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Continuous Time Trading in Markets with Adverse Selection

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Abstract: We investigate the nature of the adverse selection problem in a market for a durable good where trading and entry of new buyers and sellers takes place in continuous time. In the continuous time model equilibria with properties that are qualitatively different from the static equilibria, emerge. Typically, in equilibria of the continuous time model sellers with higher quality wait in order to sell and wait more than sellers of lower quality do. Among other things, we show that *for any distribution of quality* there exist an *infinite number* of cyclical equilibria where *all goods are traded within a finite time after entering the market*. This holds true even if the good is not perfectly durable or when buyers are not risk-neutral.

Key Words: Dynamic Trading, Asymmetric Information, Entry, Durable Goods.

JEL Classification: D82

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1. Introduction

Since the pioneering work of Akerlof (1970), economists have regarded the presence of asymmetric information as one of the main sources of market failure. The adverse selection problem points to the fact that prices that the uninformed side of the market is willing to pay for the expected quality are unacceptable for those agents at the informed side of the market that possess the better qualities. As a consequence, in any market equilibrium good quality products do not change ownership (see also, Wilson, 1979, 1980). Subsequent work has investigated the extent to which non-market institutions can reveal information about quality. The role that is played by certification intermediaries, leasing, guarantees and other institutions in durable goods markets that suffer from adverse selection has recently been studied by Guha and Waldman (1997), Lizzeri (1999) and Waldman (1999), among others. This paper, in contrast, is driven by a more basic question, namely: to what extent can the market mechanism itself, by changing prices over time, provide adequate incentives for sellers of different qualities to sort themselves over time? This question is relevant in markets where goods have a use value that extends over some time periods and where high quality goods have a higher use value than low quality goods. In such an environment preferring not to sell can be a signal of having good quality.[1]

Durability introduces some additional factors that are not explicitly taken into account in the static model: goods not traded in any period can be offered for sale in the future and, in addition, new cohorts of potential sellers may enter the market over time. Janssen and Roy (1999a, 1999b) and Janssen and Karamychev (2000) have investigated some of the issues that arise when durability is explicitly taken into account in a dynamic model. All these papers study models where markets open and close in different time periods. Janssen and Roy (1999a) address the issue whether a *given stock of goods* can be traded over time. They show that in any dynamic competitive equilibrium all goods eventually will be traded. The main idea behind this result is that given a sequence of prices high quality sellers have more incentives to wait (and enjoy a higher use value before selling) than low quality sellers do. Once certain (low) qualities are sold, only relatively high qualities remain in the market. Risk-neutral consumers can predict that sellers of different qualities

[1] Taylor (1999) shows that when potential buyers can inspect quality, time on the market may be a signal of low, rather than of high quality. We show in the context of a pure adverse selection model where quality inspection is too costly, a reverse result.

will sort themselves into different time periods and, hence, they are willing to pay higher prices in later periods. The equilibrium is thus one in which higher qualities are sold in later periods at higher prices.

Janssen and Roy (1999b) and Janssen and Karamychev (2000) address the same issue in the context of markets where identical cohorts of perfectly durable goods enter the market in each time period. In such markets, the infinite repetition of the static equilibrium under adverse selection is an equilibrium in the dynamic model. In fact, it is the unique stationary equilibrium and also the only equilibrium where prices and average quality traded are (weakly) monotonic over time. These papers show that there exist other equilibria, however, where *all* goods are traded within finite time after they have entered the market. These equilibria are cyclical in prices and quantities in the sense that once all goods are traded, prices (and quantities) will fall. Up to the moment all goods are sold, however, the dynamic process of prices and quantities is monotonically increasing.

This paper investigates which factors are driving the results obtained in Janssen and Roy (1999b) and Janssen and Karamychev (2000). A first issue is whether it matters that trade takes place in discrete time. Some markets (like financial markets) are characterized by continuous time trading and a continuous time model provides a better description of such markets than a discrete time model.[2] We show that the continuous time model exhibits qualitatively similar properties as the discrete time model. Second, in the discrete time dynamic models it is difficult to assess the role of the discount factor in determining the equilibrium path. In the continuous time model, we are able to show that changes in the discount parameter only effect the speed with which goods can be traded rather than the qualitative properties of equilibrium trading. Finally, as the continuous time model is easier to analyze, we are able to consider important extensions showing that similar results hold true when consumers are not risk-neutral, which is important in insurance markets, or when the good is not perfectly durable.

Our basic model is as follows. We consider a competitive market for a perfectly durable good where potential sellers are privately informed about the quality of the goods they own. Each moment in time a constant flow of sellers with an identical but arbitrary distribution of quality enters the market. In order to concentrate on the possibility of time acting as a sorting mechanism, the demand side is kept as simple as possible in the basic model. All buyers are identical, have unit demand and for any given quality, a buyer's willingness to pay exceeds the reservation price of a seller for that quality. As buyers do not know the quality, their willingness to pay equals the expected valuation of goods traded at a certain time. The flow of such buyers into the market is larger than the flow of sellers so that, in equilibrium, prices are equal to the expected buyers' valuation. Once traded, goods are not re-sold in the same market.[3]

The Akerlof-Wilson model can be considered the static version of our basic model and adverse selection implies that in equilibrium only a certain range of low qualities is traded. The infinitely repeated version of a static equilibrium outcome is also an equilibrium in our dynamic model. In this particular equilibrium of the continuous time model high quality goods remain unsold forever.

We concentrate on the existence of other equilibria with more interesting properties. A first result says that changes in the interest rate, which is related to changes in the discount factor in discrete models, do not affect the nature of equilibria in any way. Interest rates only determines the speed of evolution along an equilibrium path and in particular, higher interest rates implies a higher volume of trade at each moment as it is easier to separate goods of different quality. Next, we argue that there are infinitely many equilibria where the range of quality which is eventually traded in the market exceeds that of the stationary (static) outcome. Moreover, sellers of different qualities within the inflow of entrants separate themselves out over time. As the use value of low quality goods is lower than that of high quality goods, low quality sellers sell earlier than high quality sellers. A third, more powerful, result says that there exist an infinite number of equilibria where *every* potential seller entering the market trades within a finite time after entering the market. When the quality distribution is such that there are relatively few sellers around the static equilibrium quality such equilibria only exist when we allow price to be a discontinuous function of time before all goods are sold.

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[2] Inderst and Muller (2000) study a continuous time model where a market consists of several sub-markets. They show in an endogenous search model, that high quality goods may be traded in sub-markets with high prices and low probability of selling.

[3] For example, in car markets, it is publicly observable how many owners a car has had up to particular point in time. Hence, second hand markets may be distinguished from third-hand markets, and so on.

We also consider two extensions. First, we show that the results obtained for the basic model easily extend to models that allow for risk averse or risk-loving behavior. Second, we consider the case where goods are not perfectly durable but depreciate over time. The main result we obtain here is that if the depreciation rate is low enough, i.e., if goods are "almost perfectly durable", the results of the basic model roughly hold true: eventually, all goods are sold (even those high quality goods that just came to the market and did not depreciate very much). If the depreciation rate is higher, owners of high qualities will first wait until the quality of the good becomes worse before selling. Stationary equilibria, different from the static equilibria, may then emerge where low quality "new" goods and depreciated goods that originally were of high quality are traded at the same time. A full analysis of the depreciation case turns out to be rather complicated, however. We indicate, mainly by resorting to examples, possible equilibrium phenomena.

The results obtained in the paper provide a different perspective on the adverse selection problem. In the static Akerlof-Wilson model, the adverse selection problem manifests itself in the fact that high quality goods cannot be traded despite the potential gains from trade. When trade takes place in time and goods are durable, the *lemons problem* is not so much the impossibility of trading relatively high quality goods, but rather the fact that sellers of high quality goods need to wait in order to trade. When the good is imperfectly durable, sellers of good qualities may have to wait until their good has depreciated enough before selling.

The paper is organized as follows. Section 2 sets out the basic model and the equilibrium concept. Section 3 shows that for all distributions of goods entering the market, cyclical equilibria exist where, within a cycle, price and marginal quality are continuously increasing functions of time. The main result of the paper relating to the existence of an infinite number of equilibria where all goods are traded within finite time after entering the market is outlined in section 4. This section also shows that for a special class of distributions there exist equilibria such that, within a cycle, price and marginal quality are continuously increasing functions of time. Section 5 discusses the extensions and section 6 concludes. Proofs are contained in the Appendix.

2. The Basic Model

Consider a Walrasian market for a perfectly durable good whose quality, denoted by q, varies between $\underline{q}$ and $\bar{q}$, where $0<\underline{q}<\bar{q}<\infty$. Time, denoted by t, is continuous and runs from zero up to infinity. For every time moment t a constant flow of sellers I enters the market. Let t_i be the entry time of seller i and let q_i be the quality he is endowed with. The set of all sellers, therefore, is $\mathbf{I}=\{i\}=\{(q_i,t_i)\}$. We denote by $m(q)$ the Lebesgue measure of sellers in the flow I who own a good of quality less than or equal to q. We assume that $m(q)$ is strictly increasing absolutely continuous with respect to the Lebesgue measure and constant over time.

Each seller i knows the quality q_i of the good he is endowed with and derives flow utility from ownership of the good until he sells it. Therefore, the seller's reservation price is the present discounted value of the flow of gross utility and we normalize this to be equal to q_i. This implies that the gross utility flow is rq_i, where r is the discount rate.

On the demand side there is an inflow of new buyers, which is larger in size than $m(\bar{q})$. All buyers are identical and have unit demand. A buyer's valuation of quality q is equal to vq, where $v>1$. Thus, under full information, there are gains from trade. All buyers know the *ex ante* quality distribution $m(q)$ but do not know the quality of the good offered by a particular seller. Goods that are once bought are not re-sold in the same market. Buyers and sellers discount the future at the common rate r and maximize their expected utility.

We will denote by $h(\{I'\})$ the expected quality of a good from seller i that belongs to a certain subset $I'\subset\mathbf{I}$. It follows that $h(I')\equiv\frac{1}{m(I')}\int_{i\in I'}q_i\,dm(I')$, where $m(\{I'\})=m(\{i|i\in I'\})$. Adverse selection implies that $vh(\{I\})<\bar{q}$, i.e., the willingness to pay for the average quality in the population is lower then the reservation price of the seller of the bast quality. Thus, the static Akerlof-Wilson version of the model has a largest equilibrium quality, which we will denote by q_S.

To simplify our analysis we introduce the following regularity assumption. Throughout this paper, we assume that this assumption holds.

Assumption 2.1. The measure function $m(q)$ is differentiable on $\overline{U}=[q_S - e_m, \bar{q}]$ for some $e_m > 0$. Let $f(q)=\frac{dm}{dq}$. The function $f(q)$ is strictly positive and Lipschitz-continuous function, i.e., $f(q) \geq e_f > 0$ and for some M_f $|f(q') - f(q'')| < M_f |q' - q''|$ for all $q', q'' \in \overline{U}$.

Given an evolution of market prices $p(t)$, $t \in [0, \infty)$, each seller i chooses whether or not to sell and if he decides to sell, the selling time. If he chooses not to sell his gross surplus is equal to q_i and, therefore, his net surplus equals zero. On the other hand, if he decides to sell at time $t \geq t_i$ his gross surplus becomes

$$\int_{t_i}^{t} r q_i \, \mathrm{e}^{-r(t-t_i)} dt + \mathrm{e}^{-r(t-t_i)} p(t) = q_i + \mathrm{e}^{-r(t-t_i)}(p(t) - q_i),$$

and, therefore, his net discounted surplus is equal to

$$s_i(t) = \mathrm{e}^{-rt}(p(t) - q_i).$$

The set of time moments in which it is optimal to sell for a seller i is given by

$$T_i(p(t)) \equiv \arg\max_{t \geq t_i} \{s_i(t) | s_i(t) \geq 0\} = \arg\max_{t \geq t_i} \{\mathrm{e}^{-rt}(p(t) - q_i) | p(t) \geq q_i\}.$$

If $p_t - q_i < 0$ for all $t \geq t_i$ then $T_i(p(t)) = \varnothing$.

