TI 2000-013/1
Tinbergen Institute Discussion Paper

Consistent Expectations Equilibria and Complex Dynamics in Renewable Resource Markets

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September 2000 (Macroeconomic Dynamics, forthcoming)

Abstract. Price fluctuations under adaptive learning in renewable resource markets such as fisheries are examined. Optimal fishery management with logistic fish population growth implies a backward-bending, discounted supply curve for bioeconomic equilibrium sustained yield. Higher discount rates bend supply backwards more to generate multiple steady state rational expectations equilibria. Under bounded rationality adaptive learning of a linear forecasting rule generates steady state, 2-cycle as well as chaotic consistent expectations equilibria (CEE), which are self-fulfilling in sample average and autocorrelations. The possibility of "learning to believe in chaos" is robust and even enhanced by dynamic noise.

Keywords: bounded rationality, adaptive learning, cobweb dynamics, chaos, optimal resource management, fishery model.

Acknowledgements. During the preparation of this paper we have benefitted from many useful discussions. In particular we would like to thank Peter Boswijk, Buz Brock, Jean-Paul Chavas, Colin W. Clark, Jon Conrad, Nancy Dávila, Cees Diks, Joanne Doyle, Gerwin Griffioen, Jean-Michel Grandmont, Michael Kopel, Scott Milliman, Michael Neugart, Scott Page, Charles Perrings, Tönu Puu, Tom Sargent and Gerhard Sorger. Comments by an anonymous referee and the special issue editor Jim Bullard have also been very useful. None of the above are responsible for any errors in this paper. Financial support by the Netherlands Organization for Scientific Research (NWO) under a Pionier grant is gratefully acknowledged.

1 Introduction

In the past decade adaptive learning models have been proposed as an alternative to rational expectations, see e.g. Evans and Honkapohja (1999) and Sargent (1993,1999) for recent surveys. In contrast to rational expectations, adaptive learning models assume that agents do not have perfect knowledge about market equilibrium equations, but have some belief, the perceived law of motion, about the true unknown actual law of motion. Usually the perceived law of motion is some parameterized model and adaptive learning simply means updating of the parameters of the perceived law of motion, e.g. by ordinary least squares, as additional observations become available. The implied actual law of motion under adaptive learning is thus a time-varying self-referential or feedback system depending upon the perceived law of motion. In this framework, a rational expectations equilibrium is simply a situation where the implied actual law of motion *exactly* coincides with the perceived law of motion, and adaptive learning may converge to such a rational expectations equilibrium.

However, convergence to a rational expectations equilibrium can only occur when the perceived law of motion is correctly specified and in the same class as the (unknown) actual law of motion. When agents believe in a misspecified model convergence to rational expectations can never occur, and the best one can hope for is that the adaptive learning process converges to an 'approximate rational expectations equilibrium' with optimal misspecified forecasts (Sargent, 1999). Such an approximate rational expectations equilibrium may be a useful concept describing a situation where agents do not understand the world in its full complexity, but have some simple perception of this complex world and try to minimize their forecasting errors within their simple view of the world.

The present paper applies the notion of *consistent expectations equilibrium (CEE)* to an optimal harvesting model of renewable resource markets, in particular fisheries. A CEE refers to a situation where the unknown law of motion is nonlinear, but agents try to forecast the complex nonlinear world with linear forecasting rules. In equilibrium these linear forecasting rules are consistent however, that is, they are correct in terms of sample mean and sample autocorrelations. A CEE may thus be seen as an 'approximate rational expectations equilibrium', where the misspecified perceived law of motion is the best linear approximation, within the class of perceived laws of motion, of the unknown true nonlinear law of motion.

At this point, let us relate the contribution of this paper to some recent literature. In his Presidential address to the Econometric Society in the early 1990s, Grandmont (1998) introduced the concept of a self-fulfilling mistake. This phenomenon can emerge when economic agents cannot distinguish between randomness and determinism, a situation that can occur when the underlying true dynamics are chaotic (e.g. Brock and Dechert (1991) and Radunskaya (1994)). Agents believe mistakenly that say prices follow a stochastic law of motion, and given their belief the actual law of motion becomes deterministically chaotic. If agents can not distinguish between randomness and chaos, their mistake becomes self-fulfilling. Bunow and Weiss (1979) and Sakai and Tokumaru (1980) showed that a simple stochastic AR(1) model can mimic the behavior of a chaotic tent map. Hommes and Sorger (1998), following earlier work in Hommes (1998) and Sorger (1998), applied these results and introduced the notion of a consistent expectations equilibrium (CEE), where agents believe that prices follow a linear AR(1) stochastic process, whereas the implied actual law of motion is a deterministic chaotic map. Along a CEE price realizations have the same sample mean and sample autocorrelation coefficients at all leads and lags as the AR(1) process. Hommes and Sorger (1998) find three types of CEE, steady state, 2-cycle as well as chaotic CEE, and they show that agents can learn to converge on each of these CEE's.

Related work on complicated 'approximate rational expectations' equilibria under learning includes work by Bullard (1994) and Schönhofer (1999, 2000), who show periodic and even chaotic dynamics under adaptive learning in an OLG-model with inflation. In another related macroeconomic application, Arifovic (1996) reports evidence of fluctuations in exchange rates driven by genetic algorithm learning in an overlapping generations economy with two currencies. Sögner and Mitlöhner (2000) have recently applied the concept of CEE in a standard asset pricing model.

In microeconomics it has long been understood that cobweb adjustment models can generate chaotic dynamics when agents have adaptive expectations, even when demand and supply are monotonic curves (Chiarella, 1988; Hommes, 1994). Another related model is the backward-bending supply curve of labor due to Bolle and Neugart (1998). They demonstrate the possibility of chaotic dynamics in this model, but do not fully work out the self-fulfilling mistake implications. However, as stressed in Hommes (1998), along these chaotic fluctuations expectational errors have significant autocorrelations, and boundedly rational agents might take advantage of those and revise expectations accordingly. Hommes and Sorger show that in the cobweb model with monotonic demand and supply curves with agents using an AR(1) forecasting rule, the only CEE is the rational expectations steady state. However, Hommes and Sorger (1998) also show for the case of a backward-bending supply curve in which the true underlying dynamics are an asymmetric tent map that a simple AR(1) cobweb behavior can mimic the true dynamics in the manner of a self-fulfilling mistake. They also show the possibility that an adaptive learning process converges to chaotic CEE cobweb dynamics. Even if the agents do not initially select the specific parameters that generate such a chaotic CEE, there can exist positive Lebesgue measure sets of initial values for those for which a simple adaptive learning scheme, such as sample autocorrelation learning, will lead the agent to adjust those values so that they converge on the parameters that do generate such a chaotic CEE. Such a process may start out with very regular behavior that then becomes more complex as the system converges on the chaotic implied actual law of motion. This phenomenon has been coined *learning to believe in chaos* in Hommes (1998, p.360), and explicit examples have been given in Sorger (1998) and Hommes and Sorger (1998).

The present paper makes two contributions. Firstly, we investigate CEE in a fishery model where the backward bending supply curve is derived from optimal management of the fish resources, by a sole owner maximizing discounted revenues. A backward bending supply curve in a renewable resource market appears to be natural when the discount rate is sufficiently high. In the fishery model the implied law of motion becomes a smooth one-dimensional non-monotonic map. Our results show that the possibility of chaotic CEE is not restricted to piecewise linear tent map dynamics, but occurs for general smooth non-monotonic mappings. Secondly, we investigate the effect of dynamic noise upon the learning dynamics and show that learning to believe in chaos is robust with respect to noise. In the presence of noise, the adaptive learning process can easily settle down to a chaotic CEE in which errors of the AR(1) forecasting rule have no significant autocorrelations. In fact, in a noisy environment it becomes even more difficult for the agents to distinguish between their stochastic AR(1) belief and the unknown underlying chaotic law of motion. Along a noisy CEE, agents using linear statistical techniques are not able to reject the hypothesis that prices follow a stochastic AR(1) process.

