TI 2000-063/3
Tinbergen Institute Discussion Paper

Adding Risks

Kin Lam
Liang Zou

Tinbergen Institute
The Tinbergen Institute is the institute for economic research of the Erasmus Universiteit Rotterdam, Universiteit van Amsterdam and Vrije Universiteit Amsterdam.

Tinbergen Institute Amsterdam
Keizersgracht 482
1017 EG Amsterdam
The Netherlands
Tel.: +31.(0)20.5513500
Fax: +31.(0)20.5513555

Tinbergen Institute Rotterdam
Burg. Oudlaan 50
3062 PA Rotterdam
The Netherlands
Tel.: +31.(0)10.4088900
Fax: +31.(0)10.4089031

Adding Risks: Some General Results About Time Diversification*

Kin LAM

Hong Kong Baptist University, Kowloon Tong, Hong Kong

Liang ZOU†

University of Amsterdam, 1018 WB Amsterdam, The Netherlands

Abstract: We show in general that risky investments become more attractive as the investment horizon (n) lengthens. Specifically, any investor's maximal expected utility directly increases with n, as well as the investor's willingness to allocate more capital to the risky assets if his optimal strategy is bounded by the leverage constraint.

Journal of Economic Literature code: G11.

Key words: Time diversification, expected utility, long-term capital allocation.

*Correspondence Address: Dr. Liang Zou, Associate Professor of Finance, Finance Group, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands. Phone: (31)-20-5254160; Fax: (31)-20-5255285. Email: liang@fee.uva.nl.

†This article was written while Liang Zou visited the Hong Kong Baptist University. He acknowledges the gracious support of the host.

1 Introduction

Time diversification is a notion that risky investments become more attractive as the investment horizon lengthens. This notion has remained to be a controversial issue to date. Viewing long-term investments as repeated gambles, for instance, Samuelson (1963) shows that if one would reject a bet at all wealth levels then one should reject any sequence of N such bets. Indeed, there is a fundamental difference between across-asset and across-time diversification. In the former, one risks only a fraction of his capital in each asset (cutting up risk); in the latter, one risks his entire capital at the start of each play of the game (adding up risks) – therefore risk is always there no matter how long one holds the risky assets.

On the other hand, Ross (1999) recently shows that providing the bet is favorable (with positive mean) there exists a large class of utility functions that will *eventually* accept a sequence of N such bets as N goes to infinity – even if a single such bet will be rejected. In order to obtain this result, Ross must restrict the utility functions to be less risk averse than the exponential in the lower tail.

Since virtually all investors confront the decisions of long-term investments, it is an important issue to understand how repeated gambles will affect the investors' expected utilities. Although both analyses of Samuelson and Ross are correct, they look at only a restricted model in which the bets are *indivisible*. In practice, investments involve also the decision of capital allocation, i.e., the proportion of capital invested in risky assets. Thus the amount of stake put in each bet in each period can vary. In this article, we look at the case of *divisible* bets and show how adding up (uncorrelated) risks generally, and unambiguously, improves investors' expected utility.

2 The long-term investment problem

The problem of long-term investment we consider here is that of an investor who invests his initial wealth W_0 for a finite horizon and consumes only his wealth at the end. The time between present and his horizon consists of n periods. Let W_i denote the investor's wealth at the end of period i, $i = 1, 2, \ldots, n$. The investor's objective is to maximize his expected utility of his terminal wealth by choosing a dynamic asset-allocation strategy that specifies the composition of his portfolio in each period.

Although risks can be "added up" or "multiplied up", their analyses are essentially the same via an exponential or a log transformation. We choose here a setting where risks are multiplied up for its direct relevance to long-term investments. Since we do not restrict our analysis to risk-averse investors, all the results obtained can be readily transformed so that they apply to the case of additive gambles as well. For ease of exposition, we shall interpret n as the investment horizon.

Assume that the investor's investment feasibility set consists of one stock (risky) and one bond (risk-free). Let a_i denote the proportion of the investor's wealth, W_{i-1}, invested in the stock and $1-a_i$ in the bond for period i. Assume that the bond yields a constant (continuously compounded) risk-free rate of return r per period for the entire investment horizon, and that the stock's return per period is $x_i \in (-\infty, \infty)$, also continuously compounded.[1] Given a strategy $\{a_i; i = 1, 2, \ldots, n\}$, in which the choices of a_i may depend on the realizations of $x_1, x_2, \ldots, x_{i-1}$ as well as W_0, the terminal wealth is given by

