TI 2000-050/2
Tinbergen Institute Discussion Paper

Inherent Reward and Risk (Part I): Towards a Universal Paradigm for Investment Analysis

Liang Zou

Tinbergen Institute
The Tinbergen Institute is the institute for economic research of the Erasmus Universiteit Rotterdam, Universiteit van Amsterdam and Vrije Universiteit Amsterdam.

Tinbergen Institute Amsterdam
Keizersgracht 482
1017 EG Amsterdam
The Netherlands
Tel.: +31.(0)20.5513500
Fax: +31.(0)20.5513555

Tinbergen Institute Rotterdam
Burg. Oudlaan 50
3062 PA Rotterdam
The Netherlands
Tel.: +31.(0)10.4088900
Fax: +31.(0)10.4089031

INHERENT REWARD AND RISK (PART I): TOWARDS A UNIVERSAL PARADIGM FOR INVESTMENT ANALYSIS

By Liang Zou*

University of Amsterdam, 1018 WB Amsterdam, The Netherlands

Correspondence Address: Dr. Liang ZOU, Associate Professor of Finance, Finance Group, University of Amsterdam, Roetersstraat 11, 1018 WB Amsterdam, The Netherlands. Phone: (31)-20-5254160; Fax: (31)-20-5255285. Email: liang@fee.uva.nl.

*Part of this paper was written while I was visiting Hong Kong Baptist University. The generous support of my host, the stimulating discussions with Rod Aya, Henk Koster, Kin Lam, Wei Li, Michael Prince, Laixiang Sun, colleagues at the Finance Group, seminar participants at University of Amsterdam, Hong Kong Baptist University, University of Hong Kong, Tinbergen Institute and, in particular, the detailed comments by Florian Wagener and Peter Wakker, are gratefully acknowledged. All errors remain to be my own, of course.

Abstract: In this paper, a new paradigm is developed for analyzing investment strategies and pricing financial assets. This paradigm assumes that any investment strategy has its own "inherent reward" and "inherent risk" that can be judged with common sense. I justify axiomatically the existence and uniqueness (ratio scale) of inherent reward (U) and inherent risk (D) that could be regarded as universal measures of reward and risk for any given investment strategy. Incorporating the notion of "inherent efficiency" in a portfolio context, I show that the inherent reward-to-risk ratio ($Z = U/D$) is capable of ranking all the investment strategies with any return distributions, while being consistent with the fundamental principles of no-arbitrage and first-order stochastic dominance. If there exists an inherently efficient benchmark portfolio within any given set of feasible strategies, then the risk premium on any of these strategies must satisfy a simple relationship with the benchmark risk premium (the Inherent CAPM). Sophisticated securities such as options or portfolios with imbedded options can then be priced without having to assume that the market is complete or that the security price follows a specific process. Other issues discussed in the paper include prospect theory, the Allais paradox, the computation of inherent reward and risk, the mean-variance CAPM, and performance evaluation.

Key words: inherent reward, inherent risk, inherent dominance, stochastic dominance, inherent efficiency.

"*Nature is the realization of the simplest conceivable mathematical ideas.*"

ALBERT EINSTEIN, *Ideas and Opinions* (1954, p. 274)

1 INTRODUCTION

ONE FUNDAMENTAL ISSUE in investment analysis under uncertainty is the choice of valid investment objectives. There are two distinct approaches to this issue. The first one, which I call the dichotomous approach, assumes that investment under uncertainty involves two basic parameters – reward and risk (e.g., Knight, 1921; and Hicks, 1939). Investments with higher reward or lower risk will be preferred by all individuals, and investment decisions boil down to finding an optimal trade-off between reward and risk. The second one, the expected utility approach, assumes that the objective of investment should be one of maximizing the expected value of some utility function (e.g., von Neumann and Morgenstern, 1947; Marschak, 1950; Samuelson, 1952; and Herstein and Milnor, 1953; and Savage, 1954). The two approaches do not in general overlap, and they both have their merits and weaknesses.

In the dichotomous models, reward is conventionally measured as the expected return (or the expected return over a risk-free rate, i.e., the risk premium) on investment. However, risk has not received any consensus as to what constitutes its proper measure. Risk can be viewed as the variance of investment return (Markowitz, 1952, 1959; Tobin, 1958; Sharpe, 1964; and Lintner, 1965a, 1965b; Mossin, 1966); a mean-preserving spread (Rothschild and Stiglitz, 1970, 1973); the probability of shortfalls (e.g., Roy, 1952; Markowitz, 1959; Pruitt, 1962; Samuelson, 1963); the semi-variance (e.g., Porter, 1974); the lower partial-moments of return (e.g., Bawa, 1974, 1975, 1976; Fishburn, 1977; and Harlow and Rao, 1989); the so-called value at risk (e.g., Morgan, 1995); and so on. A general form of risk measures that includes the above as special cases has been proposed by Stone (1973):

$$L(W_0, k, A) = \int_{-\infty}^{A} |W - W_0|^k dF(W) \quad k \geq 0 \tag{1}$$

where $F(W)$ is the cumulative distribution of the investor's uncertain terminal wealth W. Hence the selection of any particular risk measure involves the selection of three parameters, namely, W_0 (reference level of wealth), k (the relative importance of deviations of final wealth from the reference level), and A (the outcome that should be included in the risk measure).

The dichotomous analysis of reward and risk is appealing for its intuitive simplicity as well as for its applicability. The mean-variance model is, undoubtedly, the most successful of the dichotomous models on which modern portfolio theory is built. This model uses the return variance or standard deviation as a measure of risk. Among the most important results of this theory are the separation theorem, the role of correlation among assets in the selection of optimal portfolios (Markowitz, 1952, 1959; and Tobin, 1958, 1965), and the capital asset pricing model or CAPM (Sharpe, 1963, 1964; and Lintner, 1965a, 1965b).[1] On the other hand, the limitations of the mean-variance model are also well documented (e.g., Fishburn, 1977 and the references therein). In addition to criticisms of using the return variance or standard deviation as a measure of risk, there have been challenges to the validity of the results of the mean-variance theory. For instance Dybvig and Ingersoll (1982) show that CAPM necessarily involves risk-free arbitrage opportunities if it is to hold for all assets in a complete capital market; moreover, the security market line of CAPM can also be problematic if it is used to measure investment performance (Dybvig and Ross, 1985a, 1985b).

Indeed, when asked what the risk of an investment really is, most investors will probably say that it is the risk of losing one's money. This common notion of risk is reflected in the mean-

[1] I assume that the reader is familiar with these results.

downside risk literature. In this literature, investment risk is defined as the risk of underperforming a fixed and deterministic benchmark return. In terms of the general form of (1), for instance, Domar and Musgrave (1944) measure risk as the probability-weighted losses ($k = 1$, $A = W_0$) in their study of the effect of progressive income taxation on risk taking; Porter (1974), Bawa (1975, 1976, 1978), and Fishburn (1977) show that the lower-partial moments as a measure of risk is consistent with the stochastic dominance rules ($k = 0, 1, 2$); and Bawa and Lindenberg (1977), and Harlow and Rao (1989) examine the capital asset pricing relationship in the lower-partial moment framework ($k = 1, 2$). Unfortunately these mean-downside risk models have not gained much popularity. Lacking a theoretical foundation, the choices of k, W_0 and A in (1) remain to be a matter of opinion. Critics of the downside-risk models would also point to their added complexity in both theoretical and empirical analysis.

In contrast to the dichotomous view of investment reward and risk, the expected utility theory offers a much more rigorous approach to making investment decisions. According to this theory, optimal investment decisions should be derived by maximizing the expected value of some utility functions, provided that investors' preferences agree with the basic axioms upon which the theory is built. Thanks to the intuitive appeal of these axioms, the expected utility theory is successfully adopted in almost every field of study involving risk: capital markets, game theory, incentive contracts, security design, optimal taxation, etc. Apparently the expected utility models are more general than the dichotomous models since the former makes use of the information of distributions on their whole support whereas the latter only a few statistical parameters of the distributions. This is perhaps the reason why much attention has been devoted to justifications of the dichotomous models in the framework of expected utility theory. It is logical to find out that, for whatever dichotomous model one selects, it can be justified in the expected utility framework only for a small class of utility functions (e.g., quadratic utility function for the mean-variance model), or for a small class of the return distributions (e.g., normal distribution for the mean-variance model). The dichotomous models restrict the investors' objective functions to a smaller class than what would be available in the more general expected utility maximization problem. Trading off generality for simplicity seems unavoidable if we confine ourselves to the paradigm of expected utility.

Nevertheless, there is some cause for concern about the empirical validity of the expected utility hypothesis (e.g., Tversky, 1967a, 1967b, 1969, 1975; De Bondt and Thaler, 1995). It has been observed that there are circumstances where the individual choices can systematically violate the expected utility hypothesis (e.g., the well known Allais paradox that we shall discuss later). Individuals also exhibit a preference for flexibility to change their tastes – thereby to change their utility functions – as they face new uncertainty (e.g., Kreps, 1979). In view of the limited rationality of individuals (e.g., Simon, 1955, 1956), the dichotomous approach remains attractive if only for its simplicity. It has the potential to resolve some of the puzzles raised by the expected utility theory as well. But the dichotomous approach first needs a more rigorous theoretical foundation before it can be confidently adopted.

This paper offers a theoretical foundation for the dichotomous view of investment uncertainty, represented by the investment reward and risk. I develop a system of axioms that involves intuitive *common (or objective) judgement* of how the investment reward and risk should be defined. I call such reward and risk the investment's *inherent reward* and *inherent risk*. Thus, by virtue of common judgement, inherent reward and risk can be objectively defined and not affected by any change in tastes of an individual. Provided that their underlying axioms are universally accepted, the inherent reward and risk measures can be used as *universal measures of investment reward and* *risk*. In comparison, the expected utility theory concerns individual *behavior* towards risk. It is developed from a system of axioms that describes *individual preferences*. When risk is defined in terms of individual preferences, it is natural that different individuals with different preferences will come up with different judgements about what they prefer. In the expected utility paradigm, it is well known that when choosing between two investment alternatives, the only criterion for all individuals who prefer more to less (wealth) to arrive at the same choice is the first-order stochastic dominance criterion (e.g., Quirk and Saposnik, 1962; and Hadar and Russell, 1969). The class of distributions that are comparable by the first-order stochastic dominance criterion, unfortunately, is rather small and many interesting investment strategies are not comparable even by the higher-order stochastic dominance criteria that limit the class of utility functions.

At the bottom of the new paradigm that will be developed in this paper is the notion that any investment strategy, broadly defined, has its own *inherent reward* and *inherent risk* – the reward and risk that are independent of any particular investor's preference or behavior. Inherent reward and risk differ from subjectively construed reward and risk in that they are directly related to the investment's returns, rather than to the investor's utility. They are actual; independent of one's preference or taste. Inherent risk, for instance, exists for everyone including risk-lovers – only they require less reward to compensate for the risk. "A new-born calf does not fear tigers," goes a Chinese saying. The danger of a tiger for the calf, however, is there despite the calf's ignorance. If some investors refuse to buy stocks, it is not because they are more liable to losses than those who do; they are simply more risk-averse and do not think that the reward is sufficient for them to take the risk. An asset's risk of yielding less than the risk-free rate of return, for example, is inherent in the asset's actual return distribution, and not as a personal matter. Likewise, there exists an inherent reward in a risky investment to earn more than the risk-free rate of return.

Section 2 is devoted to the development of the axioms that lead to the existence of a unique measure for inherent reward (U) and a unique measure for inherent risk (D) of any investment strategy (unique in the sense that U and D are ratio scale measures, invariant up to a positive multiplicative transformation). These measures are shown to possess the desired properties of being simple, transitive, and complete (able to measure all investment strategies). The inherent reward-to-risk ratio ($Z = U/D$) is shown to be capable of ranking all the investment strategies with *any* return distributions, while respecting the fundamental principles of no-arbitrage and stochastic dominance. The contribution of this section is the development of what might be called an "inherent dominance theory" for investment analysis.

