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Price versus Quantity Discrimination in Optimal IPOs

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Price versus Quantity Discrimination in Optimal IPOs.*

Moez Bennouri† **Sonia Falconieri**‡

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Abstract

This paper addresses the issue of the choice of the optimal instrument to sell new shares, this choice being *price* versus *quantity discrimination* (rationing). Previous results in the literature (Benveniste and Wilhelm, 1990) show that the issuing firm would be better off if allowed to use both price and quantity discrimination. This is not consistent with what we observe in practice.

Using a mechanism design approach, we derive endogenously the optimal IPO mechanism and show that it can be implemented through a uniform price across institutional investors and a *uniform* rationing, when appropriate.

Keywords: Initial Public Offering, Price Discrimination, Rationing, Optimal Auction.

JEL Classification: D8, G2.

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1 Introduction

The wide literature on Initial Public Offerings (IPOs) of new issues has so far mainly focused on the explanation of some apparent market pathologies such as underpricing, oversubscription and long-term underperformances (Rock, 1986, Allen and Faulhaber, 1989, Benveniste and Spindt, 1989, Cornelli and Goldreich, 1998). Relatively few papers have focused on the optimal role of price and quantity discrimination in IPOs. The only paper that does so (Benveniste and Wilhelm, 1990) concludes that both price and quantity discrimination should be applied to achieve an optimal performance[1]. This is however inconsistent to what we observe in practice. Typically, IPOs use quantity discrimination (*rationing*), which then results in a conflict between theory and practice. In this paper, we address the issue of the choice of the optimal instrument to perform an IPO, this choice being price versus quantity discrimination. Using an optimal mechanism design approach, we derive endogenously the optimal IPO mechanism and show that it is indeed one that solely employs quantity discrimination. Thus, we provide the first theoretical model to support the existing practice of IPOs.

IPOs are characterized by the many informational asymmetries arising among the involved agents. The players usually participating in an IPO are a) the issuing firm; b) the intermediary/ries managing the sale; c) the institutional investors which are regularly involved in such market operations and, d) a number of small and occasional investors usually called the *retail investors.* The relevant information is spread among all the players: the issuing firm is well informed about his economic prospects, but lacks of information about the market structure and market demand. Professional investors and intermediaries receive information about the market demand and the market value of the issue. And, retail investors, because of their occasional participation in IPOs, are in general supposed to be uninformed.

Benveniste and Wilhelm (1990) are the first investigating the effects of the regulatory constraints imposed on the intermediary's marketing effort. They compare several institutional environments in which intermediaries may be constrained by the possibility of using price-discrimination or/and *rationing* (i.e. quantity-discrimination). They consider a standard IPO where there exists an agency problem between the seller and the institutional investors who receive a signal about the market value of the issue.[2] A group of retail investors take part in the IPO as well. They may also receive, with some probability, a signal about the market

[1] The maximization of the seller's profit.

[2] The seller denotes indifferently the issuing firm and the intermediary.

value of the issue and their demand function, assumed to be flat, is common knowledge.

The authors show that, the less discretion the intermediary has, the smaller the issuing firm's profit is. They conclude that, the seller is better off when he can use both instruments, price and quantity discrimination. This however is not what we observe in practice. The current regulatory scheme does not allow the seller to price-discriminate across the buyers whereas rationing is widely used.

Benveniste and Wilhelm use comparative statics to compare the different regulatory regimes. In this paper, we instead apply a mechanism design approach to describe IPOs. In other words, we consider IPOs as an optimal auction design problem (Myerson, 1981; Maskin and Riley, 1990). This allows us to derive endogenously the optimal IPO mechanism and to analyze its features. We find that the optimal IPO - the one maximizing the seller's profit - can be implemented through a uniform price across all the institutional investors. Furthermore, in equilibrium, the seller applies a *uniform* rationing scheme whenever oversubscription of shares occurs. Our result seems to be consistent with what we observe in practice.

IPOs are a very natural example of an optimal auction design problem because of their peculiar informational structure which exhibits a number of agency problems and because the currently used IPO mechanisms are either auction or auction-like mechanisms (first price auctions, bookbuilding, *mise en vente).* As a matter of fact, the idea of using an optimal mechanism design approach to describe initial offerings is not new in the literature. Bias, Bossaerts and Rochet (1996) employ the same approach to show that fixed price offering is not an optimal selling procedure in presence of asymmetries of information. They derive the optimal selling mechanism that enables the seller to extract all the information from the coalition and find that this mechanism specifies a price schedule decreasing in the quantity allocated to retail investors. This price schedule eliminates the winner's curse problem faced by the uninformed investors since, when the stock of shares they are allotted is quite large the price is set at a relatively low level. They also find that the optimal mechanism exhibits underpricing to institutional investors. This underpricing represents the informational rents left by the selling firm to the informed investors.

The main difference between their model and this one is on the role of the intermediary in the IPO. Bias, Bossaerts and Rochet assume that, because of their long-term relationship, the intermediary forms a (perfect) coalition with the informed investors. We instead assume that the intermediary in charge with the marketing of the issue serves the interests of the issuing firm, in line with Benveniste and Wilhelm. We denote this coalition firm/intermediary as the seller. The role an intermediary plays in an IPO depends on the financial market we refer to. The assumption of a coalition between the intermediary and the institutional investors can be realistic in European financial markets where firms and intermediaries usually interact only occasionally for the initial offering whereas the intermediary and the professional investors are often engaged in long term and repeated relationships. However, the same assumption is not reasonable in the American financial markets where, on the contrary, firms rely on intermediaries for a number of other financial operations besides the issuing procedures. This makes their relationship much more stable over time.

The other players taking part in the IPO are n institutional investors and, a group of atomistic investors (*retail investors*) who only occasionally participate in the offering. Each institutional investor receives a signal about the market value of the asset. Signals are i.i.d. and the true value of the asset is given by the average of all signals (common value auction). Institutional investors are risk averse. Retail investors have no information about the value of the asset on sale. They are characterized by an initial endowment W they are willing to invest in the new shares which is common knowledge in the IPO.

The seller tries to elicit the information from institutional investors by offering them an optimal contract which fixes the quantity and the price investors have to pay for each share. We consider linear price schedule. Institutional investors' preferences are described by a concave utility function. The signals they receive are iid according to a uniform distribution function.[3] We first compute the optimal quantities for both institutional and retail investors. Then, we derive the optimal price schedule and show that it is uniform for all institutional investors. Also, whenever an oversubscription of shares occurs, the optimal contract is such that a) all the issues are assigned to institutional investors; and b) institutional investors are *uniformly* rationed, in the sense that they are rationed by the same amount of shares.

In conclusion, we show that the seller does not need to use both price and quantity discrimination to achieve an optimal performance. Quantity discrimination alone is enough to implement the optimal IPO. Our result are consistent with the empirical evidence on the performances of the bookbuilding procedure as well as the *mise en vente,* an auction-like

[3] The optimal contract can also be derived in a more general setting with generic concave utility function and distribution of the signals and non linear price schedules (see Bennouri and Falconieri, 2000).

mechanism widely used in France to sell new shares.[4]

The paper is organized as follows. In the next section we set up the model and the mechanism design problem for the seller. In Section 3, we derive the optimal quantities from the relaxed problem in which the monotonicity constraint is ignored and checked ex-post. Section 4 analyses the case of an oversubscription of shares and in Section 5 we prove the two keen results of the paper on the existence, in equilibrium, of a uniform price schedule and a uniform rationing scheme. Section 6 links our theoretical result up to the empirical evidence on current IPO mechanisms. The last section summarizes and concludes.

