudo concavity together with some monotonicity property are sufficient properties to establish that result. In this section of the paper, we demonstrate that any fixed point of the operator A is also a fixed point of another operator $\widehat{A}$ which is shown to be pseudo concave and x_0-monotone. This method, also used in Coleman (2000) and Datta et al. (2002), is based on a theorem in Coleman (1991), which we generalize for a non-compact state space.

Theorem 3.[16] *An operator $\widehat{A}$ that is pseudo concave and x_0-monotone has at most one strictly positive fixed point.*

The operator $\widehat{A}$ is constructed as follows. First define the set of functions $m : \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{R}$ such that:
(i). m is continuous;
(ii). For all $(K, z) \in \mathbb{R}_+ \times \mathbb{Z}$, $0 \leq m(K, z) \leq F(K, z)$;
(iii). For any $K = 0$, $m(K, z) = 0$.
Denote M this set, which is endowed with the standard pointwise partial ordering. Consider the function $\Psi(m(K, z))$ implicitly defined by:

$$u'[\Psi(m(K,z))] = \frac{1}{m(K,z)} \text{ for } m > 0, \text{ 0 elsewhere.}$$

Naturally, Ψ is continuous, increasing, $\lim_{m \to 0} \Psi(m) = 0$, and $\lim_{m \to F(K,z)} \Psi(m) = F(K, z)$. Using the function Ψ, we denote:

$$\widehat{Z}(m, \widetilde{m}, K, z) = \frac{1}{\widetilde{m}} - \beta E_z \left\{ \frac{H(F(K,z) - \Psi(\widetilde{m}(K,z)), z')}{m(F(K,z) - \Psi(\widetilde{m}(K,z)), z')} \right\},$$

and consider the operator $\widehat{A}$:

$$\widehat{A}m = \{\widetilde{m} \mid \widehat{Z}(m, \widetilde{m}, K, z) = 0 \text{ for } m > 0, \text{ 0 elsewhere}\}.$$

Since $\widehat{Z}$ is strictly increasing in m and strictly decreasing in $\widetilde{m}$, and since $\lim_{\widetilde{m} \to 0} \widehat{Z} = +\infty$ and $\lim_{\widetilde{m} \to F(K,z)} \widehat{Z} = -\infty$, for each $m(K, z) > 0$,with $K > 0$, and $z \in Z$ there exists a unique $\widehat{A}m(K, z)$.

Note that we can relate each orbit of the operator A to a specific orbit of the operator $\widehat{A}$ in the following manner. Given any c_0 in the order interval $[0, F]$ of E, there exists a unique m_0 in M such that:

$$m_0(K,z) = \frac{1}{u'(c_0(K,z))}.$$

By construction, there exists a unique $\widehat{A}m_0$ that satisfies $\widehat{Z}(m_0, \widehat{A}m_0, K, z) = 0$, that is:

$$\frac{1}{\widehat{A}m_0(K,z)} = \beta E_z \left\{ \frac{H(F(K,z) - \Psi(\widehat{A}m_0(K,z)), z')}{m_0(F(K,z) - \Psi(\widehat{A}m_0(K,z)), z')} \right\}$$

[16] The proof of Coleman (1991) is sligthly amended in the Appendix to address the case of a non-compact state space.

or, equivalently (from the definition of c_0):

$$\begin{aligned} \frac{1}{\widehat{A}m_0(K,z)} &= \beta E_z\{H(F(K,z)-\Psi(\widehat{A}m_0(K,z)),z') \\ &\quad \cdot u'(c_0(F(K,z)-\Psi(\widehat{A}m_0(K,z)),z'))\}. \end{aligned}$$

By construction, Ac_0 satisfies:

$$\begin{aligned} u'((Ac_0)(K,z)) &= \beta E_z\{H(F(K,z)-Ac_0(K,z),z') \\ &\quad \cdot u'(c_0(F(K,z)-Ac_0(K,z),z'))\}. \end{aligned}$$

Therefore, by uniqueness of $\widehat{A}m_0$ it must be that $1/\widehat{A}m_0 = u'(Ac_0)$ (or, equivalently, that $\Psi(\widehat{A}m_0) = Ac_0)$). By induction, it is trivial to demonstrate that for all $n = 1, 2, ...$ $A^n c_0 = \Psi(\widehat{A}^n m_0)$.