We will apply the CEE concept to fishery models, but it can apply more generally to any renewable resource market with an open access problem as well. Fisheries have long presented great difficulties of understanding to both biologists and economists, as well as to policymakers. This has been especially the case as there have been many collapses of fisheries around the world as well as serious disputes between fishers from different countries. Understanding fisheries involves modeling both the biological aspect as well as the economic aspect and integrating the two in a sound manner, a fusion labeled bioeconomics by Colin W. Clark (1990). Clark (1985, 1990) emphasizes that the modeling of fishery dynamics is among the most complicated and difficult of all such cases of bioeconomic modeling. As a scientist who has also been involved with advising the Canadian government regarding managing the now-collapsed Grand Banks cod fishery, he is also acutely aware of the policy difficulties involved as well.

A number of special peculiarities arise in the case of fisheries. One, understood since the work of Copes (1970), is that supply curves in fisheries may be backward-bending, one of the few markets where this can happen. Copes did not present a rigorous derivation of this result but rather imputed it from the problem of open access that had long been identified as a serious problem aggravating the overharvesting problem in many fisheries (Gordon, 1954). However Clark (1990) shows that such a backward-bending outcome can occur in an optimally managed fishery without open access, as long as there is a sufficiently high discount rate, a result that will be explicitly derived below. In such cases we already know that chaotic dynamics can arise in fairly simple models with discrete dynamic adjustments. Conklin and Kolberg (1994) have provided a specific model of chaotic dynamics for the Pacific halibut fishery with such a backward-bending supply curve, although without using a CEE framework. Chaotic dynamics, and chaotic CEE in particular, are more likely to happen when there is either open access, high discount rates, or relatively inelastic demand curves, the latter a result emphasized more broadly in more general nonlinear bioeconomic models by Chavas and Holt (1995). Although some of the details of the models are somewhat different, these basic problems also arise in other renewable resource situations such as managing grazing pastures and wild game hunting preserves (Rosser, 1995).

The paper is organized as follows. Section 2 presents the model for optimal management of the fish resources and derives the discounted equilibrium supply curve. Section 3 focuses on adaptive learning and consistent expectations equilibria in the fishery model, both without and with noise. Finally, section 4 concludes.

2 The Clark-Gordon-Schaefer Fishery Model

We shall use an optimal control theoretic version of the Gordon-Schaeffer fishery model, following Clark (1990). The presentation of the optimal equilibrium supply and demand will be in terms of a continuous model, whereas the price fluctuations in the corresponding speculative cobweb dynamics will be in discrete time.

Let us first introduce some notation. Let x denote population or stock of fish (measured in terms of biomass units), h harvest of fish and $F(x) = \frac{dx}{dt}$ growth of fish population without harvest. In the Schaefer (1957) model the sustained yield, with sustained yield holding if harvesting equals population growth, is given by a logistic function:[1]

$$h = F(x) = rx(1 - \frac{x}{k}), \tag{1}$$

with r intrinsic growth rate of the fish population and k the ecological carrying capacity for the fishery, that is, the maximum possible steady-state level of x. This yield function admits a level of the stock x at which a maximum sustained yield (MSY) will occur which will be at $x = k/2$. We note that there has long been a conflict between biologists and economists, with many biologists favoring the MSY level of the population as being the goal of optimal public policy, whereas when economic considerations are added to the biological ones in a combined bioeconomic analysis, it is highly unlikely that the MSY is optimal from the economic standpoint.

Following Gordon (1954) the harvest equation is given by

$$h(x) = qEx, \tag{2}$$

where E is the catch effort (measured in standardized vessel time) and q is catchability (measured per vessel per day) reflecting technology, and labor and capital. Denoting the price of the fish per biomass by p, total revenue will be $pqEx$. Marginal cost of effort c are assumed to be constant here so that total cost will equal cE. Clark (1990) has studied more general cost functions with congestion effects and

[1] Clark (1990) and Rosser (1991, chapter 13) consider more complicated yield functions that can involve catastrophic collapses of populations below certain critical levels. Such a non-sustained yield outcome does not arise with the Schaefer logistic yield function. Dávila and Martín-Gonzáles (1997) show yield functions for multi-species fisheries that generate backward bending supply curves.

dynamic models in which capital stock has inertia in the form of the unwillingness of fishers to retire their vessels even when a fishery is obviously being overfished. This latter phenomenon can play an especially important role in the collapse of actual fisheries.

Gordon (1954) solved for the open access equilibrium in which all positive rents would be fished away by the continuing entry of fishers into the fishery until that point is reached. Such entry behavior can be shown to arise from a certain kind of externality in which the individual fisher perceives his private marginal product to equal the social average product, the amount currently being caught per vessel per day. Essentially the entering fishers do not take into account the effect of their entry on the fishery and so too many of them enter and the fishery is overfished from an economic perspective. This open access equilibrium is now called a bionomic equilibrium and will occur where total revenue equals total cost at

$$E_\infty = \frac{r}{q}(1 - \frac{c}{pqk}) \tag{3}$$

$$x_\infty = \frac{c}{pq} \tag{4}$$

$$h_\infty = r\frac{c}{pq}(1 - \frac{c}{pqk}) \tag{5}$$

These equations have the ∞-subscript on the left-hand terms because this solution is identical to the socially optimal solution to be presented below when the discount rate is infinity, with the discount rate being given by δ. That is the same as saying that the fishers are totally myopic and are paying no attention whatsoever to the future in their decision making. That can be seen intuitively as being essentially what happens in the open access situation where no individual fisher is taking into account the effects of his own actions. At the other extreme, when $\delta = 0$ the fishers will treat the far distant future as being equally valuable as today, that is they will be very farsighted.

Clark (1990) presents both an optimal control solution and an optimal social utility solution which yield the same result. Here we follow the optimal control fishery management solution. Assume that there is a 'sole owner', say a government agency or a private firm, who owns all rights to the exploitation of the fish population. The sole owner's objective is to maximize discounted net revenues, that is, finding a harvesting policy $h(t)$ solving the following maximization problem:

$$max_{h(t)} \int_0^\infty e^{-\delta t}(p - c[x(t)])h(t)dt, \tag{6}$$

subject to $x(t) \geq 0$ and $h(t) \geq 0$, where p is the fish price and $c(x)$ the unit harvesting cost when the population level is x. Substituting $h(t) = F(x) - \dot{x}$ into (6) yields

$$max_{h(t)} \int_0^\infty e^{-\delta t}(p - c[x(t)])(F(x) - \dot{x})dt, \tag{7}$$

which is of the form $\int \phi(t,x,\dot{x})dt$ so that we can apply the classical Euler necessary condition for a maximum $\frac{\partial \phi}{\partial x} = \frac{d}{dt}\frac{\partial \phi}{\partial \dot{x}}$. Using the Euler equation, a straightforward computation yields

$$F'(x) - \frac{c'(x)F(x)}{p - c(x)} = \delta. \tag{8}$$

Equation (8) is an implicit equation for the *optimal equilibrium population level* x^*. At this optimal population level, the corresponding optimal sustained yield is

$$h = F(x^*). \tag{9}$$

Assuming as in Gordon (1954) and Schaefer (1957) a cost function $c(x) = c/qx$ and the logistic sustained yield equation, we get

$$c'(x) = -\frac{c}{qx^2} \tag{10}$$

$$F'(x) = r - \frac{2rx}{k} \tag{11}$$

Substituting these into (8) yields a quadratic equation for the optimal fish population x^*, whose positive solution is given by

$$x^*_\delta(p) = \frac{k}{4}\{1 + \frac{c}{pqk} - \frac{\delta}{r} + \sqrt{(1 + \frac{c}{pqk} - \frac{\delta}{r})^2 + \frac{8c\delta}{pqkr}}\}. \tag{12}$$

This optimal solution x^*_δ is usually referred to as the *bioeconomic equilibrium*, and is a function of the discount rate δ, the fish price p, and the other parameters such as the carrying capacity k, the catchability q, the marginal cost of effort c and the growth rate of fish r. The corresponding optimal sustained yield is given by

$$S_\delta(p) = h = F(x^*_\delta(p)). \tag{13}$$

We will refer to $S_\delta(p)$ in (13) as the *discounted equilibrium supply curve*, and writing it as a function of the fish price p will be convenient when we study cobweb dynamics under adaptive learning in section 3. A straightforward computation shows that, in the limit as the discount rate tends to infinity, the discounted supply curve reduces to the open access supply curve

$$S_\infty(p) = \frac{rc}{pq}(1 - \frac{c}{pqk}). \tag{14}$$

The reader may easily check that at the minimum price $p_{min} = c/(qk)$ the (discounted) equilibrium supply becomes 0; we will assume that below this minimum price the equilibrium supply equals zero. For consumer demand for fish, we will choose a simple, linear form

$$D(p) = A - Bp. \tag{15}$$

Figure 1a shows plots of the equilibrium demand and supply system, for different values of the discount rate δ, with the other parameters of the discounted supply curve fixed at

- $k = 400.000$, $q = 0.000013$, $c = 5000$ and $r = 0.05$,

as suggested for several specific fisheries by Clark (1985, pp. 25, 45 and 48), and the parameters of the demand curve fixed at

- $B = 0.25$ and $A = \frac{kr}{4} + \frac{Ac}{qk} = 5240.5$.