2 The Model

2.1 Agents

Consider a firm offering a fixed amount of shares Q in an IPO, which, without loss of generality, is normalized to 1.[5] An intermediary is in charge of marketing the issues. He serves the firm's interests, so, hereafter, we denote the coalition firm-intermediary as the *seller.*[6] The seller's objective is to maximize the proceeds from the sale.

There are two classes of investors participating into the IPO, institutional and retail investors. There is a set N of $n > 2$ institutional investors. Their preferences are described by the following utility function:

$$u_i(p_i, q_i, v) = q_i[(\alpha v - \frac{\delta}{2}q_i) - p_i] \qquad \textit{with} \quad i = 1, 2, ...n \quad \text{and} \quad \alpha > 1$$

where q_i is the quantity assigned to investor i and p_i is the price per share investor i has to pay. We denote by $T_i = p_i q_i$ the payment from investor i to the seller for the whole quantity q_i. Notice that, the above utility function is linear in the transfer T_i.[7] Also, define $z(q_i, v) = q_i(\alpha v - \frac{\delta}{2}q_i)$, which is concave in the quantity q_i. Notice that, by the concavity of $z(.)$, institutional investors exhibit an increasing risk aversion in the quantity they receive. A possible reason for this kind of preferences is that, investors are averse to the *inventory risk,*

[4]For a detailed description of the bookbuilding procedure see Cornelli and Goldreich (1998). For details on the *mise en vente* see Biais, Bossaerts and Rochet (1998) and Biais and Faugeron-Crouzet (2000).

[5]The quantity to be sold is not always fixed in reality. In American IPOs, for instance, the seller can exploit the so called *over allotment option:* he can increase the quantity announced at the beginning of the IPO by at most 10%. For the sake of simplicity we do not allow this option. It is likely however that introducing the possibility to vary the quantity during the auction would allow the design of more efficient mechanisms.

[6]The need for an intermediary is often due to the fact that the firm cannot reach directly the investors.

[7]This kind of utility functions exhibiting linearity in the total payment, are very common in the auction literature. (See for instance Crèmer and McLean, 1988).

that is the risk associated to the composition of their portfolio. Furthermore, the function z looks very much like a Mean-Variance utility function with the only difference that the expected value v is weighed more than the transfer T_i. The parameter α measures the regular investor's aggressiveness: the higher α is, the more aggressive the investor is in the IPO procedure.

After the intermediary has completed his investigation of the issuing firm, the information acquired is made public in the prospectus.[8] Some additional information about the future market value of the asset may however become available to institutional investors. In particular, we assume that each institutional investors receive a signal s_i about the future market value of the shares on the stock market. The value of the shares v is given by the average of all the signals, that is[9]

$$v = \frac{1}{n} \sum_i s_i$$

Signals are i.i.d. according to a uniform distribution defined on $\Omega_i = [\underline{s}; \overline{s}]$, so the cumulative distribution function is $F_i(s_i) = \dfrac{s_i - \underline{s}}{\Delta s}$, and the density function $f_i(s_i) = \dfrac{1}{\Delta s}$.[10] Let us also denote by $f(s)$ the joint density function so that

$$f(s) = f(s_i, s_{-i}) = \prod_i f_i(s_i) \qquad \text{with } s = (s_i, s_{-i}) \in \Omega = \bigotimes_i \Omega_i = [\underline{s}, \overline{s}]^n .$$

We assume a continuum of competitive, risk neutral small or *retail* bidders. The total mass of these retail bidders is normalized to one. Each of them has a maximum amount of wealth W he is willing to invest in the issued asset. We set $W \geq \overline{s}$ so that retail investors are never cash constrained, even when they are assigned the whole quantity.[11] We denote by q_R the quantity they receive in equilibrium; by p_R the price per unit of share they are asked and by $T_R = q_R p_R$ the total transfer to the seller.

The seller wants to maximize the proceeds of the sale, computed in expected value on Ω . Its objective function is thus,

$$U_F = E_s \left[\sum_i T_i + T_R \right]$$

[8] We assume here that this information can be revealed credibly to the public, due for instance to reputational concerns.

[9] Biais, Martimort and Rochet (2000) assume that the final value of the asset is $v_i = s_i + \varepsilon_i$, with s_i indipendent of ε_i. Our assumption is a generalization of this one to the case of multiple agents.

[10] With utility functions linear in payments, we need to consider i.i.d. signals, otherwise the seller could design a fully extracting mechanism (Crémer and McLean, 1988; McAfee and Reny, 1992).

[11] Introducing cash constraints to retail investors would not qualitatively change the results.

where T_i is the transfer paid by the *ith* informed bidders and T_R is the transfer paid by retail bidders.

Retail investors share the same information as the seller. They do not receive any signal about the market value of the asset and only observe the density $f_i(s_i)$ of signals.

In order to extract the information from the institutional investors, the seller offers a contract specifying the quantity and the price per share to pay to each institutional investor and to retail investors. By using the *revelation principle,* we can focus on *direct mechanisms* in which the firm asks each investor to announce his signal and then fixes the quantity and the price as a function of their announcements in such a way to induce them to truthfully reveal their information (see Fudenberg and Tirole, 1991).

The contract the seller proposes can thus be written in the following way:

$$\Gamma = \{p_i(\widehat{s}_i, \widehat{s}_{-i}),\ q_i(\widehat{s}_i, \widehat{s}_{-i});\ p_R(\widehat{s}_i, \widehat{s}_{-i}),\ q_R(\widehat{s}_i, \widehat{s}_{-i})\} \qquad \forall i$$

with $\widehat{s}_i$ being institutional investor i's announcement and $\widehat{s}_{-i}$ the vector of the announcements of all the other investors.

We assume that all the shares issued must be allocated between institutional and retail investors.[12] Consequently, by choosing the vector $\{q_i\}_i$ the firm implicitly determines the number of shares to allocate to retail investors, which is given by:

$$q_R = 1 - \sum_i q_i$$

2.2 The IPO Design Problem

The optimal IPO mechanism for the firm, denoted by $\Gamma^* = \{\ p_i^*(\widehat{s}_i, \widehat{s}_{-i}),\ q_i^*(\widehat{s}_i, \widehat{s}_{-i}),\ \forall i;\ p_R^*(\widehat{s}_i, \widehat{s}_{-i})\}$, is the solution to the following optimization program:

$$\max_{q_i} U_F = E_s \left[\sum_i T_i(\widehat{s}_i, \widehat{s}_{-i}) + T_R(\widehat{s}_i, \widehat{s}_{-i}) \right]$$

subject to the standard constraints (see Fudenberg and Tirole, 1991):

- *Retail Investors' Participation Constraint* (RPC):

$$E_S\left[q_R(s_i, s_{-i})(v - p_R(s_i, s_{-i})\right] \geq 0$$

[12] This means that we do not allow the so-called *firm-commitment contract* of underwriting widely used in practice, in particular in the US financial market. With this kind of contract, the underwriter (the intermediary) commits to buy all the shares not sold during the initial offering (see Ritter, 1987 and Biais and Faugeron, 1995, for more details respectively on the US and the European markets).

which requires their expected payoff to be larger than their reservation utility which, without loss of generality, may be set equal to zero.