It is easy to show that to each fixed point of the operator A corresponds a fixed point of the operator $\widehat{A}$. Indeed, consider x such that $Ax = x$ and define $y = 1/u'(x)$ (or, equivalently $\Psi(y) = x$). By definition, x satisfies:

$$\begin{aligned} u'(x(K,z)) &= \beta E_z\{H(F(K,z)-x(K,z),z') \\ &\quad \cdot u'(x(F(K,z)-x(k,z),z'))\} \text{ for all } (K,z). \end{aligned}$$

Substituting the definition of y into this expression, this implies that:

$$\frac{1}{y} = \beta E_z \frac{H(F(K,z)-\Psi(y(K,z)),z')}{y(F(K,z)-\Psi(y(k,z),z'))},$$

which shows that y is a fixed point of $\widehat{A}$. We have the important following result:

Lemma 3. The operator $\widehat{A}$ is pseudo concave and x_0-monotone.

This Lemma, in conjunction Theorem 3, stated in the beginning of this section, implies that $\widehat{A}$ has at most one fixed point. Thus, A also has at most one fixed point, although at least one (obtained as $\lim_{n\to\infty} A^n F$). Therefore, A has exactly one fixed point.

5 EXAMPLES AND CONCLUDING REMARKS

This paper provides an important extension of the work of Coleman (1991, 2000) and Greenwood and Huffman (1995) to the case of unbounded growth. Such an extension is important, as many models studied in the applied growth and macroeconomics literature are formulated on unbounded state spaces. It is not trivial, since all the standard fixed point results used in the literature do not apply to our problem, because relaxing the assumption of a compact state space makes it very difficult to establish suitable algebraic or topological structures on particular spaces of functions. However, the mapping corresponding to the recursive problem, expressed in the form of iterations on an operator defined via an Euler equation, maps a complete lattice into itself and has critical monotone properties. Exploiting these order-theoretic notions, we prove existence of equilibrium by applying Tarski's theorem. To prove uniqueness, we demonstrate that the fixed points of this mapping are also fixed points of another mapping, which we know has at most one fixed point. Finally, we show that the unique equilibrium can be obtained as the limit of a simple algorithm, and we discuss some issues related to comparative statics. In particular, we show how to conduct some simple comparative statics on the space of economies considered in the paper.

We now briefly demonstrate how to apply the results to some standard frameworks used extensively in the macroeconomic literature. We begin with the case of unbounded growth with nonconvex technologies.

Example 1. Endogenous growth with constant income tax and unbounded utility. Consider the simple growth economy where a representative agent's preferences are represented by $\sum_{t=0}^{\infty} \beta^t u(c_t)$, the production technology is $zk^{\alpha}K^{1-\alpha}$ with $0 < \alpha < 1$, and income is taxed at the constant rate τ. Define $\Psi(k, K, z) = [0, (1-\tau)(zK^{\alpha}K^{1-\alpha} + (k-K)\alpha z K^{\alpha-1}K^{1-\alpha} + t_2(K, z)]$. Existence and uniqueness of a value function satisfying Bellman's equation:

$$\begin{aligned} v(k_t, K_t, z_t) &= \sup_{k_{t+1} \in \Psi(k_t, K_t, z_t)} \{u[(1-\tau)(z_t K_t^{\alpha} K_t^{1-\alpha} + (k_t - K_t)\alpha z_t K_t^{\alpha-1} K_t^{1-\alpha}) + t_2(K_t, z_t) - k_{t+1}] \\ &\quad + \beta E_t(v(k_{t+1}, h(K_t, z_t), z_{t+1}))\} \end{aligned}$$

can be established when $u(c)$ satisfies Assumption 2, following a standard argument in Stokey et al. (1989). Note that Assumption 1 is satisfied: In particular the after tax marginal product of capital , $(1 - t_1(K, z))f_1(K, K, z) = (1-\tau)\alpha z$ is independent of K.

For any c in E, define Ac as the solution of the following nonlinear equation:

$$u'(Ac(k, z)) = \beta E_z\{u'(c(zK - Ac(K, z), z')) * (1-\tau)\alpha z'\}.$$

A direct application of our results shows that the unique fixed point can be computed as the limit of the sequence $(F, AF, A^2F, ...)$ and that it is decreasing in β.