In Section 3, I apply the inherent dominance theory to the study of various important issues. Among these issues are the prospect theory, the Allais paradox, the stochastic dominance criteria, investment performance evaluation, and comparisons of the inherent dominance criterion with that of mean-variance, etc. Section 4 shows the computation of inherent reward and risk when the investment returns follow a binomial process or when the payoff distribution is normal.

In Section 5, the inherent analysis is extended to the portfolio context where I introduce the notion of *inherent efficiency* of a portfolio or an investment strategy. I show that inherent efficiency of any portfolio, in particular that of the capital market, directly implies a capital asset pricing formula that exhibits both rigor and simplicity. It is capable of pricing sophisticated securities such as options without having to assume that the market is complete or that the stock price follows a specific process. Section 6 concludes the paper with some suggestions for future research in this inherent analytical framework.

In a companion paper (Zou, 2000), I show how the basic capital asset pricing model can be extended to a multi-period framework and how it can be used to price sophisticated financial products. Moreover, I derive a multi-period capital market equilibrium model which is consistent with both expected utility maximization and inherent efficiency of the capital market.

2 THE AXIOMATIC DEVELOPMENT OF INHERENT REWARD AND INHERENT RISK MEASURES

2.1 Preliminaries

Much of the development in this section is similar in form to the early axiomatic development of the expected utility theory (e.g., von Neumann and Morgenstern, 1947; Marschak, 1950; Samuelson, 1952; Herstein and Milnor, 1953; and Debreu, 1960).[2] I try to stay away from the issue of individual preference and behavior, focusing instead on what is likely to be agreed upon by all investors – that is, by common judgement. This allows me to add a few intuitive axioms that eventually lead to the simple measures of inherent reward and risk, and develop a simple dominance criterion that is able to rank all investment strategies in the sense of inherent dominance.

The problem I consider is one of investment under uncertainty with a known investment horizon. Assuming away cash inflows and outflows during the holding period, any initial investment, combined with any well-defined investment strategy, conceivably leads to a probability distribution of the monetary payoff at the end of the horizon.[3] The return distribution of any investment

[2] Some axioms, lemmas and theorems are straightforward extensions of the expected utility theory. We include the proofs for ease of reference and completeness. The line of presentation follows largely the textbook of DeGroot, 1970 (Chapter 7).

[3] Thus any dividends, interest payments, rents, etc., that will be earned during the holding period are assumed to be reinvested and storage costs of the assets, if any, will be financed by selling partially the invested funds. For any individuals with interim income and consumption during the holding period, we can assume in the spirit of MM's theorem (Modigliani and Miller, 1958) that their surplus or shortage for consumption can be saved or financed with

strategy is assumed to be independent of the amount invested, i.e., to exhibit *stochastic constant returns to scale* (Arrow, 1974). Thus the focus is on investment returns, rather than total payoffs, of investment strategies.[4] Further, any two identical return distributions will be assumed to be the same no matter *how* they are generated. For instance, after the ticket is bought, the inherent reward and risk of a bet on a racehorse is the same whether or not one watches the race. This assumption is crucial for ensuring the legitimacy of the use of Bayesian rules in the development of the inherent reward and risk measures, as it is in the development of expected utility theory. In summary, my purpose is to study the inherent reward and risk of *probability distributions of investment returns* for an arbitrarily given, and fixed, investment horizon.[5]

A *well-defined investment strategy* is understood as a decision, or a sequence of "if then" decisions that can be implemented by delegation or via a computer program. The decision to buy a mutual fund, for instance, is a well-defined strategy that delegates the interim trading decisions to the mutual fund manager. Although investors may have different beliefs about the probability distribution of their investment returns, it is conceivable that there is only one *true* (or objective) distribution for any well defined strategy. To avoid ambiguity, I shall analyze inherent reward and risk of the investment strategies *as though* such true distributions of the random returns are known. A lottery's payoff distribution, for example, can be made known to all players. The return distribution of a stock, on the other hand, may not be perfectly known by any individual. The axiom system to be developed here, however, has little to rely on personal beliefs.[6]

---

lending or borrowing at the risk-free interest rate. We do not explore these issues here, however.

[4]A further justification for focusing on returns rather than wealth is that an asset's inherent reward and risk must be common to all investors: investment returns are common, whereas invested wealth is not.

[5]Most of the research in the expected utility paradigm focus on *terminal-wealth distributions*. See, for instance, Dybvig (1988a) for more discussions.

[6]It might help by imagining the existence of an omniscient creature who is willing to show us the return dis-

There are several advantages of such an objective approach. First, subjective beliefs although important for the (experimental) study of human behavior have little empirical content since all empirical studies of investments rely on data drawn from the "true" sample distributions. Second, it is the true distribution of returns that determines an investor's potential gains or losses, and not the hope or belief of any individual investor. In evaluating the performance of a fund manager, for instance, we look at what he actually does and not at what he claims he can do. Third, analytically, this approach avoids the potentially complicating issues that may arise with subjective beliefs and expectations, such as the distinction between risk and uncertainty (Knight, 1921); between known and unknown probabilities (e.g., Ellsberg, 1961; Schmeidler, 1989; Fishburn, 1991; and Sarin and Wakker, 1992); and between true probabilities and subjectively weighted probabilities (e.g., the references in Machina, 1989).

The potential for application of the inherent dominance theory to practical issues, on the other hand, are not limited to this perception of true distributions. An investor with his own subjective beliefs about the probability distributions of different investment strategies, for instance, can assess the inherent reward and risk of these strategies that *he believes*. These decision issues are investigated in Section 3.

2.2 The Formal Set-up

Uncertainty: For 1 dollar of initial ($t = 0$) capital invested and a given time horizon $t > 0$, let r_t denote the random variable, the return on this investment that will be evaluated at time t. Uncertainty of any such investment is represented by a probability space $\{\mathbb{R}, \mathcal{B}, P_t\}$ where $\mathbb{R}$ (the real line) denotes the space of all possible returns $r_t \in \mathbb{R}$ on the investment, $\mathcal{B}$ denotes the

tributions of all available strategies for all horizons, and who resolves our uncertainty over time by following these distributions until the end of our chosen horizon.

σ-field of Borel sets of $\mathbb{R}$, and P_t denotes the probability distribution of r_t on $(\mathbb{R}, \mathcal{B})$. Let $\mathcal{P}$ denote the class of all probability distributions P on $(\mathbb{R}, \mathcal{B})$ whose mean exists, and we focus only on investments whose return distribution is in $\mathcal{P}$.[7] I would like to simultaneously discuss discrete distributions, continuous distributions, and a mixture of both kind of distributions with minimum notations. Thus, $P(x)$ can mean either the probability that $r = x$ for the discrete case or the density function at $r = x$ for the continuous case. For any Borel set $X \subseteq \mathbb{R}$, $P(X)$ will denote the probability that $r \in X$. More generally, for any probability distribution P over $\mathbb{R}$, any integrable function y on $\mathbb{R}$, and any subset $X \in \mathcal{B}$, the abstract integral $\int_X y(r)dP(r)$ will be interpreted as the Lebesgue-Stieltjes integral $\int_X y(r)dF(r)$ where F is the distribution function that corresponds to P.

Strategy: For a given time horizon $t > 0$, an investment strategy S is defined as any decision or a sequence of decisions that matches the initial investment capital (1 dollar at time 0) to a probability distribution of returns $P_t^S \in \mathcal{P}$ on horizon t. For instance, the decision to buy and hold an asset A that matches the initial capital to the return distribution of the asset $P_t^A \in \mathcal{P}$ is a well-defined strategy. An example of an ill-defined strategy for horizon t is one of holding an option that expires before time t, if subsequent actions are not specified for the remaining time after the expiration of the option up to time t. A strategy is feasible if under the current investment environment it can be actually implemented. Some strategies may not be feasible in practice. For one thing, if short-selling of a security is prohibited then

[7]Thus, we avoid discussion on investments that generate infinite expected returns, such as St. Petersburg Paradox (e.g., Samuelson, 1977). We do allow for cases of risk-free arbitrage, however. An arbitrage strategy can be perceived as an investment that generates an abnormal (finite) return without any downside risk, and in order to generate infinitely high returns one needs to be able to replicate the strategy infinitely many times. This is a special case that will be dealt with after the definition of inherent reward and risk.

any strategy involving short-selling of this security is not feasible. Let Ω denote the class of all strategies that are feasible at $t = 0$. All assets, funds, indices, or buy-and-hold portfolio strategies that are feasible in the market place, for example, form a subset of Ω. A dynamic trading strategy, e.g., a periodically rebalanced portfolio that keeps the proportion of each asset in the portfolio constant, is also an element of Ω. More generally, strategies in which sequential actions will be taken upon realized contingencies or upon new information arrivals can be seen as an element of Ω, provided that at time $t = 0$ the return distributions of such strategies on horizon t can be assessed. Again, recall that even if an investor does not have a perfect idea about the return distribution of his strategy, an objective distribution can be perceived to exist from which his actual return will be drawn.[8] Since the return on investment is the only concern here, two strategies with identical return distributions on horizon t will be considered the same despite their possible differences in actions and payoffs before or after t. Thus, we can also directly refer to any return distribution P_t as a strategy provided that P_t is attainable by some strategy in Ω.

Inherent dominance relationship: Let the binary relation $\succ$ denote the notion "inherently dominates", and let $\sim$ denote "inherently equivalent".[9] The symbol $\succeq$ will mean "either $\succ$ or $\sim$". For any two probability distributions $P_t^A \in \mathcal{P}$ and $P_t^B \in \mathcal{P}$, $P_t^A \succ P_t^B$ means that P_t^A *inherently dominates* P_t^B, $P_t^B \prec P_t^A$ means the same as $P_t^A \succ P_t^B$, and $P_t^B \preceq P_t^A$ means

[8] Although the true return distribution of a trading strategy may never be known to us, just as that of a particular asset, we have at least statistical methods to estimate the distribution – especially where a strategy can be frequently repeated.

[9] It bears remarking that here $\succ$ is not meant to be an individual preference relationship but rather a relationship that is based on common judgement. By analogy, imagine two dishes of food served in a restaurant. Although it might be impossible to have all individuals prefer one dish to the other, it is easier to commonly agree on which dish is more salty, spicy, etc.

the same as $P_t^A \succeq P_t^B$. These binary relations are defined also for the case of certainty. Thus, given any two returns r_1 and r_2, $r_1 \succ r_2$ indicates that receiving a return r_1 for sure inherently dominates receiving r_2 for sure. Such a relation $\succeq$ is assumed to be complete and transitive. That is, it is able to completely rank all the probability distributions in $\mathcal{P}$ in the following sense. First, if P_t^A and P_t^B are any two probability distributions in $\mathcal{P}$, then exactly one of the following relations must hold: $P_t^A \succ P_t^B$, $P_t^A \sim P_t^B$, or $P_t^B \succ P_t^A$. Second, let any three probability distributions P_t^A, P_t^B and P_t^C be given. If $P_t^A \succeq P_t^B$ and $P_t^B \succeq P_t^C$, then $P_t^A \succeq P_t^C$; if $P_t^A \succ P_t^B$ and $P_t^B \succeq P_t^C$, then $P_t^A \succ P_t^C$; and if $P_t^A \succeq P_t^B$ and $P_t^B \succ P_t^C$, then $P_t^A \succ P_t^C$.