The marginal demand B has been chosen small, to allow for the possibility of multiple equilibria. The constant A has been chosen such that at the minimum price $p_{min} = c/(qk)$ consumer demand would be exactly equal to the maximum sustained yield. This is a convenient way of parameterizing the demand curve in such a way that the price dynamics under adaptive learning in section 3 will be well defined and remain bounded for all time; other nearby choices of the demand parameters lead to similar results as those presented below.

At the extreme case $\delta = 0$, that is when the sole owner treats the far distant future as equally valuable as today, the supply curve is upward sloping and approaches the maximum sustained yield (MSY), as illustrated in figure 1a. For positive values of the discount factor δ, the supply curve (13) is backward bending. This follows easily from the observation that the bionomic equilibrium $x^*_\delta(p)$ is a decreasing function of the fish price p and the population growth map F is non-monotonic. Figure 1a shows that, as the discount rate δ increases, the supply curve becomes more backward bending. The most backwardly bent supply curve corresponds with the totally myopic case of $\delta = \infty$, which corresponds to the open access bionomic equilibrium case studied by Gordon (1954) and which is associated with overfishing behavior. We note that the supply curve bends backwards quite quickly at values of the discount rate that are empirically and socially meaningful, in contrast with the kinds of discount rates that are necessary to generate chaotic dynamics in golden rule neoclassical growth models (Nishimura and Yano, 1996; Montrucchio and Sorger, 1996; Mitra, 1998).

Figure 1a also contains plots of the (linear) demand curve, illustrating the fact that a backward-bending supply curve together with a sufficiently inelastic demand curve may lead to multiple steady state equilibria even for the static case. In the extreme case $\delta = 0$ there is a unique steady state equilibrium price, whereas at the other extreme $\delta = +\infty$ there are three different steady state equilibrium prices. The two additional steady states are created through a tangent bifurcation at $\delta = \delta^* \approx 0.085$. This shows the original argument of Copes (1970) who argued that in the case of a strongly backward bending supply curve, increasing demand could lead to a collapse of a fishery and a jump in the equilibrium. Such a result can be modeled by using catastrophe theory and was done so for the collapse of the Antarctic fin and blue whale stocks (Jones and Walters, 1976), the latter falling from over 150,000 to less than 1,000 within the space of a few years during the 1960s (Clark, 1985, p. 6). Clark (1985) provides a comprehensive (and lengthy) list of fisheries that have collapsed around the world, although, as we have already noted, a variety of other factors including capital stock inertia have been involved in these tragedies.

3 Price Dynamics under Adaptive Learning

We now shall consider cobweb type price fluctuations under adaptive learning, with the equilibrium supply of fish derived from optimal fishery management as described in the previous section. Section 3 is divided into three subsections. The first subsection describes price fluctuations under naive price expectations, and argues that the naive forecasts can be improved in a linear statistical sense, even when price fluctuations are chaotic. Subsection 3.2 recalls the notion of consistent expectations equilibria (CEE) as introduced by Hommes and Sorger (1998). Finally, subsection 3.3 investigates CEE in the optimal control fishery model without and with noise.

3.1 Cobweb Dynamics under Naive Expectations

We now turn to the price fluctuations in this renewable resource market, assuming that producers have to make their investment decision for fishery equipment some fixed time period ahead. Given producers' price expectation, the optimal production decision is derived from the discounted equilibrium supply curve (13). Price expectations are formed one fixed time period ahead, which may be viewed as an investment lag. The price dynamics induced by these investment decisions then reduce to the usual cobweb 'hog cycle' model, with a fixed production lag[2]. The market equilibrium price at date t is determined by demand and supply, i.e.,

$$D(p_t) = S_\delta(p_t^e), \tag{16}$$

with D the consumer demand (15) and S_δ the discounted supply curve (13). It will be instructive to discuss the case of naive expectations first, where producers believe that last year's price will also prevail this year, i.e., $p_t^e = p_{t-1}$. Given that producers have naive price expectations, the implied actual law of motion becomes

$$p_t = G_\delta(p_{t-1}) = D^{-1}S_\delta(p_{t-1}) = \frac{A - S_\delta(p_{t-1})}{B}. \tag{17}$$

Figure 1b shows graphs of the implied actual law of motion G_δ under naive price expectations, for different values of the discount rate. At the extreme case $\delta = 0$, supply is increasing so that the map G_δ is decreasing, and under naive expectations prices diverge from an unstable steady state and converge to a stable 2-cycle. Along this 2-cycle, expectations are systematically wrong. When producers expect a low (high) price, they decide to produce a low (high) quantity, which will induce a high

[2]Another complication is the speed of adjustment of the fish population. Implicitly we assume a rapid adjustment of the population compared to price adjustment. But adjustment speeds of fish stocks vary considerably for different fisheries, with some apparently having adjustment lags as long as one year. For investment we assume that one year is the time horizon of decision and thus is the appropriate lag. Therefore, there will be disequilibrium production that we are not fully capturing in our simplified model. In subsection 3.3 we show that important features of the adaptive learning process are persistent under dynamic noise. These dynamic noise terms may be interpreted as noise in supply (and/or demand).

(low) market equilibrium price. For positive discount rates, beyond a critical price the graph of G_δ is increasing, so that the implied actual law of motion becomes a non-monotonic map. As the discount rate increases, the implied actual law of motion becomes strongly upward sloping for high prices and the price dynamics under naive expectations become more complicated. It is not hard to show by graphical analysis that, e.g. for the tangent bifurcation value $\delta^* \approx 0.085$ the dynamics under naive expectations is (topologically) chaotic. In fact, under naive expectations and with the other parameters fixed as before, complicated dynamics arises for relatively small values of the discount rate, say for $0.02 \leq \delta \leq 0.085$. For sufficiently high values of the discount rate, e.g. for $\delta \approx 0.1$, naive expectations drives the system to the "bad" stable steady state equilibrium, with a high price and low fish stock.

Let us now discuss what would happen under rational expectations (perfect foresight). Recall that there is a critical parameter value $\delta^* \approx 0.085$ for which a tangent bifurcation occurs at which the number of steady states changes from one, to two steady states at the bifurcation value and three steady states beyond the bifurcation value. For small discount rates $0 \leq \delta \leq \delta^*$ the unique steady state solution $p_t \equiv p_1^*$, for all t, is the only rational expectations equilibrium (REE). For large discount rates $\delta > \delta^*$, three different steady states p_i, $i = 1$, 2 or 3, coexist, and there are multiple stationary REE. For example, three constant steady state REE exist, where $p_t \equiv p_i$, for all t, and i is fixed at 1, 2 or 3. In addition however, infinitely many non-constant REE exist, since any solution $p_t = p_i^*$, for all t, and i switching between $1, 2$ and 3, is a REE. For high discount rates infinitely many REE coexist, for which prices are switching arbitrarily between the three different steady state price levels p_i^*, with the fish stock switching between the corresponding high and low levels.