It is important to note that, in some cases, it is possible to relax the assumption of bounded utility. In particular, when utility is of the form $u(c) = (1/\theta)c^{\theta}$, where $0 < \theta < 1$, the existence and uniqueness of a value function can be established, as demonstrated in the Appendix. The case $\theta = 0$, that is, $u(c) = \ln c$, addressed in Greenwood and Huffman (1995), is facilitated by the property that it generates a setup that has a stationary representation. As noted before, there is no general theory that guarantees existence of a value function, let alone existence of an Euler equation, when an unbounded return function is combined with a unbounded state space. Additional extensions of this example should be easily obtain using the recent results of LeVan and Morhaim (2002).

Example 2. Government spending and endogenous growth. As a second example, consider a discrete time version of Barro (1990) where preferences of a typical agent are represented by $\sum_{t=0}^{\infty} \beta^t u(c_t)$, the production technology is $K^{1-\alpha} g^{\alpha}$, where government spending g is financed contemporaneously by a flat-rate income tax τ. If we asume a constant tax rate τ and a balanced government budget each period this setup is simply an AK model, as pointed out by Barro, since balancing the budget requires $g_t = \tau K_t^{1-\alpha} g_t^{\alpha}$, that is, $g_t = BK_t$ where $B = (\tau)^{1/(1-\alpha)}$. Consequently, disposable income of an agent in period t is $(1-\tau)[B^{\alpha} K_t + (k_t - K_t) B^{\alpha}(1-\alpha)]$.

With utility of the form $u(c) = (1/\theta)c^{\theta}$, where $0 < \theta < 1$, the existence and uniqueness of a value function can be established following a similar argument as for Example 1 above (See Appendix). The Bellman's equation associated with an agent's maximization problem is:

$$v(k_t, K_t) = \sup_{k_{t+1} \in \Psi(k_t, K_t)} \{u[(1-\tau)(B^{\alpha} K_t + (k_t - K_t) B^{\alpha}(1-\alpha)) - k_{t+1}] + \beta v(k_{t+1}, h(K_t))\}$$

where $\Psi(k_t, K_t) = [0, (1-\tau)(B^{\alpha} K_t + (k_t - K_t) B^{\alpha}(1-\alpha))]$. The Euler equation associated with the previous Bellman's equation generates the operator A defined implicitly by:

$$u'(Ac(K)) = \beta u'(c((1-\tau)B^{\alpha} K - Ac(K))) * (1-\tau)(1-\alpha) B^{\alpha}$$

Example 3. Cash in Advance Economy. As a third example, we show how to use our methodology to study an unbounded monetary economy. In this case, we assume for convenience that utility is bounded, although (given the above examples) we could consider a class of unbounded return functions often used in the literature. Our example is basically a cash-in-advance economy with exogenous financial constraints, but the method can be adapted to endogenous cash in advance models such as Lacker and Scheft (1996), to shopping time models, and also to the recent models such as Alvarez et al. (2000).

Consider a deterministic version of the economy described in Stockman (1981), for which the primitive data satisfy the unbounded growth assumptions in Jones and Manuelli (1990).[17] That is, for simplicity f is deterministic, and sufficiently productive relative to a "inflation tax" t_1. The primitive data satisfy Assumption 1, 2 and:

Assumption 1'. $f(K)$ is such that $f(K) = AK + f(K)$ where $f(K)$ satisfies the following limiting conditions $\lim_{K \to \infty} f'(K) = 0$, *and A satisfies $\beta(A(1 - t_1(K) + 1 - \delta) > 1$.*

[17] A version of this economy for bounded growth using the equivalence between monetary economies and economies with state contingent taxes is studied in Coleman (1992). One can generalize the results in this application to endogenous cash-credit models with elastic labor supply following the construction in Datta et al. (2002), using the main results in our present paper within an economy with unbounded growth. Finally, as in Datta, Mirman, and Reffett, the underlying fixed point argument takes place on an closed equicontinuous subset of continuous functions in the uniform topology (which is a complete lattice), and arguments of the main theorem of the paper can be generalized to problems with elastic labor supply.