Each potential seller i chooses a time $t_i \in T_i$ when to sell. Let $\hat{\mathbf{o}} = \{t_i\}_{i \in \mathbf{I}}$ be a set of all selling decisions. This implies that there is a flow of goods being offered for sale. We will denote this flow by J_t and it follows that $J_t \equiv \{i | t_i = t\}$. This generates a certain distribution of qualities in that flow and the expected quality of the goods offered for sale in time moment t is $h_t = h(\{J_t\})$.

A dynamic equilibrium is an equilibrium where all players rationally maximize their objectives, expectations are fulfilled and market always clears. As all buyers are identical they have identical expectations about quality at any time t, which are denoted by $E(t)$.

Definition 2.1. A dynamic equilibrium is described in terms of a path of prices $p(t)$, buyers' quality expectations $E(t)$ and a set of selling decision $\hat{\mathbf{o}} = \{t_i\}_{i \in \mathbf{I}}$ such that:

a) **Sellers maximize:** $t_i \in T_i(p(t))$ for all $i \in I$, i.e., every seller i chooses time t_i to trade optimally.

b) **Buyers maximize and market clear**: If $m(\{J_t\})>0$ then $p(t)=vE(t)$, i.e., if at time t there is a strictly positive flow of goods offered for sale, then each buyer earns zero net surplus so that he is indifferent between buying and not buying and market clears. If $m(\{J_t\})=0$ then $p(t)\geq vE(t)$, i.e., if there are (almost) no goods for sale at time t then each buyer can earn at most zero net surplus. Hence, it is optimal for him not to buy at that time as well.

c) **Expectations are fulfilled when trade occurs**: If $m(\{J_t\})>0$ then $E(t)=h_t$.

d) **Expectations are reasonable even if no trade occurs:** For all t $E(t)\geq \underline{q}$.

Given the set-up described above, conditions (a)-(c) are quite standard. Condition (d) is introduced for the formal reason that expected quality is not defined when no trade occurs. The condition says that even if the flow of goods offered for sale is zero, buyers should believe that the expected quality is larger than the *apriori* lowest possible quality. This condition assures that autarky, i.e., no trade at any time, cannot be sustained in an equilibrium of the dynamic model. Given the condition, the willingness to pay, hence the price at any time, is restricted from below by $v\underline{q}$ and sellers with low enough qualities prefer to sell against this price rather than not sell.

It is easily seen that the infinitely repeated outcome of the static model, i.e., $p(t)=q_S$ and sellers with quality $q_i\in[\underline{q},q_S]$ sell immediately upon entering the market, is a dynamic equilibrium of our model. Hence, an equilibrium always exists. In the next section we will show that in the dynamic model there are infinitely many other equilibria. In all these other equilibria eventually more goods with higher qualities are sold than in the static equilibrium.

3. Multiple Equilibria

We will now show that for any distribution of quality entering the market and for all values of the parameters v and r there exists an infinite number of dynamic equilibria trading all goods from a certain range $[\underline{q},\hat{q}]$, where $\hat{q}\in(q_S,\overline{q}]$. The arguments we provide show how to find a price path $p(t)$ that entirely determines the dynamic equilibrium. All the equilibria are cyclical in the sense that the function $p(t)$ is periodical, i.e., for some $T>0$ and for all $t\in[0,\infty)$ $p(t+T)=p(t)$. Within the cycle $p(t)$ is strictly increasing.

We start our analysis by arguing that if a good of certain quality sells at time t, then all goods with lower qualities that have entered the market before (and are still in the market) will also sell at that time. Given any $p(t)$ a seller i of quality $\boldsymbol{q}_i$ by selling at time t earns a net discounted surplus $\mathrm{e}^{-rt}(p(t)-\boldsymbol{q}_i)$. Maximizing this expression[4] yields the first order conditions:

a) $\quad \mathbf{t}(\boldsymbol{q}_i)=t_i \qquad\qquad$ if $\dot{p}(t_i)+r\boldsymbol{q}_i \le rp(t_i)$, or

b) $\quad \dot{p}(\mathbf{t}(\boldsymbol{q}_i))+r\boldsymbol{q}_i = rp(\mathbf{t}(\boldsymbol{q}_i))$ if $\dot{p}(t_i)+r\boldsymbol{q}_i > rp(t_i)$.

The second order condition is simply $\ddot{p}(\mathbf{t})< r\dot{p}(\mathbf{t})$ if $\mathbf{t}(\boldsymbol{q}_i)>t_i$.

We first will look for equilibria that satisfy the second order condition for all $\boldsymbol{q}_i$. This implies that for any given $\boldsymbol{q}_i$ the optimal selling time $\mathbf{t}(\boldsymbol{q}_i)$ is unique if it exists.[5] Then, the first order condition (a) says that a seller should sell immediately upon entering the market, i.e., at time t_i, if the benefit of using a good rather than selling at time t_i, i.e., the use value of the good $r\boldsymbol{q}_i$ plus capital gain $\dot{p}(t_i)$, is smaller than the opportunity cost of owing the good at the entry time, which is equal to $rp(t_i)$. If, on the other hand, the benefit is larger than the cost, then the seller should wait until the moment they are equal to each other and sell at that time (condition b).

It follows that if a seller of quality $\boldsymbol{q}_i$ sells at time t then all sellers with qualities from the range $[\boldsymbol{q},\boldsymbol{q}_i]$, who are in the market at time $\mathbf{t}$, also prefer to sell at that time $\mathbf{t}$. This allows us to define for any t a marginal seller $\boldsymbol{q}(t)$ as the seller of the highest quality at time t:

$$\boldsymbol{q}(t)=\sup_i\{\boldsymbol{q}_i \mid i\in J_t\}=p(t)-\tfrac{1}{r}\dot{p}(t), \text{ or}$$

$$\dot{p}(t)=r(p(t)-\boldsymbol{q}(t)). \qquad (1)$$

Differentiating (1) gives

$$\ddot{p}-r\dot{p}=r(\dot{p}-\dot{\boldsymbol{q}})-r\dot{p}=-\dot{\boldsymbol{q}},$$

which implies that the second order condition requires $\boldsymbol{q}(t)$ to be an increasing function.

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[4] Implicitly, we have assumed that $p(t)$ is twice differentiable. As we will see, the solution that we obtain is such that this assumption is satisfied.

[5] We will see that there are equilibria such that high quality sellers will never sell. If this is the case then the first order conditions are never satisfied for them and the optimal selling time does not exist.

Having established that a flow of goods offered for sale at any time t is a range of qualities $[\underline{q}, q(t)]$, we denote by $h(q)$ the expected quality of goods from I conditional on those goods having a quality in the range $[\underline{q}, q\,]$ and it follows that $h(q) = \frac{1}{m(q)} \int_{\underline{q}}^{q} q\,dm$.

Now we are able to derive the main equation that must be satisfied along the equilibrium path. Let us consider an infinitely short time interval $(t, t+dt)$ such that $q(t) \in \overline{U}$. All qualities that entered before and at time t from the interval $[\underline{q}, q(t)]$ have already been traded and all qualities from $(q(t), \overline{q}\,]$ that have entered before are still in the market. Then the measure of goods with quality less than q, which are in the market at the moment $t + dt$, becomes

$$m_{t+dt}(q) = \begin{cases} m(q)dt & \text{for } q < q(t) \\ m(q(t))dt + (t+dt)(m(q) - m(q(t))) & \text{for } q > q(t) \end{cases}, \tag{2}$$

and the expected quality from the range $[\underline{q}, q(t+dt)]$ will be

$$h_t = \frac{dt \int_{\underline{q}}^{q(t)} V dm(V) + (t+dt) \int_{q(t)}^{q(t+dt)} V dm(V)}{dt \int_{\underline{q}}^{q(t)} dm(V) + (t+dt) \int_{q(t)}^{q(t+dt)} dm(V)} = \frac{m(q(t))h(q(t)) + tf(q(t))\dot{q}(t)q}{m(q(t)) + tf(q(t))\dot{q}(t)},$$

as $\dot{q}(t) > 0$. Therefore, price at time t must be equal to

$$p = v\frac{m(q)h(q) + tf(q)\dot{q}q}{m(q) + tf(q)\dot{q}}.$$

Rewriting gives

$$\dot{q} = \frac{m(q)(p - vh(q))}{tf(q)(vq - p)}.$$

Together with (1) we have finally obtained the following system

$$\begin{cases} \dot{p} = r(p - q) \\ \dot{q} = \dfrac{m(q)(p - vh(q))}{tf(q)(vq - p)} \end{cases}, \tag{3}$$

which describes the evolution of price and marginal quality along an equilibrium path. Note that $\dot{q} > 0$ and $\dot{p} > 0$ as long as $p > vh(q)$ and $q < p < vq$.

We first argue that the set of dynamic equilibria is independent of the interest rate. The parameter r only determines the speed with which prices and marginal qualities change over time. To this end, we rescale time by the parameter r as $y = rt$, where y is a new "time" variable. Then the net discounted surplus of a seller i if he decides to sell at "time" $y = rt \geq rt_i \equiv y_i$ against the price $\tilde{p}(y)$, becomes

$$\tilde{s}_i(y) = \mathrm{e}^{-(y-y_i)}(\tilde{p}(y) - q_i).$$

Hence, the marginal seller at "time" y will be a function $\tilde{q}(y)$ of y. Finally, along an equilibrium path $\tilde{p}(y) = vh(\{\tilde{J}_y\})$, where $\tilde{J}_y$ is the set of sellers with qualities smaller than $\tilde{q}(y)$ who have entered before or just at "time" y. So, we have closed the system in y while r drops out from all conditions to be satisfied except one: now the inflow of agents (buyers and sellers) become $\frac{1}{r}$ times larger (as the new "time" variable y is $\frac{1}{r}$ times denser than the old one) and we have to use a new measure function $\tilde{m}(q) = \frac{1}{r}m(q)$. On the other hand, expected quality $h_y = h(\tilde{J}_y)$ does *not* depend on the size of the flow $\tilde{J}_y$, but only on the distribution of quality. Thus, r disappears completely from our analysis and we have our first result.

Proposition 3.1. The set of dynamic equilibria of the model does not depend on the interest rate r. The interest rate only determines the speed of the evolution along an equilibrium path.

Using Proposition 3.1 we restrict the analysis without loss of generality to the case $r = 1$ and consider system (3) for that case only. Figure 3.1 shows the vector field of the system for some $t > 0$,[6] which is given by

$$\frac{dp}{dq} = \frac{\dot{p}}{\dot{q}} = t\frac{f(q)(vq - p)(p - q)}{m(q)(p - vh(q))}.$$

As $p(t) = vh_t$ and $h_t < q(t)$ no dynamic path that is a solution to system (3) can be above the line $p = vq$. On the other hand, for any solution to be a dynamic equilibrium it must satisfy $p \geq q$, i.e., the surplus of the marginal seller may not be negative. For all intermediate values of prices p, $q < p < vq$, $\frac{dp}{dq} > 0$. Finally, if $p = vh(q)$ then $\dot{q} = 0$ and tangents at such points are vertical for any $t > 0$.

[6] This is non-autonomous system and the vector field changes over time.