Inherently more rewarding and more risky relationships: Similar to $\succeq$, the binary relation $\succeq_g$ denotes either $\succ_g$:"inherently more rewarding" or $\sim_g$:"inherently equally rewarding". And the binary relation $\preceq_\ell$ denotes either $\prec_\ell$:"inherently more risky" or $\sim_\ell$:"inherently equally risky" (thus $\succ_\ell$ denotes "inherently less risky"). The subscripts g and ℓ refer to the gain and loss, respectively. For any $P_t^A \in \mathcal{P}$ and $P_t^B \in \mathcal{P}$, $P_t^B \preceq_g P_t^A$ means the same as $P_t^A \succeq_g P_t^B$, and $P_t^B \preceq_\ell P_t^A$ means the same as $P_t^A \succeq_\ell P_t^B$. Also, the relations $\succeq_g$ and $\succeq_\ell$ are assumed to be complete and transitive in the same sense as $\succeq$.

Inherent reward and risk: The inherent reward and inherent risk of any return distribution are perceived as two functionals with domain of definition $\mathcal{P}$ that maps the return distribution to a pair of real numbers on $\mathbb{R}^+$. Let $U : P_t \in \mathcal{P} \longrightarrow U(P_t) \in \mathbb{R}^+$ denote the inherent reward and let $D : P_t \in \mathcal{P} \longrightarrow D(P_t) \in \mathbb{R}^+$ denote the inherent risk. Thus, for any given time horizon $t > 0$ and any given investment strategy or return distribution P_t, $U(P_t)$ and $D(P_t)$ are the measures of the inherent reward and risk of the strategy. My goal is to axiomatize some properties of such inherently more rewarding and more risky relationships $\succeq_g$ and $\preceq_\ell$ in

$\mathcal{P}$, and identify a unique (ratio scale) pair of measures U and D that map the relationships $\succeq_g$ and $\preceq_\ell$ in $\mathcal{P}$ to the $\geq$ relationship in $\mathbb{R}^+$. That is, for any two probability distributions $P_t^A \in \mathcal{P}$ and $P_t^B \in \mathcal{P}$, $P_t^A \succeq_g P_t^B$ if and only if $U(P_t^A) \geq U(P_t^B)$ and $P_t^A \preceq_\ell P_t^B$ if and only if $D(P_t^A) \geq D(P_t^B)$.

Inherent reward to risk ratio: The inherent reward to risk ratio, denoted Z $(=U/D)$, is also a unique functional (up to a positive linear transformation) $Z : P_t \in \mathcal{P} \longrightarrow Z(P_t) \in \mathbb{R}^+$ that maps the inherent dominance relationship $\succeq$ in $\mathcal{P}$ to the $\geq$ relationship in $\mathbb{R}^+$. For any two probability distributions $P_t^A \in \mathcal{P}$ and $P_t^B \in \mathcal{P}$, $P_t^A \succeq P_t^B$ if and only if $Z(P_t^A) \geq Z(P_t^B)$.

Investment objectives of the investors: Assuming that the return distributions are known for any given horizon and any feasible investment strategies, I propose a general formulation of the investment problem as follows.[10]

$$\max_{P_t \in \Omega} E[V_t(W_0, U(P_t), D(P_t), W_t); P_t] \tag{2}$$

where $P_t \in \Omega$ is the return distribution of a feasible investment strategy, $E[\cdot; P_t]$ is the expectation operator under distribution P_t, W_0 is the investor's current wealth, W_t is the investor's time-t wealth (the random variable), and $U(P_t)$ and $D(P_t)$ are the inherent reward and risk of P_t. Clearly, this formulation includes expected utility maximization as a special case and it allows the value function V_t to vary with horizon t. It is natural to assume that V_t increases in U and decreases in D, for any all t, W_0, W_t and P_t. In the companion paper

[10]In an investment context in which wealth is the major concern, a general form of the investment problem may be represented as one of choosing the optimal return distribution function of a feasible strategy, P, given one's initial wealth W_0, such that a functional $V(W_0, P)$ is maximized (see Machina, 1982). Expected utility theory boils down to restricting the form of the objective functional V to being linear in P.

(Zou, 2000), this general form of investment objectives is adopted to develop a general capital market equilibrium theory and dynamically consistent investment strategies.

From now on, the subscript t will be dropped for notational convenience.

2.3 The Basic Notion of Inherent-Dominance

Given any return distributions $P_1 \in \mathcal{P}$, $P_2 \in \mathcal{P}$, and any $\theta \in (0,1)$, let $\theta P_1 \oplus (1-\theta)P_2$ denote a lottery that assigns probability θ to distribution P_1 and $1-\theta$ to distribution P_2 for the investment returns. In particular, if r_1 and r_2 are any two deterministic levels of returns, then $\theta r_1 \oplus (1-\theta)r_2$ denotes a lottery in which the chance that the return equals r_1 is θ and that the return equals r_2 is $1-\theta$. On the other hand, let $\theta P_1 + (1-\theta)P_2$ denote a portfolio with θ and $1-\theta$ being the weight of capital invested in two assets whose return distributions are P_1 and P_2 respectively; in particular, $\theta r_1 + (1-\theta)r_2$ denotes a realized return on such a portfolio (mix of outcomes). When θ denotes the percentage of capital invested in a portfolio, it may be allowed to be smaller than zero or greater than one if short selling and leverage are feasible.

Put differently, let δ_x denote the degenerate one-stage lottery that assigns probability one to the outcome x. Then $\theta r_1 \oplus (1-\theta)r_2 = \theta\delta_{r_1} \oplus (1-\theta)\delta_{r_2}$ and $\theta r_1 + (1-\theta)r_2 = \delta_{\theta r_1+(1-\theta)r_2}$. For a numerical example, suppose $\theta = 0.8$, $r_1 = 0.2$, $r_2 = 0.1$. Then $\theta r_1 \oplus (1-\theta)r_2$ equals 0.2 with probability 0.8 and equals 0.1 with probability 0.2. Whereas $\theta r_1 + (1-\theta)r_2 = 0.8 * 0.2 + 0.2 * 0.1 = .18$ for sure.

Axiom 1 (Gain-Loss Partition) : *(i) Let any $r_0 \in \mathbb{R}$ and any two distributions P_1 and P_2 be given such that $P_1([r_0,\infty)) = 1$ and $P_2((-\infty, r_0]) = 1$, then $P_1 \succeq P_2$; and if $P_1(\{r_0\})P_2(\{r_0\}) < 1$, then $P_1 \succ P_2$. (ii) The set of all possible returns $\mathbb{R}$ can be partitioned into a set of gains G, a set of neutral returns N, and a set of losses L such that $G \neq \emptyset$, $N \neq \emptyset$, $L \neq \emptyset$, and $G \cup N \cup L = \mathbb{R}$. If*

$r_1 \in N$ and $r_2 \in N$, then $r_1 \sim r_2$. If $r_1 \in G$, $r_2 \in N$, and $r_3 \in L$, then $r_1 \succ r_2 \succ r_3$.

Underlying Axiom 1(i) is the basic notion that more wealth is better than less. Thus, if an investment yields a (weakly) higher return than another investment under all the contingencies, then the former inherently dominates the latter. Axiom 1(ii) further postulates that gains and losses can be separated by some neutral returns that are neither gains nor losses.

Lemma 1 *: For any two returns r_1 and r_2, $r_1 \succeq r_2$ if and only if $r_1 \geq r_2$, and $r_1 \sim r_2$ if and only if $r_1 = r_2$.*

Proof: Straightforward from Axiom 1 (i), choosing r_0 such that $r_1 \geq r_0 \geq r_2$. □

Lemma 2 *There exists $r_0 \in \mathbb{R}$ such that $N = \{r_0\}$, $L = (-\infty, r_0)$ and $G = (r_0, \infty)$.*

Proof: Since all returns in N are equivalent, from Lemma 1 they must be equal. Since N is not empty, there must exist a unique number r_0 in N. Thus, from Axiom 1 and Lemma 1 any $r > r_0$ must belong to G and any $r < r_0$ must belong to L. □

Intrinsically, any realized return cannot be a gain and a loss at the same time. Moreover, any investment return must be either a gain or a loss, unless it breaks even by some standard. I call r_0 the *neutral-return benchmark*. In practice, of course, investors may have different goals and different standards. Fund managers, for instance, are usually evaluated against the performance of a benchmark portfolio and investors may have different target rates. For practical purposes, then, one may interpret the probability distributions discussed here as those of the deviations of any fund's return from that of a benchmark portfolio, and speak about the inherent reward and risk of deviations from the benchmark.

For notational convenience, we let $\overline{G} = G \cup N$ and $\overline{L} = L \cup N$.

2.4 The "Inherently More Rewarding" and "Inherently More Risky" Relationships

Having partitioned the outcome space $\mathbb{R}$ into the gain and loss domains that are separated by a neutral-return benchmark, I now partition the distributions in $\mathcal{P}$ as follows.[11]

Definition 1 *For any $P \in \mathcal{P}$, let P^+ be defined as a distribution over $\overline{G}$ such that $P^+ = P$ over G and $P^+(r_0) = P(\overline{L})$. Similarly, let P^- be defined as a distribution over $\overline{L}$ such that $P^- = P$ over L and $P^-(r_0) = P(\overline{G})$.*

Axiom 2 (Reward-Risk Partition) *Let any distributions $P_1 \in \mathcal{P}$ and $P_2 \in \mathcal{P}$ be given. Then $P_1 \succeq_g P_2$ if and only if $P_1^+ \succeq_g P_2^+$; $P_1 \succeq_\ell P_2$ if and only if $P_1^- \succeq_\ell P_2^-$. In particular, if $P_1(\overline{G}) = P_2(\overline{G}) = 1$ then $P_1 \succeq P_2$ if and only if $P_1 \succeq_g P_2$; and if $P_1(\overline{L}) = P_2(\overline{L}) = 1$ then $P_1 \succeq P_2$ if and only if $P_1 \succeq_\ell P_2$.*

In other words, Axiom 2 postulates that the inherent reward of a distribution P is uniquely determined by its upper part over $r \geq r_0$; and the inherent risk of a distribution P is uniquely determined by its lower part over $r \leq r_0$. Therefore, the inherent reward and inherent risk of any investment strategy can be analyzed separately. The distributions P^+ and P^- will be called the *upside equivalent* of P and the *downside equivalent* of P, respectively.[12]

Axiom 3 (Lottery Independence) *: Let any distributions $P_1 \in \mathcal{P}$, $P_2 \in \mathcal{P}$, and $P \in \mathcal{P}$ be*

[11] Although it is common in prospect theory to partition the outcome distributions according to gains and losses, that partition is usually based on individual preferences. Here, we are seeking a commonly perceived partition in which the neutral-return benchmark is a universal r_0, i.e., is the same for all investors.