- *Institutional Investors' Participation Constraint* (IPC):

$$U_i(s_i, s_i) = E_{s_{-i}} \left\{ q_i(s_i, s_{-i}) \left[(\alpha v - \frac{\delta}{2} q_i(s_i, s_{-i})) - p_i(s_i, s_{-i}) \right] \right\} \geq 0 \qquad \forall s_i, \forall i$$

which has the same meaning as the RPC.

- *Institutional Investors' Incentive Compatibility Constraint* (IIC):

$$U_i(\widehat{s}_i, s_i) \leq U_i(s_i, s_i) \qquad \forall s_i, \forall \widehat{s}_i \text{ and } \forall i,$$

with

$$U_i(\widehat{s}_i, s_i) = E_{s_{-i}} \left\{ q_i(\widehat{s}_i, s_{-i}) \left[(\alpha v - \frac{\delta}{2} q_i(\widehat{s}_i, s_{-i})) - p_i(\widehat{s}_i, s_{-i}) \right] \right\}$$

which ensures that institutional investors do not have incentive to mis-represent their type - the signal they receive - to the firm.

- Last, the *Full Allocation Constraint* (FAC):

$$\sum_i q_i(s_i, s_{-i}) + q_R(s_i, s_{-i}) = 1$$

and the *feasibility constraint:*

$$q_i(s_i, s_{-i}) \geq 0 \qquad \forall i, \forall s_i \text{ and } s_{-i}$$

Before turning to the solution of this optimization problem, we make the following assumptions on the function $z(q_i, v) = q_i(\alpha v - \frac{\delta}{2} q_i)$ and on the hazard rate:

- **A.1** z is increasing in q_i and v thus $z_1 = \alpha v - \delta q_i > 0$ and $z_2 = \alpha q_i > 0$;

  this assumption holds under the condition that $\underline{s} > \dfrac{\delta}{\alpha}$ which we assume hereafter.

- **A.2** $z_{11} = -\delta < 0$;

  which implies that informed buyers take into account the effect of the inventory risk when evaluating the quantity of shares they are assigned.

- **A.3** $z(0, v) = 0$ for all v;

- **A.4** $z_{12} = \alpha > 0$;

  this is the usual single-crossing condition used in the mechanism design literature. It states that the agent's type affects the marginal rate of substitution between the allocations and the payment in a monotonic way. The single-crossing condition is a necessary condition for the mechanism to be implementable (see Fudenberg and Tirole, 1991).

- **A.5** $z_{122} = z_{112} = 0$;

  this assumption puts a restriction on the marginal rate of substitution between the quantity and the asset value which must be non-increasing in v and non-decreasing in q.

- **A.6** $z_{12}(0, v) = \alpha \geq 1$

  this assumption is made for technical reasons. It allows to characterize implementable mechanisms as those satisfying the monotonicity condition (Guesnerie and Laffont, 1984, and Fudenberg and Tirole, 1991).It implies that the marginal rate of substitution between allocations and the transfers does not increase too fast when the transfer goes to infinity.

- **A.7:** Monotonic Hazard Rate: $\frac{\partial}{\partial s_i}\left[\frac{f_i(s_i)}{1 - F_i(s_i)}\right] \geq 0.$

  in our setting, the hazard rate of the uniform distribution function is $\frac{1}{(\bar{s} - s_i)}$ and clearly verifies the monotonicity condition. This assumption allows us to focus on the relaxed problem, that is to check the SOC - the monotonicity constraint - ex-post (see the next section).

By the RPC and the maximand we know that the seller's profit is increasing in the retail investors' payments, therefore at the optimum he will make the RPC binding. We can then rewrite the RPC in the following way:

$$\int_{\Omega} p_R(s)q_R(s)f(s)ds = \int_{\Omega} vq_R(s)f(s)ds = \int_{\Omega} v\left(1 - \sum_i q_i(s)\right) f(s)ds \tag{1}$$

where we have put $s = (s_i, s_{-i})$ and replaced to $q_R(s)$ from the FAC.

Now, let us consider the IPC which can be rewritten as follows:

$$\int_{\Omega_{-i}} p_i(s)q_i(s)f_{-i}(s_{-i})ds_{-i} = \int_{\Omega_{-i}} z(q_i(s), v)f_{-i}(s_{-i})ds_{-i} - U_i(s_i, s_i) \geq 0 \quad \text{for all } s_i \text{ and } i. \tag{2}$$

by integrating over Ω_i (that is taking the expectations over s_i) we find:

$$\int_{\Omega} p_i(s)q_i(s)f(s)ds = \int_{\Omega} z(q_i(s), v)f(s)ds - \int_{\Omega_i} U_i(s_i, s_i)f_i(s_i)ds_i. \tag{3}$$

From the IIC, after some computations (see the Appendix for the details), we can write the following equation:

$$\int_{\Omega_i} U_i(s_i, s_i)f_i(s_i)ds_i = \frac{1}{n}\int_{\Omega} \alpha(\overline{s} - s_i)q_i(s_i, s_{-i})f(s)ds. \tag{4}$$

where $(\overline{s} - s_i)$ is the inverse of the *hazard rate.*

By replacing equation (4) into equation (3) and using equation (1), the firm's optimization program can be rewritten in the following way:

$$\max_{\{q_i\}_1^n} \int_{\Omega} \left\{ v + \sum\nolimits_i \left[q_i(s) \left((\alpha - 1)\, v - \frac{\delta}{2} q_i(s) - \frac{\alpha}{n}(\overline{s} - s_i) \right) \right] \right\} f(s)ds$$

$s.t$:

$$\frac{\partial q_i(s)}{\partial s_i} \geq 0 \qquad \text{for all } i \text{ and all } s$$

$$q_i(s) \geq 0 \qquad \text{for all } i \text{ and all } s$$

$$\sum_i q_i(s) \leq 1 \qquad \text{for all } s.$$

where the first constraint is *the monotonicity constraint* which ensures that the SOC are met and the mechanism is implementable.[13]

[13] This monotonicity condition is stronger than what is sufficient for the implementability of the optimal mechanism. By the second derivatives of $U_i(s_i, s_i)$, it would be sufficient the following inequality to hold,

$$\frac{\alpha}{n} E_{s_{-i}}\left[\frac{\partial q_i(s)}{\partial s_i}\right] \geq 0.$$

for each i and s_i.

In the next section, the optimal quantities are derived. We consider the relaxed problem in which we drop the monotonicity constraint and check it ex post. We will then compute the optimal price schedules and the rationing rate.

3 The optimal quantities

The relaxed problem is set by ignoring the monotonicity condition. The objective function becomes then an ordinary maximand with the constraints defined in each point and can be maximized pointwise on Ω.

We now derive the optimal quantity $q_i(s)$ for each i and each s (hereafter we drop the "$*$" to denote the optimal mechanism), that is the amount of shares the firm must assign to investor i in order to elicit his information. $q_i(s)$ is the solution of the following maximization problem, for each $s \in \Omega$:

$$
\begin{aligned}
&\max_{\{q_i,\}_{i=1,..n}} \sum\nolimits_i \left[q_i(s) \left((\alpha - 1)\, v - \frac{\delta}{2} q_i(s) - \frac{\alpha}{n}(\overline{s} - s_i) \right) \right] \\
&s.t: \\
&U_i(\underline{s}, \underline{s}) = 0 \qquad \forall\, i \\
&q_i(s) \geq 0 \qquad \text{and} \qquad \sum_i q_i(s) \leq 1 \qquad \forall i,\ \forall s.
\end{aligned}
$$

The Kuhn-Tucker conditions for this maximization program are the following:

$$
\begin{cases}
(\alpha - 1)v - \delta q_i - \dfrac{\alpha}{n}(\overline{s} - s_i) + \lambda_i(s) - \beta(s) = 0, & \text{for all } i \\
\lambda_i(s) q_i(s) = 0 \\
\beta(s)[1 - \sum\limits_i q_i(s)] = 0.
\end{cases}
\tag{5}
$$

with λ_i and β are the Kuhn-Tucker multipliers associated respectively to the feasibility constraint and to the FAC.