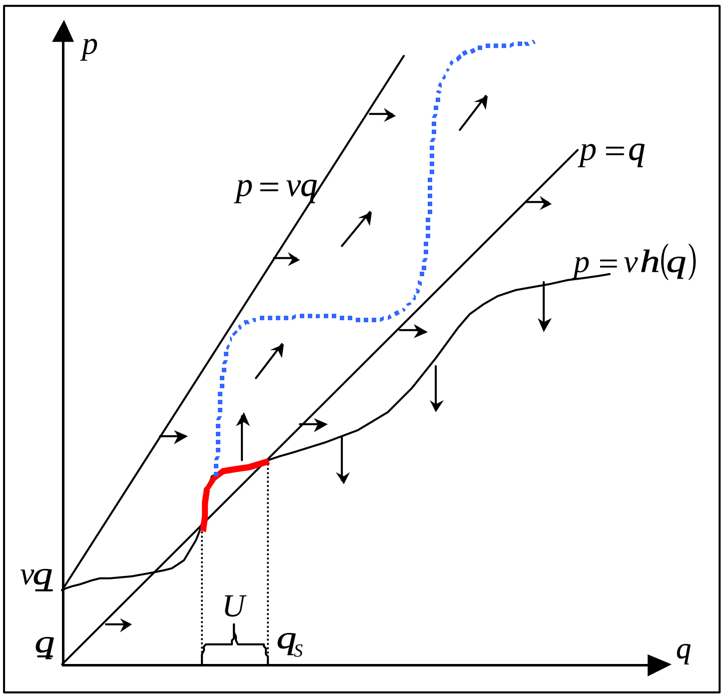

Figure 3.1.

Our objective, therefore, is to show that there exists a neighborhood U such that for any $q_0 \in U$ system (3) with initials $q(0)=q_0$ and $p(0)=p_0=vh(q_0)$ has a solution $(q(t,q_0), p(t,q_0))$, like the solution denoted by the dotted line in Figure 3.1, with $p \geq q$. What we will prove then in Proposition 3.2 is that for all $q_0 \in U$ there exists a time $T(q_0)>0$ such that all equilibrium conditions are fulfilled, prices and marginal qualities increase over the time interval $(0,T(q_0))$ and either $q(T,q_0)=\bar{q}$ or $q(T,q_0)=p(T,q_0)>q_S$. In both cases $q(T,q_0)$ is the largest quality that can be traded and we can extend $(q(t,q_0), p(t,q_0))$ in a periodic way, namely by defining $p(t+T)=p(t)$ and $q(t+T)=q(t)$, in this way we obtain a dynamic equilibrium where all goods from the range $[\underline{q},\hat{q}]$ are traded with $\hat{q}=q(T)$.

Proposition 3.2. There exists an infinite number of dynamic equilibria trading all goods from a certain range $[\underline{q},\hat{q}]$, where $\hat{q} \in (q_S,\bar{q}]$.

Proposition 3.2 implies that the repetition of the static equilibrium is the only stationary equilibrium. If we choose any arbitrary $q_0 < q_S$, the dynamic path will be such that eventually more than the static equilibrium amount of goods will be sold.

4. Equilibria Trading All Goods

So far, we have shown that for all distributions we can trade more than the static equilibrium quality if we allow for trade to take place over time. In this section we extend this result by showing that all goods can be traded if we relax the assumption about continuity of $p(t)$.

In the following Proposition 4.1 we show that there exists an infinite number of cyclical dynamic equilibria where all goods are traded at time $T, 2T, 3T, \ldots$.

Proposition 4.1. There exists an infinite number of dynamic equilibria $(\boldsymbol{q}(t), p(t))$ such that for some T:

a) $p(t+T) = p(t)$ and $\boldsymbol{q}(t+T) = \boldsymbol{q}(t)$;

b) $\boldsymbol{q}(T) = \bar{\boldsymbol{q}}$;

c) $\boldsymbol{q}(t)$ and $p(t)$ are strictly increasing functions for all $t \in (0, T)$ except (at most) at a finite number of points $\{t^{(k)}\}_{k=1}^{K}$ where both functions are discontinuous.

Figure 4.1 represents a typical equilibrium path $\boldsymbol{q}(t)$. Within each cycle n, where $t \in (nT, (n+1)T]$, the path is piecewise continuous, i.e., $\boldsymbol{q}(t)$ is a solution of (3) for every subcycle $t \in (t^{(k)}, t^{(k-1)}]$, $k = 1, 2, \ldots, K$, where K is a finite number defined in the proof of Proposition 4.1. The equilibrium construction is such that all sellers of quality $\boldsymbol{q}_S$ earn the same discounted surplus by selling at $t = t^{(k)}$, $k = 0, 1, \ldots, K-1$. Hence, they are indifferent between selling at each of these moments.

The discontinuities described in Figure 4.1 are used to build up enough time and high quality goods to allow the expected quality to improve enough to trade all goods. One may

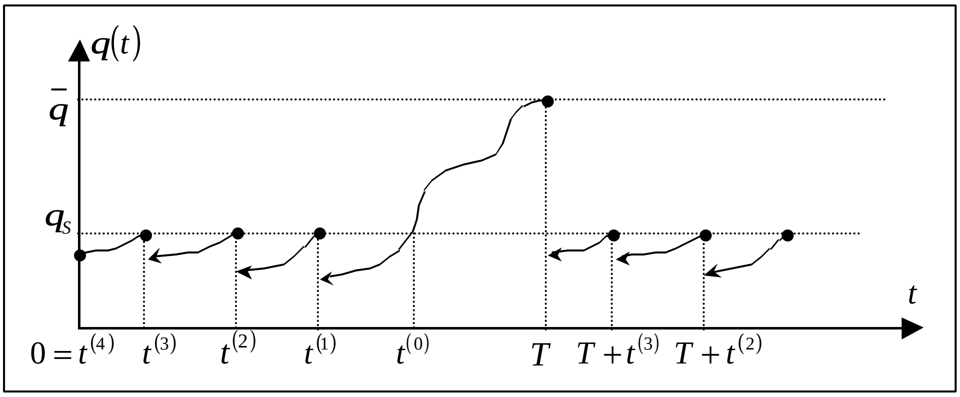

Figure 4.1.

wonder whether these discontinuities are required for all distributions of quality. Next, we will show that for certain distributions we can construct infinitely many equilibrium paths with $q(t)$ and $p(t)$ being continuous and strictly increasing over the whole cycle $(0,T)$. In the proof of Proposition 3.2 we have defined $a = a(q_S)$ and a function $a(q)$ on $\overline{U}$ as

$$a(q) \equiv \tfrac{d}{dq} v h(q) = \tfrac{1}{m(q)} v f(q)(q - h(q)), \qquad (4)$$

We will now show that this parameter a plays a crucial role in analyzing when continuous price equilibria exist. First, we will provide an economic interpretation of the parameter a and argue that generically, it must be that $0 < a < 1$. To this end, consider the surplus of the marginal seller in the *static model*, denoted by $s^{(S)}$, as a function of q:

$$s^{(S)}(q) \equiv p(q) - q = v h(q) - q = v \tfrac{1}{m(q)} \int_{\underline{q}}^{q} q\, dm(q) - q \text{, and}$$

$$\frac{ds^{(S)}(q_S)}{dq} = v \frac{d}{dq}\left(\tfrac{1}{m(q)} \int_{\underline{q}}^{q} q\, dm(q) \right)(q_S) - 1 = -(1-a).$$

Hence, $1-a$ can be interpreted as the way in which the surplus of the marginal seller changes in the neighborhood of the largest static equilibrium quality. Using a Tailor expansion, the surplus of the marginal seller can be written as

$$s^{(S)}(q) \equiv s^{(S)}(q_S) + (q - q_S)\tfrac{d}{dq} s^{(S)}(q_S) + o(q - q_S) = -(1-a)(q - q_S) + o(q - q_S).$$

Suppose then that $a > 1$.[7] This would imply that $s^{(S)}(q) > 0$ in some right neighborhood of q_S. But this contradicts the assumption that q_S is the highest static equilibrium quality. Hence, generically, $a < 1$. Lastly, $a(q) > 0$ under Assumption 2.1.

It turns out that the value $\frac{1-a}{a}$ determines the qualitative behavior of $(\hat{x}, \hat{y})$ and that the functions $\hat{x}(t)$ and $\hat{y}(t)$ behave quite differently depending on whether $\frac{1-a}{a}$ is smaller or larger than 1, i.e., whether a is larger or smaller than $\frac{1}{2}$.[8] Figure 4.2 shows the solution $(\hat{x}, \hat{y})$ as a parametric function $\hat{y}(\hat{x})$ with the parameter t, for two different values of a, $a = 0.1$ and $a = 0.6$. One can see that in the former case $\hat{y}(\hat{x})$ oscillates around the origin so that the second order condition ($\dot{q} > 0$) is not satisfied. In case $a = 0.6$ $\hat{x}(t)$ and $\hat{y}(t)$ are increasing functions so that in the neighborhood of the static equilibrium quality, prices and marginal qualities are increasing functions as well.

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[7] The case where $a = 1$ is a non-generic case.

[8] In the proof of Proposition 4.1 we use the so-called *Kummer's function* (Abramowitz and Stegun, 1972, pp. 504-515) with $\frac{1-a}{a}$ as one of the parameters.

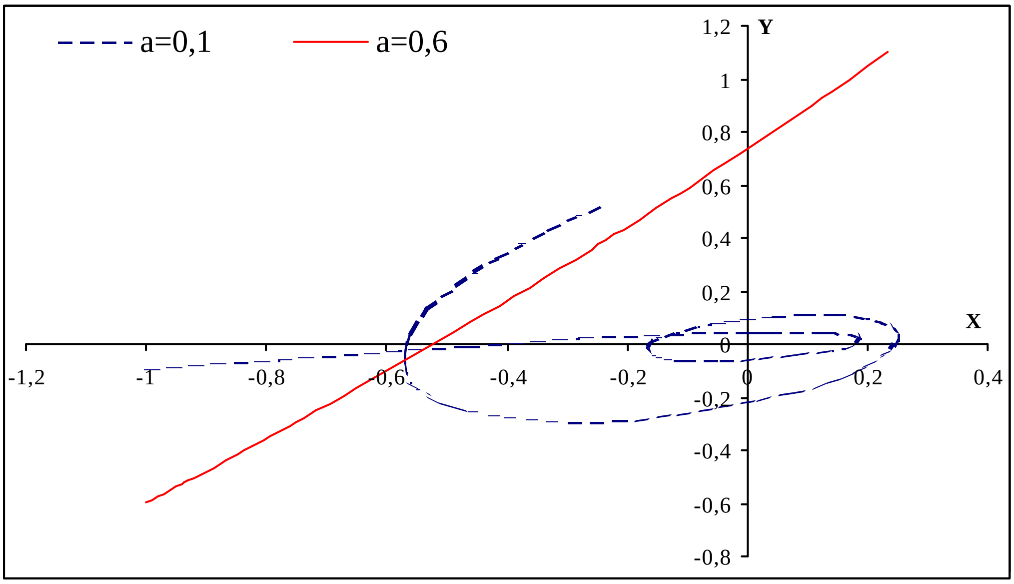

Figure 4.2.

Proposition 4.2 constructs for quality distributions with $a > \frac{1}{2}$ equilibria trading all goods where price and marginal quality are continuous in every cycle. It is easy to see that in case quality is uniformly distributed $s^{(S)}(q) \equiv \frac{v}{2}(q + \underline{q}) - q$ and $a = \frac{v}{2}$. As, adverse selection implies $1 < v < 2$, it is clear that the uniform distribution satisfies this condition.