[12] It is worth noting that except for special cases, P^+ and P^- are not conditional distributions; that is, for all $r \in \Re$, $P^+(r) \neq P(r|\overline{G})$ and $P^-(r) \neq P(r|\overline{L})$. The shape of a distribution on a subset of events does not in general overlap with the conditional distribution obtained by restricting the events to be in the subset.

given. Then for all $\theta \in (0,1]$,

$$P_1^+ \succ_g P_2^+ \text{iff} \theta P_1^+ \oplus (1-\theta)P^+ \succ_g \theta P_2^+ \oplus (1-\theta)P^+, \tag{3}$$

$$P_1^- \succ_\ell P_2^- \text{iff} \theta P_1^- \oplus (1-\theta)P^- \succ_\ell \theta P_2^- \oplus (1-\theta)P^-. \tag{4}$$

Axiom 3 is analogous to the well known (traditional) independence axiom in the development of expected utility theory.[13] Here, independence is extended to a distribution's upside and downside equivalences on the gain and loss domains respectively. As I shall show later, the implications of this lottery-independence axiom are not likely to suffer the empirical problems that confront the traditional independence axiom (e.g., the Allais paradox). In words, Axiom 3 postulates that, if distribution P_1 is inherently more rewarding (or less risky) than distribution P_2, then a lottery whose return follows with probability θ the distribution P_1 and otherwise P is also more rewarding (or less risky) than a lottery with probability θ to follow distribution P_2 and otherwise P. Since two other independence axioms will be introduced later, I call this axiom *the lottery-independence axiom.*

Lemma 3 *Let any distributions* P_1, P_2, Q_1 *and* Q_2 *in* $\mathcal{P}$ *be given. Then for all* $\theta \in (0,1]$,

$$\theta P_1^+ \oplus (1-\theta)Q_1^+ \succeq_g \theta P_2^+ \oplus (1-\theta)Q_2^+ \tag{5}$$

if $P_1^+ \succeq_g P_2^+$ *and* $Q_1^+ \succeq_g Q_2^+$, *and*

$$\theta P_1^- \oplus (1-\theta)Q_1^- \succeq_\ell \theta P_2^- \oplus (1-\theta)Q_2^- \tag{6}$$

[13] See, e.g., Marschak (1950), Malinvaud (1952), and Samuelson (1952). Implying that the functional form of the preference function is linear in probabilities, this axiom was critical in the early development of the expected utility theory. Machina (1982) later showed that the expected utility analysis can be justified locally without this axiom, provided that preference functionals are smooth in a differentiable sense. We do not investigate such extensions here in the inherent dominance context.

if $P_1^- \succeq_\ell P_2^-$ *and* $Q_1^- \succeq_\ell Q_2^-$*.*

Further, the strict $\succ_g$ *holds in* (5) *if either* $P_1^+ \succ_g P_2^+$ *or* $Q_1^+ \succ_g Q_2^+$*, and the strict* $\succ_\ell$ *holds in* (6) *if either* $P_1^- \succ_\ell P_2^-$ *or* $Q_1^- \succ_\ell Q_2^-$*.*

Proof: Applying Axiom 3 twice yields

$$\theta P_1^+ \oplus (1-\theta)Q_1^+ \succeq_g \theta P_2^+ \oplus (1-\theta)Q_1^+ \succeq_g \theta P_2^+ \oplus (1-\theta)Q_2^+.$$

One of the $\succeq_g$ above must be strict if either $P_1^+ \succ_g P_2^+$ or $Q_1^+ \succ_g Q_2^+$. The same holds if $\succeq_g$ is replaced with $\succeq_\ell$. □

Lemma 4 *Let any distributions* $P_1 \in \mathcal{P}$ *and* $P_2 \in \mathcal{P}$ *be given. Then for all* $\theta \in (0,1)$*,*

$$P_1^+ \succ_g \theta P_1^+ \oplus (1-\theta)P_2^+ \succ_g P_2^+ \text{if} P_1^+ \succ_g P_2^+, \tag{7}$$

$$P_1^- \succ_\ell \theta P_1^- \oplus (1-\theta)P_2^- \succ_\ell P_2^- \text{if} P_1^- \succ_\ell P_2^-. \tag{8}$$

Proof: Note that for all P, $P \sim \theta P \oplus (1-\theta)P$. Suppose $P_1^+ \succ_g P_2^+$. It follows from Lemma 3 that

$$\begin{aligned} P_1^+ &\sim_g \theta P_1^+ \oplus (1-\theta)P_1^+ \\ &\succ_g \theta P_1^+ \oplus (1-\theta)P_2^+ \\ &\succ_g \theta P_2^+ \oplus (1-\theta)P_2^+ \\ &\sim_g P_2^+. \end{aligned}$$

The same holds for (8) with $\succ_\ell$ instead of $\succ_g$. □

A particular case of Lemma 4 is that for any deterministic returns r_1 and r_2, and for all $\theta \in (0,1)$,

$$r_1 \succ_g \theta r_1 \oplus (1-\theta)r_2 \succ_g r_2 \quad \text{if} \quad r_1 > r_2 \geq r_0,$$

$$r_1 \succ_\ell \theta r_1 \oplus (1-\theta)r_2 \succ_\ell r_2 \quad \text{if} \quad r_0 \geq r_1 > r_2.$$

Lemma 5 *Let any returns $r_1 \in \mathbb{R}$ and $r_2 \in \mathbb{R}$ be given, and let any $\xi \in [0,1]$ and $\psi \in [0,1]$ be given. Then for $r_1 > r_2 \geq r_0$,*

$$\xi r_1 \oplus (1-\xi) r_2 \succ_g \psi r_1 \oplus (1-\psi) r_2 \text{ iff } \xi > \psi, \tag{9}$$

and for $r_2 < r_1 \leq r_0$,

$$\xi r_2 \oplus (1-\xi) r_1 \prec_\ell \psi r_2 \oplus (1-\psi) r_1 \text{ iff } \xi > \psi. \tag{10}$$

Proof: It suffices to show that if $\xi > \psi$, then (9) and (10) hold. Let $\alpha = (1-\xi)/(1-\psi)$ and assume that $1 \geq \xi > \psi \geq 0$ so that $\alpha \in [0,1)$. Suppose first that $r_1 > r_2 \geq r_0$. It can be verified that

$$\begin{aligned}
& \xi r_1 \oplus (1-\xi) r_2 \\
\sim_g \quad & \alpha[\psi r_1 \oplus (1-\psi) r_2] \oplus (1-\alpha) r_1 \\
\succ_g \quad & \alpha[\psi r_1 \oplus (1-\psi) r_2] \oplus (1-\alpha)[\psi r_1 \oplus (1-\psi) r_2] \quad \text{(by Lemma 4)} \\
\sim_g \quad & \psi r_1 \oplus (1-\psi) r_2.
\end{aligned}$$

Suppose next that $r_2 < r_1 \leq r_0$. Then,

$$\begin{aligned}
& \xi r_2 \oplus (1-\xi) r_1 \\
\sim_\ell \quad & \alpha[\psi r_2 \oplus (1-\psi) r_1] \oplus (1-\alpha) r_2 \\
\prec_\ell \quad & \alpha[\psi r_2 \oplus (1-\psi) r_1] \oplus (1-\alpha)[\psi r_2 \oplus (1-\psi) r_1] \quad \text{(by Lemma 4)} \\
\sim_\ell \quad & \psi r_2 \oplus (1-\psi) r_1.
\end{aligned}$$

□

Axiom 4 *: Let any P, P_1 and P_2 in $\mathcal{P}$ be given. If $P_1^+ \succ_g P^+ \succ_g P_2^+$, then there exists $\psi \in (0,1)$ and $\xi \in (0,1)$ such that*

$$\psi P_1^+ \oplus (1-\psi) P_2^+ \succ_g P^+ \succ_g \xi P_1^+ \oplus (1-\xi) P_2^+. \tag{11}$$

Similarly, for $P_1^- \succ_\ell P^- \succ_\ell P_2^-$, there exists $\psi \in (0,1)$ and $\xi \in (0,1)$ such that

$$\psi P_1^- \oplus (1-\psi)P_2^- \succ_\ell P^- \succ_\ell \xi P_1^- \oplus (1-\xi)P_2^-. \tag{12}$$

Theorem 1 *: For any r_1, r, and r_2 in G such that $r_1 > r > r_2$, there exists a unique number $\theta \in (0,1)$ such that $r \sim_g \theta r_1 \oplus (1-\theta)r_2$. Similarly, for any r_1, r, and r_2 in L such that $r_1 < r < r_2$, there exists a unique number $\phi \in (0,1)$ such that $r \sim_\ell \phi r_1 \oplus (1-\phi)r_2$.*

Proof: Suppose $r_1 > r > r_2 > r_0$. Let the sets X and Y be defined as follows.

$$\begin{aligned} X &= \{\theta | \theta r_1 \oplus (1-\theta)r_2 \succ_g r; \;\; \theta \in (0,1)\} \\ Y &= \{\theta | \theta r_1 \oplus (1-\theta)r_2 \prec_g r; \;\; \theta \in (0,1)\} \end{aligned}$$

Axiom 4 implies that these sets are open and not empty. It follows from Lemma 5 that for all $\psi \in X$ and $\phi \in Y$, $\psi > \phi$. Thus there exists α such that $\inf\{\theta | \theta \in X\} \geq \alpha \geq \sup\{\theta | \theta \in Y\}$. By Axiom 4, this α does not belong to either X or Y, thus $r \sim_g \alpha r_1 \oplus (1-\alpha)r_2$. Lemma 5 further implies that this α is unique. The $\sim_\ell$ part of the theorem can be proved analogously. □

2.5 The Gain and Loss Functions

Similar to the utility functions, there are now defined two auxiliary functions that will be useful for the development of inherent reward and inherent risk measures. Let any function $g(\cdot) : r \in \overline{G} \rightarrow g(r) \in \mathbb{R}^+$ be called a *gain function* provided $g(r_1) \geq g(r_2)$ if $r_1 \geq r_2$ for any $r_1 \in \overline{G}$ and $r_2 \in \overline{G}$. Let any function $\ell(\cdot) : r \in \overline{L} \rightarrow \ell(r) \in \mathbb{R}^+$ be called a *loss function* provided $\ell(r_1) \geq \ell(r_2)$ if $r_1 \leq r_2$ for any $r_1 \in \overline{L}$ and $r_2 \in \overline{L}$.

Theorem 2 *: There exists a gain function $g(\cdot) : r \in \overline{G} \rightarrow g(r) \in \mathbb{R}^+$ that strictly increases in r, and there exists a loss function $\ell(\cdot) : r \in \overline{L} \rightarrow \ell(r) \in \mathbb{R}^+$ that strictly decreases in r with*

$g(r_0) = \ell(r_0) = 0$. *They satisfy that, for all* r_1, $r_2 \in \overline{G}$ *and* $\theta \in [0,1]$ *such that* $r_1 \sim_g \theta r_2 \oplus (1-\theta) r_0$, $g(r_1) = \theta g(r_2)$; *and for all* r_1, $r_2 \in \overline{L}$ *and* $\theta \in [0,1]$ *such that* $r_1 \sim_\ell \theta r_2 \oplus (1-\theta) r_0$, $\ell(r_1) = \theta \ell(r_2)$.

Proof: I first prove the existence. Let any $\overline{r} \in \mathbb{R}$ and $\underline{r} \in \mathbb{R}$ be given such that $\overline{r} > r_0 > \underline{r}$. By Theorem 1 a unique gain function $g(\cdot)$ on $\overline{G}$ and a unique loss function $\ell(\cdot)$ on $\overline{L}$ (for given $\overline{r}$ and $\underline{r}$) can be defined as follows. For all $r \in [r_0, \overline{r}]$, define $g(r) \in [0,1]$ such that

$$r \sim_g g(r)\overline{r} \oplus (1 - g(r)) r_0 \tag{13}$$

And for $r \in (\overline{r}, \infty)$, define $g(r) \in (1, \infty)$ such that

$$\overline{r} \sim_g \frac{1}{g(r)} r \oplus \frac{g(r) - 1}{g(r)} r_0 \tag{14}$$

Similarly, for $r \in [\underline{r}, r_0]$ define $\ell(r) \in [0,1]$ such that

$$r \sim_\ell \ell(r)\underline{r} \oplus (1 - \ell(r)) r_0 \tag{15}$$

And for $r \in (-\infty, \underline{r})$ define $\ell(r) \in (-\infty, 0)$ such that

$$\underline{r} \sim_\ell \frac{-\ell(r)}{1 - \ell(r)} r \oplus \frac{1}{1 - \ell(r)} r_0 \tag{16}$$

Thus a gain function $g(r)$ on $\overline{G}$ and a loss function $\ell(r)$ on $\overline{L}$ are defined. They are unique for any given values of $\overline{r}$ and $\underline{r}$.

I now verify the stated properties of g and ℓ. Let any r_1 and r_2 from $\overline{G}$ be given and assume that for some $\theta \in [0,1]$, $r_1 \sim_g \theta r_2 \oplus (1-\theta) r_0$. The property to be verified is $g(r_1) = \theta g(r_2)$.

