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On the Duality Theory of Convex Objects

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Introduction

The most remarkable property of convex objects is that they can be viewed in two ways, primal and dual. The transition between these viewpoints is given by so called duality operators. The oldest example is the well-known duality between points and lines in the two-dimensional plane. There are many other examples of duality operators such as the polar of a convex set containing the origin, the dual norm, the Legendre-Young-Fenchel-transform of a convex function and the conjugate of a convex cone. Moreover convex sets and sublinear functions are dual objects; the correspondence is given in one direction by the support function and in the other one by the subdifferential. The theory of each of these operators is of great interest, not only because of its fundamental character and inherent beauty, but also because there are many applications.(see cite: M--T)

In this paper we offer a new unified approach to the theories of these duality operators. The resulting theorems are sharper than those in the literature. All results in this paper hold for arbitrary vector spaces, finite or infinite dimensional, which moreover do not have to be provided with a topology. Furthermore the only assumption which has to be made is the existence of relatively internal points for the convex objects involved.

We proceed as follows: we prove a generalized Hahn-Banach theorem for convex cones and derive from this result generalized Fenchel-Moreau and Dubovitsky-Milyutin theorems. These three results form the duality theory for convex cones. Then we derive the duality theory for other types of convex objects in the following simple way. For each type we give a construction to associate to an object of that type a convex cone, which still contains all the information of the original object. For this convex cone the results of the duality theory for convex cones are available. It is for each type straightforward to decode these results in terms of the original convex objects. To be more specific, on inspection, the conjugate of the associated convex cone turns out to be the convex cone associated to another type of convex object. Then the resulting transition from one type of convex object to another turns out to be the classical duality operator for that pair of types. Moreover, for the purpose of decoding, one has to translate the operations 'sum' and 'intersection' from the associated convex cones to operations on the underlying convex objects.

Standard results and definitions

Let V be a vectorspace (we consider only *real* vectorspaces). For a linear subspace W of V the quotientspace V/W can be characterized as follows: there is a pair $(V/W, \pi)$ consisting of a vectorspace V/W and a surjective linear mapping π from V to V/W with kernel W. The codimension of W in V, denoted by mbox $_V(W)$, is defined to be the dimension of the quotientspace V/W.

The dual vectorspace V' is the vectorspace consisting of all linear functionals on V. Taking duals preserves exactness. Explicitly, for each linear subspace W of a vector space V one can view $(V/W)'$ as a linear subspace of V' and the quotientspace is isomorphic to W'; to be more precise, the inflation map *inf* from $(V/W)'$ to V' is injective, the restriction map *res* from V' to W' is surjective and the image of *inf* equals the kernel of *res*. Moreover the natural map i from V to its bidual $V'' = (V')'$ given by $i(v)(\varphi) = \varphi(v) ::: \forall v \in V$ $\forall \varphi \in V'$ is injective.

Let V be a vectorspace and let S be an arbitrary subset of V. The linear span of S, denoted by mbox (S), is the smallest linear subspace of V containing S. An element s of S is a *relatively internal point* of S if

$$\forall d \in \text{ mbox } (S)\ \exists \varepsilon > 0\ \forall \tau \in (0, \varepsilon] : s + \tau d \in S.$$

The set of all relatively internal points of S is denoted by mbox (S). If S is moreover a convex

set, then an element $s \in S$ belongs to mbox (S) if and only if
$\forall d \in$ mbox $(S){:}\exists \varepsilon > 0 : s + \varepsilon d \in S$.

The – algebraic – closure $\bar{S}$ of S is defined to be the set consisting of all vectors $v \in V$ such that

$$\exists s \in S{:}\forall \alpha \in (0,1){:}\alpha v + (1-\alpha)s \in S$$

The set S is called – algebraically – closed if $\bar{S} = S$. We observe that $\bar{S}$ is closed for each set S.

Let V be a vectorspace. A nonempty subset C of V is called a convex cone in V if it is closed under addition and under multiplication with nonnegative scalars. The convex cone C is called *pointed* if it does not contain any line through 0. One calls C *solid* if mbox $(C) = V$

Let V be a vectorspace and let C be a convex cone in V. The *conjugate cone* or *dual cone* $(C,V)'$, corresponding to this cone C, is the convex cone in the dual vector space V' which consists of all functionals $\varphi : V \to \mathbb{R}$ satisfying $\varphi(c) \geq 0{:}\forall c \in C$.