$$W_n = W_0 \prod_{i=1}^{n} [a_i \exp\{x_i\} + (1 - a_i) \exp\{r\}]. \tag{1}$$

Let $U(W_n)$, $U'(\cdot) > 0$, denote the investor's utility of his terminal wealth. The investor's problem can then be stated as

$$\max_{\substack{a_i \in [0,1] \\ i=1,\ldots,n}} EU(W_n) = \int U(W_n) dF(x_1, x_2, \ldots x_n) \tag{2}$$

where E is the expectation operator, and $F(\cdot)$ is the cumulative distribution function of $(x_1, x_2, \ldots, x_n)$. Since in practice default is costly, we restrict attention to the case where $a_i \in [0, 1]$ so that leverage and shortselling are not feasible.[2] Because U is continuous and a_i are chosen from closed intervals, an optimal solution to problem (2) exists.[3]

3 Horizon effect on investor's expected utility

Proposition 1 *Assume that the stock's returns are serially independent (not necessarily identical). Then, for all $i = 1, \ldots, n$, the allocation to the stock in period i is strictly positive if and only if the stock in period i yields a strictly positive risk premium over the risk-free asset, i.e., $a_i > 0$ if and only if $E(\exp\{x_i\}) > \exp\{r\}$.*

Proof: The derivative of EU w.r.t. a_i is given by

$$\frac{dEU}{da_i} = E[U'(W_n)(\exp\{x_i\} - \exp\{r\})].$$

Evaluating at $a_i = 0$, we have

$$\frac{dEU}{da_i}|_{a_i=0} = E[U'(W_{n/i} \exp\{r\})][E(\exp\{x_i\}) - \exp\{r\}] > 0 \text{ iff } E(\exp\{x_i\}) > \exp\{r\}$$

where the subscript n/i means "excluding the ith term", i.e.,

$$W_{n/i} = W_0 \prod_{\substack{j=1 \\ j \neq i}}^{j=n} [a_j \exp\{x_j\} + (1 - a_j) \exp\{r\}].$$

It follows from $U' > 0$ that the investor would like to choose a strictly positive allocation to the stock at all realized wealth levels, i.e., $a_i > 0$ if and only if $E(\exp\{x_i\}) > \exp\{r\}$. Q.E.D.

From Proposition 1, if a favorable bet ($E(\exp\{x_i\}) > \exp\{r\}$ here) is divisible then no investor who prefer more wealth to less would reject it completely. This shows a limitation of the analysis of Samuelson (1963) when it is applied to actual investment problems. That is, when the investor can choose the *amount* of stake to be put in a favorable bet, then the set of increasing utility functions that would completely reject the bet is empty. Allowing the bet to be divisible, thus, is important for analyzing the effect of adding up risks. The next proposition shows the unambiguous positive effect of a larger n.

Proposition 2 *Assume that the stock's returns are serially independent (not necessarily identical), and that* $E(\exp\{x_i\}) > \exp\{r\}$ *for all* $i = 1, ..., n$. *Then all investors who prefer more wealth to less strictly prefer a larger* n. *Specifically,*

$$\max EU(W_n) > \max EU(W_{n-1}\exp\{r\}) > \ldots > \max EU(W_1\exp\{(n-1)r\}) > U(W_0\exp\{nr\}).$$

Proof: For any $n \geq 1$, we have from Proposition 1 that

$$\max_{\substack{a_i\in[0,1]\\ i=1,\ldots,n}} EU(W_n) > \max_{\substack{a_i\in[0,1],a_n=0\\ i=1,\ldots,n-1}} EU(W_n) = \max_{\substack{a_i\in[0,1],a_n=0\\ i=1,\ldots,n-1}} EU(W_{n-1}\exp\{r\}).$$

It follows that $\max EU(W_{n-j}\exp\{jr\}) > \max EU(W_{n-\ell}\exp\{\ell r\})$ for all $n \geq 1$, and $0 \leq j < \ell \leq n$. Q.E.D.

These propositions shed light on the issue of time diversification. If time diversification means that investors are better off with a longer investment horizon provided that the risk premium on the stock is positive, then Proposition 2 offers a rigorous and general proof of this assertion.[4]

4 Horizon effect on investor's strategy

Of course, being better off does not directly imply that the investor should invest more in the stock when investment horizon lengthens. For one thing, if the stock returns are independently and identically distributed over time, and if the investor has a constant relative risk aversion (CRRA) utility function, then his allocation strategy is invariant with n (e.g., Samuelson 1971). However, in the case where the choice of allocation, a_i, is bounded from above, there is an important effect of investment horizon on the investor's willingness to relax the upper bound – in other words, the price the investor is willing to pay in order to *increase* a_i goes up with n.