What happens when agents are boundedly rational and do not have exact knowledge about underlying market equilibrium equations? Would boundedly rational agents be able to learn the "good" steady state equilibrium with low prices and high fish stock? Would boundedly rational agents be able to discover regularities in their forecasting errors under naive expectations and change expectations accordingly? In the simple case of convergence to a stable period 2 price cycle agents should, at least in theory, be able to learn from their systematic forecasting errors and improve their forecasts[3]. But what about the case of chaotic equilibrium price fluctuations? Are boundedly rational agents able to learn in a chaotic environment and detect regularities from time series observations to improve their forecasts?

Figure 2 shows a chaotic price series under naive expectations and the corresponding chaotic forecasting errors, for $\delta = 0.02$. Table I contains the sample

[3]Hommes, Sonnemans and van de Velden (2000) have recently done laboratory experiments where participants had to predict prices of an unknown cobweb model, with feedback from their own forecasts. Only in about one third of the single agent experiments was the participant able to learn the unique steady state REE. In similar multi-agent experiments in Hommes, Sonnemans, Tuinstra and van de Velden (2000), the sample mean of realized market prices was close to the REE price, but all multi-agent experiments exhibit significant excess volatility driven by heterogeneous expectations.

autocorrelations of the forecasting errors, with the first lags being highly significant. In particular, the chaotic forecasting errors have a strongly significant negative first order autocorrelation coefficient $\rho_1 \approx -0.646$. Using standard linear statistical tools a boundedly rational agent would thus conclude that naive expectations are 'systematically wrong', even when prices fluctuate chaotically. As a first step agents might try to improve their forecast accuracy by using a simple linear AR(1) rule with a negative first order coefficient, and try to optimize the forecast parameters by adaptive learning as additional observations become available.

3.2 Consistent Expectations Equilibria

In order to be self-contained, we briefly recall the notion of *Consistent Expectations Equilibrium*, as introduced and discussed extensively in Hommes and Sorger (1998). Consider a dynamic market equilibrium model

$$p_t = G(p_t^e), \tag{18}$$

where G is a function relating the realized market price p_t as a function of the expected price p_t^e. The cobweb model discussed above is of this type, with $G = D^{-1}S_\delta$. To complete the dynamic model, one has to specify how the agents form their price expectation p_t^e. We assume that agents do *not* know market equilibrium equations and form expectations based only upon time series observations. We assume that the agents know all past prices $p_0, p_1, \ldots, p_{t-1}$ and use these in their forecast p_t^e. Notice that we assume that the equilibrium price p_t is *not* known to the agents when making their forecast p_t^e, since it has not been revealed by the market equilibrium equation yet. We further assume that agents believe that prices follow a simple linear stochastic process, and that expectations are homogeneous across agents. More specifically, we assume that all agents believe that prices are generated by a stochastic AR(1) process. Given this perceived law of motion and prices known up to p_{t-1}, the unique predictor or forecasting rule for p_t which minimizes the mean squared prediction errors is given by

$$p_t^e = \alpha + \beta(p_{t-1} - \alpha), \tag{19}$$

where α and β are real numbers, $\beta \in [-1, 1]$. The expected price thus equals a constant α (the unconditional mean of the AR(1) process) plus the constant β (the first order autocorrelation coefficient) times the deviation of the previous price from the unconditional mean. Given that agents use the linear predictor (19), the *implied actual law of motion* becomes

$$p_t = G_{\alpha,\beta}(p_{t-1}) := G(\alpha + \beta(p_{t-1} - \alpha)). \tag{20}$$

Now recall that the empirical or sample average of a time series $(p_t)_{t=0}^{\infty}$ is defined as (see e. g. Brockwell and Davis, 1991)

$$\bar{p} = \lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} p_t \tag{21}$$

and the empirical or sample autocorrelation coefficients are given by

$$\rho_j = \lim_{T \to \infty} \frac{c_{j,T}}{c_{0,T}}, \qquad j \geq 1, \tag{22}$$

where

$$c_{j,T} = \frac{1}{T+1} \sum_{t=0}^{T-j} (p_t - \bar{p})(p_{t+j} - \bar{p}), \qquad j \geq 0. \tag{23}$$

In the special case where the time series is constant, the definition of ρ_j involves an indeterminate expression and all sample autocorrelations can be defined as β^j for some $\beta \in [-1, 1]$. We are now ready for the definition of a CEE.

Definition 1 A triple $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$, where $(p_t)_{t=0}^{\infty}$ is a sequence of prices and α and β are real numbers, $\beta \in [-1, 1]$, is called a *consistent expectations equilibrium* (CEE) if

1. the sequence $(p_t)_{t=0}^{\infty}$ satisfies the implied actual law of motion (20),
2. the sample average $\bar{p}$ exists and is equal to α, and
3. the sample autocorrelation coefficients ρ_j, $j \geq 1$, exist and the following is true:
   a. if $(p_t)_{t=0}^{\infty}$ is a convergent sequence, then $\text{sgn}(\rho_j) = \text{sgn}(\beta^j)$, $j \geq 1$;
   b. if $(p_t)_{t=0}^{\infty}$ is not convergent, then $\rho_j = \beta^j$, $j \geq 1$.

A CEE is a price sequence together with an AR(1) belief process such that the expectations are self-fulfilling in terms of the observable sample average and sample autocorrelations. Along a CEE expectations are thus correct in a linear statistical sense, and using time series observations only agents would have no reason to deviate from their belief[4].

Given an AR(1) belief, there are at least three possible types of CEE: (i) a *steady state CEE* in which the price sequence $(p_t)_{t=0}^{\infty}$ converges to a steady state price p^*; (ii) a *2-cycle CEE* in which the price sequence $(p_t)_{t=0}^{\infty}$ converges to a period two cycle $\{p_1^*, p_2^*\}$ with $p_1^* \neq p_2^*$; and (iii) a *chaotic CEE* in which the price sequence $(p_t)_{t=0}^{\infty}$ is chaotic. Which of these cases occurs in a particular model depends on the mapping G in (20).

Sample autocorrelation (SAC) learning.
The definition of a CEE involves a fixed AR(1) belief described by the parameters α and β. Agents are supposed to stick to this belief over the entire time horizon and the consistency of the implied actual dynamics with the belief can only be verified if the entire price sequence is known. Now consider the more flexible situation

[4] Hommes and Sorger (1998) focus attention on the case of AR(1) beliefs, but emphasize that the definition of CEE can easily be generalized to higher order belief processes, e. g., AR(k) processes with $k \geq 2$.

of adaptive learning in which agents change their forecasting function over time within the class of AR(1) beliefs, and update their belief parameters α_t and β_t, as additional observations become available. A natural learning scheme which nicely fits the framework of CEE is based upon sample average and sample autocorrelation coefficients.

For any finite set of observations $\{p_0, p_1, \ldots, p_t\}$ the sample average is given by

$$\alpha_t = \frac{1}{t+1} \sum_{i=0}^{t} p_i \, , \qquad t \geq 1 \tag{24}$$

and the first order sample autocorrelation coefficient is given by (see Brockwell and Davis, 1991)

$$\beta_t = \frac{\sum_{i=0}^{t-1} (p_i - \alpha_t)(p_{i+1} - \alpha_t)}{\sum_{i=0}^{t} (p_i - \alpha_t)^2} \, , \qquad t \geq 1. \tag{25}$$

When, in each period, the belief parameters are updated according to their sample average and their first order sample autocorrelation, the (temporary) law of motion (20) becomes

$$p_{t+1} = G_{\alpha_t, \beta_t}(p_t) = G(\alpha_t + \beta_t(p_t - \alpha_t)) \, , \qquad t \geq 0. \tag{26}$$

We call the dynamical system (24) - (26) the actual dynamics with *sample autocorrelation learning* (SAC-learning)[5]. The initial state for the system (24) - (26) can be any triple (p_0, α_0, β_0) with $\beta_0 \in [-1, 1]$.