The main point of the following lemma is that each convex cone is the sum of a pointed convex cone and a linear subspace. Moreover it describes how this decomposition carries over to the conjugate cone.

Lemma *Let V be a vector space and C a convex cone in V. Then the following statements hold true.*

i *There are linear subspaces W_1, W_2, W_3 of V and there exists a pointed,solid convex cone D in W_2 such that $V = W_1 \oplus W_2 \oplus W_3$ and $C = W_1 + D$. Here the vectorspace W_1 (resp. $W_1 \oplus W_2$) is uniquely determined as the maximal (resp. minimal) linear subspace of V which is contained in C (resp. contains C).*

ii *$(\bar{C},V)'$ equals $(C,V)'$.*

iii *Assume $\bar{C} = C$ and let W_1, W_2, W_3 and D be as in (**i**). Then $V' = W_1' \oplus W_2' \oplus W_3'$ and $(C,V)' = W_3' + (D,W_2)'$; moreover $(D,W_2)'$ is a pointed, convex cone in W_2'.*

This result suggests that for the analysis of the conjugate cone it is useful to view the set of its nonzero elements $(C,V)' \setminus \{0\}$ as the union of two disjoint subsets: its *singular part* $S(C,V)$ defined by

$$S(C,V) := \{\varphi \in V' \setminus \{0\} \mid \varphi(c) = 0{:}\forall c \in C\}$$

and its *regular part* $R(C,V)$ defined by

$$R(C,V) := (C,V)' \setminus S(C,V) \cup \{0\}.$$

The connection between these definition and the lemma above is as follows: $S(C,V) \cup \{0\}$ is the maximal linear subspace of V' which is contained in $(C,V)'$. Therefore it follows from the lemma above that $S(C,V) \cup \{0\}$ is isomorphic to the dual vector space $(V/$ mbox $(C))'$.

The conjugate cone and its regular and singular elements allow the following geometrical interpretation. The set of rays of the conjugate cone $(C,V)'$ corresponds bijectively to the set of pairs consisting of a linear subspace in V of codimension 1 with C on one of its two sides together with a choice of side which contains C: for each $\varphi \in (C,V)'\backslash\{0\}$ we associate to the ray $\mathbb{R}^+\varphi$ the linear subspace $\{v \in V | \varphi(v) = 0\}$ together with its side $\{v \in V | \varphi(v) \geq 0\}$. For each $\varphi \in (C,V)'\backslash\{0\}$ one has $\varphi \in S(C,V)$ precisely if $C \subseteq \ker\varphi$. If mbox $(C) \neq \emptyset$ one has $\varphi \in R(C,V)$ precisely if $\ker\varphi \cap$ mbox $(C) = \emptyset$.

Now we are going to recall the well-known classification of closed convex cones in a two-dimensional vectorspace. Moreover we recall the explicit description of their conjugates. It is convenient to do this for the complex plane $\mathbb{C}$ viewed as a two-dimensional vector space. We define for $\phi_1, \phi_2 \in \mathbb{R}$ with $0 \leq \phi_2 - \phi_1 \leq \pi$ the convex cone C_{ϕ_1,ϕ_2} in $\mathbb{C}$ by
$C_{\phi_1,\phi_2} = \{re^{i\phi} | r \in \mathbb{R}^+{:}\&{:}\phi_1 \leq \phi \leq \phi_2\}$.

We identify the dual vector space $(\mathbb{C})'$ with $\mathbb{C}$ by letting $T \in (\mathbb{C})'$ and $w \in \mathbb{C}$ correspond if and only if $T(z) = Re(w\bar{z}):\forall w \in W$. Here $^-$ denotes complex conjugate and Re denotes 'real part'.