Now let us denote by $H(q, v)$ our objective function, that is:

$$
H(q, v) = \sum_i \left[q_i(s) \left((\alpha - 1)\, v - \frac{\delta}{2} q_i(s) - \frac{\alpha}{n}(\overline{s} - s_i) \right) \right]
$$

with $q_i \in [0,1]$ and $s \in \Omega = [\underline{s}, \overline{s}]^n$. This function is concave in q_i for all i, since $\dfrac{\partial^2 H}{\partial q_i^2} \leq 0$ by assumptions (A.2) and (A.5).

If $q_i(s) \neq 0$, then $\lambda_i(s) = 0$, so from equation (5) it is:

$$\frac{\partial H}{\partial q_i} = (\alpha - 1)v - \delta q_i - \frac{\alpha}{n}(\overline{s} - s_i) = \beta(s) \geq 0$$

given the sign of the Kuhn-Tucker multiplier. Let us denote by $h_i(s)$ the first derivative of H w.r.t. q_i, i.e. $\dfrac{\partial H}{\partial q_i}_{|q_i=0}$. A sufficient condition for the inequality above to hold is that $h_i(s) > 0$. That is:

$$\begin{aligned} h_i(s) &= (\alpha - 1)v - \frac{\alpha}{n}(\overline{s} - s_i) = \\ &= \frac{1}{n}\left[(\alpha - 1)v_{-i} + (2\alpha - 1)s_i - \alpha\overline{s}\right] > 0 \end{aligned} \qquad (6)$$

where we have put $v_{-i} = \sum_{j \neq i} s_i$.[14] And, consequently, $q_i(s) = 0$ if $h_i(s) \leq 0$.

The solution of the optimization program depends on the sign of the function $h_i(s)$. We have to distinguish the two possible cases:

- $h_i(s) \leq 0$ if and only if

$$\frac{1}{n}\left[(\alpha - 1)v_{-i} + (2\alpha - 1)s_i - \alpha\overline{s}\right] \leq 0 \Longleftrightarrow$$

$$s_i \leq \frac{\alpha\overline{s} - (\alpha - 1)v_{-i}}{(2\alpha - 1)} = s^\circ{}_i(v_{-i})$$

  so the optimal quantity for all i such that $s_i \leq s_i{}^\circ(v_{-i})$ would be zero. Consequently,

- $h_i(s) > 0$ for $s_i > s_i{}^\circ(v_{-i})$ and the optimal quantity for all i for which this condition holds, would be positive.

Notice also that for values of $v_{-i} > \dfrac{\alpha\overline{s} - \underline{s}(2\alpha - 1)}{(\alpha - 1)} = v^\circ$, it is $s^\circ{}_i(v_{-i}) < \underline{s}$ and then $s_i > s^\circ{}_i(v_{-i})$ for all i and consequently the optimal quantity is positive for all institutional investors.[15]

[14] This result uses the fact that given a function Ψ continuous and concave on an interval $[a,b]$, if $\Psi'(a) \leq 0$, then Ψ is decreasing all over on $[a,b]$. In our case, this means that if $h_i(s) \leq 0$, the first derivative of H with respect to q_i, $\dfrac{\partial H}{\partial q_i}$, is also negative and so is the Kuhn-Tucker multiplier $\beta(s)$ what is impossible. By contradiction, it must be that $q_i(s) = 0$ if $h_i(s) \leq 0$.

[15] It is instead always true that $s^\circ{}_i(v_{-i}) < \overline{s}$, since $s^\circ{}_i(v_{-i}) > \overline{s}$ if and only if

$$\begin{aligned} \alpha\overline{s} - (\alpha - 1)v_{-i} > (2\alpha - 1)\overline{s} \Leftrightarrow \\ v_{-i} < -\overline{s} \end{aligned}$$

which is impossible for $\overline{s} > 0$. This proves the result.

Furthermore, to ensure that $q_i(\underline{s}, s_{-i}) = 0$ we assume that $h_i(\underline{s}, s_{-i}) \leq 0$ which is true if $v_{-i} \leq v^\circ$ for all s_{-i}.[16]

The next lemma states a sufficient condition for this to be true:

Lemma 1 *: A sufficient condition for $v_{-i} \leq v^\circ$ to hold for all s_{-i}, is that:*

$$\overline{s} \leq \frac{2\alpha - 1}{n(\alpha - 1)} \Delta s$$

Proof: (See the Appendix).

Now, assume that $\overline{s} \leq \dfrac{2\alpha - 1}{n(\alpha - 1)} \Delta s$. Then, for any $s \in \Omega = [\underline{s}; \overline{s}]^n$, we can split the set N into two sub-sets

$$\begin{aligned} N^-(s) &= \{i \in N \mid s_i \leq s^\circ{}_i(v_{-i})\} \\ N^+(s) &= \{i \in N \mid s_i > s^\circ{}_i(v_{-i})\} \end{aligned}$$

such that $N(s) = N^+(s) \cup N^-(s)$. This partition of the set N is related to the characteristics of the investors' utility functions which in turn depend on the value of the signal they receive. This becomes clear if we look at risk neutral institutional investors. In this case, in fact, the set $N^+(s)$ is empty (see Section 3.1).

We can then state the following result:

Proposition 1 *: Assume that institutional investors' utility function is $u_i(q_i, p_i, v) = q_i[(\alpha v - \frac{\delta}{2} q_i) - p_i]$ with $\alpha > 1$ and that signals are i.i.d. according to a uniform distribution with support $\Omega_i = [\underline{s}; \overline{s}]$. Then, for any $s \in \Omega = [\underline{s}; \overline{s}]^n$ the optimal mechanism is such that:*

- *for any $i \in N^-(s)$, the optimal quantity is $q_i(s) = 0$;*
- *for any $i \in N^+(s)$ we can compute the quantity $\widetilde{q}_i(s) > 0$ which is equal to:*

$$\widetilde{q}_i = \frac{(\alpha - 1)nv - \alpha(\overline{s} - s_i)}{n\delta} \tag{7}$$

then,

a) if $\sum_{i \in N^+(s)} \widetilde{q}_i(s) < 1$ the optimal quantity each institutional investor receives is exactly $\widetilde{q}_i(s)$.

[16] This condition is sligthly more restrictive than what we need, but it considerably simplifies the analysis.

b) if, instead, $\sum_{i \in N^+(s)} \widetilde{q}_i(s) > 1$*, the optimal quantity is* $q_i(s) = \widehat{q}_i(s) < \widetilde{q}_i(s)$ *solving the following system of equations:*

$$\begin{cases} \widehat{q}_i(s)[(\alpha - 1)v - \delta \widehat{q}_i(s) - \frac{\alpha}{n}(\overline{s} - s_i) - \beta(s)] = 0 \\ \sum_{i \in N^+(s)} \widehat{q}_i(s) = 1; \qquad \beta(s) > 0, \end{cases} \tag{8}$$

where $\beta(s)$ *is as usual the Kuhn-Tucker multiplier.*

Proof: (See the Appendix).