Proposition 4.2. If $a > \frac{1}{2}$, then there exists an infinite number of cyclical dynamic equilibria $(q(t), p(t))$ such that:

a) $p(t+T) = p(t)$ and $q(t+T) = q(t)$;

b) $q(T) = \overline{q}$;

c) $q(t)$ and $p(t)$ are strictly increasing and continuous functions on $(0, T)$.

The result obtained in Proposition 4.2 says that in case $a > \frac{1}{2}$ we can choose $q(0)$ sufficiently close to q_S such that we do not need to build more than one subcycle in order to build up enough time and high quality goods to allow the expected quality to improve enough to trade all goods. Basically, the condition $a > \frac{1}{2}$ says that in a neighborhood of q_S there is a sufficient mass of goods so that at the moment when the marginal quality becomes larger then q_S, the marginal seller is able to make a positive surplus.

5. Extensions

So far, we have mainly focussed on the issue whether the evolution of market prices can be such that sellers of different qualities sort themselves over time. To this end, the demand side of the model has been kept as simple as possible. Also, we have considered the case of perfectly durable goods. In this section we relax these assumptions. We first consider a much larger class of preferences, including risk averse and risk loving behavior. In the context of adverse selection applications risk aversion is important when considering insurance markets. Next, we focus on the case where goods depreciate over time allowing us to address the case of imperfectly durable goods.

5.1 More General Demand Structure

The basic model can be easily extended to incorporate more general demand structures. Suppose that a buyer's valuation of quality q is equal to $v(q)$, where $v(q)-q > e_v$ and $\frac{dv}{dq} > e_v$ for some $e_v > 0$ and all $q \in [\underline{q}, \overline{q}]$, i.e., a buyers' valuation of a good of quality q is given by an arbitrary function such that (i) there are gains from trade under the full information, and (ii) higher quality goods are valued more than lower qualities. Having bought a good of quality q at time $t \geq t_i$ against price $p(t)$, a buyer derives utility $u(v(q) - p)$, where $u(0) = 0$ and $u' > e_u > 0$. This utility function allows for risk averse, risk neutral and risk loving preferences. The rest of the model remains as before.

In any equilibrium the expected buyers' utility must be equal to zero, i.e.,

$$\int_{\underline{q}}^{q(t)} u(v(q) - p(t))dm_t(q) = 0, \tag{5}$$

where $m_t(q)$ is the distribution of qualities within the flow of goods being offered for sale at time t. In this environment the static equilibrium is defined by $\int_{\underline{q}}^{q_S} u(v(q) - q_S)dm = 0$.

As the supply side is modeled in the same way the first differential equation (1) remains the same as well as the second order condition saying $\dot{q}(t) > 0$. In order to get the second differential equation we consider an infinitely short time interval $(t, t+dt)$ such that $q(t) \in \overline{U}$. Given the quality distribution (2), we can rewrite (5) as

$$\int_{\underline{q}}^{q(t)} u(v(x) - p(t))dm(x) + u(v(q(t)) - p(t))tf(q(t))\dot{q}(t) = 0, \text{ or}$$

$$\dot{\theta}(t) = -\frac{\int_{\underline{\theta}}^{\theta(t)} u(v(x) - p(t))d\mu(x)}{tf(\theta(t))u(v(\theta(t)) - p(t))} = \frac{F(\theta(t), p(t))}{tf(\theta(t))u(v(\theta(t)) - p(t))},$$

where $F(\theta, p) \equiv -\int_{\underline{\theta}}^{\theta} u(v(x) - p)d\mu(x)$ is differentiable in both arguments and strictly increasing in p. By definition $F(\theta_S, \theta_S) = 0$.

Thus, we have the system

$$\begin{cases} \dot{p}(t) = r(p(t) - \theta(t)) \\ \dot{\theta}(t) = \dfrac{F(\theta(t), p(t))}{tf(\theta(t))u(v(\theta(t)) - p(t))} \end{cases}.$$

In this case we define $a(\theta) \equiv \frac{dp_0(\theta_0)}{d\theta_0}(\theta)$ and using the definition of $F(\theta, p)$ it follows that

$$a(\theta) = \frac{u(v(\theta) - p(\theta))f(\theta)}{\int_{\underline{\theta}}^{\theta} u'(v(x) - p_0(\theta))d\mu(x)}.$$

As in the basic model it can be shown that generically it must be that $a \equiv a(\theta_S) \in (0,1)$. It can be checked that all propositions from sections 3 and 4 are still valid for the extended model when we use the new notion of $a(\theta)$.

5.2 Depreciation

The situation changes when the good under consideration is not perfectly durable and depreciates over time. Let $\delta \in [0, \infty)$ be the rate at which goods depreciate. This implies that the quality owned by a seller i, who entered the market at t_i, becomes a function of time: $\theta_i(t) = \theta_i \mathrm{e}^{-\delta(t - t_i)}$. We normalize sellers' gross utility flow to be equal to $(r + \delta)\theta_i(t)$, instead of $r\theta_i$, as in the basic model. The rest of the model remains as in the basic model.

If a seller i sells at time $t \geq t_i$ against $p(t)$ his net discounted surplus equals

$$s_i(t) = \mathrm{e}^{-rt}\left(p(t) - \mathrm{e}^{-\delta(t - t_i)}\theta_i\right).$$

Maximizing surplus with respect to the selling time $t \geq t_i$, assuming $p(t)$ is twice differentiable, yields $\theta = \frac{1}{r+\delta}(rp - \dot{p})$ as the marginal quality traded at time t. The second order condition in this case becomes $\dot{\theta} + \delta\theta > 0$.

In a similar way as we have done in section 3, one can derive the system describing the dynamic behavior of prices and marginal qualities:

$$\begin{cases} \dot{p} = r\left(p - \left(1 + \frac{d}{r}\right)q\right) \\ \dot{q} = dq\left(\dfrac{m(q)(p - v h(q))}{(vq - p)\left(m\left(q e^{dt}\right) - m(q)\right)} - 1\right) \end{cases}.$$

Proposition 3.1 remains valid for the extension considered here.[9] Taking without loss of generality $r = 1$, we get the following system:

$$\begin{cases} \dot{p} = p - (1 + d)q \\ \dot{q} = dq\left(\dfrac{m(q)(p - v h(q))}{(vq - p)\left(m\left(q e^{dt}\right) - m(q)\right)} - 1\right) \end{cases}. \tag{6}$$

The main new feature of (6) is that the system becomes autonomous for large t when $q e^{dt} \geq \bar{q}$, which is guaranteed by the second order condition $\dot{q} + dq > 0$, and, therefore, $m\left(q e^{dt}\right) = m(\bar{q})$. In other words, for large t the system (6) becomes independent of t.

We start our analysis arguing that if $v < 1 + d$, then in any continuous equilibrium path $(q(t), p(t))$ price decreases over time as $\dot{p} = p - (1 + d)q \leq vq - (1 + d)q = (v - 1 - d)q < 0$, and, therefore, $q(t)$ must drop below $\underline{q}$ at a certain time. From this moment on q decreases exponentially with rate d, which implies that all sellers with quality higher than q will never sell and sellers with quality lower than q have already sold. Hence, in case $v < 1 + d$ there is no trade after a certain time. Note that this condition is somewhat similar to the case $v < 1$ in the basic model. Buyers do not value the good enough and there can be no trade. From now on we assume $v > 1 + d$.

Now we will show that there exists a steady state $\left(q^*, p^*\right)$, not necessarily unique, such that for all $t \geq \frac{1}{d}\ln\frac{\bar{q}}{q^*}$[10] $(q(t), p(t)) = \left(q^*, p^*\right)$ is a solution of system (6). Indeed, solving for $\dot{q}(t) = \dot{p}(t) = 0$ yields:

$$\begin{cases} p^* = (1 + d)q^* \\ m\left(q^*\right)\left(q^*(1 + d) - v h\left(q^*\right)\right) - q^*(v - 1 - d)\left(m(\bar{q}) - m\left(q^*\right)\right) = 0 \end{cases}. \tag{7}$$

Note that at $q^* = \underline{q}$ the left-hand side of the second equation is negative, while at $q^* = \bar{q}$ it is positive. Therefore, there exists at least one point $q^* \in \left(\underline{q}, \bar{q}\right)$ such that the left-hand side equals zero. Hence, there is at least one steady state.

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[9] If we introduce $d^* = \frac{d}{r}$ as a "relative depreciation rate" the structure of the system allows us to get rid of r by substitution $y = rt$.

[10] This condition follows from setting $q e^{dt} \geq \bar{q}$ and solving for t.

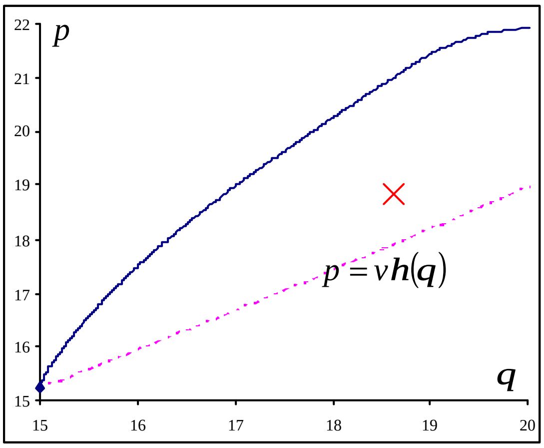

Figure 5.1.

The natural question about the local stability of the singular point (q^*, p^*) can be resolved by taking a linear analysis of (6) in the neighborhood of (q^*, p^*). One can derive the following sufficient conditions for (q^*, p^*) to be stable: $d > \frac{v-1}{v+1}$.

In the scope of this paper we will not fully investigate the solution of (6). We first observe that the conclusions reached concerning the basic model generalize when the depreciation rate d is small enough. Let us consider an example, where $\underline{q} = 10$, $\bar{q} = 20$, $f(q) = e^{0.1(q - \underline{q})}$, $v = 1.2$ and $d = 0.01$. Figure 5.1 shows that all goods can be sold in finite time even if there exists a stationary equilibrium (which is unstable) denoted by a cross in the figure. Proposition 5.1 generalizes this example and argues that we can extend the conclusion of Proposition 4.2 to the case where d is small enough.

Proposition 5.1. If $a > \frac{1}{2}$, then there exists a $\bar{d} > 0$ such that for all $d \in [0, \bar{d})$ there exist an infinite number of cyclical dynamic equilibria $(q(t,d), p(t,d))$ such that for some $T^d > 0$:

a) $p(t + T^d) = p(t)$ and $q(t + T^d) = q(t)$;

b) $q(T^d) = \bar{q}$;

c) $q(t,d)$ and $p(t,d)$ are continuous functions on $t \in (0, T^d)$.