Lemma

i *Each closed convex cone C in $\mathbb{C}$ with $C \neq$ mbox (C) is of the form C_{ϕ_1,ϕ_2} for suitable $\phi_1, \phi_2 \in \mathbb{R}$ with $0 \leq \phi_2 - \phi_1 \leq \pi$.*

ii *Let $\phi_1, \phi_2 \in \mathbb{R}$ with $0 \leq \phi_2 - \phi_1 \leq \pi$, then the conjugate cone $(C_{\phi_1,\phi_2}, \mathbb{C})'$ equals $C_{\phi_2 - \frac{1}{2}\pi, \phi_1 + \frac{1}{2}\pi}$*

Furthermore we record the following easy fact.

Lemma *Let V be a vector space and C a convex cone in V with mbox $(C) \neq \emptyset$. Then $C +$ mbox $(C) \subseteq C$.*

Finally we introduce some more notation. For each function $f : V \to \overline{\mathbb{R}} = \mathbb{R} \cup [\infty, -\infty]$ let the function $\overline{f} : V \to \overline{\mathbb{R}}$ be defined by $epi{:}\overline{f} = \overline{epi{:}f}$. For each subset S of a vectorspace V its convex hull coS is the smallest convex subset of V containing S.

Convex cones

We begin by stating and proving the central result of this paper.

Theorem (Generalized Theorem of Hahn-Banach) *Let V be a vector space and let C be a convex cone in V satisfying $C \neq V$, and mbox $(C) \neq \emptyset$. Then the following statements hold.*

i *There exists at least one regular element, i.e. $R(C,V) \neq \emptyset$, if and only if mbox $(C) \neq C$.*

ii *There exists at least one singular element, i.e. $S(C,V) \neq \emptyset$, if and only if mbox $(C) \neq V$.*

iii *The dual cone $(C,V)'$ has at least one nonzero element.*

iv *Let L be a linear subspace of V which contains at least one relatively internal point of C, i.e. $L \cap$ mbox $(C) \neq \emptyset$. Then $L \subsetneq C$ implies $(C \cap L, L)' \neq 0$ and furthermore, each element of the dual cone $(C \cap L, L)'$ can be extended to an element of the dual cone $(C,V)'$.*

Proof *The heart of the proof is the verification of the following statement.*

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*Choose a linear subspace W of V with $H \subset W$, mbox $_W(H) = 2$ and $c_0 \in W \cap$ mbox (C). Let j be the natural map from W to the quotient space W/H. We claim that the convex cone $j(C \cap W)$ is a strict subset of the space W/H. Indeed, from $C +$ mbox $(C) \subseteq$ mbox (C) and $H \cap$ mbox $(C) = \emptyset$ we conclude $(c_0 + C) \cap H = \emptyset$. Hence, we find $j(-c_0) \notin j(C \cap W)$. Therefore, $j(C \cap W)$ is a convex cone in the twodimensional vectorspace $j(W)$ with $j(c_0) \in$ mbox $(j(C \cap W))$ and $-j(c_0) \notin j(C \cap W)$. Now we use some standard facts from the previous section. In the first place it follows that there exists an element $\varphi \in (j(C \cap W), j(W))'$ with $\varphi(j(c_0))\ > 0$. As $c_0 \in$ mbox (C) it follows that $\varphi \circ j(c) > 0{:}{:}{:}\forall c \in$ mbox (C). Therefore the choice $K = ker\,\varphi \circ j$ satisfies $K \cap$ mbox $(C) = \emptyset$. Moreover $H \subseteq K$ as $H = ker\,j$ and $K = ker\,\varphi \circ j$. Furthermore mbox $_W(H) = 2$ and mbox $_V(H) = 1$ and so mbox $_K(H) = 1$. This finishes the verification of statement (ref: *).*

In addition, we have to verify the following two statements.

$$C \neq \text{ mbox } (C) \quad \Leftrightarrow \quad \text{mbox } (C) \cap (-C) = \emptyset \qquad \#$$

and

$$L \cap \ mbox\,(C) \neq \emptyset \quad \Rightarrow \quad L \cap \ mbox\,(C) \ = \ mbox\,(L \cap C). \qquad \#$$

*To prove (ref: **) it suffices to check that* $mbox\,(C) \cap (-C) \neq \emptyset$ *implies* $C = \ mbox\,(C)$*. The rest is obvious. Choose* $c_0 \in \ mbox\,(C) \cap (-C)$*. As* $0 = (-c_0) + c_0$ *it follows that* $0 \in C + \ mbox\,(C)$*. Therefore, using the inclusions* $C + \ mbox\,(C) \subseteq C$ *– see section* 2 *– we can conclude that* $0 \in \ mbox\,(C)$*. As* C *is a convex cone it follows that* $C = \ mbox\,(C)$*.*