Notice that, *a)* the optimal quantity $\widetilde{q}_i(s)$ is increasing in the signal s_i. The seller rewards good signals with a larger quantity. *b)* The case in which $\sum_{i \in N^+(s)} \widetilde{q}_i(s) > 1$ corresponds to an *oversubscription* of shares and implies that institutional investors are rationed in equilibrium.

The next step is to prove that the optimal mechanism obtained by solving the relaxed problem is implementable, that is it satisfies the monotonicity condition. The result is stated in the next proposition. We omit the proof, which is quite long and complicated.[17]

Proposition 2 *The optimal mechanism derived in proposition 1 is implementable, that is the optimal quantity* $q_i(s)$ *satisfies the following monotonicity condition:*

$$\frac{\partial q_i(s)}{\partial s_i} \geq 0$$

for all i *and* s_i*.*

3.1 The Risk-Neutral Case

We get interesting result in the special case of risk-neutral institutional investors.

Suppose institutional investors are risk neutral so that their utility function becomes $u_i(.) = q_i(v - p_i)$. It is easy to check that the function $h_i(s) = \frac{\partial H}{\partial q_i}_{|q_i=0}$ reduces to

$$h_i(s) = -\frac{1}{n}\left(\frac{1 - F_i(s_i)}{f_i(s_i)}\right) = -\frac{\overline{s} - s_i}{n} < 0$$

for all i and s.

Consequently, the set $N^+(s) = \emptyset$ so all $i \in N^-(s)$, for all s.

[17] See Bennouri and Falconieri (2000) for a detailed proof in a general model of auction design.

Therefore, by the result stated in Proposition 1, in this case, the optimal mechanism assigns all the shares to retail investors at a marginal price equal to $p_R = v$.

In conclusion, with risk neutral institutional investors, the seller is able to implement a full extracting mechanism to the extent that, he leaves no informational rents to informed investors.

4 Oversubscription and Rationing

The case $\sum_{i \in N^+(s)} \widetilde{q}_i(s) > 1$ corresponds to a situation of oversubscription of shares, because institutional investors demand overall more shares than the amount on sale. Consequently, in equilibrium, they do not receive their optimal quantity $\widetilde{q}_i(s)$, but instead, each institutional investor is assigned the lower amount $\widehat{q}_i(s)$ determined by the system of equation (8). Retail investors receive nothing in equilibrium.

Proposition 3 *In the case of oversubscription of the new shares,* $\sum_{i \in N^+(s)} \widetilde{q}_i(s) > 1$, *in equilibrium, institutional investors will be uniformely rationed. The quantity they are assigned is*

$$\widehat{q}_i(s) = \widetilde{q}_i(s) - \beta(s)/\delta$$

where, the Kuhn-Tucker multiplier is given by

$$\beta(s) = \frac{\delta}{Card^+} \left(\sum_{i \in N^+(s)} \widetilde{q}_i(s) - 1 \right)$$

Proof: By the system of equations (8), the quantity each institutional investor is assigned, $\widehat{q}_i(s)$, is

$$\begin{aligned} \widehat{q}_i(s) &= \frac{(\alpha - 1)v - \frac{\alpha}{n}(\overline{s} - s_i) - \beta(s)}{\delta} = \\ &= \widetilde{q}_i(s) - \beta(s)/\delta \end{aligned} \tag{9}$$

where $\widehat{q}_i(s) < \widetilde{q}_i(s)$, being $\beta(s) > 0$.

The value of the Kuhn-Tucker multiplier, $\beta(s)$, can be explicitly computed by using the condition $\sum_{i \in N^+(s)} \widehat{q}_i(s) = 1$. By equation (9), it is

$$\sum_{i \in N^+(s)} \widehat{q}_i(s) = \sum_{i \in N^+(s)} [\widetilde{q}_i(s) + \beta(s)/\delta] = 1$$

Define now $Card^+ \equiv Card[N^+(s)]$, re-arranging the above equality then yields

$$\beta(s) = \frac{\delta}{Card^+} \left(\sum_{i \in N^+(s)} \widetilde{q}_i(s) - 1 \right)$$

■

Notice that, the rationing scheme is uniform in the sense that, in equilibrium, the quantity assigned to each institutional investor is reduced of the same amount with respect to what would be otherwise optimal for them. This is consistent with monotonicity of the optimal quantity schedule according to which the seller assigns more shares to the investors with the better signals.

Nonetheless, the rationing rate as defined by the ratio $R_i = \dfrac{\widehat{q}_i}{\widetilde{q}_i} < 1$, varies across investors and, by replacing from equation (9), is equal to

$$R_i = 1 - \frac{\beta(s)}{\delta \cdot \widetilde{q}_i}$$

which implies that better signals, lower rationing. This is consistent with what is observed in the bookbuilding procedure, where investment banks reward investors who convey more information through their bids (e.g. limit orders) (Cornelli and Goldreich, 1999). Here, the seller rewards - by applying a lower rationing rate - the better information.

5 The optimal prices

We can now compute the optimal prices for both institutional and retail investors.

Recall that retail investors are left with a quantity $q_R = \left[1 - \sum_{i \in N^+} \widetilde{q}_i(s) \right]$ which is always strictly positive. The optimal price for them is thus determined by the following equation:

$$\int_\Omega p_R(s) q_R(s) f(s) ds = \int_\Omega v \left(1 - \sum_i \widetilde{q}_i(s) \right) f(s) ds$$

which is an integral equation admitting infinite solutions. One of the possible solutions of this equation is the following,

$$p_R(s) = v \tag{10}$$

Notice that the seller uses the information extracted from the informed investors to set the price to retail investors, who are charged the true value of the asset. As a consequence, whenever retail investors are allotted all the shares, namely when institutional investors reveal very bad information, the price is likely to be lower than the expected value, $v < E(v)$.[18] In a sense, this can be interpret as *underpricing* which is needed to compensate small investors for the winner's curse (Rock, 1986).

We now compute the optimal price for institutional investors. The next result is a crucial one: we show that the optimal price schedule to institutional investors is uniform.

Proposition 4 *The optimal IPO mechanism can be implemented through a (unique) uniform price schedule to institutional investors.*

Proof (Sketch): The linear transfer must satisfy the following equation

$$\int_{\Omega_{-i}} p_i(s) q_i(s) f_{-i}(s_{-i}) ds_{-i} = \int_{\Omega_{-i}} \left\{ z(q_i(s), v) - \frac{1}{n} \left[\int_{\underline{s}}^{s_i} z_2(q_i(\widetilde{s}_i, s_{-i}), v(\widetilde{s}_i, s_{-i})) d\widetilde{s}_i \right] \right\} f_{-i}(s_{-i}) ds_{-i}; \tag{11}$$

Now, consider the following price function

$$p_i^0(s) = \frac{z(q_i(s), v) - \frac{1}{n} \left[\int_{\underline{s}}^{s_i} z_2(q_i(\widetilde{s}_i, s_{-i}), v(\widetilde{s}_i, s_{-i})) d\widetilde{s}_i \right]}{q_i(s)} \tag{12}$$

for each s_i and each $q_i(s)$ such that $q_i(s) \neq 0$.[19]

Any price satisfying equation (1) can be also written as $p_i(s) = p_i^0(s) + \Phi_i(s)$ with Φ_i satisfies the following equation

$$\int_{\Omega_{-i}} \Phi_i(s) q_i(s_i, s_{-i}) f_{-i}(s_{-i}) ds_{-i} = 0, \qquad \forall i, s_i.$$

We say that, a uniform price function exists if and only if $p_i(s)$ satisfies the following partial differential equation

$$\frac{\partial p_i(s)}{\partial s_i} = \frac{\partial p_i(s)}{\partial s_j} \qquad \forall i, j$$

which is equivalent to

$$\frac{\partial p_i^0(s)}{\partial s_i} + \frac{\partial \Phi_i(s)}{\partial s_i} = \frac{\partial p_i^0(s)}{\partial s_j} + \frac{\partial \Phi_i(s)}{\partial s_j}. \tag{13}$$

[18] This also depends on the number of institutional investors which determines the accuracy of the information revealed. More precisely, it can be shown that there exists a n^* such that the inequality holds for all $n \geq n^*$.