By the construction of $g(r)$, for $r_1 < r_2 \le \overline{r}$ it follows from (13) that

$$r_1 \quad \sim_g \quad g(r_1)\overline{r} \oplus (1 - g(r_1)) r_0 \tag{17}$$

$$r_2 \quad \sim_g \quad g(r_2)\overline{r} \oplus (1 - g(r_2)) r_0 \tag{18}$$

Thus

$$\begin{aligned} r_1 &\sim_g \theta r_2 \oplus (1-\theta) r_0 \\ &\sim_g \theta[g(r_2)\overline{r} \oplus (1-g(r_2))r_0] \oplus (1-\theta)r_0 \\ &\sim_g [\theta g(r_2)]\overline{r} \oplus [\theta(1-g(r_2)) + (1-\theta)]r_0 \end{aligned} \tag{19}$$

From Lemma 5, the probability of reaching $\overline{r}$ in (19) must be the same as that in (17). Thus $g(r_1) = \theta g(r_2)$.

For $r_2 \geq \overline{r} \geq r_1$, it follows from (14) that

$$\overline{r} \sim_g \frac{1}{g(r_2)} r_2 \oplus [\frac{g(r_2)-1}{g(r_2)}] r_0. \tag{20}$$

Consequently,

$$\begin{aligned} r_1 &\sim_g g(r_1)\overline{r} \oplus (1-g(r_1))r_0 \\ &\sim_g g(r_1)[\frac{1}{g(r_2)} r_2 \oplus (\frac{g(r_2)-1}{g(r_2)}) r_0] \oplus (1-g(r_1))r_0 \\ &\sim_g g(r_1)\frac{1}{g(r_2)} r_2 \oplus [g(r_1)\frac{g(r_2)-1}{g(r_2)} + 1 - g(r_1)] r_0. \end{aligned} \tag{21}$$

By Lemma 5, the probability of reaching r_2 in (21) must be the same as that given as in $r_1 \sim_g \theta r_2 \oplus (1-\theta) r_0$. Thus $g(r_1) = \theta g(r_2)$.

Finally, for $r_2 \geq r_1 \geq \overline{r}$, it follows from (14) and $r_1 \sim_g \theta r_2 \oplus (1-\theta) r_0$ that both (20) and the following relationship must hold.

$$\overline{r} \sim_g \frac{1}{g(r_1)} r_1 \oplus [\frac{g(r_1)-1}{g(r_1)}] r_0 \tag{22}$$

$$\sim_g \frac{1}{g(r_1)}[\theta r_2 \oplus (1-\theta) r_0] \oplus [\frac{g(r_1)-1}{g(r_1)}] r_0 \tag{23}$$

$$\sim_g [\frac{1}{g(r_1)}\theta] r_2 \oplus [\frac{1}{g(r_1)}(1-\theta) + \frac{g(r_1)-1}{g(r_1)}] r_0. \tag{24}$$

By Lemma 5 again, the probability of reaching r_2 in (24) must be the same as that in (20). Thus $g(r_1) = \theta g(r_2)$. Lemma 5 also implies that, following the condition that $r_1 \sim_g \theta r_2 \oplus (1-\theta) r_0$, $\theta \in (0,1)$ if $r_0 < r_1 < r_2$, $\theta = 0$ if $r_1 = r_0$, and $\theta = 1$ if $r_1 = r_2$. In other words, $g(r)$ is a strictly increasing function of r on $\overline{G}$. Finally, substituting r_0 for r_1 in $g(r_1) = \theta g(r_2)$ yields $g(r_0) = 0$.

The proof with r_1 and r_2 in $\overline{L}$ is similar. $\square$

2.6 Expected Gain and Loss as Representations of Inherent Reward and Risk

Extending the expected utility theory, this subsection shows that the inherent reward and risk can be represented by the expected values of the gain and loss functions. To motivate the subsequent analysis, I first show a theorem that holds for the discrete case with finite possible outcomes.

Theorem 3 *Let any $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ be given and let $\tilde{r}$ denote a random variable. If the outcome spaces of both P and Q contain only a finitely many possible returns $\tilde{r} \in \{r_1, r_2, \ldots, r_n\} \subset G$, then $P \succeq_g Q$ if and only if $E(g(\tilde{r}); P^+) \geq E(g(\tilde{r}); Q^+)$. Similarly, if the outcome spaces of both P and Q contain only a finitely many possible returns $\tilde{r} \in \{r_1, r_2, \ldots, r_n\} \subset L$, then $P \succeq_\ell Q$ if and only if $E(\ell(\tilde{r}); P^-) \leq E(\ell(\tilde{r}); Q^-)$.*

Proof: I prove only the $\succeq_g$ part of the theorem; the $\succeq_\ell$ part of the theorem can be proved analogously. Without loss of generality, assume that P and Q have the same upside support $\{r_1, r_2, \ldots, r_n\} \subset G$ and consider P first. Let $\tilde{r} \in \{r_1, r_2, \ldots, r_n\}$ denote the random return variable with distribution P where r_j occurs with probability $p_j \geq 0$, for $j = 1, 2, \ldots, n$. Let $\phi = 1 - \sum_{j=1}^{n} p_j$ denote the probability that $\tilde{r} \leq r_0$ (thus no restriction on the downside). Without loss of generality, assume that $r_0 < r_1 < r_2 < \ldots < r_n$. By definition,

$$P^+ \sim_g p_1 r_1 \oplus p_2 r_2 \oplus \ldots \oplus p_n r_n \oplus \phi r_0. \tag{25}$$

In other words, P^+ can be seen as a lottery with $n+1$ possible outcomes. It assigns probability p_j for $\tilde{r} = r_j$ and $1 - \sum_{j=1}^{n} p_j$ for $\tilde{r} = r_0$. By Theorem 1, for all r_j, $j = 0, 1, \ldots, n$, there exists a unique $\theta(r_j)$ such that

$$r_j \sim_g \theta(r_j) r_n \oplus (1 - \theta(r_j)) r_0 \tag{26}$$

with $\theta(r_0) = 0$ and $\theta(r_j) \in (0, 1)$ for $j = 1, 2, \ldots, n$. Note also that

$$E(\theta(\tilde{r}); P^+) = \sum_{j=1}^{n} p_j \theta(r_j) + \phi\theta(r_0) = \sum_{j=1}^{n} p_j \theta(r_j).$$

Consequently, substituting (26) into (25) and by n applications of Axiom 3 yield

$$\begin{aligned} P^+ \quad &\sim_g \quad \bigoplus_{j=1}^{n} p_j [\theta(r_j) r_n \oplus (1 - \theta(r_j)) r_0] \oplus \phi r_0 && (27) \\ &\sim_g \quad [\sum_{j=1}^{n} p_j \theta(r_j)] r_n \oplus [1 - \sum_{j=1}^{n} p_j \theta(r_j)] r_0 && (28) \\ &\sim_g \quad E(\theta(\tilde{r}); P^+) r_n \oplus [1 - E(\theta(r); P^+)] r_0. && (29) \end{aligned}$$

The right hand side of (27) is a two stage lottery that first chooses a r_j with probability p_j and r_0 with probability ϕ. If r_0 is chosen in the first stage, the lottery yields r_0 with certainty in the second stage. If r_j is chosen in the first stage, the lottery then chooses r_n or r_0 with probability $\theta(r_j)$ in the second stage. The final result of the lottery is thus either r_n or r_0, as expressed in (28). The total probability that r_n will be chosen is $\sum_{j=1}^{n} p_j \theta(r_j)$ and that r_0 will be chosen is $\sum_{j=1}^{n} p_j (1 - \theta(r_j)) + \phi = 1 - \sum_{j=1}^{n} p_j \theta(r_j)$.

From Theorem 2, however, there is a relationship between $\theta(r_j)$ and $g(r_j)$ given by $g(r_j) = \theta(r_j) g(r_n)$. Thus, substituting $g(\tilde{r})/g(r_n)$ for $\theta(\tilde{r})$ in (29) yields

$$P^+ \sim_g \frac{E(g(\tilde{r}); P^+)}{g(r_n)} r_n \oplus [1 - \frac{E(g(\tilde{r}); P^+)}{g(r_n)}] r_0. \tag{30}$$

The same holds also for Q; that is,

$$Q^+ \sim_g \frac{E(g(\tilde{r}); Q^+)}{g(r_n)} r_n \oplus [1 - \frac{E(g(\tilde{r}); Q^+)}{g(r_n)}] r_0. \tag{31}$$

It follows from Lemma 5, then, that $P \succeq_g Q$ if and only if $E(g(\tilde{r}); P^+) \geq E(g(\tilde{r}); Q^+)$. $\square$

I now extend the result in Theorem 3 to the more general set-up involving both discrete and continuous distributions over possible unbounded outcome spaces. For any given $P \in \mathcal{P}$, its upside equivalent P^+ and its downside equivalent P^- on the sets $\overline{G}$ and $\overline{L}$, respectively, have been previously defined. Consider now the conditional distributions obtained by further restricting P^+ to the set $G_n = [r_0, r_n]$ for some $r_n > r_0$, and by restricting P^- to the set $L_m = [r_m, r_0]$ for some $r_m < r_0$. Let $P_n^+ \in \mathcal{P}$ denote the conditional distribution of P^+ on G_n such that $P_n^+(r) = P^+(r|G_n)$, and let $P_m^- \in \mathcal{P}$ denote the conditional distribution of P^- on L_m such that $P_m^-(r) = P^-(r|L_n)$.

Lemma 6 *For any $r_m < r_1 < r_0 < r_2 < r_n$, $P_n^+ \succeq_g P_2^+$ and $P_1^- \succeq_\ell P_m^-$.*

Proof: Let $A_1 = (r_2, r_n]$ and $A_2 = [r_0, r_2]$. Since $P^+(A_1|A_1) = P^+(A_2|A_2) = 1$, from Axiom 1 it follows that $P^+(\cdot|A_1) \succ_g P^+(\cdot|A_2)$. Further, since $P_n^+(A_1) + P_n^+(A_2) = 1$ and $P_n^+(\cdot) = P^+(\cdot|A_1)P_n^+(A_1) + P^+(\cdot|A_2)P_n^+(A_2)$, it follows from Lemma 4 that $P_n^+ \succeq_g P^+(\cdot|A_2) = P_2^+$.

Similarly, let $B_1 = [r_1, r_0]$ and $B_2 = [r_m, r_1)$. Since $P^-(B_1|B_1) = P^-(B_2|B_2) = 1$, it follows from Axiom 1 that $P_1^- = P^-(\cdot|B_1) \succ_\ell P^-(\cdot|B_2) = P_2^-$. Since $P_m^- = P_1^- P_m^-(B_1) + P_2^- P_m^-(B_2)$, it follows from Lemma 4 that $P_1^- \succeq_\ell P_m^-$. $\square$

By Theorem 1, for all $r \in [r_0, r_n]$ there exists a unique $\psi(r)$ such that

$$r \sim_g \psi(r) r_n \oplus (1 - \psi(r)) r_0, \tag{32}$$

and for all $r \in [r_m, r_0]$ there exists a unique $\xi(r)$ such that

$$r \sim_\ell \xi(r) r_m \oplus (1 - \xi(r)) r_0, \tag{33}$$

with $\psi(r_0) = \xi(r_0) = 0$. The next axiom postulates that for all distributions $P \in \mathcal{P}$, what has been shown in (29) is generally true with P_n^+ over G_n and with P_m^- over L_m.

Axiom 5 *For any $P \in \mathcal{P}$, let $\psi(r)$ be defined on G_n as in (32) and let $\xi(r)$ be defined on L_m as in (33). Then*

$$P_n^+ \sim_g E(\psi(r); P_n^+) r_n \oplus [1 - E(\psi(r); P_n^+)] r_0, \tag{34}$$

$$P_m^- \sim_\ell E(\xi(r); P_m^-) r_m \oplus [1 - E(\xi(r); P_m^-)] r_0. \tag{35}$$

A variation of the (traditional) continuity axiom is further adopted as follows.