*To prove (ref: ***) it suffices to check that* $L \cap \ mbox\,(C) \neq \emptyset$ *implies the inclusion* $L \cap \ mbox\,(C) \subseteq \ mbox\,(L \cap C)$*. The other inclusion is obvious. Choose* $c_0 \in L \cap \ mbox\,(C)$*. Then for each* $v \in L \cap \ mbox\,(C)$ *there exists* $\varepsilon > 0$ *with* $c_0 + \varepsilon v \in L \cap C$*; therefore* $v = \frac{1}{\varepsilon}((c_0 + \varepsilon v) - c_0)$ *lies in* $mbox\,(L \cap C)$*.*

Now we are ready to prove all the statements of the theorem.

We start with the proof of i. Assume $C \neq \ mbox\,(C)$*. Then statement (ref: **) implies* $mbox\,(C) \cap (-C) = \emptyset$*, so we can choose* $c_0 \in \ mbox\,(C)$ *with* $-c_0 \notin C$*. The collection of linear subspaces of* V *which are disjoint from* $mbox\,(C)$ *can be ordered by inclusion. Zorn's lemma implies that this collection has a maximal element. By (ref: *), the codimension, with respect to* V*, of each such maximal element is equal to* 1*. It follows, using the geometric interpretation of the elements of* $R(C,V)$ *given in section* 2*, that* $R(C,V)$ *is nonempty. Now we prove the converse implication* $R(C,V) \neq \emptyset \quad \Rightarrow \quad C \neq \ mbox\,(C)$*. Assume* $R(C,V) \neq \emptyset$*. Choose* $\varphi \in R(C,V)$ *and* $c_0 \in \ mbox\,(C)$*. Then* $\varphi(c_0) > 0$ *i.e.* $\varphi(-c_0) < 0$ *and so* $-c_0 \notin C$*, using* $\varphi \in (C,V)'$*. Moreover* $-c_0 \in \ mbox\,(C)$*. It follows that* $C \neq \ mbox\,(C)$*.*

In order to prove statement ii of the theorem, we assume $mbox\,(C) \neq V$*. Then* $(V/\ mbox\,(C), V)' \neq \{0\}$*, that is,* $S(C,V) \neq \emptyset$*. The converse implication* $S(C,V) \neq \emptyset \quad \Rightarrow \quad mbox\,(C) \neq V$ *is obvious.*

In order to prove statement iii of the theorem, we argue by contradiction. Assume $(C,V)' = 0$*, then* $R(C,V) = \emptyset$ *and* $S(C,V) = \emptyset$*. Therefore, by* (i) *and* (ii)*,* $C = \ mbox\,(C)$ *and* $mbox\,(C) = V$*. Hence* $C = V$*.*

Now, for the proof of statement iv of the theorem, let L *be a linear subspace of* V *with* $L \cap \ mbox\,(C) \neq \emptyset$*.*

In order to prove the first part of statement iv, assume $L \subsetneq C$*. Then* $C \cap L$ *is a convex cone in* L *satisfying both* $C \cap L \neq L$ *and* $mbox\,(C \cap L) = \ mbox\,(C) \cap L \neq \emptyset$*. Hence, statement iii implies that* $(C \cap L, L)' \neq 0$*.*

Next, choose $\psi \in (C \cap L, L)' \backslash \{0\}$*. We want to prove the existence of an extension of* ψ *to an element* $\varphi \in (C,V)'$*. Consider the following two cases.*

***Case* 1:** $\boldsymbol{\psi \in R(C \cap L, L)}$

The collection of linear subspaces of V *which are disjoint from* C *and which contain* $\ker\psi$ *can be ordered by inclusion. By Zorn's lemma this collection has a maximal element. By (ref: *) this maximal element has codimension* 1 *with respect to* V*. Therefore this maximal element corresponds to a ray of* $R(C,V)$*, using the geometric interpretation of* $R(C,V)$ *(see section* 2*). It is readily seen that one of the elements of this ray is the required extension* $\varphi \in R(C,V)$ *of the given element* $\psi \in R(C \cap L, L)$*.*