We will specify how the mapping t_1 is constructed in a monetary economy in a moment. The state of the aggregate economy is $S = K$ and a monetary agent introduces money into the economy according to a state dependent technology $j(K)$ so that:

$$M' = j(K)M$$

where j is continuous and M is the per capita level of the money stock. Let $l(K)$ be a lump sum monetary transfer, and therefore $M' = M + l(K)$. Assume that the (normalized) price level is given by a continuous mapping $p(K) = P(K)/M$ and the recursive structure for per capita measurement of capital follows:

$$K' = h(K).$$

A household enters each period with an individual stock of fiat money m and a capital stock k. It can be shown that the representative consumer solves the following functional equation:

$$v(k, m, K) = \sup_{c,k',m' \in \Psi(k,m,K)} u(c) + \beta v(k', m', K')$$

where $\Psi(k, m, S)$ is the household feasible correspondence and consists of the set of (c, k, m) satisfying the following restrictions:

$$p(K)(c + k') + m'/M \leq p(K)[f(K) + (k - K)f'(K)] + [m + l(K)]/M, \quad (5)$$

$$p(K)(c + k') \leq [m + l(K)]/M, \quad (6)$$

$$c, m', k' \geq 0.$$

Notice that, given our assumptions, $\Psi(k, m, K)$ is well-behaved (it is a continuous, compact and convex valued non-empty correspondence that admits measurable selections). In addition, a standard argument shows that there exists a bounded, continuous, strictly concave, strictly increasing value function $v(k, m, K)$ satisfying the household's Bellman's equation, and that v admits an envelope condition for both k and m for fixed S.

Letting λ and ϕ be the multipliers on (5) and (6) respectively. The following Euler equations are obtained by substituting the envelope conditions into the original first-order conditions, and also recalling the feasibility conditions:

$$u'(c) \leq p(\lambda + \phi) \text{ with equality if } m > 0, \quad (7)$$

$$p(\lambda + \phi) = \beta u'(c')f'(K'), \quad (8)$$

$$\lambda = \beta \frac{(\lambda' + \phi')}{j(K')}. \quad (9)$$

Define $\lambda(S) = \lambda(k, m, K)$ and $\phi(K) = \phi(k, m, K)$ by imposing the equilibrium conditions $k = K$ and $m = M$. Define also the following tax:

$$t_1(K) = \frac{\phi}{\lambda + \phi}.$$

Then, assuming that $\frac{\beta}{j} < 1$, it can be shown that (7) binds, and we can rewrite (7)-(9) as:

$$u'(c) = \beta u'(c')(1 - t_1(K'))f'(K') \tag{10}$$

and,

$$1 - t_1(K) = \beta \frac{u'(c')}{u'(c)j(K')}. \tag{11}$$

This economy grows in equilibrium asymptotically under Assumption 1, 1', and 2, as it does in the models discussed in Jones and Manuelli (1990). We can then define an operator in Section 3 as in the tax economy, apply the main results of the paper to solve (10) for a tax equilibrium for a fixed t_1, and then use this equilibrium to determine the class of monetary economies indexed by $j(K)$ such that that tax economy associated with t_1 can be written as a monetary economy associated with a side condition like (11). In addition, if we place the distortions t_1 in a partially ordered set T with the pointwise Euclidean partial order on a space of functions, we can derive comparative statics for the set of fixed points of the nonlinear fixed point operator as in Section 3, and in addition prove the existence of unique Markov equilibrium.

6 APPENDIX: PROOFS

Proof of Lemma 1. Recall that a lattice E is complete if any subset G of E is such that G has a sup and an inf. Consider any family G of elements of E. Clearly (i) $0 \leq \sup G \leq F$, (ii) $\sup G$ is weakly increasing, and (iii) $F - \sup G = \inf\{F - g\}_{g \in G}$ is also weakly increasing. A similar argument applies for $\inf G$. Thus E is a complete lattice.

By the double monotonicity assumption (ii) and (iii), for all $h \in E$, and for all $s' \geq s$:

$$0 \leq h(s') - h(s) \leq F(s') - F(s).$$

Therefore, since F is continuous on its domain,

$$\forall \varepsilon > 0, \exists \delta > 0, \mid s - s_0 \mid < \delta \text{ implies } \mid h(s) - h(s_0) \mid \leq \mid F(s) - F(s_0) \mid < \varepsilon.$$

Proof of Lemma 2. The proof that Ac exists, is unique, weakly increasing, and that $F - Ac$ is weakly increasing follows the construction in Coleman (1991), as does the monotonicity of A. First, recall that Ac is defined as:

$$u'[Ac(K,z)] = \beta E_z\{u'[c(F(K,z) - Ac(K,z))] * H(F(K,z) - Ac(K,z))\}.$$

While the LHS is strictly decreasing in $Ac(K,z)$ (from ∞ to a finite quantity), the RHS is strictly increasing in $Ac(K,z)$ under Assumption 1(ii) and (iv) and 2 (from a finite quantity to ∞). Thus for each (K,z), $Ac(K,z)$ exists and is unique.