The proof relies on the fact that the system (6) can be written as

$$\begin{cases} \dot{p} = p - q - dq \\ \dot{q} = \frac{1}{vq - p}\left(\frac{1}{tf(q)} m(q)(p - vh(q)) + dG(q, p, t)\right) \end{cases} \tag{8}$$

where

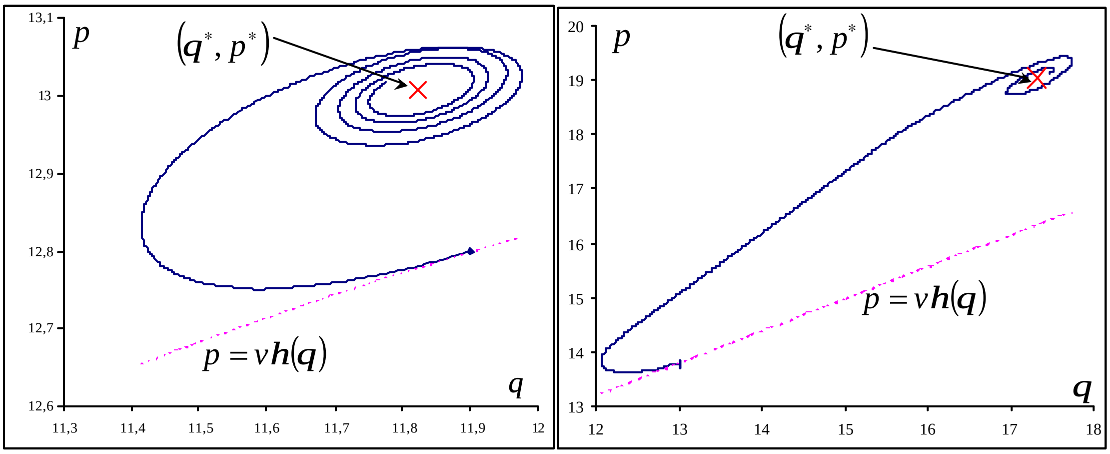

Figure 5.2.

$$G(q, p, t) \equiv \frac{m(q)(p - vh(q))}{f(q)} \cdot \frac{f(q)q - \dfrac{m(q\,\mathrm{e}^{dt}) - m(q)}{td}}{m(q\,\mathrm{e}^{dt}) - m(q)} - q(vq - p).$$

As $G(q, p, t)$ is finite for all p, $t > 0$, $d > 0$ and $q \in [\underline{q}, \bar{q} - e]$, system (3) is an approximation of (6) for small d.

When d is larger qualitatively new phenomena may emerge in equilibrium. This is shown in Figure 5.2 and Figure 5.3 by means of examples. Figure 5.2 shows that the stable steady state can be either below the static equilibrium (left graph, where $\underline{q} = 10$, $\bar{q} = 20$, $f(q) = \mathrm{e}^{-(q-\underline{q})}$, $v = 1.2$, $d = 0.1$, $q^* \approx 11.82$ and $q_S \approx 13.01$) or above the static equilibrium (right graph, where $\underline{q} = 10$, $\bar{q} = 30$, $f(q) = \mathrm{e}^{-0.01(q-\underline{q})}$, $v = 1.2$, $d = 0.1$, $q^* \approx 17.32$ and $q_S \approx 14.94$). Unlike the static equilibrium in the basic model, in these stationary equilibria all qualities are eventually traded in the market. However, owners of qualities $q_i > q^*$ first wait until their good has depreciated to q^* before selling.

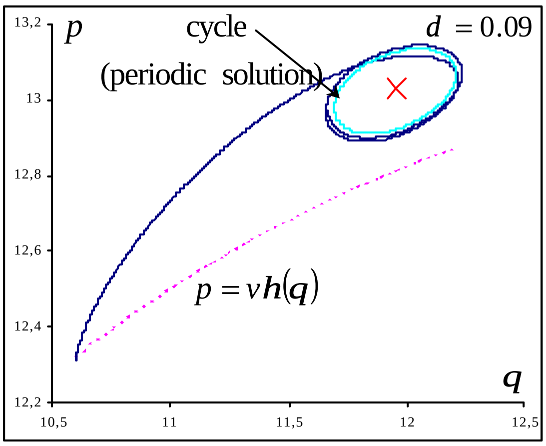

Figure 5.3.

In the second case, Figure 5.3, where $\underline{q}=10$, $\bar{q}=20$, $f(q)=\mathrm{e}^{-(q-\underline{q})}$, $v=1.2$ and $d=0.09$, d is "slightly" below $\frac{v-1}{v+1}$. Then (q^*, p^*) is not stable but there exists a cycle, i.e., a periodical solution of the corresponding autonomous system. In the long run, prices as well as marginal quality fluctuate with an asymptotically constant period.

6. Conclusions

In this paper, we have provided a different perspective on the way the adverse selection problem may manifest itself in durable good markets, where entry takes place and trading occurs in continuous time. In the static Akerlof-Wilson model, adverse selection results in high quality goods not being able to trade despite the potential gains from trade. The infinite repetition of this static equilibrium is also an equilibrium in the dynamic model where a durable good is traded in a competitive market. One result of this paper, however, says that there are infinitely many other equilibria where all goods are sold within finite time after entering the market. This result holds true even if consumers are not risk-neutral or when the good is not perfectly durable. When the depreciation rate is above a critical value, however, new stationary equilibria emerge where all goods are eventually traded. In this type of equilibrium, owners of high quality goods sell only after the good has depreciated enough.

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Appendix.

Proof of Proposition 3.2.

Under Assumption 2.1 for any $t_0 > 0$ and (q_0, p_0) such that $vq_0 \neq p_0$ system (3) has a unique solution $(q(t,q_0,p_0,t_0), p(t,q_0,p_0,t_0))$ with initial conditions $q(t_0)=q_0$ and $p(t_0)=p_0$, which is continuous w.r.t. q_0 and p_0. Considering $\dot{q}$ as a function of q, p and t, i.e., $\dot{q} = \Theta(q,p,t) \equiv \frac{m(q)(p - vh(q))}{tf(q)(vq - p)}$, allows us to take the following limit

$$\lim_{t\to 0}(t\Theta(q,p,t)) = \lim_{t\to 0}\frac{m(q)(p - vh(q))}{f(q)(vq - p)} = \frac{m(q(0))(p(0) - vh(q(0)))}{f(q(0))(vq(0) - p(0))} < \infty .$$

Therefore, system (3) has a solution even for $t_0 = 0$, but not necessarily unique. Uniqueness of the solution is guaranteed by the fact that $t\Theta(q,p,t) \not\equiv q + l$, where l is a constant. Finally, that solution is differentiable at $t = 0$ as long as $\lim_{t\to 0}(t\Theta(q,p,t)) = 0$, i.e., $p_0 = vh(q_0)$, and we will denote it as $(q(t,q_0), p(t,q_0)) \equiv (q(t,q_0,vh(q_0),0), p(t,q_0,vh(q_0),0))$. Indefiniteness of $\dot{q}(0,q_0)$ is resolved by continuity:

$$\dot{q}(0,q_0) = \lim_{t\to 0}\dot{q}(t,q_0) = \lim_{t\to 0}\frac{m(q)(p - vh(q))}{tf(q)(vq - p)} = \frac{m(q_0)}{f(q_0)(vq_0 - p_0)}\lim_{t\to 0}\frac{p - vh(q)}{t}.$$

Using the mean value theorem we have that the later expression equals

$$\frac{1}{a(q_0)}\lim_{t\to 0}\frac{[p_0 + \dot{p}(x_1 t,q_0)t] - [vh(q_0) + a(q(x_2 t,q_0))\dot{q}(x_2 t,q_0)t]}{t} =$$

$$= \frac{1}{a(q_0)}\lim_{t\to 0}\frac{\dot{p}(x_1 t,q_0)t - a(q(x_2 t,q_0))\dot{q}(x_2 t,q_0)t}{t} = \frac{1}{a(q_0)}(\dot{p}_0 - a(q_0)\dot{q}_0) = \frac{\dot{p}_0}{a(q_0)} - \dot{q}_0,$$

where $a(q_0) \equiv \left(\frac{d}{dq}vh(q)\right)_{q=q_0} = \frac{vf(q_0)(q_0 - h(q_0))}{m(q_0)}$,

$x_1 \in (0,1)$, $x_2 \in (0,1)$ and $\dot{q}_0 \equiv \dot{q}(0,q_0)$, $\dot{p}_0 \equiv \dot{p}(0,q_0)$. Rewriting yields $\dot{q}_0 = \frac{1}{2a(q_0)}\dot{p}_0$.[11]

Hence, $\left(\frac{dp}{dq}\right)_{t=0} = \frac{\dot{p}_0}{\dot{q}_0} = 2a(q_0) > a(q_0) = \left(\frac{d}{dq}vh(q)\right)_{t=0}$.

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[11] If it had been that $t\Theta(q,p,t) \equiv q + l$, or equivalently, $\dot{q} = \frac{q+l}{t}$, then we would have had $\dot{q}(0) = \lim_{t\to 0}\frac{q+l}{t} = \lim_{t\to 0}\frac{\dot{q}(0)}{1} = \dot{q}(0)$, that is identity. Therefore, in this case $\dot{q}(0)$ is not defined and can be chosen arbitrarily, that gives rise to multiple solutions.

This implies that for small $t > 0$ $p(t, \mathbf{q}_0) > v\mathbf{h}(\mathbf{q}(t, \mathbf{q}_0))$ as long as $\dot{p}_0 = v\mathbf{h}(\mathbf{q}_0) - \mathbf{q}_0 > 0$.

Now we define $\underline{\mathbf{q}_0}$ as $\underline{\mathbf{q}_0} = \inf\{\mathbf{q}' | \forall \mathbf{q} \in (\mathbf{q}', \mathbf{q}_S) : v\mathbf{h}(\mathbf{q}) > \mathbf{q}\}$ and $\hat{U}$ as $\hat{U} = (\underline{\mathbf{q}_0}, \mathbf{q}_S) \cap \overline{U}$ such that for all $\mathbf{q}_0 \in \hat{U}$ $v\mathbf{h}(\mathbf{q}_0) > \mathbf{q}_0$.

Finally, we will show that there exists a neighborhood $U \subset \hat{U}$ such that for any solution $(\mathbf{q}(t, \mathbf{q}_0), p(t, \mathbf{q}_0))$ of (3), where $\mathbf{q}_0 \in U$, there exists a time $T(\mathbf{q}_0) > 0$ such that for all $t \in (0, T)$ $p > \mathbf{q}$ and $(p(T) - \mathbf{q}(T))(\overline{\mathbf{q}} - \mathbf{q}(T)) = 0$, i.e., either all goods are sold or the marginal surplus is zero at time T. If this were not the case then there would have been $\lim_{t \to \infty} p = \lim_{t \to \infty} \mathbf{q} = \mathbf{q}'$[12]. But then the equation $\dot{\mathbf{q}} = \dfrac{m(\mathbf{q})(p - v\mathbf{h}(\mathbf{q}))}{tf(\mathbf{q})(v\mathbf{q} - p)}$ for large t becomes

$$\dot{\mathbf{q}} = \frac{1}{t}\left(\frac{m(\mathbf{q}')(\mathbf{q}' - v\mathbf{h}(\mathbf{q}'))}{f(\mathbf{q}')(v-1)\mathbf{q}'} + \mathbf{e}(t) \right),$$ where $\lim_{t \to \infty} \mathbf{e}(t) = 0$, and, therefore,

$$\mathbf{q}(t) > \frac{m(\mathbf{q}')(\mathbf{q}' - v\mathbf{h}(\mathbf{q}'))}{2f(\mathbf{q}')(v-1)\mathbf{q}'} \ln t + const$$

for sufficiently large t. Hence, $\lim_{t \to \infty} \mathbf{q}(t) = \infty$ unless $\mathbf{q}' - v\mathbf{h}(\mathbf{q}') = 0$, i.e., $\mathbf{q}' = \mathbf{q}_S$.