***Case* 2:** $\boldsymbol{\psi \in S(C \cap L, L)}$

*Statement ref: *** implies that the natural map from $L/$ mbox $(C \cap L)$ to $V/$ mbox (C) is injective. Therefore the induced restriction map from $(V/\text{ mbox }(C))'$ to $(L/\text{ mbox }(C \cap L))'$ is surjective. That is, the natural restriction map from $S(C, V)$ to $S(C \cap L, L)$ is surjective, hence $\psi \in S(C \cap L, L)$ can be extended, as desired, to an element $\varphi \in S(C, V)$.*

Now, we derive two results from this theorem.

For a a convex cone C in a vector space V, it is sometimes convenient to write C' for $(C, V)'$ and C'' for $(C', V')'$. We recall that i is the natural injective map from V to V''.

Theorem (Generalized Theorem of Fenchel-Moreau) *Let V be a vector space and let $C \subset V$ be a convex cone which has a relatively internal point, i.e. mbox $(C) \neq \emptyset$. Then*

$$C'' \cap i(V) = i(\overline{C}).$$

Proof *Since $i(\overline{C}) \subseteq C'' \cap i(V)$ is obvious, it suffices to prove that for each $v \in V \setminus \overline{C}$ there exists $\varphi \in (C, V)'$ with $\varphi(v) > 0$. We only consider the case $C \neq$ mbox (C), since the other case is obvious. By statement (ref: **) in the proof of theorem 3.1, we have mbox $(C) \cap (-C) = \emptyset$. Choose $c_0 \in$ mbox $(C) \setminus (-C)$ and consider the convex cone $C \cap (\mathbb{R}v + \mathbb{R}c_0)$ in the vector space $\mathbb{R}v + \mathbb{R}c_0$. Choose $\psi \in (C \cap (\mathbb{R}v + \mathbb{R}c_0), \mathbb{R}v + \mathbb{R}c_0)'$ with $\psi(v) < 0$ and $\psi(c_0) > 0$. This can be extended to an element φ of $(C, V)'$ by the Generalized Theorem of Hahn-Banach.*

Remark *Let C_1, C_2 be convex cones in a vectorspace V, then $(C_1 + C_2, V)' = (C_1, V)' \cap (C_2, V)'$. This is immediate from the definitions.*

For the dual cone $(C_1 \cap C_2, V)'$ one has the following result.

Theorem (Generalized Theorem of Dubovitsky-Milyutin) *Let C_1, C_2 be convex cones in a vector space V with mbox $(C_1) \cap$ mbox $(C_2) \neq \emptyset$. Then*

$$(C_1 \cap C_2, V)' = (C_1, V)' + (C_2, V)'.$$

Proof *Let Δ be the diagonal subspace $\{(v, v) \mid v \in V\}$ in $V \times V$. It suffices to prove that the natural restriction map from $(C_1 \times C_2, V \times V)'$ to $((C_1 \times C_2) \cap \Delta, \Delta)'$ is surjective, as $((C_1 \times C_2) \cap \Delta, \Delta)'$ is isomorphic to $(C_1 \cap C_2, V)'$ by the definitions and as this restriction map factorizes by way of the inclusion map from $(C_1, V)' + (C_2, V)'$ into $(C_1 \cap C_2, V)'$. The desired surjectivity follows from the Generalized Theorem of Hahn-Banach with $V := V \times V$, $C := C_1 \times C_2$ and $L := \Delta$: indeed, choose $c \in$ mbox $(C_1) \cap$ mbox (C_2) then $(c, c) \in$ mbox $(C_1 \times C_2) \cap \Delta$, so mbox $(C_1 \times C_2) \cap \Delta \neq \emptyset$ and so the Generalized Theorem of Hahn-Banach can be applied.*

Linear subspaces

Let V be a vectorspace and let L be a linear subspace of this vectorspace. Let $(L, V)^{\perp}$ be the linear subspace of V' which consists of all linear functionals on V which are zero on L, that is, it is the kernel of the restriction map from V' to L'. One calls $(L, V)^{\perp}$ the *annihilator* of L in V.