Second, considering c_1 and c_2 such that $c_1 \leq c_2$, we have:

$$u'[Ac_2(K,z)] = \beta E_z\{u'[c_2(F(K,z) - Ac_2(K,z))] * H(F(K,z) - Ac_2(K,z))\}$$

and,

$$u'[Ac_2(K,z)] \leq \beta E_z\{u'[c_1(F(K,z) - Ac_2(K,z))]H(F(K,z) - Ac_2(K,z))\}. \quad (3)$$

Assume that $Ac_1 \geq Ac_2$. Then, $Ac_1(K,z) \geq Ac_2(K,z)$ and $F(K,z) - Ac_2(K,z) \geq F(K,z) - Ac_1(K,z)$. Because c_1 is increasing:

$$c_1(F(K,z) - Ac_1(K,z)) \leq c_1(F(K,z) - Ac_2(K,z)).$$

With both u' and H decreasing functions, the previous inequality implies that:

$$\begin{aligned} u'[Ac_1(K,z)] = \beta E_z\{u'[c_1(F(K,z) - Ac_1(K,z))]H(F(K,z) - Ac_1(K,z))\} \\ \geq \\ \beta E_z\{u'[c_1(F(K,z) - Ac_2(K,z))]H(F(K,z) - Ac_2(K,z))\}. \end{aligned} \quad (4)$$

Combining (2) and (3) leads to:

$$u'[Ac_1(K,z)] \geq u'[Ac_2(K,z)],$$

which contradicts the hypothesis that $Ac_1 \geq Ac_2$. It must therefore be that $Ac_1 \leq Ac_2$, that is, A is a monotone operator.

Proof of Proposition 2. For any s in X, the sequence $A^n F(s)$ is decreasing and bounded, and therefore converges. Denoting $\widetilde{c}(s)$ the pointwise limit, necessarily, $\widetilde{c} = \inf\{A^n F\}_{n \in N}$. Thus $\widetilde{c}$ belongs to E (as E is a complete lattice), which implies by Lemma 1 that $\widetilde{c}$ is continuous. It remains to show that $A\widetilde{c} = \widetilde{c}$.

Pick any $K = (x, y)$ in $\mathbb{R}_+ * \mathbb{R}_+$ and consider $s = (K, z)$. Assume, without loss of generality, that $x \geq y$. The sequence $\{c_{n+1}\}_{n=0}^{\infty} = \{Ac_n\}_{n=0}^{\infty}$ converges to $\widetilde{c}$ pointwise, so that:

for all z in Z, $F(s) - Ac_n(s)$ converges to $F(s) - \widetilde{c}(s)$

and, since H is continuous:

for all z in Z, $H(F(s) - Ac_n(s))$ converges to $H(F(s) - \widetilde{c}(s))$.

Since $\widetilde{c}$ is the pointwise limit, we know that the convergence of sequence $\{c_n\}_{n=0}^{\infty}$ toward $\widetilde{c}$ is uniform on the compact space $Y = [0, F(x, x, z^{\max})] * [0, F(x, x, z^{\max})] * Z$. Consequently,

for all z in Z, $c_n(F(s) - Ac_n(s))$ converges to $\widetilde{c}(F(s) - \widetilde{c}(s))$.

Note that the uniform convergence toward $\widetilde{c}$ is essential in establishing this result. Indeed, for all z:

$$|c_n(F(s) - Ac_n(s)) - \widetilde{c}(F(s) - \widetilde{c}(s))|$$

$$\leq$$

$$|c_n(F(s) - Ac_n(s)) - \widetilde{c}(F(s) - Ac_n(s))|$$

$$+$$

$$|\widetilde{c}(F(s) - Ac_n(s)) - \widetilde{c}(F(s) - \widetilde{c}(s))|\,.$$

The first absolute value on the right side of the inequality above is bounded above by $\sup|c_n - \widetilde{c}|$ on the compact Y, which can be made arbitrarily small because of the uniform convergence on the compact Y. The second absolute value can be made arbitrarily small by equicontinuity of $\widetilde{c}$.