Axiom 6 *Let any $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ be given. If $P \succ_\ell Q$, then there exists $r_1 \leq r_0$ such that for all $r_m \leq r_1$, $P_m^- \succ_\ell Q$. If $P \succ_g Q$, then there exists $r_2 \geq r_0$ such that for all $r_n \geq r_2$, $P_n^+ \succ_g Q$.*

This Axiom means that, for all distributions, the binary relationship $\succ_g$ is continuous in the limit as $r_n \to \infty$, and the binary relationship $\succ_\ell$ is continuous in the limit as $r_m \to -\infty$. The next theorem shows that the inherent reward and risk measures can in general be represented by the expected value of the gain and loss functions constructed in Theorem 2.

Theorem 4 (Dichotomous Expected Utility) *Let any $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ be given. Then $P \succeq_g Q$ if and only if $E(g(r); P^+) \geq E(g(r); Q^+)$, and $P \succeq_\ell Q$ if and only if $E(\ell(r); P^-) \leq E(\ell(r); Q^-)$.*

Proof: Substituting $g(r)/g(r_n)$ for $\psi(r)$ in (34) and $\ell(r)/\ell(r_m)$ for $\xi(r)$ in (35), it follows from Axiom 5 that

$$P_n^+ \sim_g \frac{E(g(r); P_n^+)}{g(r_n)} r_n \oplus [1 - \frac{E(g(r); P_n^+)}{g(r_n)}] r_0, \tag{36}$$

$$Q_n^+ \sim_g \frac{E(g(r); Q_n^+)}{g(r_n)} r_n \oplus [1 - \frac{E(g(r); Q_n^+)}{g(r_n)}] r_0. \tag{37}$$

Similarly,

$$P_m^- \sim_\ell \frac{E(\ell(r); P_m^-)}{\ell(r_m)} r_m \oplus [1 - \frac{E(\ell(r); P_m^-)}{\ell(r_m)}] r_0, \tag{38}$$

$$Q_m^- \quad \sim_\ell \quad \frac{E(\ell(r);Q_m^-)}{\ell(r_m)} r_m \oplus [1 - \frac{E(\ell(r);Q_m^-)}{\ell(r_m)}] r_0. \tag{39}$$

If both P and Q are bounded distributions, a sufficiently small r_m and a sufficiently large r_n can be chosen so that $P_n^+ = P^+$, $P_m^- = P^-$, $Q_n^+ = Q^+$, and $Q_m^- = Q^-$. It then follows from Lemma 5 that the theorem holds.

In general, it can be shown that $P^- \prec_\ell Q^-$ implies $E(\ell(r);P^-) > E(\ell(r);Q^-)$. By Axiom 6 and Lemma 6 there exists $r_1 < r_0$ such that for all $r_m \leq r_1$,

$$P_m^- \preceq_\ell P_1^- \prec_\ell Q^- \preceq_\ell Q_m^- \preceq_\ell Q_1^-.$$

Since P_1^- and Q_1^- are bounded, $E(\ell(r);P_1^-) > E(\ell(r);Q_1^-)$. From Axiom 4, there exists $\theta \in (0,1)$ such that

$$\theta P_1^- \oplus (1-\theta) Q_1^- \prec_\ell Q^- \preceq_\ell Q_m^-.$$

Since the above distributions are bounded,

$$E(\ell(r);P_m^-) \geq E(\ell(r);P_1^-) > \theta E(\ell(r);P_1^-) + (1-\theta)E(\ell(r);Q_1^-) \geq E(\ell(r);Q_m^-).$$

Taking the limit as $r_m \to -\infty$ yields $E(\ell(r);P^-) > E(\ell(r);Q^-)$.

That $P^+ \succ_g Q^+$ implies $E(g(r);P^+) > E(g(r);Q^+)$ can be shown analogously as follows. Similar to the above proof, there exists $r_2 > r_0$ such that for all $r_n \geq r_2$,

$$P_n^+ \succeq_g P_2^+ \succ_g Q^+ \succeq_g Q_n^+ \succeq_g Q_2^+.$$

which implies that $E(g(r);P_2^+) > E(g(r);Q_2^+)$. From Axiom 4, there exists $\theta \in (0,1)$ such that

$$\theta P_2^+ \oplus (1-\theta) Q_2^+ \succ_g Q \succeq_g Q_n^+$$

It follows that

$$E(g(r);P_n^+) \geq E(g(r);P_2^+) > \theta E(g(r);P_2^+) + (1-\theta)E(g(r);Q_2^+) \geq E(g(r);Q_n^+).$$

Taking limit as $r_n \to \infty$ yields $E(g(r); P^+) > E(g(r); Q^+)$. $\square$

I call this theorem the dichotomous expected utility theorem for its relation (in form) to the von Neumann-Morgenstern expected utility theory. To see this, interpret r and r_0 as an investor's time-t wealth (random) and the benchmark wealth (fixed) respectively. Call $g(r)$ the investor's upside utility and $\ell(r)$ the investor's downside utility. Then, the investor's investment objective can be postulated as follows.

$$\max_{P \in \Omega} E(g(r); P^+) - E(\ell(r); P^-) \tag{40}$$

In general, $g(r)$ and $\ell(r)$ depend on the benchmark r_0 and horizon t. As the investor's benchmark wealth changes and/or investment horizon changes, the functional forms of $g(r)$ and $\ell(r)$ can change as well. It is easily seen that the expected utility is a special case of (40) if the benchmark r_0 is held fixed for all wealth levels and the functional forms of $g(r)$ and $\ell(r)$ are fixed. A similar uniqueness result can be derived such that for all $\overline{g}(r)$ and $\overline{\ell}(r)$ having the property stated in Theorem 4, they are a positive linear transformation of $g(r)$ and $\ell(r)$.

Further, if the distribution function P is interpreted as one's subjective probability distribution of wealth, then (40) gives a general form of objectives of which many special cases have been studied in the non-expected utility models (see, e.g., a list of such models in Machina, 1989, p. 1631). In this regard, Theorem 4 contributes a normative foundation to the non-expected utility analysis of choice under uncertainty.

2.7 The Form of The Gain and Loss Functions

So far I have shown only that the gain and loss functions $g(r)$ and $\ell(r)$ exist and are monotonic in r. It is desirable to know more about plausible forms of these functions that may result from the requirement of common judgement. In this subsection, I propose two additional axioms that together with the preceding axioms imply a simple linear form for both $g(\cdot)$ and $\ell(\cdot)$.

The next axiom, the allocation independence axiom, is similar in form to the lottery independence axiom (Axiom 3) but it concerns the mix of outcomes rather than the mix of distributions. For any $\phi \in [0, 1]$, let $\phi r + (1 - \phi)r_0$ denote the weighted average of returns r (random) and r_0 (deterministic), where ϕ is the weight. If $P \in \mathcal{P}$ is the distribution of r and r_0 is the risk-free interest rate, then $\phi r + (1 - \phi)r_0$ can be interpreted as the return on a portfolio with proportion of capital ϕ allocated to P and $1 - \phi$ allocated to the risk-free asset. For notational convenience, let P_ϕ denote the distribution of the return on such a combined strategy over $\mathbb{R}$, i.e., for all r, $P_\phi(r) = P(\phi r + (1 - \phi)r_0)$. Similarly, let P_ϕ^+ denote the upside equivalence of P_ϕ over $\overline{G}$, and P_ϕ^- the downside equivalence of P_ϕ over $\overline{L}$ (see Definition 1).

Axiom 7 (Allocation Independence) : *Let any two distributions $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ be given. If $P^+ \succeq_g Q^+$, then for all $\phi \in [0, 1]$, $P_\phi^+ \succeq_g Q_\phi^+$. If $P^- \succeq_\ell Q^-$, then for all $\phi \in [0, 1]$, $P_\phi^- \succeq_\ell Q_\phi^-$.*

It will be shown later in Theorem 7 that if investors can borrow or lend a risk-free asset, then r_0 *must* be the risk-free interest rate under what will be called the leverage independence axiom. In such a situation, Axiom 7 postulates that the inherently more rewarding and more risky relationships are invariant with the proportion of capital allocated to the strategies being compared (hence the label of the axiom).[14] Intuitively, since individuals may have different risk tolerance and thus may allocate their capital to risky assets differently, in order for all to commonly agree on

[14]This view is held quite commonly. For instance, in a practitioner's journal, Modigliani and Modigliani (1997) proposed that any measures used for ranking investment performances should meet the criteria of: "1) leverage changes the reward and risk of portfolios in the same direction, and 2) leverage does not change the ranking of portfolios at any level of risk (p.51)." Our axiom postulates a weaker condition on capital allocation with θ required only to be between 0 and 1. It is also worth remarking that the stochastic dominance criteria fail to meet this second condition above.

any "inherently more risky" relationship this relationship must be independent of any individual's capital allocation. This holds also for the "inherently more rewarding" relationship. In general, Axiom 7 is a postulate about a reasonable *perception* of inherent reward and risk. Even in the absence of a risk-free asset, the axiom would still have the same connotation *as if* there were such an asset that yields a neutral return.

The next lemma is a straightforward implication of Axiom 7.

Lemma 7 *Let any two distributions $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ be given. If $P^+ \sim_g Q^+$, then for all $\phi \in [0,1]$, $P^+_\phi \sim_g Q^+_\phi$. If $P^- \sim_\ell Q^-$, then for all $\phi \in [0,1]$, $P^-_\phi \sim_\ell Q^-_\phi$.*

Proof: If $P^+ \sim_g Q^+$, then $P^+ \succeq_g Q^+$ and $Q^+ \succeq_g P^+$. Thus by Axiom 7 both $P^+_\phi \succeq_g Q^+_\phi$ and $Q^+_\phi \succeq_g P^+_\phi$ must hold for all $\phi \in [0,1]$, which implies $P^+_\phi \sim_g Q^+_\phi$. The second part of the lemma with respect to the $\sim_\ell$can be proved analogously. □

Lemma 8 *Any gain function $\hat{g}(\cdot) : \overline{G} \to \mathbb{R}$ and loss function $\hat{\ell}(\cdot) : \overline{L} \to \mathbb{R}$ having the corresponding property stated in Theorem 4 are differentiable on G and L, respectively, with $\hat{g}'(r) > 0$ on G and $\hat{\ell}'(r) < 0$ on L.*

Proof: I look at $\hat{g}(r)$ only; the proof concerning $\hat{\ell}(r)$ is analogous.

Let any r and r_1 be given from G and assume that $r < r_1$. From Theorem 1, there exists $\theta \in (0,1)$ such that

$$r \sim_g \theta r_1 \oplus (1-\theta) r_0. \tag{41}$$

From Axiom 7 and Theorem 4 it follows that

$$\begin{aligned} \hat{g}(r_0 + \phi(r - r_0)) &= \theta\hat{g}(r_0 + \phi(r_1 - r_0)) + (1-\theta)\hat{g}(r_0) \;\; \forall \phi \in [0,1] & (42) \\ \hat{g}(r_0 + \phi_1(r - r_0)) &= \theta\hat{g}(r_0 + \phi_1(r_1 - r_0)) + (1-\theta)\hat{g}(r_0) \;\; \forall \phi_1 \in [0,1]. & (43) \end{aligned}$$

Subtracting (43) from (42) yields

$$\begin{aligned}
& \hat{g}(r_0 + \phi_1(r - r_0)) - \hat{g}(r_0 + \phi(r - r_0)) \\
= \; & \theta[\hat{g}(r_0 + \phi_1(r_1 - r_0)) - \hat{g}(r_0 + \phi(r_1 - r_0))] \;\; \forall \phi \in [0,1], \;\; \forall \phi_1 \in [0,1].
\end{aligned}$$

For $\phi \in [0,1)$, define $x(\phi) = r_0 + \phi(r - r_0)$ and $y(\phi) = r_0 + \phi(r_1 - r_0)$. Since $\hat{g}(r)$ is monotonically increasing in r on G, by the Lebesgue differentiation theorem (e.g., Edwin Hewitt and Karl Stromberg, 1955, p. 264) $\hat{g}$ is differentiable almost everywhere on any closed interval of G . This implies that for any $x > r_0$, there exists $y > x$ such that $\hat{g}'(y)$ exists.