To show this, assume from now on that the stock's return, $x_i \in (-\infty, \infty)$, $i = 1, 2, \ldots, n$, are identically and independently distributed with mean μ and variance σ^2. We focus on the CRRA utility functions so that the constant-allocation strategy is optimal. Let α denote the proportion of the investor's *initial* wealth in the stock, and $1 - \alpha$ in the risk-free asset. Since at $\alpha = 1$, the buy-and-hold strategy and constant-allocation strategy are identical, we look at the (slightly more difficult) buy-and-hold strategies. The study of buy-and-hold strategies has its own merit as well, since in practice there would be a concern of transaction costs that prohibit frequent rebalancing.

The terminal wealth is now given by

$$W_n(\alpha) = W_0[[\alpha \exp\{\sum_{i=1}^{n} x_i\} + (1-\alpha)\exp\{nr\}]. \tag{3}$$

The investor's problem is

$$\max_{\alpha} EU(W_n(\alpha)) \tag{4}$$

$$\text{s.t.} \qquad 0 \leq \ \alpha \leq 1. \tag{5}$$

Letting $g_n(\alpha)$ denote the derivative of $EU(W_n(\alpha))$ w.r.t. α, we have

$$g_n(\alpha) = \frac{\partial EU(W_n(\alpha))}{\partial \alpha} = E[U'(W_n(\alpha))(\exp\{S_n\} - \exp\{nr\})],$$

where $S_n = \sum_{i=1}^{n} x_i$. In particular, we have

$$g_n(1) = E[U'(W_0 \exp\{S_n\})(\exp\{S_n\} - \exp\{nr\})],$$

and

$$g_n(0) = U'(W_0 \exp\{nr\}) E(\exp\{S_n\} - \exp\{nr\}).$$

Clearly, $\alpha > 0$ if and only if $E(\exp\{S_n\}) > \exp\{nr\}$. This is a special case of our proposition 1.

In the remaining of this section, we assume that x follows a normal distribution with mean μ and standard deviation σ. Let $M_x(\theta)$ denote the moment generating function: $M_x(\theta) = E(\exp\{\theta x\})$, and we focus on distribution functions whose moment generating functions exist for some $\theta \neq 0$. For normal distributions, it is known that

$$M_x(\theta) = \exp\{\theta\mu + \frac{1}{2}\sigma^2\theta^2\}$$

Consider the class of utility functions with constant relative risk aversion (CRRA), i.e., where $U(W) = W^\theta/\theta$ for $\theta \neq 0$ and $U(W) = \ln(W)$ for $\theta = 0$. Rewrite the investor's problem (4) in the form of a Lagrange function:

$$\max_\alpha E(\frac{W_n^\theta(\alpha)}{\theta}) + \lambda_n(1-\alpha) + \eta_n\alpha \tag{6}$$

where λ_n and η_n are the Lagrange multipliers associated with constraints $1 - \alpha \geq 0$ and $\alpha \geq 0$, respectively. We first show a lemma about the optimal buy-and-hold strategies.

Lemma 1 *Assume that x follows a normal distribution with (continuously compounded) mean μ and standard deviation σ, and that $U(W) = W^\theta/\theta$ for $\theta \neq 0$ and $U(W) = \ln(W)$ for $\theta = 0$. Then, the optimal α solving the problem in (4) satisfies*

$$\begin{aligned}
\mu < r - \frac{\sigma^2}{2} \quad &\Rightarrow \quad \alpha = 0, \quad \eta_n > 0, \quad \lambda_n = 0 \\
r - \frac{\sigma^2}{2} \leq \mu \leq r + \frac{\sigma^2}{2}(1-2\theta) \quad &\Rightarrow \quad 0 \leq \alpha \leq 1, \eta_n = 0, \lambda_n = 0 \\
r + \frac{\sigma^2}{2}(1-2\theta) < \mu \quad &\Rightarrow \quad \alpha = 1, \quad \eta_n = 0, \quad \lambda_n > 0
\end{aligned}$$

Proof: Since S_n is normally distributed with mean $n\mu$ and variance $n\sigma^2$,

$$\begin{aligned} g_n(1) &= W_0^{\theta-1}E(\exp\{S_n(\theta-1)\}(\exp\{S_n\}-\exp\{nr\})) \\ &= W_0^{\theta-1}[M_{S_n}(\theta)-\exp\{nr\}M_{S_n}(\theta-1)] \\ &= W_0^{\theta-1}\exp\{n\}[\exp\{\mu\theta+\frac{\sigma^2\theta^2}{2}\}-\exp\{r+\mu(\theta-1)+\frac{\sigma^2(\theta-1)^2}{2}\}] \end{aligned}$$