Then, by continuity of u':

for all z in Z, $u'[c_n(F(s) - Ac_n(s))]$ converges to $u'[\widetilde{c}(F(s) - \widetilde{c}(s))]$.

Thus,

$$\beta E_z\{u'[c_n(F(s) - Ac_n(s))]H(F(s) - Ac_n(s))\}$$

converges to:

$$\beta E_z\{u'[\widetilde{c}(F(s) - \widetilde{c}(s))]H(F(s) - \widetilde{c}(s))\}.$$

The former term is exactly $u'(Ac_n(s))$, which we know converges to $u'(\widetilde{c}(s))$. By uniqueness of the limit:

$$u'(\widetilde{c}(s)) = \beta u'[\widetilde{c}(F(s) - \widetilde{c}(s))]\widetilde{c}(F(s) - \widetilde{c}(s)),$$

which demonstrates that, for all s, $A\widetilde{c}(s) = \widetilde{c}(s)$.

Proof of Theorem 2. Suppose $t' \geq_T t$. Consider $s^{t'}$ the minimal fixed point of $A_{t'}$. Because A_t is increasing in t:

$$s^{t'} = A_{t'}s^{t'} \geq A_t s^{t'}.$$

That is, $s^{t'}$ is an excessive point of A_t, which implies that $s^{t'} \geq s^t$, where s^t is the minimal fixed point of A_t, since the minimal fixed point is the least excessive point of by Theorem 1. Similarly, the maximal fixed point of A_t denoted x^t satisfies:

$$x^t = A_t x^t \leq A_{t'} x^t.$$

That is, x^t is a deficient point of $A_{t'}$. By Theorem 1, necessarily $x^t \leq x^{t'}$, where $x^{t'}$ is the maximal fixed point of $A_{t'}$ since the maximal fixed point is the greatest deficient point.

Proof of Theorem 3. Suppose that $\widehat{A}$ has two distinct strictly positive fixed points, which we denote c_1 and c_2. Assume without loss of generality that there exists $(\widehat{k}, \widehat{z})$ with $\widehat{k} > 0$ such that $c_1(\widehat{k}, \widehat{z}) < c_2(\widehat{k}, \widehat{z})$. Choose $0 < k_1 \leq \widehat{k}$ and $0 < t < 1$ such that:

$$c_1(k, z) \geq tc_2(k, z) \text{ for all } k_1 \leq k \leq \sup(\widehat{k}, 2k_1), \text{ all } z, \tag{i}$$

with equality for some (k, z). Note that such t exists because the interval $[k_1, \sup(\widehat{k}, 2k_1)]$ is compact.[18] Combining the x_0-monotonicity of $\widehat{A}$ and (i) implies:

$$c_1(k, z) \geq (\widehat{A}tc_2)(k, z) \text{ for all } k_1 \leq k \leq \sup(\widehat{k}, 2k_1), \text{ all } z.$$

We therefore have that, for all z and for all $k_1 \leq k \leq \sup(\widehat{k}, 2k_1)$:

$$c_1(k, z) \geq \widehat{A}tc_2(k, z) > t\widehat{A}c_2(k, z) = tc_2(k, z),$$

in which the strict inequality, which follows from the hypothesis of pseudo concavity of $\widehat{A}$, contradicts (i). Therefore, there is at most one fixed point.

Proof of Lemma 3. Recall that $\widehat{A}$ is pseudo concave if, for any strictly positive m and any $0 < t < 1$, $\widehat{A}tm(K, z) > t\widehat{A}m(K, z)$ for all $K > 0$ and for all $z \in Z$. Since $\widehat{Z}$ is strictly decreasing in its second argument, a sufficient condition for this to be true is that:

$$\widehat{Z}(tm, t\widehat{A}m, K, z) > \widehat{Z}(tm, \widehat{A}tm, K, z) = 0. \tag{ii}$$

By definition:

$$\widehat{Z}(tm, t\widehat{A}m, K, z) = \frac{1}{t\widehat{A}m} - \beta E_z \left\{ \frac{H(F(K, z) - \Psi(t\widehat{A}m(K, z)), z')}{tm(F(K, z) - \Psi(t\widehat{A}m(K, z)), z')} \right\}$$

so that:

$$t\widehat{Z}(tm, t\widehat{A}m, K, z) = \frac{1}{\widehat{A}m} - \beta E_z \left\{ \frac{H(F(K, z) - \Psi(t\widehat{A}m(K, z)), z')}{m(F(K, z) - \Psi(t\widehat{A}m(K, z)), z')} \right\}.$$