Theorem (Hahn-Banach for linear subspaces) *Let L be a linear subspace of a vectorspace V.*

i $(L, V)^{\perp} \neq 0$ *if and only if* $L \neq V$.

ii *Let M be another linear subspace of V. Then $M \subsetneq L$ implies that $(L \cap M, M)^{\perp} \neq 0$. Moreover each element of $(L \cap M, M)^{\perp}$ can be extended to an element of $(L, V)^{\perp}$.*

We write $L^{\perp}$ for $(L, V)^{\perp}$ and $L^{\perp\perp}$ for $(L^{\perp}, V')^{\perp}$ in a context where the ambient vectorspace does not change. We recall that i denotes the natural injective map from V to V''.

Theorem (Fenchel-Moreau for linear subspaces) *Let L be a linear subspace in a* *vectorspace V. Then* $L^{\perp\perp} \cap i(V) = i(L)$.

Theorem (Dubovitsky-Milyutin for linear subspaces) *Let* L_1, L_2 *be linear subspaces in a vectorspace V. Then* $(L_1 \cap L_2)^{\perp} = L_1^{\perp} + L_2^{\perp}$.

Remark *These results follow from the results on convex cones using that* $(L, V)^{\perp} = (L, V)'$

Affine subspaces

For each affine subspace $A = L + b$ of a vectorspace V – where L is a linear subspace and $b \in V$ – we let $(A, V)^{\bullet}$ be the affine subspace of V' defined as follows

$$(A, V)^{\bullet} = L^{\perp} \text{ if } b \in L$$
$$= \{\varphi \in L^{\perp} \mid \varphi(b) = -1\} \text{ if } b \notin L$$

Theorem (Hahn-Banach for affine subspaces) *Let* $A = L + b$ *be an affine subspace in a vectorspace V with* $L \neq V$. *Here L is a linear subspace of V and* $b \in V$. *Then the following statements hold.*

- $(A, V)^{\bullet} \neq 0$
- *Let M be a linear subspace of V with* $M \cap A \neq \emptyset$. *Then each element of* $(A \cap M, M)^{\bullet}$ *can be extended to an element of* $(A, V)^{\bullet}$.

Now we write $A^{\bullet}$ for $(A, V)^{\bullet}$ and $A^{\bullet\bullet}$ for $(A^{\bullet}, V')^{\bullet}$.

Theorem (Fenchel-Moreau for affine subspaces) *Let A be an affine subspace in a vectorspace V. Then* $A^{\bullet\bullet} \cap i(V) = i(A)$

Theorem (Dubovitsky-Milyutin for affine subspaces) *Let* A_1, A_2 *be affine subspaces in a vectorspace V with* $A_1 \cap A_2 \neq \emptyset$. *Then* $(A_1 \cap A_2)^{\bullet} = [co(A_1 \cup A_2)]^{\bullet}$.

Remark *These results follow from the results on linear subspaces using that*

$$(A, V)^{\bullet} \times 1 = (span(A \times 1), V \times \mathbb{R})^{\perp} \cap (V' \times 1)$$

if A is not a linear subspace.

Norms

For each norm $\| \bullet \|$ on a vectorspace V, let $V^{\star}$ be the linear subspace of all $\varphi \in V'$ for which $\sup_{\|v\|=1} \varphi(v)$ is finite and let $\| \bullet \|^{\star}$ be the norm on $V^{\star}$ defined by $\|\varphi\|^{\star} = \sup_{\|v\|=1} \varphi(v)$ for all $\varphi \in V^{\star}$.

Theorem (Hahn-Banach for norms) *Let* $(V, \| \bullet \|)$ *be a normed vectorspace and W a linear subspace of V Then each* $\psi \in W^{\star}$ *can be extended to an element* $\varphi \in V^{\star}$ *with* $\|\varphi\|_V^{\star} = \|\psi\|_W^{\star}$.

Theorem (Fenchel-Moreau for norms) *Let* $(V, \| \bullet \|)$ *be a normed vectorspace. Then the natural map* $i : V \to V''$ *sends* V *to* $V^{\star\star}$ *and* $\|i(v)\|^{\star\star} = \|v\| : \forall v \in V$.