In order to rule out the possibility that $\mathbf{q}' = \mathbf{q}_S$ (and, hence, that $\mathbf{q}(t, \mathbf{q}_0)$ and $p(t, \mathbf{q}_0)$ converge to $\mathbf{q}' < \overline{\mathbf{q}}$) we rewrite system (3) as follows

$$\begin{cases} \dot{p} = (p - \mathbf{q}_S) - (\mathbf{q} - \mathbf{q}_S) \\ \dot{\mathbf{q}} = \dfrac{1}{at}\left((p - \mathbf{q}_S) - a(\mathbf{q} - \mathbf{q}_S) + \begin{pmatrix} (\mathbf{q} - \mathbf{q}_S) \\ (p - \mathbf{q}_S) \end{pmatrix}' \mathbf{B}(\mathbf{q}, p) \begin{pmatrix} (\mathbf{q} - \mathbf{q}_S) \\ (p - \mathbf{q}_S) \end{pmatrix} \right) \end{cases},$$

where $a = a(\mathbf{q}_S)$ and $\|\mathbf{B}(\mathbf{q}, p)\| < \infty$ uniformly[13] in a certain neighborhood of $\mathbf{q}_S$ $U \subset \hat{U}$. Thus, for all $\mathbf{q}_0 \in U$ the solution $(\mathbf{q}(t, \mathbf{q}_0), p(t, \mathbf{q}_0))$ can be written as

$$\begin{cases} \mathbf{q}(t, \mathbf{q}_0) = \mathbf{q}_S + (\mathbf{q}_S - \mathbf{q}_0)\hat{x}(t) + o(\mathbf{q}_S - \mathbf{q}_0) \\ p(t, \mathbf{q}_0) = \mathbf{q}_S + (\mathbf{q}_S - \mathbf{q}_0)\hat{y}(t) + o(\mathbf{q}_S - \mathbf{q}_0) \end{cases},$$

where $(\hat{x}(t), \hat{y}(t))$ solves the corresponding linearized system

$$\begin{cases} \dot{y} = y - x \\ \dot{x} = \frac{1}{at}(y - ax) \end{cases}, \tag{9}$$

[12] As p and $\mathbf{q}$ are increasing and bounded: $\mathbf{q} < \overline{\mathbf{q}}$, $p < v\mathbf{q} < v\overline{\mathbf{q}}$, and $\lim_{t \to \infty} \dot{p} = 0 = \lim_{t \to \infty}(p - \mathbf{q})$.

[13] The proof of this claim is available from the authors.

with initials $x(0) = -1$, $y(0) = -a$. Defining $k(t) = \frac{y(t)}{x(t)}$ allows us to rewrite (9) as $\dot{k} = -\left(1 - k - k\frac{a-k}{at}\right)$. Obviously, for all $t \in (0,\infty)$ $x < 0$, $y < 0$ and $y > ax$.[14] Hence, $k \in (0,a)$ and for sufficiently large t $\dot{k} < -\frac{1-a}{2} < 0$ and, therefore, $\lim_{t\to\infty} k(t) = -\infty$, which is in contradiction with $k \in (0,a)$.

So, for any $q_0 \in U$ $\exists T(q_0) > 0$ such that either $q(T,q_0) = \bar{q}$ or $q(T,q_0) = p(T,q_0)$. In both cases we extend (q,p) in a periodic way, namely $p(t+T) = p(t)$ and $q(t+T) = q(t)$. In order to show that $q(T) > q_S$ when $q(T) = p(T) < \bar{q}$ let us consider two cases.

a) $q(T) = q_S$. This contradicts with the uniqueness of the solution with initials $q(T) = q_S$. Indeed, we always have a static solution $q(t,q_S) = p(t,q_S) = q_S$ and we have found another, namely $(q(t,q_0), p(t,q_0))$, such that $q(T,q_0) = p(T,q_0) = q_S$.

b) $q(T) < q_S$. This implies that $vh(q(T)) > q(T)$, which can never happen as for small t $vh(q(t)) < q(t)$ and no solution may cross the curve $p = vh(q)$ from above at $q < q_S$.

Hence, $q(T) > q_S$.

As we have obtained a discontinuous function $p(t)$, we lost the sufficiency of the first and second order conditions. So, we must check the optimality of the stipulated sellers behavior directly.

Let us take any seller i with quality q_i and entry time $t_i \in [nT,(n+1)T)$, where n is the entry cycle's number. If $q_i > q(T)$ then he will never sell, which is clearly optimal for him. If, on the other hand, $q_i < q(T)$ then there are two possibilities.

a) $q_i > q(t_i)$. In this case he maximizes his surplus by selling in the current cycle n at time $\boldsymbol{t}_i(q_i)$, where $q(\boldsymbol{t}_i) = q_i$, and getting $s_i(\boldsymbol{t}_i) = s(\boldsymbol{t}_i)$, see Figure A.1(a), and it follows that $\boldsymbol{t}_i > t_i$.[15] Indeed if he had been waiting for the next cycle $(n+1)$ he would have chosen time $\boldsymbol{t}_i'(q_i) = \boldsymbol{t}_i + T$ to sell, where $q(\boldsymbol{t}_i') = q_i$, and got $s_i(\boldsymbol{t}_i') = s(\boldsymbol{t}_i')$. But $p(\boldsymbol{t}_i) = p(\boldsymbol{t}_i + T) = p(\boldsymbol{t}_i')$ and the seller i will certainly choose the earlier time $\boldsymbol{t}_i$.

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[14] As $q < q_S$, $p < q_S$ and $p > vh(q)$.

[15] Within a cycle the first and second order conditions still work so there is a unique optimal selling time $\boldsymbol{t}_i$.

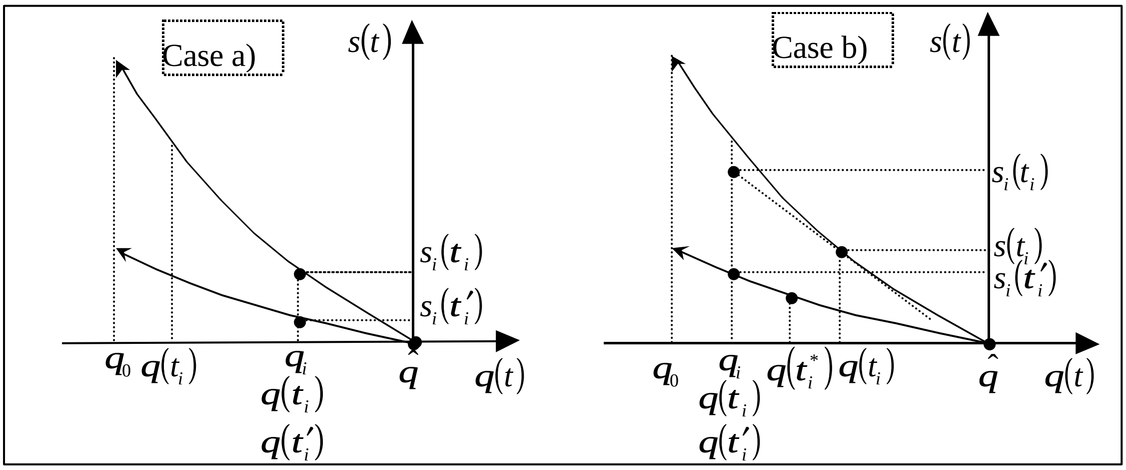

Figure A.1.

b) $q_i < q(t_i)$. In this case let us first investigate the marginal surplus function $s(t) \equiv e^{-t}(p - q)$. From (3) it follows that

$$\frac{ds}{dq} = \frac{\dot{s}}{\dot{q}} = -e^{-t}.$$

Although the above expression has been obtained only for $t \in [0, T]$ it holds for any $t \in [0, \infty)$. To see this, suppose $t \in (nT, (n+1)T]$. It then follows that

$$\frac{ds}{dq}(t) = -e^{-nT}\frac{ds}{dq}(t - nT) = -e^{-nT}e^{-(t-nT)} = -e^{-t} \text{ for all } t \in [0, \infty). \quad (10)$$

Hence, $s(t)$ is a positive, decreasing and convex function on $(nT, (n+1)T)$. These properties allow us to validate the maximum principle across different continuous segments of an equilibrium path.

Now let us define $\mathbf{t}_i(q_i)$ such that $\mathbf{t}_i \in (nT, (n+1)T]$ and $q(\mathbf{t}_i) = q_i$, in this case $\mathbf{t}_i < t_i$, see Figure A.1(b). If seller i sells immediately after the entry, i.e., at time t_i, he gets

$$s_i(t_i) = e^{-t_i}(p(t_i) - q_i) = e^{-t_i}(p(t_i) - q(t_i)) + e^{-t_i}(q(t_i) - q_i) = s(t_i) + e^{-t_i}(q(t_i) - q_i),$$

while if he waits until the next cycle $(n+1)$ he will choose time $\mathbf{t}_i'(q_i) = \mathbf{t}_i + T$ to sell, where $q(\mathbf{t}_i') = q_i$, and, therefore, his surplus becomes

$$s_i(\mathbf{t}_i') = e^{-t_i'}(p(\mathbf{t}_i') - q_i) = e^{-t_i}(p(\mathbf{t}_i') - q(\mathbf{t}_i')) = s(\mathbf{t}_i').$$

In order to show that $s_i(t_i) > s_i(\mathbf{t}_i)'$ for all $q_i < q(t_i)$ let us consider $\Delta(q_i) \equiv s_i(t_i) - s_i(\mathbf{t}_i')$ as a function of q_i and apply the mean-value theorem:

$$s(\mathbf{t}_i') = s(t_i + T) + (q(\mathbf{t}_i') - q(t_i + T))\tfrac{ds}{dq}(\mathbf{t}_i^*) = e^{-T} s(t_i) + (q(t_i) - q_i)e^{-t_i^*},$$

for some $t_i^* \in (t'_i, t_i + T)$. Then

$$\Delta(q_i) = s(t_i) + e^{-t_i}(q(t_i) - q_i) - e^{-T} s(t_i) - (q(t_i) - q_i)e^{-t_i^*} =$$
$$= s(t_i)(1 - e^{-T}) + (q(t_i) - q_i)(e^{-t_i} - e^{-t_i^*}) > 0.$$

Therefore, we have shown that for any $q_0 \in U$ $(q(t,q_0), p(t,q_0))$ constitutes a dynamic equilibrium trading all goods from the range [$\underline{q}, \hat{q}$], where $\hat{q} = q(T) > q_S$.

In order to prove the following propositions we need the following lemma.

Lemma 1. For any Suppose that for any numbers $q_0 \in (q_S, \bar{q})$ and $p_0 \in (q_0, vq_0)$ there exists $\hat{t} > 0$, depending on $p_0 - q_0$, such that for all $t_0 > \hat{t}$ system (3) has a unique solution with initials $q(t_0) = q_0$ and $p(t_0) = p_0$. Moreover, there exists a finite time $T > \hat{t}$ such that $q(T) = \bar{q}$.

Proof. Under Assumption 2.1 for any t_0 system (3) has a unique solution passing through (q_0, p_0). All we need to show then is the existence of T if t_0 is taken to be sufficiently large. We define $a \equiv \min\{p_0 - q_0, \frac{v-1}{2} q_S\}$ and

$$\hat{t} \equiv \frac{2vm(\bar{q})\bar{q}}{(v-1)e_f a q_S}. \tag{11}$$

As a is a function of $p_0 - q_0$ so is $\hat{t}$. Now let us consider the solution mentioned above when $t_0 > \hat{t}$ and suppose that $p(t) - q(t) = a$ for some $t \geq t_0$. Then

$$\frac{d}{dt}(p(t) - q(t)) = \dot{q}(t)\frac{d}{dq}(p(t) - q(t)) = \dot{q}(t)\left(\frac{dp}{dq} - 1\right) = \dot{q}(t)\left(\frac{\dot{p}(t)}{\dot{q}(t)} - 1\right).$$

Using (3) yields

$$\frac{d}{dt}(p(t) - q(t)) = \dot{q}\left(t\frac{f(q)(p-q)(vq-p)}{m(q)(p - vh(q))} - 1\right) > \dot{q}\left(t_0 \frac{e_f a((v-1)q - a)}{m(\bar{q})p} - 1\right) >$$
$$> \dot{q}\left(t_0 \frac{e_f a((v-1)q_S - \frac{v-1}{2} q_S)}{m(\bar{q})v\bar{q}} - 1\right) > \dot{q}\left(\hat{t}\,\frac{(v-1)e_f a q_S}{2vm(\bar{q})\bar{q}} - 1\right) = 0.$$

Thus $p(t) - q(t) > a > 0$ for all $t \geq t_0$. Now it becomes clear (see Proof of Proposition 3.2.) that for some T we must have $q(T) = \bar{q}$.