[19] Otherwise $p_i^0(s) = 0$.

The proof consists in two main steps: first, we reduce the above condition on the uniform price into a condition on the existence of the function Φ; and next, we prove the existence of such a function. (See the Appendix for the details).

We can then conclude that, the seller does not need to use both price and quantity discrimination to achieve an optimal performance in the IPO. Rationing alone is sufficient to implement the optimal mechanism. Furthermore, whenever oversubscription occurs, the rationing scheme (not the rate!) is uniform for all the investors.

5.1 Comparison with the Benveniste and Wilhelm

Our model differs in many respects from the Benveniste and Wilhelm paper (BW). Our model has the advantage to derive endogenously the optimal IPO mechanism. We also have a more general information structure with a continuous of possible signals, whereas in BW regular investors are either informed or uninformed. If informed, the information may be either good or bad. Moreover, regular investors are risk-neutral up to a quantity $\overline{q}$ and infinitely risk-averse for larger quantities. As a consequence, the underwriter is not able to freely ration regular investors. He can only apply a discrete rationing scheme, i.e. either he fills the full order or he gives nothing. Therefore, price discrimination is needed in addition to rationing to allow the underwriter to elicit the information from the regular investors.

Here, instead, the seller has full discretion in how to ration institutional investors. The rationing scheme is continuous and optimally chosen. Therefore, rationing alone suffices to elicit the investors' information, without any need to additionally use price discrimination.

There is a general principle lying behind the two results. One dimensional asymmetry of information, one instrument to solve it. BW need both price and quantity discrimination only because the seller cannot fully rely on the latter and, hence, an additional instrument becomes necessary to achieve the information extraction.

6 The Practice of IPOs.

From a normative standpoint, our result suggests that the current regulation of IPOs, which prevents the selling firms from price discriminating, is not inefficient, since rationing alone is enough to achieve optimal performances. This is consistent with what we observe in practice in the current IPO mechanisms.

In particular, our result seems to be confirmed by the evidence from the bookbuilding procedure. Investment banks exhibit a strong attitude to strategically use rationing as a reward to investors who transmit more and better information during the opening of the book (e.g. limit orders). Because of that, investment banks often set the price sufficiently high to induce oversubscription and, consequently, to have complete discretion in how to ration shares among the investors. (Cornelli and Goldreich, 1999, 2000).

Our result is also consistent with the literature both empirical and theoretical indicating auction mechanisms as the most efficient ones to sell new shares.

An interesting evidence comes from the Israeli IPOs which are conducted as non-discriminatory (uniform price) auctions. Kandel and Sarig (1998) find that Israeli IPOs exhibit a very low underpricing (4.5%) compared to other IPO mechanisms and only on the first day of trading. No significant abnormal return is found beyond the first trading day. Derrien and Womack (2001) use French IPO data for the 1992-1998 period to compare the efficiency of three different IPO selling procedures: fixed priced, auction and bookbuilding.[20] They find that the auction mechanism results in less underpricing as well as in a lower variability of the underpricing in hot *versus* cold market conditions.

In line with the empirical literature, Biais and Faugeron-Crouzet (2000) prove, in their paper, that auction-like mechanisms perform better than others, the reason being that auction procedures are more informationally efficient, i.e. they allow to elicit and incorporate more information from the market as well as from investors into the pricing of IPOs.

7 Conclusions

The Initial Public Offerings of shares can be described as a very special kind of auction in which the auctioneer faces two groups of bidders with different information sets. In particular, fully uninformed bidders co-exist with informed bidders. In the IPO context, the uninformed bidders are represented by the retail investors who only occasionally take part to the offering, whereas the informed bidders are institutional investors who are professional investors regularly taking part into the offerings.

In this paper, we adopt an optimal auction approach to analyze the functioning of initial offerings with the focus on the choice by the seller of the optimal instrument to allocate efficiently the issues, this choice being price *versus* quantity discrimination (rationing).

[20] Resepctively, Offre a prix ferme, Mise en Vente and Placement Garanti.

The main results of the papers are the following.

The seller does not need price discrimination to achieve an optimal performance. Instead, the optimal IPO can always be implemented through a uniform price schedule. Moreover, we also find that, when oversubscription occurs, institutional investors are uniformly rationed.

As a minor result, we confirm the existence of underpricing to retailers in the optimal IPO as a compensation for the winner's curse (Rock, 1986).

Our model can be alternatively interpreted as a model of optimal auctions. In this respect, it can be applied to all kind of auctions exhibiting the same informational structure as an IPO, namely, the auctioneer facing two groups of bidders with different information sets. The main insight we can derive from our result is that the existence of a group of uninformed bidders allows the auctioneer to extract the information from the informed bidders at a lower cost. The reason relies upon the more discretion the auctioneer enjoys in deciding to whom to allocate the good which makes the informational problem less severe. This becomes striking when we assume risk neutral informed bidders. In this case, the auctioneer can implement a full extracting mechanism by assigning all the quantity to the uninformed bidders, thereby leaving no informational rents to the informed ones (Bennouri and Falconieri, 2000).

8 Appendix

Proof of Equation 4:

By applying the envelope theorem to the IIC, we have:

$$U_i'(s_i, s_i) = \frac{1}{n}\int_{\Omega_{-i}} \alpha q_i(s) f_{-i}(s_{-i}) ds_{-i}$$

thus,

$$U_i(s_i, s_i) = U_i(\underline{s}, \underline{s}) + \int_{\underline{s}}^{s_i} \left\{ \frac{1}{n}\int_{\Omega_{-i}} \alpha q_i(\widetilde{s}_i, s_{-i}) f_{-i}(s_{-i}) ds_{-i} \right\} d\widetilde{s}_i.$$

For the IPC to be satisfied we can simply set $U_i(\underline{s}, \underline{s}) = 0$. This assumption means that the seller will leave no rents to the informed traders with the lowest evaluations.