Another perhaps better known adaptive learning process is ordinary least squares (OLS) learning. In fact, although not identical, the SAC- and the OLS-learning schemes are closely related, see the discussion in Hommes and Sorger (1998). In particular, we would like to stress that for any given bounded time series the differences in the parameter estimations of the SAC- and OLS learning schemes become arbitrarily small for large t. In the initial phase of the learning schemes there may be differences between OLS- and SAC-learning however, which in a self-referential system may in turn lead to differences in the implied realized price series in the long run. A particular problem with the OLS-learning scheme is that the estimate $\hat{\beta}$ does not necessarily lie in the interval $[-1, 1]$, which may cause global divergence of the realized price series. Marcet and Sargent (1989) have proposed to impose a so-called projection facility on OLS-learning, that is a maximum allowable interval for the OLS-estimate $\hat{\beta}$, but in general the choice of such a projection facility is arbitrary. In contrast, the SAC estimate β_t in (25) always lies in the interval $[-1, 1]$, so that the AR(1) coefficient β_t can not cause global divergence of the price series. In the next subsection we will focus on the SAC adaptive learning scheme. Simulations with the OLS-learning scheme lead to similar results and would not change our general conclusions below.

[5]In the case of an AR(1) belief the SAC-learning scheme coincides exactly with the Durbin-Levinson Algorithm, the well-known recursive form of the Yule-Walker estimators for an AR(p) process, see e.g. Brockwell and Davis (1991, pp.238-245).

3.3 CEE in the Clark-Gordon-Schaefer Fishery Model

In our simulations of the adaptive SAC-learning process (24), (25) and (26), with $G \equiv G_\delta = D^{-1}S_\delta$ we have observed three typical outcomes:

- convergence to the "good" steady state equilibrium with a low price and a high fish stock
- convergence to the "bad" steady state equilibrium with a high price and a low fish stock
- convergence to a chaotic CEE, with prices and fish stock irregularly jumping between low and high values

Simulations of the SAC-learning dynamics suggest that for low values of the discount rate convergence to the "good" equilibrium steady state is the most likely outcome of the SAC learning process, whereas for high values of the discount rate convergence to the "bad" steady state is most likely. For intermediate discount rates the outcome of the learning process is uncertain and in general depends on the initial states, i.e. on the initial belief parameters α_0, β_0 and the initial fish stock x_0. The system may settle down to either the "good" or the "bad" steady state, possibly after a long (chaotic) transient. However, it may also happen that belief parameters α_t and β_t converge to constants α^* and β^* while prices never converge to a steady state (or to a cycle), but keep fluctuating chaotically, as illustrated in figure 3 for $\delta = 0.1$. This situation is referred to as *learning to believe in chaos* and it seems to occur with positive probability, that is, for an open set of initial states (x_0, α_0, β_0). Learning to believe in chaos means that the SAC-learning dynamics converges to a chaotic system, when α_t and β_t have converged to constants α^* and β^*, while prices keep fluctuating chaotically[6]. Figure 3 shows an example with the learning parameters (α_t, β_t) converging to $(\alpha^*, \beta^*) \approx (4988, -0.87)$ and permanent chaotic price fluctuations with sample average α^* and strongly negative first order autocorrelation coefficient β^*.

In order to understand the existence of chaotic CEE for our smooth, non-monotonic implied actual law of motion it is useful to consider the graph of the corresponding one-dimensional map. Given the AR(1) forecasting rule with parameters α^* and β^*, Figure 4 shows the graph of the implied actual law of motion $G_{\delta,\alpha^*,\beta^*}(p) = D^{-1}S_\delta(\alpha^* + \beta^*(p - \alpha^*))$, and its second iterate $G^2_{\delta,\alpha^*,\beta^*}$. From these

[6] The notion *learning to believe in chaos* has been introduced by Hommes (1998, p.360), and the first examples have been given by Sorger (1998) and Hommes and Sorger (1998). The key feature is that learning parameters converge to constants whereas prices do not converge but fluctuate chaotically on a strange attractor, with the correct sample average and sample autocorrelations. Schönhofer (1999, 2000) has recently employed the notion of learning to believe in chaos in a somewhat different context, namely when the entire OLS-learning process fluctuates chaotically. In Schönhofer's examples belief parameters of the OLS-learning scheme do *not* converge but keep fluctuating chaotically, while at the same time, due to inflation, prices diverge to infinity.

graphs it follows immediately that the implied actual law of motion is a (topologically) chaotic map. A typical chaotic trajectory of the implied actual law of motion will be characterized by up and down oscillation around the unstable steady state. The graph of the implied actual law of motion thus suggest chaotic time series with strongly negative first order autocorrelation. Apparently, typical chaotic time series generated by the implied law of motion are self-fulfilling in terms of sample average and sample autocorrelations.

Recall that our boundedly rational agents have no knowledge about underlying market equilibrium equations, and therefore do not know the implied actual law of motion. They only observe time series and use linear statistical techniques. Would they be satisfied with their linear forecasting rules and stick to their AR(1) belief? Would boundedly rational agents be able to reject their stochastic AR(1) belief or perceived law of motion by linear statistical hypothesis testing?[7] Table II shows the first 10 lags of the sample autocorrelation and partial autocorrelation of 300 observations of the chaotic price series. The autocorrelation pattern of the chaotic series is indeed similar to the autocorrelation pattern of an AR(1) process with strongly negative first order autocorrelation. The first order partial autocorrelation coefficient is strongly negative; all other partial autocorrelation coefficients are small, but the lags $2-6$ are significantly different from 0. Table III contains estimation results of AR(1) model to the chaotic price series of 300 observations, implying estimated belief parameters $\hat{\beta} \approx -0.88$ and $\hat{\alpha} = C/(1-\beta) \approx 5064$. Table IV contains the first ten lags of the sample autocorrelations, together with their Q-statistics, of the residuals of the fitted AR(1) model. Although all autocorrelations of the AR(1)-residuals are small, some of them, e.g. at the first two lags, are statistically significant. Hence, a careful boundedly rational agent, based upon this linear statistical analysis, would reject the null hypothesis that prices follow an AR(1) process. Using the last 300 observations agents would discover that their AR(1) forecasting rule is not optimal and that their AR(1) model must be misspecified.

Now let us investigate the effect of noise upon the learning dynamics. SAC-learning with additive dynamic noise is given by (24), (25), as before, and adding a noise term to the implied actual law of motion, i.e.

$$p_{t+1} = G_{\delta,\alpha_t,\beta_t}(p_t) = G_\delta(\alpha_t + \beta_t(p_t - \alpha_t)) \ + \epsilon_t, \qquad t \geq 0, \tag{27}$$

where ϵ_t is an independently identically distributed (IID) random process and $G_\delta = D^{-1}S_\delta$ in (17) as before. Notice that the noise is not merely observational noise, but dynamic noise affecting the dynamic law of motion in each period of time. Figure 5 illustrates a typical example, with ϵ_t drawn from a uniform distribution over the interval $[-1000, +1000]$[8]; for this choice of the noise process, the signal to noise ratio,

[7]See also Crespo-Cuaresma and Sorger (1999) for statistical hypothesis testing of consistent expectations equilibria; Schönhofer (2000) investigates statistical hypothesis testing of chaotic equilibria under OLS adaptive learning.

[8]The graphs of $G_{\delta,\alpha^*,\beta^*}$ and $G^2_{\delta,\alpha^*,\beta^*}$ in figure 4 show that the basin of attraction of chaotic motion is bounded above by $p = \bar{p} \approx 9290$. Adding noise to the system may therefore lead to prices diverging away from the chaotic region, and lock into the "bad" steady state equilibrium with a

as measured by the ratio σ_p/σ_ϵ of standard deviations of the noise free price series to the noise, is about 5. Surprisingly, even in the presence of dynamic noise, the SAC-learning dynamics still settles down to a chaotic CEE as illustrated in figure 5. The noisy chaotic series has an autocorrelation pattern very similar to that of an AR(1) process with strongly negative first order autocorrelation, as can be seen in Table V. Table VI contains the estimation results of an AR(1) process for 300 observations (after a transient of 5000) of the noisy chaotic series. The estimated parameters are $\hat{\beta} \approx -0.87$ and $\hat{\alpha} \approx 5420$, which are fairly close to the coefficients of the chaotic CEE $\beta^* \approx -0.87$ and $\alpha^* \approx 4988$ in the noise free case. Table VII shows that the sample autocorrelations coefficients of the residuals of the fitted AR(1) model are not statistically significant, and the Q-statistics indicate that the null hypothesis that prices follow a stochastic AR(1) process can not be rejected, not even at the 10% level. Learning to believe in noisy chaos is thus a possibility which is not rejected by linear statistical theory.