Now for any given $x \geq r_0$, choose $r_1 > r$ for $y \geq x$ such that $\hat{g}'(y)$ exists. Consequently,

$$\begin{aligned}
\hat{g}'(x) &= \lim_{\Delta x \to 0} \frac{\hat{g}(x + \Delta x) - \hat{g}(x)}{\Delta x} \\
&= \lim_{\Delta \phi \to 0} \frac{\hat{g}(r_0 + (\phi + \Delta\phi)(r - r_0)) - \hat{g}(r_0 + \phi(r - r_0))}{\Delta\phi(r - r_0)} \\
&= \lim_{\Delta \phi \to 0} \frac{\theta[\hat{g}(r_0 + (\phi + \Delta\phi)(r_1 - r_0)) - \hat{g}(r_0 + \phi(r_1 - r_0))]}{\Delta\phi(r_1 - r_0)} \cdot \frac{r_1 - r_0}{r - r_0} \\
&= \theta \lim_{\Delta y \to 0} \frac{\hat{g}(y + \Delta y) - \hat{g}(y)}{\Delta y} \cdot \frac{r_1 - r_0}{r - r_0} \\
&= \theta \hat{g}'(y) \cdot \frac{r_1 - r_0}{r - r_0}.
\end{aligned}$$

It follows that $\hat{g}'(x)$ exists and $\hat{g}'(x) \geq 0$ for all $x > r_0$. Noting that $\hat{g}'(y) = 0$ would imply that $\hat{g}'(x) = 0$ for all $x \in (r_0, y)$. This would contradict the property that $\hat{g}(y) > \hat{g}(x)$ for $y > x > r_0$. Thus $\hat{g}'(x) > 0$ for all $x > r_0$. □

Theorem 5 *For any gain function $\hat{g}(\cdot) : \overline{G} \to \mathbb{R}$ and any loss function $\hat{\ell}(\cdot) : \overline{L} \to \mathbb{R}$ having the corresponding property stated in Theorem 4, they must have the form $\hat{g}(r) = C_1(r - r_0)^a$ and $\hat{\ell}(r) = C_2(r_0 - r)^b$, where a, b, C_1 and C_2 are positive and constant real numbers independent of r.*

Proof: Let any $\hat{g}(r)$ be given and assume that it satisfies the property stated in Theorem 4. Let any r and r_1 be given from G and assume that $r < r_1$. By Theorem 1 there exists $\theta \in (0,1)$

such that

$$r \sim_g \theta r_1 \oplus (1-\theta) r_0. \tag{44}$$

By Axiom 7 and Theorem 4,

$$\hat{g}(r_0 + \phi(r - r_0)) = \theta \hat{g}(r_0 + \phi(r_1 - r_0)), \ \ \forall \phi [0,1]. \tag{45}$$

Applying Lemma 8, I differentiate both sides of (45) with respect to ϕ and obtain

$$\hat{g}'(r_0 + \phi(r - r_0))(r - r_0) = \theta \hat{g}'(r_0 + \phi(r_1 - r_0))(r_1 - r_0), \ \ \forall \phi [0,1]. \tag{46}$$

Dividing (46) by (45) and letting $\phi = 1$ thus yield

$$\frac{\hat{g}'(r)(r - r_0)}{\hat{g}(r)} = \frac{\hat{g}'(r_1)(r_1 - r_0)}{\hat{g}(r_1)}, \quad \forall r \in G, \forall r_1 \in G, \tag{47}$$

or, equivalently,

$$\frac{\hat{g}'(r)(r - r_0)}{\hat{g}(r)} = a, \quad \forall r \in G, \tag{48}$$

where $a > 0$ is a constant. Dividing both sides of (48) by $r - r_0$ and integrating over any interval $(r_1, r) \subset G$ yield the desired functional form for $\hat{g}$; that is, $\hat{g}(r) = C_1(r - r_0)^a$ where C_1 is a constant. Further, from the property that $\hat{g}$ is an increasing function of r, $C_1 > 0$. The proof concerning $\hat{\ell}$ is analogous. □

Theorem 5 is an important result in its own. It justifies the gain-loss objective functions that are frequently adopted in prospect theory and in the behavioral finance literature. I discuss Theorem 5 further in the next section, and continue to proceed here with the development of the inherent aspects of investment reward and risk.

From Theorem 5, the gain and loss functions are differentiable on the open sets (r_0, ∞) and $(-\infty, r_0)$ respectively. I now extend the derivatives of the gain and loss functions and define $\hat{g}'(r_0)$

and $\hat{\ell}'(r_0)$ as the left and the right limit. That is,

$$\hat{g}'(r_0) = \lim_{\epsilon \to 0} \frac{\hat{g}(r_0 + \epsilon) - \hat{g}(r_0)}{\epsilon} \qquad \epsilon \geq 0$$

and

$$\hat{\ell}'(r_0) = \lim_{\epsilon \to 0} \frac{\hat{\ell}(r_0) - \hat{\ell}(r_0 - \epsilon)}{\epsilon} \qquad \epsilon \geq 0.$$

The next axiom further restricts the functional form of the inherent reward and risk measures.

Axiom 8 (Properness) *: There exist r_1 and r_2 ($r_1 > r_0 > r_2$), $\epsilon_0 > 0$, and $\theta_1 \in (1 - \epsilon_0, 1)$ and $\theta_2 \in (0, \epsilon_0)$, such that for all $\epsilon \in (0, \epsilon_0)$,*

$$\theta_1 r_1 \oplus (1 - \theta_1) r_0 \succeq_g \theta_1 (r_1 - \epsilon) \oplus (1 - \theta_1)(r_0 + \epsilon) \tag{49}$$

$$\theta_2 r_1 \oplus (1 - \theta_2) r_0 \preceq_g \theta_2 (r_1 - \epsilon) \oplus (1 - \theta_2)(r_0 + \epsilon) \tag{50}$$

$$\theta_1 r_2 \oplus (1 - \theta_1) r_0 \preceq_\ell \theta_1 (r_2 + \epsilon) \oplus (1 - \theta_1)(r_0 - \epsilon) \tag{51}$$

$$\theta_2 r_2 \oplus (1 - \theta_2) r_0 \succeq_\ell \theta_2 (r_2 + \epsilon) \oplus (1 - \theta_2)(r_0 - \epsilon) \tag{52}$$

I term this axiom *the properness axiom* because it implies that the gain and loss functions should not have an infinite slope ($g'(r_0) = \infty$, or $\ell'(r_0) = -\infty$) or a slope of zero ($g'(r_0) = 0$ or $\ell'(r_0) = 0$) at the neutral origin r_0. In terms of preferences, intuitively, this implication means that when receiving 1 dollar for free one should (marginally) neither be infinitely happier nor be totally indifferent; likewise, when losing 1 dollar one should (marginally) neither be infinitely unhappier nor be totally indifferent.[15] Note that fixing any small amount ϵ and choosing θ_1 sufficiently close

[15]In comparison with the utility functions of wealth w that exhibit constant relative risk aversion (CRRA), i.e., $V(w) = w^\alpha/\alpha$ for $\alpha \neq 0$ and $V(w) = \ln(w)$ for $\alpha = 0$, the gain and loss functions here differ in their having a neutral-return benchmark. Thus, although the CRRA utility functions do have a slope that goes to infinity for $\alpha < 1$ and to zero for $\alpha > 1$ as wealth goes to 0, the interpretation is very different. For instance, with $\alpha < 1$ and $V'(0) = \infty$ it is natural to interpret that the investor cannot survive if deprived of everything. Whereas the properness axiom here says that *given one's current wealth level*, an additional 1 dollar gain or loss should not matter that extreme.

to 1 and θ_2 sufficiently close to 0 we can always meet the conditions in (49)–(52). Here, the axiom essentially postulates that an event that is virtually certain (with probability of reaching it close to 1) dominates the event that virtually will not happen (with probability of reaching it close to 0) in the determination of the inherent "more rewarding" and "more risky" relationships. For instance, suppose $r_1 = 1$, $r_0 = 0$, $\theta_1 = 0.999$ and $\epsilon = 0.01$. Then, the relationship in (49) says that a lottery with a probability 0.999 of realizing 1 and probability 0.001 of realizing 0 is inherently more rewarding that another lottery with the same probability 0.999 of realizing 0.99 and probability 0.001 of realizing 0.01. That is, the inherent dominance relationships are robust to (infinitesimally) small shift of contingent outcomes from the high probability state r_1 to the low probability state r_0.

Lemma 9 *For any gain function $\hat{g}(\cdot) : \overline{G} \to \mathbb{R}$ and any loss function $\hat{\ell}(\cdot) : \overline{L} \to \mathbb{R}$ having the corresponding property stated in Theorem 4, $0 < \hat{g}'(r_0) < \infty$ and $-\infty < \hat{\ell}'(r_0) < 0$.*

Proof: Let any $\hat{g}(r)$ be given and assume that it satisfies the property stated in Theorem 4. From (49) it follows for all $\epsilon \in (0, \epsilon_0)$ that

$$\theta_1 \hat{g}(r_1) + (1-\theta_1)\hat{g}(r_0) \geq \theta_1 \hat{g}(r_1 - \epsilon) + (1-\theta_1)\hat{g}(r_0 + \epsilon),$$

or, altering terms, that

$$\theta_1[\hat{g}(r_1) - \hat{g}(r_1 - \epsilon)] \geq (1-\theta_1)[\hat{g}(r_0+\epsilon) - \hat{g}(r_0)]. \tag{53}$$

Dividing both sides of (53) by $\epsilon > 0$ and taking limit as ϵ goes to zero yield

$$\hat{g}'(r_0) \leq \frac{\theta_1}{1-\theta_1}\hat{g}'(r_1) < \infty.$$

From (50) it also follows for all $\epsilon \in (0, \epsilon_0)$ that

$$\theta_2 \hat{g}(r_1) + (1-\theta_2)\hat{g}(r_0) \leq \theta_2 \hat{g}(r_1 - \epsilon) + (1-\theta_2)\hat{g}(r_0 + \epsilon),$$

or, equivalently, that

$$\theta_2[\hat{g}(r_1) - \hat{g}(r_1 - \epsilon)] \leq (1 - \theta_2)[\hat{g}(r_0 + \epsilon) - \hat{g}(r_0)]. \tag{54}$$

Dividing both sides of (54) by $\epsilon > 0$ and taking limit as ϵ goes to zero yield

$$\hat{g}'(r_0) \geq \frac{\theta_2}{1 - \theta_2}\hat{g}'(r_1) > 0.$$

Similarly, let any $\hat{\ell}(r)$ be given and assume that it satisfies the property stated in Theorem 4. The property (51) then implies that

$$\theta_1[\hat{\ell}(r_2 + \epsilon) - \hat{\ell}(r_2)] \leq (1 - \theta_1)[\hat{\ell}(r_0) - \hat{\ell}(r_0 - \epsilon)]. \tag{55}$$

Dividing both sides of (55) by $\epsilon > 0$ and taking limit as ϵ goes to zero yield

$$\hat{\ell}'(r_0) \geq \frac{\theta_1}{1 - \theta_1}\hat{\ell}'(r_2) > -\infty.$$

And finally, (52) implies that

$$\theta_2[\hat{\ell}(r_2 + \epsilon) - \hat{\ell}(r_2)] \geq (1 - \theta_2)[\hat{\ell}(r_0) - \hat{\ell}(r_0 - \epsilon)]. \tag{56}$$

Dividing both sides of (56) by $\epsilon > 0$ and taking limit as ϵ goes to zero yield

$$\hat{\ell}'(r_0) \leq \frac{\theta_2}{1 - \theta_2}\hat{\ell}'(r_2) < 0.$$

$\square$

Thus, any gain and loss functions having the desired property in Theorem 4 should not grow too fast, nor too slow, at r_0.The next theorem, the main result of this paper, states that the gain and loss functions must be a linear function of $r - r_0$.