Applying Fubini's theorem to the last equation yields:

$$U_i(s_i, s_i) = \frac{1}{n}\int_{\Omega_{-i}} \left\{ \int_{\underline{s}}^{s_i} \alpha q_i(\widetilde{s}_i, s_{-i}) d\widetilde{s}_i \right\} f_{-i}(s_{-i}) ds_{-i}.$$

Integrate over Ω_i, and apply again Fubini's theorem. We get the following

$$\int_{\Omega_i} U_i(s_i,s_i)f_i(s_i)ds_i = \frac{1}{n}\int_{\Omega_{-i}}\left\{\int_{\Omega_i}\left[\int_{\underline{s}}^{s_i}\alpha q_i(\widetilde{s}_i,s_{-i})d\widetilde{s}_i\right]f_i(s_i)ds_i\right\}f_{-i}(s_{-i})ds_{-i}.$$

Now, after having integrated by parts the integral $\int_{\Omega_i}\left[\int_{\underline{s}}^{s_i}\alpha q_i(\widetilde{s}_i,s_{-i})d\widetilde{s}_i\right]f_i(s_i)ds_i$, yields the following equation:

$$\int_{\Omega_i}\left[\int_{\underline{s}}^{s_i}\alpha q_i(\widetilde{s}_i,s_{-i})d\widetilde{s}_i\right]f_i(s_i)ds_i \quad = $$

$$\left[\left\{\int_{\underline{s}}^{s_i}\alpha q_i(\widetilde{s}_i,s_{-i})d\widetilde{s}_i\right\}(F_i(s_i)-1)\right]_{\underline{s}}^{\overline{s}} - \int_{\Omega_i}\alpha q_i(s_i,s_{-i})(F_i(s_i)-1)ds_i.$$

from which, Equation 4 is straightforward.

Proof of Lemma 1:

A sufficient condition to have

$$v_{-i} \leq \frac{\alpha\overline{s}-\underline{s}(2\alpha-1)}{(\alpha-1)} = v^{\circ} \qquad \forall s_{-i}$$

is that the following holds:

$$v_{-i} \leq (n-1)(\alpha-1)\overline{s} \leq \alpha\overline{s}-\underline{s}(2\alpha-1) \qquad \forall s_{-i}.$$

By re-arranging, the above inequality can be re-interpreted as a (sufficient) condition on the support of the signals,

$$n(\alpha-1)\overline{s} \leq (2\alpha-1)\Delta s \Longrightarrow \overline{s} \leq \frac{2\alpha-1}{n(\alpha-1)}\Delta s$$

■

Proof of Proposition 1:

By the definition of the two sets $N^-(s)$ and $N^+(s)$, we know that for each $i \in N^-(s)$ it must be $q_i(s) = 0$.

For each $i \in N^+(s)$, define the quantity $\widetilde{q}_i$ as the one solving the equation below,

$$(\alpha-1)v - \delta\widetilde{q}_i(s) - \frac{\alpha}{n}(\overline{s}-s_i) = 0$$

By the concavity of function H (with respect to q_i), $\widetilde{q}_i(s)$ is strictly positive for each $i \in N^+(s)$. Now, if

- $\sum_{i \in N^+(s)} \widetilde{q}_i(s) \leq 1$

  the quantity $\widetilde{q}_i(s)$ is the solution of our mechanism.[21]

- $\sum_{i \in N^+(s)} \widetilde{q}_i(s) > 1$,

  this case corresponds to a situation of *oversubscription* of the new shares. The quantity $\widetilde{q}_i(s)$ cannot thus be the optimal solution because it violates the FAC, so the optimal mechanism is given by the solution of the following system of equations that we denote by $\widehat{q}_i(s)$:

$$\begin{cases} \widehat{q}_i(s)[(\alpha-1)v - \delta\widehat{q}_i(s) - \frac{\alpha}{n}(\overline{s} - s_i) - \beta(s)] = 0 \\ \sum_{i \in N^+(s)} \widehat{q}_i(s) = 1; \qquad \beta(s) > 0, \end{cases}$$

  which implies that $\widehat{q}_i(s)$ is either zero or is positive and solves the following equation

$$(\alpha-1)v - \delta\widehat{q}_i(s) - \frac{\alpha}{n}(\overline{s} - s_i) - \beta(s) = 0$$

  Notice that, in this case, all the share are allotted to institutional investors. Retail investors get nothing in equilibrium. ■

Proof of Proposition 4:

A necessary and sufficient condition for the price schedule to be uniform is the following:

$$\frac{\partial p_i^0(s)}{\partial s_i} + \frac{\partial \Phi_i(s)}{\partial s_i} = \frac{\partial p_i^0(s)}{\partial s_j} + \frac{\partial \Phi_i(s)}{\partial s_j}. \tag{14}$$

From equation (15), by differentiating $p_i^0(s)q_i(s)$ with respect to s_i and replacing to $z(q,v) = q[\alpha v - (\delta/2)q]$ we get

$$\frac{\partial p_i^0(s)}{\partial s_i} q_i(s) = \frac{\partial q_i(s)}{\partial s_i}\left[-p_i^0(s) + \alpha v - \delta q_i(s)\right] \tag{15}$$

and

$$\frac{\partial p_i^0(s)}{\partial s_j} q_i(s) = \frac{\partial q_i(s)}{\partial s_j}\left[-p_i^0(s) + \alpha v - \delta q_i(s)\right] + \frac{\alpha}{n}\left[q_i(s) - \int_{\underline{s}}^{s_i} \frac{\partial q_i(\widetilde{s}_i, s_{-i})}{\partial s_j} d\widetilde{s}_i\right]. \tag{16}$$

[21] In this case, the equation above defines the FOC of our objective function H, since the Kuhn-Tucker multipliers, $\lambda_i(s)$ and $\beta(s)$ are both zero.

Now, multiplying both sides of equation (15) by $q_i(s)$ and replacing from equations (20) and (21) yields the following

$$\frac{\partial q_i(s)}{\partial s_i}\left(-p_i^0(s)+\alpha v-\delta q_i(s)\right)+\frac{\partial \Phi_i(s)}{\partial s_i}q_i(s)=$$
$$=\frac{\partial q_i(s)}{\partial s_j}\left(-p_i^0(s)+\alpha v-\delta q_i(s)\right)+\frac{\alpha}{n}\left[q_i(s)-\int_{\underline{s}}^{s_i}\frac{\partial q_i(\widetilde{s}_i,s_{-i})}{\partial s_j}d\widetilde{s}_i\right]+\frac{\partial \Phi_i(s)}{\partial s_j}q_i(s).$$

or equivalently

$$\left(-p_i^0(s)+\alpha v-\delta q_i(s)\right)\left(\frac{\partial q_i(s)}{\partial s_i}-\frac{\partial q_i(s)}{\partial s_j}\right)-\frac{\alpha}{n}\left[q_i(s)-\int_{\underline{s}}^{s_i}\frac{\partial q_i(\widetilde{s}_i,s_{-i})}{\partial s_j}d\widetilde{s}_i\right]+$$
$$\frac{\partial \Phi_i(s)}{\partial s_i}q_i(s)=\frac{\partial \Phi_i(s)}{\partial s_j}q_i(s) \tag{17}$$

We now prove the following result:

Lemma 2 *The optimal quantity schedule satisfies the following condition*

$$\frac{\partial q_i(s)}{\partial s_i}-\frac{\partial q_i(s)}{\partial s_j}=\frac{\alpha}{n\delta}$$

for each s and each i.

The result is trivial when the optimal quantity is $q_i(s)=\widetilde{q}_i(s_i)$.