Proof of Proposition 4.1.

We first define functions $(\boldsymbol{q}(t), p(t))$ for all $t \in [0, t_0]$ and some t_0 such that the condition of Lemma 1 is satisfied, i.e., $t_0 > \hat{t}\left(p(t_0) - \boldsymbol{q}(t_0)\right)$, and $\boldsymbol{q}(t_0) = \boldsymbol{q}_S$. Then we show that $(\boldsymbol{q}(t), p(t))$ actually is an equilibrium path. Lastly, Lemma 1 says that all goods are traded by a certain time T.

In section 3 we have shown that for any $\boldsymbol{q}_0 \in U$ and $p_0 = v\boldsymbol{h}(\boldsymbol{q}_0)$ system (3) has a solution $(\boldsymbol{q}(t,\boldsymbol{q}_0), p(t,\boldsymbol{q}_0))$ and $\boldsymbol{q}(T) > \boldsymbol{q}_S$ for some T. As $\dot{\boldsymbol{q}}(t,\boldsymbol{q}_0) > 0$, it follows that for all $\boldsymbol{b} \in [\boldsymbol{q}_0, \boldsymbol{q}(T)]$ there exists an inverse $t(\boldsymbol{b},\boldsymbol{q}_0)$ such that $\boldsymbol{q}(t(\boldsymbol{b},\boldsymbol{q}_0),\boldsymbol{q}_0) = \boldsymbol{b}$. Function $t(\boldsymbol{b},\boldsymbol{q}_0)$ is continuously differentiable on $\boldsymbol{b} \in [\boldsymbol{q}_0, \boldsymbol{q}(T)]$ and continuous w.r.t. $\boldsymbol{q}_0$. Hence, $p(t(\boldsymbol{b},\boldsymbol{q}_0),\boldsymbol{q}_0)$ is continuous w.r.t. $\boldsymbol{q}_0$ as well. We define $\boldsymbol{t}(\boldsymbol{q}_0)$ by $\boldsymbol{t}(\boldsymbol{q}_0) = t(\boldsymbol{q}_S, \boldsymbol{q}_0)$, so that $\boldsymbol{q}(\boldsymbol{t}(\boldsymbol{q}_0),\boldsymbol{q}_0) = \boldsymbol{q}_S$ for all $\boldsymbol{q}_0 \in U$. Note that $p(\boldsymbol{t}(\boldsymbol{q}_0),\boldsymbol{q}_0) > \boldsymbol{q}(\boldsymbol{t}(\boldsymbol{q}_0),\boldsymbol{q}_0) = \boldsymbol{q}_S$.

Now we will solve the linearized system (9). Firstly, it can be rewritten as *Kummer's equation* (see Abramowitz and Stegun, 1972, pp. 504-515):

$$t\ddot{x} + (2 - t)\dot{x} - x\left(-\tfrac{1-a}{a}\right) = 0.$$

with initials $x(0) = -1$, $\dot{x}(0) = \frac{1-a}{2a}$. The unique solution is the negative to the so-called *Kummer's function* $M(a_1, a_2, t)$, with $a_1 = -\frac{1-a}{a}$ and $a_2 = 2$. Turning back to (9):

$$\begin{cases} \hat{x}(t) = -M\left(-\tfrac{1-a}{a}, 2, t\right) = -1 + \dfrac{1-a}{a}\displaystyle\sum_{n=1}^{\infty} \dfrac{\Gamma\left(n - \frac{1-a}{a}\right)}{n!(n+1)!\Gamma\left(1 - \frac{1-a}{a}\right)} t^n \\ \hat{y}(t) = -a\tfrac{d}{dt}\left(tM\left(-\tfrac{1-a}{a}, 2, t\right)\right) = a\left(-1 + \dfrac{1-a}{a}\displaystyle\sum_{n=1}^{\infty} \dfrac{\Gamma\left(n - \frac{1-a}{a}\right)}{(n!)^2\Gamma\left(1 - \frac{1-a}{a}\right)} t^n\right) \end{cases}, \tag{12}$$

where $\Gamma(x) = \int_0^\infty \mathrm{e}^{-t} t^{x-1} dt$ is the Gamma-function, and

$$\frac{\Gamma\left(n - \frac{1-a}{a}\right)}{\Gamma\left(1 - \frac{1-a}{a}\right)} = \left(2 - \tfrac{1}{a}\right)\left(3 - \tfrac{1}{a}\right) \times \ldots \times \left(n - \tfrac{1}{a}\right).$$

Now we define $\boldsymbol{w}$ as $\hat{x}(\boldsymbol{w}) = 0$ and it follows that $\boldsymbol{w} > 0$ and $\hat{y}(\boldsymbol{w}) > 0$. As

$$\begin{cases} \boldsymbol{q}(t,\boldsymbol{q}_0) = \boldsymbol{q}_S + (\boldsymbol{q}_S - \boldsymbol{q}_0)\hat{x}(t) + o(\boldsymbol{q}_S - \boldsymbol{q}_0) \\ p(t,\boldsymbol{q}_0) = \boldsymbol{q}_S + (\boldsymbol{q}_S - \boldsymbol{q}_0)\hat{y}(t) + o(\boldsymbol{q}_S - \boldsymbol{q}_0) \end{cases},$$

the functions $\hat{x}(t)$ and $\hat{y}(t)$ describe the behavior of the solution $(\boldsymbol{q}(t,\boldsymbol{q}_0), p(t,\boldsymbol{q}_0))$ in the neighborhood of $p = \boldsymbol{q} = \boldsymbol{q}_S$ and it follows that

$$\lim_{q_0 \to q_S} \boldsymbol{t}(\boldsymbol{q}_0) = \boldsymbol{w},$$

$$\lim_{q_0 \to q_S} \frac{p(t(q_0), q_0) - q_S}{q_S - q_0} = \hat{y}(w).$$

Therefore, there exists a left neighborhood of the static equilibrium quality $U^w \equiv \left(\underline{q_0^w}, q_S\right) \in U$ such that for all $q_0 \in U^w$ $t(q_0) > \frac{1}{2} w$.

Now we are ready to construct the pair of functions $(q(t), p(t))$. Let us take any $q_0^{(1)} \in U^w$ and define $t^{(1)} \equiv t\left(q_0^{(1)}\right)$, $q^{(1)}(t) \equiv q\left(t, q_0^{(1)}\right)$ and $p^{(1)}(t) \equiv p\left(t, q_0^{(1)}\right)$ on $t \in (0, t^{(1)}]$, $s^{(1)}(t) \equiv \mathrm{e}^{-t}\left(p^{(1)}(t) - q^{(1)}(t)\right)$. By construction we have $t^{(1)} > \frac{1}{2} w$ and $s^{(1)}\left(t^{(1)}\right) > 0$.

Let us now consider the function $r^{(1)}(q_0) \equiv p(t(q_0), q_0) - q_S - s^{(1)}\left(t^{(1)}\right)$ as a function of q_0. It is continuous on $[q_0^{(1)}, q_S]$. Moreover,

$$r^{(1)}\left(q_0^{(1)}\right) = p\left(t\left(q_0^{(1)}\right), q_0^{(1)}\right) - q_S - s^{(1)}\left(t^{(1)}\right) = q_S + \mathrm{e}^{t^{(1)}} s^{(1)}\left(t^{(1)}\right) - q_S - s^{(1)}\left(t^{(1)}\right) > 0, \text{ and}$$

$$r^{(1)}(q_S) = p(t(q_S), q_S) - q_S - s^{(1)}\left(t^{(1)}\right) = q_S - q_S - \mathrm{e}^{t^{(1)}} s^{(1)}\left(t^{(1)}\right) < 0.$$

Therefore, $\exists q_0^{(2)} \in \left(q_0^{(1)}, q_S\right)$ depending on $t(q_0)$ such that $r^{(1)}\left(q_0^{(2)}\right) = 0$, i.e., $p\left(t\left(q_0^{(2)}\right), q_0^{(2)}\right) = s^{(1)}\left(t^{(1)}\right)$. Again, we define $t^{(2)} \equiv t\left(q_0^{(2)}\right)$, $q^{(2)}(t) \equiv q\left(t, q_0^{(2)}\right)$ and $p^{(2)}(t) \equiv p\left(t, q_0^{(2)}\right)$ on $t \in \left(0, t^{(2)}\right]$, $s^{(2)}(t) \equiv s\left(t, q_0^{(2)}\right)$ and, again, $t^{(2)} > \frac{1}{2} w$.

Repeating this process, we get a sequence $\left\{t^{(k)}\right\}$ such that $\lim_{k \to \infty} \sum_{j=1}^{k} t^{(j)} = \infty$. We define $K \geq 1$ to be the smallest number such that $\sum_{j=1}^{K} t^{(j)} > \hat{t}\left(p^{(1)}\left(t^{(1)}\right) - q^{(1)}\left(t^{(1)}\right)\right)$. Then, we define $t^{(k)}$ as $t^{(k)} = \sum_{j=k+1}^{K} t^{(j)}$, so that $t^{(K)} = 0$, $t^{(K-1)} = t^{(K)}$, $t^{(K-2)} = t^{(K)} + t^{(K-1)}, \ldots,$ $t^{(0)} = \sum_{j=1}^{K} t^{(j)} > \hat{t}$.

Finally, we define an equilibrium path for $t \in (0, t^{(0)}]$ as follows

$$p(t, q_0) = \left\{p^{(k)}\left(t - t^{(k)}\right), \text{if } t \in (t^{(k)}, t^{(k-1)}],\right.$$

$$q(t, q_0) = \left\{q^{(k)}\left(t - t^{(k)}\right), \text{if } t \in (t^{(k)}, t^{(k-1)}],\right.$$

$$s(t, q_0) = \left\{\mathrm{e}^{-t^{(k)}} s^{(k)}\left(t - t^{(k)}\right), \text{if } t \in (t^{(k)}, t^{(k-1)}].\right.$$

For $t > t^{(0)}$ we take the solution $\left(q\left(t, q^{(1)}\left(t^{(0)}\right), p^{(1)}\left(t^{(0)}\right), t^{(0)}\right), p\left(t, q^{(1)}\left(t^{(0)}\right), p^{(1)}\left(t^{(0)}\right), t^{(0)}\right)\right)$ of (3) as an equilibrium path.

It can be easily seen that within every interval $(t^{(k)}, t^{(k-1)}]$ a seller chooses the time to trade optimally. In order to check that he behaves optimally even across those intervals (*subcycles*) and across cycles, again, like in the proof of Proposition 3.2, we use (10) and considering a seller i with quality $\boldsymbol{q}_i$ and entry time $t_i \in (nT + t^{(k)}, nT + t^{(k-1)}]$. As the arguments are quite similar to the ones given in the proof of Proposition 3.2 we skip the details here.

Like in Proposition 3.2 any seller i optimally waits until the first moment after entry when the marginal quality is larger than or equal to his own quality. Hence, the pair of functions $(\boldsymbol{q}(t), p(t))$ constructed above satisfies all equilibrium requirements. Then, it follows from Lemma 1 that $\exists T$ such that $\boldsymbol{q}(T) = \overline{\boldsymbol{q}}$. The constructed equilibrium path is entirely determined by choosing $\boldsymbol{q}_0^{(1)}$ which is an arbitrary point from U^w. Therefore, we have obtained infinitely many (continuum of) equilibria.