It follows that $g_n(1)>0$ if and only if $r+\frac{\sigma^2}{2}(1-2\theta)<\mu$. Similarly,

$$\begin{aligned} g_n(0) &= W_0^{\theta-1}\exp\{nr(\theta-1)\}(\exp\{n\mu+n\frac{\sigma^2}{2}\}-\exp\{nr\}) \\ &= W_0^{\theta-1}\exp\{n\}[\exp\{\mu+\frac{\sigma^2}{2}+r(\theta-1)\}-\exp\{r\theta\}], \end{aligned}$$

implying that $g_n(0)<0$ if and only if $\mu<r-\sigma^2/2$. The statement of the lemma follows immediately. Q.E.D.

As a special case of the lemma, for $U(W)=\ln(W)$ the optimal $\alpha\in[0,1]$ if and only if $r-\sigma^2/2\leq\mu\leq r+\sigma^2/2$. The reason that the stock can be attractive even if $\mu<r$ stems from the fact that $E(\exp\{x\})=\exp\{\mu+\sigma^2/2\}$ when x is normal.

Proposition 3 *Assume that x follows a normal distribution with (continuously compounded) mean μ and standard deviation σ, and that $U(W)=W^\theta/\theta$ for $\theta\neq 0$ and $U(W)=\ln(W)$ for $\theta=0$. Assume further that $\mu>r+\frac{\sigma^2}{2}(1-2\theta)$. Then $\alpha=1$ for all $n\geq 1$, and $\lambda_n>\lambda_{n-1}>\ldots>\lambda_1>0$.*

Proof: From Lemma 1 and the assumption that $\mu>r+\frac{\sigma^2}{2}(1-2\theta)$, maximizing (6) yields a necessary condition that

$$g_n(1)-\lambda_n=0$$

The rest of the proof follows directly by observing that $g_n(1)$ is an increasing function of n under assumption $\mu>r+\frac{\sigma^2}{2}(1-2\theta)$ (see the proof of Lemma 1). Q.E.D.

The result of Proposition 3 can be interpreted as follows. If the investor is willing to invest more than 100% in the stock but is bounded at $\alpha = 1$, then he is more willing to increase α as his investment horizon expands. In terms of the Lagrange multiplier λ_n associated with constraint $\alpha \leq 1$, the investor with a longer horizon suffers a higher opportunity cost (or shadow cost) of being restricted from investing more in the stock, and he would be willing to pay more for this constraint to be relaxed. Thus, even for the CRRA investors, a longer investment horizon has a (potentially) positive impact on their willingness to take on more market risk.

Footnotes

1. It is inessential how the returns are compounded. We choose continuous compounding here so that the notations are consistent with those in Section 3. Note that $r = 0$ represents the special case of gambling as discussed in Ross (1999), where the result is known instantaneously.

2. In fact, the problem as formulated in (2) is only consistent with $a_i \in [0, 1]$, for else there would be a possibility of default that implicitly changes the payoff distribution F.

3. It is well known that a general characterization of the solutions to this problem is difficult (e.g., Ross, 1974), and may even suffer time inconsistency (e.g., Machina, 1989). An easy way to abstract away these complicating issues is to treat $F(\cdot)$ as a one-shot gamble so that all a_i must be determined at time 0 and cannot be changed as uncertainty is partially resolved. Restricting the class of utility functions and/or the distribution functions can also simplify the problem (see Section 3). However, because of our limited focus, the results that will be obtained are quite generally valid without these restrictions.

4. Levy and Cohen (1998) obtain a similar result under the assumption that stock returns are normally distributed.

References

Levy, Haim, and Allon Cohen, 1998, On the risk of stocks in the long run: revisited, Journal of Portfolio Management (Spring), 60-69.

Machina, M. J., 1989, Dynamic consistency and non-expected utility models of choice under uncertainty, Journal of Economic Literature XXVII, 1622-1668.

Samuelson, Paul A., 1963, Risk and uncertainty: a fallacy of large numbers, Scientia 57, no.6,1-6.

Samuelson, Paul A. 1971, The fallacy of maximizing the geometric mean in long sequences of investing or gambling, Proceedings of the National Academy of Science 68, no.10, 2493-96.

Ross, Stephen, 1974, Portfolio turnpikes theorems for constant policies, Journal of Financial Economics 1, 171-198.

Ross, Stephen A., 1999, Adding risks: Samuelson's fallacy of large numbers revisited, Journal of Financial and Quantitative Analysis 34, no. 3, 323-339.