When $q_i(s)=\widehat{q}_i(s_i)$, notice that the following holds:

$$\frac{\partial \widehat{q}_i(s)}{\partial s_i}=\frac{\partial \widetilde{q}_i(s)}{\partial s_i}-\frac{1}{\delta}\frac{\partial \beta(s)}{\partial s_i}$$

and

$$\frac{\partial \widehat{q}_i(s)}{\partial s_j}=\frac{\partial \widetilde{q}_i(s)}{\partial s_j}-\frac{1}{\delta}\frac{\partial \beta(s)}{\partial s_j}$$

By the definition of $\beta(s)$, we can easily check that $\frac{\partial \beta(s)}{\partial s_i}=\frac{\partial \beta(s)}{\partial s_j}$ which finally proves the result. ■

By Lemma 2, we can re-write equation (22) as follows

$$\frac{\alpha}{n}\left\{\frac{\left[-p_i^0(s)+\alpha v-\delta q_i(s)\right]}{\delta q_i(s)}-1+\frac{\int_{\underline{s}}^{s_i}\frac{\partial q_i(\widetilde{s}_i,s_{-i})}{\partial s_j}d\widetilde{s}_i}{q_i(s)}\right\}+\frac{\partial \Phi_i(s)}{\partial s_i}=\frac{\partial \Phi_i(s)}{\partial s_j}.$$

replacing $p_i^0(s)$ by its value, as defined by equation (14), yields

$$\frac{\alpha}{n}\left\{-\frac{3}{2}+\frac{\alpha}{n\delta}\frac{\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i,s_{-i})d\widetilde{s}_i}{[q_i(s)]^2}+\frac{\int_{\underline{s}}^{s_i}\frac{\partial q_i(\widetilde{s}_i,s_{-i})}{\partial s_j}d\widetilde{s}_i}{q_i(s)}\right\}+\frac{\partial\Phi_i(s)}{\partial s_i}=\frac{\partial\Phi_i(s)}{\partial s_j}.$$

by using again Lemma 2 to replace $\frac{\alpha}{n\delta}$ by $\frac{\partial q_i(s)}{\partial s_i}-\frac{\partial q_i(s)}{\partial s_j}$, after some simple computations, we get

$$\frac{\partial}{\partial s_i}\left[\Phi_i(s)-\frac{\alpha}{n}\frac{\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i,s_{-i})d\widetilde{s}_i}{q_i(s)}\right]=\frac{\partial}{\partial s_j}\left[\Phi_i(s)-\frac{\alpha}{n}\frac{\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i,s_{-i})d\widetilde{s}_i}{q_i(s)}\right]+\frac{\alpha}{2n}, \tag{18}$$

for each i and each j. This is a partial differential equation in $\Phi_i(s)-\frac{\alpha}{n}\frac{\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i,s_{-i})d\widetilde{s}_i}{q_i(s)}$ whose generic solution is given by

$$\Phi_i(s)=\varphi(s_1+...+s_n)+\frac{\alpha}{n}\frac{\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i,s_{-i})d\widetilde{s}_i}{q_i(s)}+\frac{\alpha}{2n}s_i,\qquad\forall\ i$$

where $\varphi(\cdot)$ is a twice differentiable function defined on the set $[n\underline{s},n\overline{s}]=n\Omega$.

Summing up, so far we have shown that the optimal mechanism may be implemented by a uniform price schedule if and only if

$$p_i(s)=p_i^0(s)+\varphi(s_1+...+s_n)+\frac{\alpha}{n}\frac{\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i,s_{-i})d\widetilde{s}_i}{q_i(s)}+\frac{\alpha}{2n}s_i$$

where φ is a twice differentiable function satisfying the following integral equation

$$\int_{\Omega_{-i}}\left[\varphi(s_1+...+s_n)+\frac{\alpha}{n}\frac{\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i,s_{-i})d\widetilde{s}_i}{q_i(s)}+\frac{\alpha}{2n}s_i\right]q_i(s_i,s_{-i})f_{-i}(s_{-i})ds_{-i}=0.$$

or alternatively

$$\begin{gathered}\int_{\Omega_{-i}}\varphi(s_1+...+s_n)q_i(s_i,s_{-i})f_{-i}(s_{-i})ds_{-i}=\\-\int_{\Omega_{-i}}\left[\frac{\alpha}{n}\frac{\int_{\underline{s}}^{s_i} q_i(\widetilde{s}_i,s_{-i})d\widetilde{s}_i}{q_i(s)}+\frac{\alpha}{2n}s_i\right]q_i(s_i,s_{-i})f_{-i}(s_{-i})ds_{-i}=g(s_i).\end{gathered} \tag{19}$$

Henceforth, proving the existence of a uniform price schedule is equivalent to prove the existence of such a function φ.

The proof proceeds as follows. The main step consists in showing that equation (19) can be written as a Volterra integral equation of the first kind.[22] Given that, by applying the general properties of this kind of integral equations we can prove the existence as well as the *uniqueness* of a function φ.

To do this, we first need to transform equation (19) into a simple integral equation, i.e. with the support defined on R.

Notice that, since $q_i(s_i, s_{-i}) = 0$ when $s_i \leq s_i^0(v_{-i})$, the support of the integral equation (19) is equal to $\Omega_{-i}^0 = \{(s_1, s_2, .., s_{i-1}, s_{i+1}, .., s_n) \mid \sum_{j \neq i} s_j = v_{-i} \geqslant v_{-i}^0(s_i)\}$ where $v_{-i}^0(s_i)$ is defined as the inverse of $s_i^0(v_{-i})$, i.e.

$$v_{-i}^0(s_i) = \frac{\alpha\overline{s} - (2\alpha - 1)s_i}{\alpha - 1}$$

Also, by using the definition of the optimal quantity, the LHS term of equation (19) is equivalent to $\int_{\Omega_{-i}^0} \varphi(s_i + v_{-i})q_i(s_i, v_{-i})f_{-i}(s_{-i})ds_{-i}$.

Now, by applying the Generalized Change Variable Theorem (GVCT) we can set $v_{-i} = \gamma_i(s_{-i})$ for each i, which finally implies that $\gamma_i(\Omega_{-i}^0) = [v_{-i}^0(s_i); (n-1)\overline{s}] \in R$.[23]

A main implication of the GVCT says that, there exists a measure λ defined over $[v_{-i}^0(s_i); (n-1)\overline{s}]$ such that

$$\int_{\Omega_{-i}^0} \varphi(s_i + v_{-i})q_i(s_i, v_{-i})f_{-i}(s_{-i})ds_{-i} = \int_{v_{-i}^0(s_i)}^{(n-1)\overline{s}} \varphi(s_i + v_{-i})q_i(s_i, v_{-i})\lambda(dv_{-i}).$$

Last, by applying the Radon-Nikodým theorem[24], we are also able to prove the existence of a density function ρ associated to the measure λ such that

$$\int_{\Omega_{-i}^0} \varphi(s_i + v_{-i})q_i(s_i, v_{-i})f_{-i}(s_{-i})ds_{-i} = \int_{v_{-i}^0(s_i)}^{(n-1)\overline{s}} \varphi(s_i + v_{-i})q_i(s_i, v_{-i})\rho(v_{-i})dv_{-i}.$$

By using the result above, the integral equation (19) reduces to

$$\int_{v_{-i}^0(s_i)}^{(n-1)\overline{s}} \varphi(s_i + v_{-i})q_i(s_i, v_{-i})\rho(v_{-i})dv_{-i} = g(s_i)$$

[22] A Volterra integral equation of the the first kind is defined in the following way:

$$\int_{y_0}^{\tau(x)} f(x, y)h(x, y)dy = g(x)$$

in other words, one of the integral extreme must depend on the variable x.

[23] See Dunford and Schwartz (1988, 3rd Ed.), chapter 3, lemma 8, page 182.

[24] See, for example, Dunford and Schwartz (1988), chapter 3, theorem 2, page 176 in the third edition.

This is a Volterra integral equation of the first kind which ensures that, as far as the function g is well behaved, a solution in φ always exists. ■

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