One major difference between the fishery management system considered here and the CEE's investigated by Hommes and Sorger (1998) is that our model is smooth, without any kinks in the curves determining the underlying law of motion. In Hommes and Sorger (1998), as in the other studies cited here that have come up with self-fulfilling chaotic mistake behavior, the underlying law of motion is given by a piecewise linear asymmetric tent map with one or more kinks. The chaotic CEE detected in the overlapping generations model in Hommes and Sorger (1999) are not piecewise linear but do have one kink. Our results for the optimal management of renewable resources shows an extension of the existence of chaotic CEE result to smooth non-monotonic mappings on the interval.

4 Summary and Conclusions

We have seen that there are a growing number of examples of systems where self-fulfilling chaotic mistakes can occur in the form of chaotic consistent expectations equilibria with the possibility of convergence through learning to those equilibria through simple learning processes, the phenomenon of learning to believe in chaos. Such cases include models where the underlying dynamics are determined by asymmetric tent maps as with the overlapping generations macroeconomic model of Sorger (1998) and the generic price adjustment model of Hommes and Sorger (1998). We have seen presented in this paper a model of fishery dynamics that shows such phenomena for reasonably realistic parameter values even for systems where the underlying dynamics are given by functions that do not have kinks of the sort found in the asymmetric tent map.

A chaotic CEE may be seen as an *approximate rational expectations equilibrium*, where agents use an optimal linear predictor to forecast an unknown nonlinear actual

high price. In the simulations with noise we have therefore chosen a bounded noise process and imposed an upper bound on (noisy) prices of 10,000. This upperbound for prices is not inconsistent with the AR(1) forecasting rule, because it always predicts a price well below this upperbound.

law of motion. We have shown that such equilibria are persistent with respect to dynamic noise. In fact, the presence of noise may increase the probability of convergence to such learning equilibria. Agents are using a simple, but misspecified model to forecast an unknown, possibly complicated actual law of motion. Without noise, boundedly rational agents using time series analysis might be able to detect the misspecification and improve their forecast model. In the presence of dynamic noise however, misspecification becomes harder to detect and boundedly rational agents using linear statistical techniques can do no better than stick to their optimal, simple linear model of the world.

With regard to the specific issues raised by consideration of the fishery dynamics model several points are in order. One is that this modeling effort certainly reinforces much that has already been known: that chaotic or irregular dynamics are more likely as myopia is greater or as there is a lack of control over access for which the solution resembles the optimal solution with total myopia. This suggests that efforts to make the markets and fishers take a longer term perspective and also to encourage systems to control access should be encouraged, by reassigning or enforcing property rights or by some collective system of access limitation in a commons fishery, although we have no specific new proposals regarding these difficult and complicated issues.

However the results in this paper do suggest at least one element of optimism that may not have been known by analysts of these problems previously. The implications of the possible existence of observable chaotic CEE's in fisheries suggests that in a world of underlying chaotic dynamics, fishers may be able to mimic the behavior implied by accurate expectations by fairly simple, boundedly rational rules of adaptation, even in the presence of dynamic noise which is certainly present in the uncertain world of fisheries. More particularly, in contrast with the catastrophic results arising from some models and situations, it should be kept in mind that chaotic dynamics remain bounded. Thus, if a group of fishers fishing a fishery are actually able to successfully follow an underlying truly chaotic dynamic, even if by doing so through a self-fulfilling chaotic mistake, the results of their doing so will not lead to the collapse of that fishery, which is certainly a desideratum.

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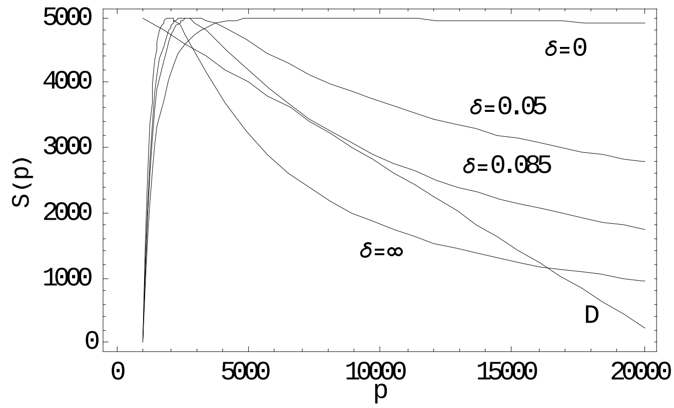
(a)



(b)

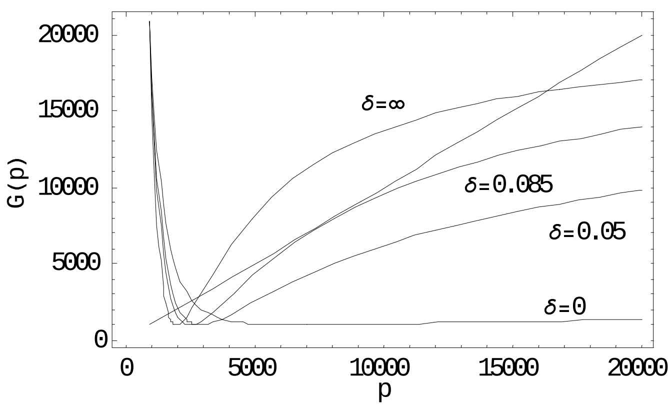


Figure 1. (a) Graphs of the demand and the discounted equilibrium supply curves S_δ in (13) and (b) graphs of the implied law of motion G_δ in (17) under naive expectations for several discount factors δ. As the discount factor δ increases two additional steady states are created at a tangent bifurcation for $\delta \approx 0.085$.

(a) (b)