Since Ψ is increasing and $H(K', z')/m(K', z')$ is decreasing in K':

$$\frac{1}{\widehat{A}m} - \beta E_z \left\{ \frac{H(F(K, z) - \Psi(t\widehat{A}m(K, z)), z')]}{m(F(K, z) - \Psi(t\widehat{A}m(K, z)), z')]} \right\}$$

$$>$$

$$\frac{1}{\widehat{A}m} - \beta E_z \left\{ \frac{H(F(K, z) - \Psi(\widehat{A}m(K, z)), z')}{m(F(K, z) - \Psi(\widehat{A}m(K, z)), z')} \right\} = 0$$

and $\widehat{Z}(tm, t\widehat{A}m, K, z) > 0$, so that condition (ii) obtains.

The condition that $\lim_{k \to 0} f_1(k, K, z) = \infty$ for all $K > 0$, all z in Assumption 1(ii) implies that $H(0, z') = \infty$ for all z'. Given that $\widehat{A}$ is monotone, this latter condition is sufficient for the operator $\widehat{A}$ to be x_0-monotone (Lemma 9 and 10 in Coleman (1991).

[18] In Coleman, the existence of t such that $c_1 \geq tc_2$ is guaranteed because the strictly positive consumption functions are compared on the compact set $[k_1, \overline{k}] \times Z$, where $\overline{k}$ is the maximum maintainable capital-labor ratio.

Example 1. We prove existence and uniqueness of the value function under the assumption that $u(c) = (1/\theta)c^\theta$ with $0 < \theta < 1$ in the simple case with no uncertainty so we set $(1-\tau)z_t = A$ for all t. We closely follow Alvarez and Stokey (1997), adapting their proof to allow for an externality in the production function. The only difficulty is to establish that the function:

$$\begin{aligned} v(k_0, K_0) &= \sup_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(AK_t^\alpha K_t^{1-\alpha} + (k_t - K_t)\alpha A K_t^{\alpha-1} K_t^{1-\alpha} - k_{t+1}) \\ \text{s.t. } k_{t+1} &\in [0, AK_t^\alpha K_t^{1-\alpha} + (k_t - K_t)\alpha A K_t^{\alpha-1} K_t^{1-\alpha}], \end{aligned}$$

given (k_0, K_0) and $K_{t+1} = h(K_t)$ is bounded above, which we show is true under the assumption that $\beta(1+\alpha)^\theta A^\theta < 1$. Indeed:

$$\begin{aligned} u(AK_t^\alpha K_t^{1-\alpha} + (k_t - K_t)\alpha A K_t^{\alpha-1} K_t^{1-\alpha} - k_{t+1}) &\leq u(AK_t + \alpha A k_t) \\ &\leq A^\theta (1+\alpha)^\theta (1/\theta)(\sup(k_t, K_t))^\theta \end{aligned}$$

and,

$$\sup(k_t, K_t) \leq \sup(A(1+\alpha)\sup(k_{t-1}, K_{t-1}), AK_{t-1})$$

since $K_{t+1} = h(K_t) \leq AK_t^\alpha K_t^{1-\alpha}$. Thus,

$$\sup(k_t, K_t) \leq A(1+\alpha)\sup(k_{t-1}, K_{t-1}).$$

Consequently, if $\beta A^\theta (1+\alpha)^\theta < 1$,

$$0 \leq v(k_0, K_0) \leq (1/\theta)[1/(1-\beta A^\theta (1+\alpha)^\theta)](\sup(k_0, K_0))^\theta.$$

Notice that:

$$\lim_{t\to\infty} \beta^t |v(k_t, K_t)| \leq \lim_{t\to\infty} \beta^t |\sup(k_t, K_t)|^\theta \|v\| \leq \lim_{t\to\infty} \beta^t A^\theta (1+\alpha)^\theta |\sup(k_0, K_0)| \|v\| = 0,$$

so v satisfies Bellman's equation (See Chapter 4 in Stokey et al., 1989). The rest of the proof follows exactly Alvarez and Stokey (1997).

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