Proof of Proposition 4.2.

If $a > \frac{1}{2}$, then each term (apart from the constant) in the Tailor expansion (12) is positive as $\Gamma\left(1 - \frac{1-a}{a}\right) > 0$ and its radius of convergence is infinity. Hence, $\hat{x}$ and $\hat{y}$ are defined by (12) for all $t \in [0, \infty)$, $\lim_{t \to +\infty} \hat{y}(t) = \lim_{t \to +\infty} \hat{x}(t) = \lim_{t \to +\infty} \dot{\hat{y}}(t) = \lim_{t \to +\infty} \dot{\hat{x}}(t) = +\infty$, and

$$\lim_{t \to +\infty} \frac{\hat{y}}{\hat{x}} = \lim_{t \to +\infty} \frac{\dot{\hat{y}}}{\dot{\hat{x}}} = \lim_{t \to +\infty} \frac{\hat{y} - \hat{x}}{\frac{\hat{y} - a\hat{x}}{at}} = \lim_{t \to +\infty} \frac{\dot{\hat{y}} - \dot{\hat{x}}}{\frac{d}{dt}\frac{\hat{y} - a\hat{x}}{at}} = \lim_{t \to +\infty} \frac{\dot{\hat{y}} - \dot{\hat{x}}}{\frac{\dot{\hat{y}} - a\dot{\hat{x}}}{at} - \hat{x}\frac{1}{t}} = \lim_{t \to +\infty} at\left[1 + \frac{(2a-1)\dot{\hat{x}}}{\dot{\hat{y}} - 2a\dot{\hat{x}}}\right] = +\infty,$$

as $\dot{\hat{y}} - a\dot{\hat{x}} > \dot{\hat{y}} - \dot{\hat{x}} > \dot{\hat{y}} - 2a\dot{\hat{x}} = (1-a)\sum_{n=2}^{\infty} \frac{\Gamma\left(n - \frac{1-a}{a}\right)t^{n-1}}{(n-2)!(n+1)!\Gamma\left(1 - \frac{1-a}{a}\right)} > 0$ for all $t > 0$.

This implies that for $a > \frac{1}{2}$

$$\lim_{t \to +\infty} \lim_{q_0 \to q_S} \frac{p(t, \boldsymbol{q}_0) - \boldsymbol{q}_S}{\boldsymbol{q}(t, \boldsymbol{q}_0) - \boldsymbol{q}_S} = \lim_{t \to +\infty} \lim_{q_0 \to q_S} \frac{\hat{y}(t) + O(\boldsymbol{q}_S - \boldsymbol{q}_0)}{\hat{x}(t) + O(\boldsymbol{q}_S - \boldsymbol{q}_0)} = \lim_{t \to +\infty} \frac{\hat{y}(t)}{\hat{x}(t)} = +\infty.$$

In other words, for any $M > 0$ $\exists t'(M)$ such that for all $t > t'$ $\exists U^a(t, M) = \left(\underline{\boldsymbol{q}_0^a}(t, M), \boldsymbol{q}_S\right)$ such that for all $\boldsymbol{q}_0 \in U^a$:

$$\frac{p(t, \boldsymbol{q}_0) - \boldsymbol{q}_S}{\boldsymbol{q}(t, \boldsymbol{q}_0) - \boldsymbol{q}_S} > M\ .$$

We take $\boldsymbol{a} = \frac{v-1}{2}\boldsymbol{q}_S$, $M = \frac{v\overline{q}}{a} = \frac{2v\overline{q}}{(v-1)q_S} > 2$ and $\boldsymbol{t} = \max\left\{\frac{M}{M-1}\hat{t}(\boldsymbol{a}), t'(M)\right\}$, where $\hat{t}(\ldots)$ is as defined (11), Lemma 1. Then it follows that

$$\boldsymbol{t} \geq \frac{M}{M-1} \frac{2v\overline{\boldsymbol{q}}m(\overline{\boldsymbol{q}})}{(v-1)\boldsymbol{e}_f \boldsymbol{a}\boldsymbol{q}_S} = \frac{2m(\overline{\boldsymbol{q}})M^2}{(v-1)\boldsymbol{e}_f \boldsymbol{q}_S (M-1)}.$$

For this $\boldsymbol{t}$ there exists a neighborhood $U^a(\boldsymbol{t}, M)$ such that for all $\boldsymbol{q}_0 \in U^a$

$$\frac{p(\boldsymbol{t},\boldsymbol{q}_0)-\boldsymbol{q}_S}{\boldsymbol{q}(\boldsymbol{t},\boldsymbol{q}_0)-\boldsymbol{q}_S} > M \ .$$

We will show that $\exists t'' \geq \boldsymbol{t}$ such that $p(t'',\boldsymbol{q}_0) \geq \boldsymbol{q}(t'',\boldsymbol{q}_0)+\boldsymbol{a}$. Suppose to the contrary that $p(t,\boldsymbol{q}_0) < \boldsymbol{q}(t,\boldsymbol{q}_0)+\boldsymbol{a}$ for all $t \geq \boldsymbol{t}$. Then it must be the case that

$$\frac{p(t,\boldsymbol{q}_0)-\boldsymbol{q}_S}{\boldsymbol{q}(t,\boldsymbol{q}_0)-\boldsymbol{q}_S} \geq M \ ,$$

for all $t \geq \boldsymbol{t}$, otherwise there would have been some $t''' \geq \boldsymbol{t}$ such that

$$\frac{p(t''',\boldsymbol{q}_0)-\boldsymbol{q}_S}{\boldsymbol{q}(t''',\boldsymbol{q}_0)-\boldsymbol{q}_S} = M \ ,$$

and, therefore, for $t = t'''$

$$\frac{d\left(\frac{p-q_S}{q-q_S}\right)}{dt} = \frac{\dot{p}}{q-q_S} - M\frac{\dot{q}}{q-q_S} = \frac{\dot{q}}{q-q_S}\left(\frac{\dot{p}}{\dot{q}} - M\right) = \frac{\dot{q}}{q-q_S}\left(t'''\frac{(p-\boldsymbol{q})f(\boldsymbol{q})(v\boldsymbol{q}-p)}{m(\boldsymbol{q})(p-v\boldsymbol{h}(\boldsymbol{q}))} - M\right) >$$

$$> \frac{\dot{q}}{q-q_S}\left(\boldsymbol{t}\frac{\left(\frac{p-q_S}{q-q_S} - \frac{q-q_S}{q-q_S}\right)\boldsymbol{e}_f(v\boldsymbol{q}-\boldsymbol{q}-\boldsymbol{a})}{m(\overline{\boldsymbol{q}})\left(\frac{p-q_S}{q-q_S} - \frac{vh(q)-q_S}{q-q_S}\right)} - M\right) >$$

$$> \frac{\dot{q}}{q-q_S}\left(\frac{2m(\overline{\boldsymbol{q}})M^2}{(v-1)\boldsymbol{e}_f\boldsymbol{q}_S(M-1)}\frac{(M-1)\boldsymbol{e}_f\left((v-1)\boldsymbol{q} - \frac{v-1}{2}\boldsymbol{q}_S\right)}{m(\overline{\boldsymbol{q}})\left(M - \frac{vh(q_S)+(q-q_S)\frac{dvh}{dq}(x)-q_S}{q-q_S}\right)} - M\right) >$$

$$> \frac{M\dot{q}}{q-q_S}\left(\frac{2M\left((v-1)\boldsymbol{q}_S - \frac{v-1}{2}\boldsymbol{q}_S\right)}{(v-1)\boldsymbol{q}_S(M-a(\boldsymbol{x}))} - 1\right) > \frac{M\dot{q}}{q-q_S}\left(\frac{M}{M} - 1\right) = 0,$$

where $\boldsymbol{x} \in (\boldsymbol{q}_S, \boldsymbol{q})$. But then $p - \boldsymbol{q} > (M-1)(\boldsymbol{q}-\boldsymbol{q}_S)$ and $\lim_{t\to\infty}\boldsymbol{q} = \infty$, which is not possible.

So $p(t'',\boldsymbol{q}_0) \geq \boldsymbol{q}(t'',\boldsymbol{q}_0)+\boldsymbol{a}$ for some $t'' \geq \boldsymbol{t} > \hat{t}(\boldsymbol{a}) \geq \hat{t}(p(t'',\boldsymbol{q}_0)-\boldsymbol{q}(t'',\boldsymbol{q}_0))$ and Lemma 1 applies.

Proof of Proposition 5.1.

One can see that for any fixed $\boldsymbol{q}_0$ the solution $(\boldsymbol{q}(t,\boldsymbol{q}_0,\boldsymbol{d}), p(t,\boldsymbol{q}_0,\boldsymbol{d}))$ of (8) converges to the solution $(\boldsymbol{q}(t,\boldsymbol{q}_0), p(t,\boldsymbol{q}_0))$ of (3):

$\lim_{d\to 0}\boldsymbol{q}(t,\boldsymbol{q}_0,\boldsymbol{d}) = \boldsymbol{q}(t,\boldsymbol{q}_0,0) = \boldsymbol{q}(t,\boldsymbol{q}_0)$, and

$\lim_{d\to 0} p(t,\boldsymbol{q}_0,\boldsymbol{d}) = p(t,\boldsymbol{q}_0,0) = p(t,\boldsymbol{q}_0)$

uniformly for all t provided $m(q(t,q_0))(p(t,q_0)-vh(q(t,q_0)))>e^*$ and $p(t,q_0)-q(t,q_0)>e^*$ for an arbitrary small $e^*>0$, in other words, for all $t\in(e_t,T-e_t)$

At $t=0$ convergence is preserved as

$$\lim_{d\to 0}\dot{p}(0,q_0,d)=\lim_{d\to 0}\left(p(0,q_0,d)-(1+d)q(0,q_0,d)\right)=p_0-q_0=\dot{p}_0 \text{, and}$$

$$\lim_{d\to 0}\dot{q}(0,q_0,d)=\lim_{d\to 0}\frac{\dot{p}(0,q_0,d)-da(q_0)q_0}{2a(q_0)}=\frac{\dot{p}(0,q_0)}{2a(q_0)}=\dot{q}_0 .$$

As $a>\frac{1}{2}$ then there exists a left neighborhood of q_S, namely U^a, such that for all $q_0\in U^a$ the solution $(q(t,q_0),p(t,q_0))$ satisfies all the requirements. This implies that $q(t,d)$ and $p(t,d)$ are continuous functions on $t\in[0,T-e_t]$ and at $d=0$. On the other hand,

$$\lim_{e_t\to 0}q(T-e_t,q_0)=q(T,q_0)=\bar{q} \text{, and}$$

$$\lim_{e_t\to 0}p(T-e_t,q_0)=p(T,q_0)>\bar{q} .$$

Let us take e_t such that for all $e\in[0,e_t]$:

$$p(T-e,q_0)>\bar{q} \text{, and}$$

$$m(q(T-e,q_0))>\frac{v\bar{q}-\bar{q}}{v\bar{q}-vh(\bar{q})}m(\bar{q}).$$

Now we define $\bar{d}(e)$ as the largest $d>0$ such that

$$p(T-e,q_0,d)\geq(1+d)\bar{q} \text{, and}$$

$$m(q(T-e,q_0,d))\geq\frac{v\bar{q}-\bar{q}}{v\bar{q}-vh(\bar{q})}m(\bar{q}).$$

It can be easily seen that $\dot{p}(t,q_0,d)>0$ and $\dot{q}(t,q_0,d)>0$ for all $d\in[0,\bar{d})$ and $t>T-e$. Moreover, $\exists T^d>T-e$ such that $q(T^d,q_0,d)=\bar{q}$ and $p(T^d,q_0,d)>\bar{q}$.