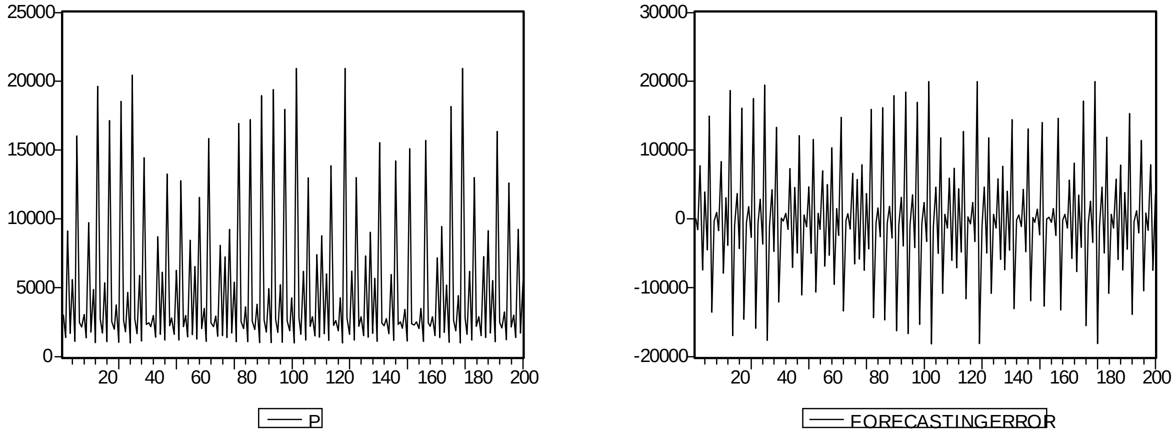
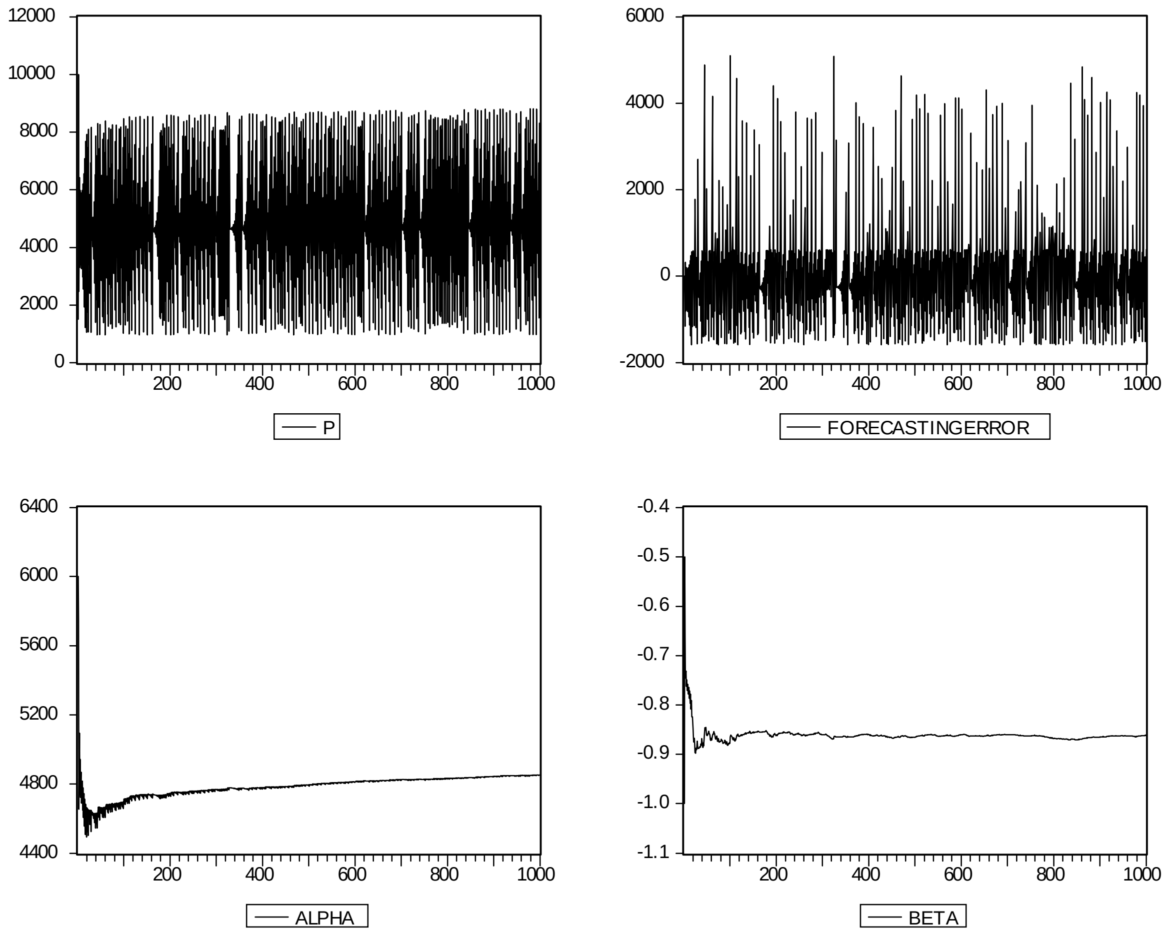


Figure 2. (a) Chaotic prices and (b) corresponding forecasting errors under naive expectations, for delta = 0.02. *The chaotic forecasting errors exhibit significant autocorrelations, especially a negative first order autocorrelation coefficient, as can be seen from Table 1.*

. (a) (b)



(c) (d)

Figure 3. Learning to believe in chaos. For $\delta = 0.1$ *and initial state* $(p_0, \alpha_0, \beta_0) = (1500, 6000, -1)$ *in the SAC-learning process prices fluctuate chaotically (a), while at the same time belief parameters* α *(c) and* β *(d) converge to constants* $\alpha^* \approx 4988$ *and* $\beta^* \approx -0.87$. *Forecasting errors (b) are chaotic and unpredictable, with amplitude e.g. much smaller than under naive expectations (cf. figure 2b).*

(a)

(b)

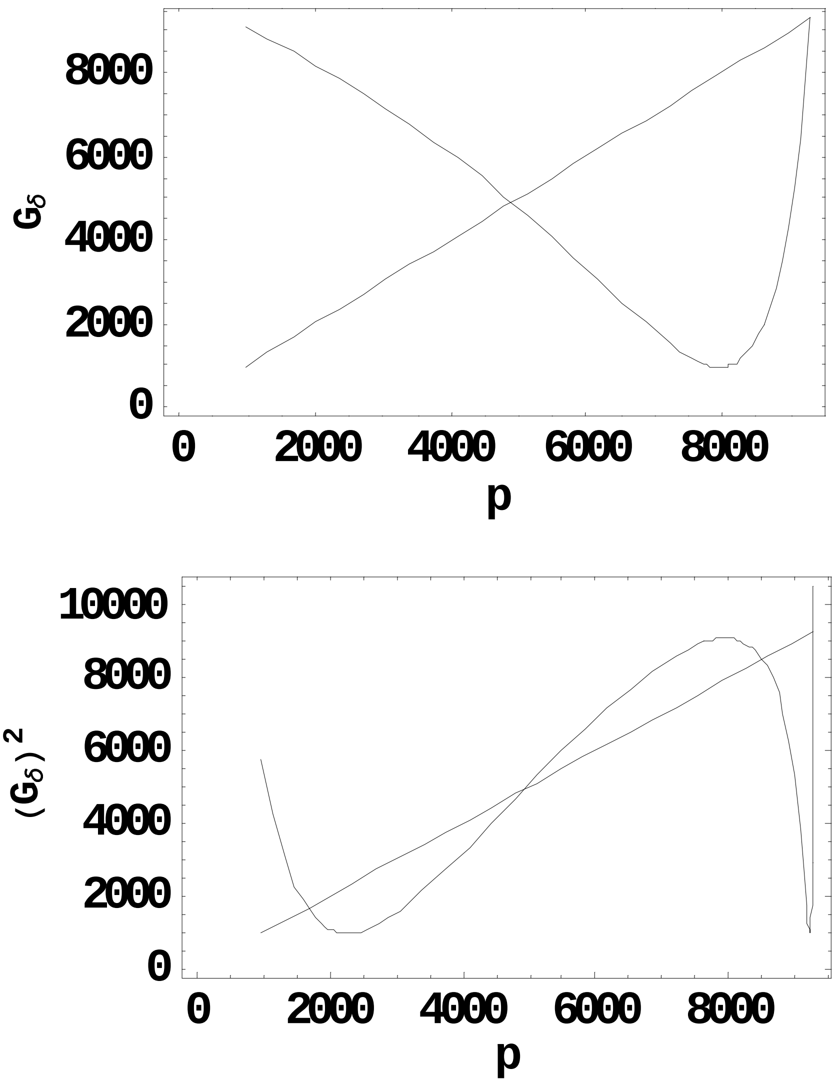


Figure 4. Graphs of the first iterate $G_{\delta,\alpha^*,\beta^*}$ *(a) and the second iterate* $G^2_{\delta,\alpha^*,\beta^*}$ *(b) of the implied actual law of motion for the chaotic CEE belief parameters* $(\alpha^*, \beta^*) = (4988, -0.87)$ *for* $\delta = 0.1$.