Let A_1, A_2 be convex subsets of one vectorspace V and let $f_1 : A_1 \to \mathbb{R}$ and $f_2 : A_2 \to \mathbb{R}$ be convex functions. Then we define functions $f_1 \vee f_2 : A_1 \cap A_2 \to \mathbb{R}$ and $conv(f_1 \wedge f_2) : co(A_1 \cup A_2) \to \mathbb{R} \cup \{-\infty\}$ as follows:
$(f_1 \vee f_2)(x) = max(f_1(x), f_2(x)) : \forall x \in A_1 \cap A_2$ and
$conv(f_1 \wedge f_2)(x) = \inf_{(\alpha, x_1, x_2)} \alpha f_1(x_1) + (1 - \alpha) f_2(x_2) ::: \forall x \in co(A_1 \cup A_2)$ where (α, x_1, x_2) runs over all triplets in $[0, 1] \times A_1 \times A_2$ with $\alpha x_1 + (1 - \alpha) x_2 = x$.

Theorem (Dubovitsky-Milyutin for norms) *Let* $\| \bullet \|_1$ *and* $\| \bullet \|_2$ *be norms on one vectorspace V, then* $(\| \bullet \|_1 \vee \| \bullet \|_2)^{\star} = conv(\| \bullet \|_1^{\star} \wedge \| \bullet \|_2^{\star})$.

Remark *These theorems follow from the results on convex cones, using the following observations*:

$(epi\| \bullet \|, V \times \mathbb{R})' = epi\| \bullet \|^\star$, *$epi\| \bullet \|$ does not contain any lines, $epi\| \bullet \|$ is algebraically closed and $epi\| \bullet \|$ has a relatively internal point*.

Convex sets containing 0

For each convex subset A of a vectorspace V containing 0 the *polar* $(A, V)^\circ$ is defined to be the following convex subset of V' containing 0:

$$(A, V)^\circ = \{\varphi \in V' | \varphi(a) \leq 1 : \forall a \in A\}$$

Theorem (Hahn-Banach for convex sets containing 0) *Let A be a convex subset of a vectorspace V containing 0 and with mbox $(A) \neq \emptyset$ and let L be a linear subspace of V with $L \cap$ mbox $(A) \neq \emptyset$. Then each element of $(A \cap L, L)^\circ$ can be extended to an element of $(A, V)^\circ$.*

We write A° for $(A, V)^\circ$ and $A^{\circ\circ}$ for $(A^\circ, V')^\circ$.

Theorem (Fenchel-Moreau for convex sets containing 0) *Let A be a convex set containing 0 in a vectorspace V with mbox $(A) \neq \emptyset$. Then $A^{\circ\circ} \cap i(V) = \overline{A}$.*

Theorem (Dubovitsky-Milyutin for convex sets containing 0) *Let A_1, A_2 be two convex sets containing 0 in a vectorspace V with mbox $(A_1) \cap$ mbox $(A_2) \neq \emptyset$. Then $(A_1 \cap A_2, V)^\circ = \overline{co((A_1, V)^\circ \cup (A_2, V)^\circ)}$*

Remark *These theorems follow from the results on convex cones using* $(\overline{\mathbb{R}^+ \bullet (A \times 1)})' = \overline{\mathbb{R}^+ \bullet (-A^\circ \times 1)}$

Convex sets and sublinear functions

For each nonempty convex subset A of a vectorspace V the *support function* $s(A)$ is the sublinear function on the convex cone $D(A) = \{\varphi \in V' | sup_{a \in A}\varphi(a) < \infty\}$ in V' defined by

$$s(A)(\varphi) = sup_{a \in A}\varphi(a) : \forall \varphi \in D(A).$$

For each convex cone D in a vectorspace V and each sublinear function $p : D \to \mathbb{R}$ the *subdifferential* ∂p is the convex subset of V' defined by

$$\partial p = \{\varphi \in V' | \varphi(v) \leq p(v) : \forall v \in V\}.$$

For each subset S of a vectorspace V we define its affine relatively internal set $afrint(S)$ to be $\{s \in S | 0 \in$ mbox $(S - s)\}$.