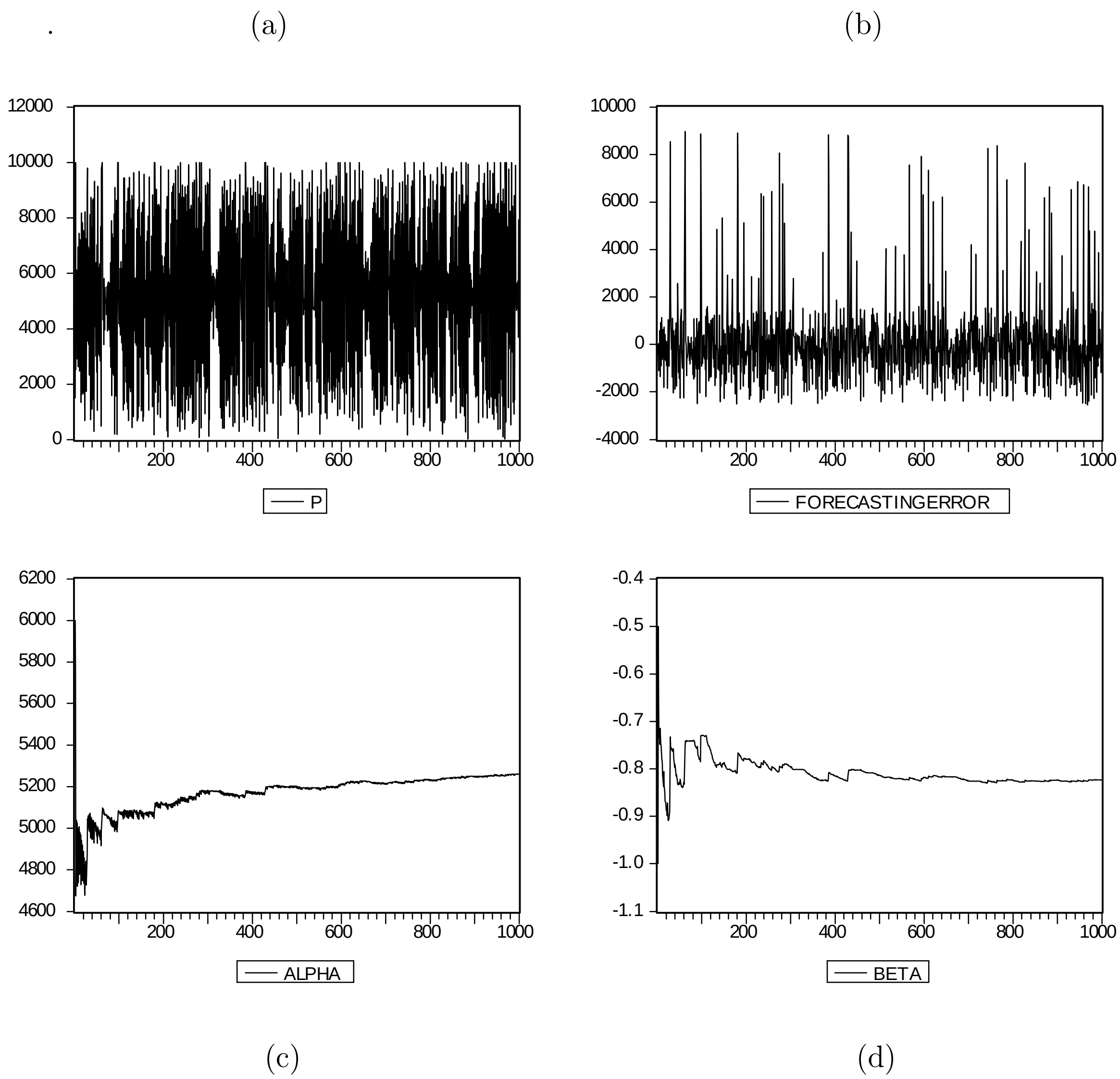


Figure 5. Learning to believe in noisy chaos, for $\delta = 0.1$ and initial state $(p_0, \alpha_0, \beta_0) = (1500, 6000, -1)$. In the presence of noise, the SAC-learning converges to a (noisy) chaotic CEE, with chaotic price flucutations (a) and at the same time convergence of the belief parameters α_t (c) and β_t (d). Forecasting errors (b) are (noisy) chaotic and seemingly unpredictable. Tables V-VII show that the null hypothesis that prices follow a stochastic AR(1) process is not rejected.

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lag	AC	PC	Q-Stat.	Prob.
1	-0.646	-0.646	84.68	0.000
2	0.230	-0.321	95.46	0.000
3	-0.151	-0.310	100.12	0.000
4	-0.079	-0.611	101.42	0.000
5	0.301	-0.408	120.25	0.000
6	-0.142	-0.141	124.46	0.000
7	-0.013	-0.154	124.50	0.000
8	-0.016	-0.071	124.55	0.000
9	-0.101	-0.179	126.71	0.000
10	0.230	-0.139	137.96	0.000

Table I: *Autocorrelations and partial correlations of forecasting errors under naive expectations.*

lag	AC	PC	Q-Stat.	Prob.
1	-0.877	-0.877	233.08	0.000
2	0.737	-0.141	398.05	0.000
3	-0.661	-0.212	531.52	0.000
4	0.578	-0.139	633.62	0.000
5	-0.527	-0.193	718.91	0.000
6	0.451	-0.257	781.63	0.000
7	-0.352	-0.023	819.97	0.000
8	0.295	-0.001	846.89	0.000
9	-0.256	-0.061	867.32	0.000
10	0.211	-0.077	881.23	0.000

Table II: *Autocorrelations and partial correlations of prices of chaotic CEE under SAC-learning.*

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Model: $p_t = C + \beta p_{t-1}, \qquad (C = \alpha(1-\beta))$				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	9538.893	157.6361	60.51211	0.0000
β	-0.883612	0.027461	-32.17699	0.0000
R-squared	0.777087	Mean dependent var		5061.651
Adjusted R-squared	0.776336	S.D. dependent var		2708.605
S.E. of regression	1280.983	Akaike info criterion		17.15531
Sum squared resid	4.87E+08	Schwarz criterion		17.18006
Log likelihood	-2562.719	F-statistic		1035.358
Durbin-Watson stat	2.261127	Prob(F-statistic)		0.000000

Table III: *Estimation results for AR(1) model on chaotic CEE (300 observations, after transient of 5000).*

lag	AC	PC	Q-Stat.	Prob.
1	-0.134	-0.134	5.3925	0.020
2	-0.221	-0.243	20.136	0.000
3	-0.083	-0.164	22.221	0.000
4	-0.084	-0.203	24.358	0.000
5	-0.119	-0.276	28.689	0.000
6	0.118	-0.087	32.979	0.000
7	0.131	-0.012	38.306	0.000
8	-0.063	-0.106	39.524	0.000
9	-0.034	-0.078	39.884	0.000
10	-0.057	-0.137	40.899	0.000

Table IV: *Autocorrelations, partial correlations and Q-statistics of residuals of fitted AR(1) model on chaotic CEE under SAC-learning.*

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lag	AC	PC	Q-Stat.	Prob.
1	-0.871	-0.871	229.91	0.000
2	0.742	-0.071	397.17	0.000
3	-0.653	-0.095	527.25	0.000
4	0.563	-0.063	624.31	0.000
5	-0.475	0.028	693.53	0.000
6	0.373	-0.121	736.36	0.000
7	-0.290	-0.017	762.40	0.000
8	0.209	-0.081	775.95	0.000
9	-0.144	-0.027	782.38	0.000
10	0.075	-0.086	784.16	0.000

Table V: *Autocorrelations, partial correlations and Q-statistics of noisy chaotic CEE under SAC-learning.*

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Model: $p_t = C + \beta p_{t-1}, \qquad (C = \alpha(1-\beta))$				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	10141.58	175.9158	57.65017	0.0000
β	-0.871095	0.028454	-30.61460	0.0000

R-squared	0.759369	Mean dependent var	5421.955
Adjusted R-squared	0.758559	S.D. dependent var	2981.958
S.E. of regression	1465.235	Akaike info criterion	17.42409
Sum squared resid	6.38E+08	Schwarz criterion	17.44884
Log likelihood	-2602.901	F-statistic	937.2537
Durbin-Watson stat	2.090609	Prob(F-statistic)	0.000000

Table VI: *Estimation results for AR(1) model on noisy chaotic CEE (300 observations, after transient of 5000).*

lag	AC	PC	Q-Stat.	Prob.
1	-0.058	-0.058	1.0013	0.317
2	-0.097	-0.101	3.8737	0.144
3	-0.048	-0.061	4.5656	0.207
4	0.031	0.015	4.8645	0.301
5	-0.081	-0.091	6.8860	0.229
6	-0.042	-0.053	7.4308	0.283
7	-0.015	-0.038	7.5024	0.379
8	-0.048	-0.075	8.2286	0.411
9	-0.019	-0.038	8.3453	0.500
10	-0.092	-0.123	10.976	0.359

Table VII: *Autocorrelations, partial correlations and Q-statistics of residuals of fitted AR(1) model on noisy chaotic CEE under SAC-learning.*