Theorem (Hahn-Banach for convex sets and sublinear functions) *Let A be a convex subset of a vectorspace V with $afrint(A) \neq \emptyset$. Let L be an affine subspace of V with $L \cap afrint(A) \neq \emptyset$. Then $s(A \cap L)(x) \leq s(A)(x) :: \forall x \in D(A \cap L)$.*

Let D be a convex cone in a vectorspace V with mbox $(D) \neq \emptyset$, let $p : D \to \mathbb{R}$ be a sublinear function and let L be a linear subspace of V with $L \cap$ mbox $(D) \neq \emptyset$. Then each linear functional on L which is majorized by p can be extended to a linear function on V which is majorized by p.

Theorem (Fenchel-Moreau for convex sets and sublinear functions) *Let A be a convex subset of a vectorspace V with $afrint(A) \neq \emptyset$. Then $\partial s(A) \cap i(V) = i(\overline{A})$.*

Let D be a convex cone in a vectorspace V with mbox $(D) \neq \emptyset$ and let

$p : D \to \mathbb{R}$ *be a sublinear function. Then* $s(\partial p) \circ i = \overline{p}$.

Theorem (Dubovitsky-Milyutin for convex sets and sublinear functions) *Let* A_1, A_2 *be convex subsets of a vectorspace V with* $afrint(A_1) \cap afrint(A_2) \neq \emptyset$. *Then* $s(A_1 \cap A_2) = s(A_1) \vee s(A_2)$.

Let D_1, D_2 *be convex cones in a vectorspace V and let* $p_1 : D_1 \to \mathbb{R}$ *and* $p_2 : D_2 \to \mathbb{R}$ *be sublinear functions with mbox* $(D_1) \cap$ *mbox* $(D_2) \neq \emptyset$. *Then* $\partial(p_1 \cup p_2) = \overline{co(\partial p_1 \cup \partial p_2)}$.

Remark *These theorems follow from the results on convex cones, using* $(\mathbb{R}^+.(A \times 1))' = epi(s(A) \circ \sigma)$ *where* $\sigma(\varphi) = -\varphi :: \forall \varphi$ *and* $(epi{:}p)' = \overline{\mathbb{R}^+(-\partial p \times 1)}$.

Convex functions

For each convex set A in a vectorspace V and each convex function $f : A \to \mathbb{R}$ we define the convex set $D(A)$ in V' by

$$D(A) = \{\varphi \in V' | sup_{a \in A}(\varphi(a) - f(a)) \ < \infty\}$$

and the convex function $f^\star : D(A) \to \mathbb{R}$ by $f^\star(\varphi) = sup_{a\in A}(\varphi(a) - f(a))$. The function $f^\star$ is called the (*Legendre-Young*)-*Fenchel transform* of *f*.

Theorem (Hahn-Banach for convex functions) *Let A be a convex subset of a vectorspace V with* $afrint(A) \neq \emptyset$, *let* $f : A \to \mathbb{R}$ *be a convex function and let* $L \subseteq V$ *be an affine subspace with* $L \cap afrint(A) \neq \emptyset$. *Then the function* $(f|L)^\star \circ \kappa$ *is majorized by the function* $f^\star$ *where* κ *is the mapping* $V' \to L'$ *induced by the inclusion of L into V.*

Theorem (Fenchel-Moreau for convex functions) *Let A be a convex subset of a vectorspace V with* $afrint(A) \neq \emptyset$ *and let* $f : A \to \mathbb{R}$ *be a convex function. Then* $f^{\star\star} \circ i = \overline{f}$.

Theorem (Dubovitsky-Milyutin for convex functions) *Let* A_1, A_2 *be convex subsets of a vectorspace with* $afrint(A_1) \cap afrint(A_2) \neq \emptyset$ *and let* $f_1 : A_1 \to \mathbb{R}$ *and* $f_2 : A_2 \to \mathbb{R}$ *be convex functions. Then* $(f_1 \vee f_2)^\star = conv\ (f_1^\star \ \wedge \ f_2^\star)$.

Remark *These theorems follow from the results for convex cones using*

$$(\mathbb{R}^+.(epif \times 1))' = \beta(\overline{\mathbb{R}^+.(epif^\star \times 1)})$$

where $\beta(\varphi, \mu, \nu) = (-\varphi, \nu, \mu) : \forall(\varphi, \mu, \nu)$.

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addcontentsline

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