Theorem 6 *: For any gain function $\hat{g}(\cdot) : \overline{G} \to \mathbb{R}$ and any loss function $\hat{\ell}(\cdot) : \overline{L} \to \mathbb{R}$ having the corresponding property stated in Theorem 4, there exists constants $C_1 > 0$ and $C_2 > 0$ such that $\hat{g}(r) = C_1(r - r_0)$ and $\hat{\ell}(r) = C_2(r_0 - r)$.*

Proof: Simply note from Theorem 5 that $\hat{g}'(r_0) = aC_1(r-r_0)^{a-1}$ and $\hat{\ell}'(r_0) = -bC_2(r_0-r)^{b-1}$. Lemma 9 implies then $a = 1$, $b = 1$, $C_1 > 0$ and $C_2 > 0$. □

2.8 Inherent Dominance

The next, and last, axiom associates the inherent dominance relationship to a capital market. It has an important implication for the level of the neutral-return benchmark

Axiom 9 (Leverage Independence) *: If investors can borrow and lend at a deterministic interest rate r_f, then $r \in \overline{G} \Longrightarrow \phi r + (1-\phi)r_f \in \overline{G}$ and $r \in \overline{L} \Longrightarrow \phi r + (1-\phi)r_f \in \overline{L}$, for all $\phi > 1$.*

Since $\phi r + (1-\phi)r_f$ is deterministic, Axiom **??** simply postulates that leverage cannot turn a "sure gain" into a "sure loss" or vice versa.

Theorem 7 *If investors can borrow and lend at a deterministic interest rate r_f, then $r_0 = r_f$.*

Proof: Suppose, contrary to what is stated in the theorem, that $r_f < r_0$. Then there exists $r \in (r_f, r_0)$ such that realizing r will be considered a loss. We have then $r_0 \succ r$ but $r_0 \prec \alpha r + (1-\alpha)r_f$ for α sufficiently large by Lemma 1. Similarly, if $r_f > r_0$, there exists $r \in (r_0, r_f)$ such that $r \succ r_0$ but $\alpha r + (1-\alpha)r_f \prec r_0$ for α sufficiently large. Both of these would contradict Axiom **??**. Thus the theorem must hold. □

It is worth noting that Theorem 7 may not apply to *individual* benchmarks; it applies only to the commonly perceived, deterministic neutral-return benchmark r_0. In a sense, Axiom **??** is related to risk-free arbitrage in a well-functioning capital market. Arbitrage typically involves shortselling or leverage to lock in risk-free profits (sure gain). The common perception of a sure gain should thus be consistent with the existence of an arbitrage opportunity. For instance, if there exists a strategy that generates a deterministic return $r > r_f$ then arbitrage profits can be realized by borrowing at r_f and investing in this strategy.

With the preceding preparations, I am now ready to define the inherent reward, risk, and dominance measures.

Definition 2 *The inherent reward of a return distribution $P \in \mathcal{P}$, denoted $U(P)$, is defined as*

$$U(P) = E[\max(r - r_0, 0); P]$$

Definition 3 *The inherent risk of a return distribution $P \in \mathcal{P}$, denoted $D(P)$, is defined as*

$$D(P) = E[\max(r_0 - r, 0); P]$$

The term $\max(r - r_0, 0)$ is the profit on a one-dollar investment relative to the neutral return r_0 *when gain actually occurs*, and the term $\max(r_0 - r, 0)$ gives the loss on a one-dollar investment relative to the neutral return r_0 *when loss actually occurs.* Thus, U and D can also be interpreted as the expected payoff on a call option, and the expected payoff on a put option respectively, both with a strike price equal to r_0 on an underlying asset that is worth one dollar today.

Theorem 8 *Let any $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ be given. Then $P \succeq_g Q$ if and only if $U(P) \geq U(Q)$, and $P \succeq_\ell Q$ if and only if $D(P) \leq D(Q)$.*

Proof: Suppose $P \succeq_g Q$. From Theorems 4 and 6 it follows that

$$C_1 E(r - r_0; P^+) = C_1 U(P) \geq C_1 E(r - r_0; Q^+) = C_1 U(Q),$$

which implies $U(P) \geq U(Q)$ since $C_1 > 0$. Similarly, if $P \succeq_\ell Q$ then

$$C_2 E(r_0 - r; P^-) = C_2 D(P) \leq C_2 E(r_0 - r; Q^-) = C_2 D(Q),$$

which implies $D(P) \leq D(Q)$ since $C_2 > 0$. $\square$

The above theorem confirms the assumption I made at the start that the relationships $\succeq_g$ and $\succeq_\ell$ exist. It is easy to see as well that U and D completely rank all the probability distributions $P \in \mathcal{P}$. Their rankings are transitive, and satisfy all the axioms postulated in this paper. This completes the development of the "inherently more rewarding" and "inherently more risky" relationships.

I now go on to build a link between the inherently more-rewarding and more-risky relationships and the inherent dominance relationship.

Definition 4 *Let any $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ be given. $P \succeq Q$ if and only if there exists $\phi \in [0,1]$ such that*

$$\text{either:} \quad P_\phi^+ \ \succeq_g \ Q^+ \text{ and } P_\phi^- \succeq_\ell Q^- \ , \tag{57}$$

$$\text{or:} \quad Q_\phi^+ \ \preceq_g \ P^+ \text{ and } Q_\phi^- \preceq_\ell P^- \tag{58}$$

$P \succ Q$ if and only if the relationships in either (57) or (58) hold and at least one of them holds strictly.

Axioms 1-9 help justify the legitimacy of this definition. Implicit in this definition, moreover, is the notion that an investment strategy being inherently more rewarding and inherently less risky than another strategy is inherently superior. Note that, by the definitions of U and D, for any $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ there always exists a $\phi \in [0,1]$ such that at least one of the conditions in (57) and (58) holds.

Definition 5 *The inherent reward to risk ratio of any distribution $P \in \mathcal{P}$, denoted $Z(P)$, is defined* *as*

$$Z(P) = \begin{cases} \infty & \text{if } D(P) = 0,\ U(P) > 0 \\ \frac{U(P)}{D(P)} & \text{if } D(P) \neq 0 \\ 1 & \text{if } D(P) = 0,\ U(P) = 0. \end{cases} \tag{59}$$

Theorem 9 *Let any $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ be given. Then $P \succeq Q$ if and only if $Z(P) \geq Z(Q)$.*

Proof: I only need to show that $Z(P) \geq Z(Q)$ implies $P \succeq Q$. Although not necessary, I assume for expositional convenience that investors can borrow and lend at the risk-free rate r_0. Consider a portfolio with ϕ per dollar invested in an arbitrarily given risky asset with random return r that follows distribution P and $1 - \phi$ in the risk-free asset yielding r_0 with certainty. Let U, D, and Z denote the risky asset's inherent reward, risk, and reward-to-risk ratio, respectively. The total return on this portfolio over the holding period is $r_0 + \phi(r - r_0)$. Thus, letting $U(P_\phi)$, $D(P_\phi)$, and $Z(P_\phi)$ denote the inherent reward, risk, and reward-to-risk ratio of the portfolio with random return $r_0 + \phi(r - r_0)$, we have $U(P_\phi) = \phi U(P)$, $D(P_\phi) = \phi D(P)$ and $Z(P_\phi) = Z(P)$. As ϕ increases, $U(P_\phi)$ and $D(P_\phi)$ clearly increase in the same direction; whereas Z remains constant, suggesting that the portfolio's relative attractiveness is independent of the allocation of capital.

Thus, if $Z(P) \geq Z(Q)$ there exists $\phi \in [0, 1]$ such that

$$\begin{aligned} \text{either}: \quad & \phi U(P) \geq U(Q) \ \text{ and } \ \phi D(P) \leq D(Q), \\ \text{or}: \quad & \phi U(Q) \leq U(P) \ \text{ and } \ \phi D(Q) \geq D(P) \end{aligned}$$

By Definition 4, then, $P \succeq Q$. □

In the special case where $U(P) > 0$ and $D(P) = 0$, P is a risk-free arbitrage strategy. Alternatively, any risk-free strategy must have an inherent reward-to-risk ratio $Z = \infty$. Two risk-free arbitrage strategies are considered inherently equivalent by assuming that $\infty = \infty$.

The above theorem confirms the assumption that the inherent dominance relationship $\succeq$ exists on $\mathcal{P}$, since the measure Z exists for any $P \in \mathcal{P}$. That the measure Z completely ranks all the probability distributions $P \in \mathcal{P}$ is obvious. This ranking is also transitive, and satisfies all the axioms postulated in this paper. This completes my development of the inherent dominance relationship.

In summary, I have shown that for any investment strategy whose return distribution is given and whose mean exists, its U exists on $[0, \infty)$, D exists on $(-\infty, 0]$, and Z exists on $[0, \infty]$. Thus all such strategies can be ranked by their inherent reward, risk, and reward-to-risk ratio. These inherent measures are also *unique* because for any investment strategy whose return distribution is given of which the mean exists, its U, D, and Z are uniquely determined. Finally, inherent measures are transitive simply because the inherent measures are real numbers.

Having the above uniqueness and transitivity properties is important for any criterion that ranks investment alternatives because these properties avoid ambiguity and ensure consistency. The completeness property is desirable because the measures can be used for ranking all feasible assets, not just a subset of the feasible assets. For one thing, it is well known that the stochastic dominance criteria for ranking investment alternatives are not complete and thus they have limited applicability.

The form of the inherent risk measure D has been assumed, although not theoretically justified, as a measure of risk in some existing works (e.g., Domar and Musgrave, 1944; Bawa, 1975, 1976, and 1978; and Fishburn, 1977; Bawa and Lindenberg, 1977; and Harlow and Rao, 1989). In these studies, investment reward is implicitly assumed to be the expected return (or wealth) also without justification. Bernardo and Ledoit (2000), as an exception, define “gain” and “loss” similar in form to our inherent reward and risk measures U and D. They study the relationship between two asset pricing approaches – the model-based pricing and arbitrage-free pricing. However, Bernardo and Ledoit's gain and loss measures are defined by taking expectations under only a benchmark risk-adjusted probability measure. Thus their gain and loss measures are equivalent to the inherent measures U and D of the benchmark portfolio only, and they do not define the reward and risk of investment strategies in general. Artzner et al. (1999) propose a set of properties to restrict the class of risk measures, which they call the class of "coherent risk measures". The main limitation of the coherent risk measures is that they are not unique, and thus individuals may still differ in their perception of risk among these coherent risk measures.

Concerning the investment reward, since for all $P \in \mathcal{P}$, $U(P) - D(P) = E(r; P)$, the expected return of an investment can now be given another meaning. That is, $E(r; P)$ can be interpreted as the *net* inherent reward of the investment – the reward minus the risk. Thanks to the equivalence between $Z(P) \geq Z(Q)$ and $Z(P) - 1 \geq Z(Q) - 1$, the inherent dominance relationship $P \succeq Q$ can also be expressed in terms of $E(r; P)/D(P) \geq E(r; Q)/D(Q)$. This *happens* to help justify, and only in the case when the form D is chosen to measure risk, the use of expected return as a measure of reward in the past.

To the best of my knowledge, the axiomatic development of inherent reward and risk measures in this paper is the first to theoretically validate a *unique* pair of measures (invariant up to a positive multiplicative transformation) for investment reward and risk under uncertainty.

3 INHERENT ANALYSIS AND INVESTMENT DECISIONS

So far I have not attempted to associate the inherent reward and risk analysis with individual preferences. It is time now to take a look at how the inherent dominance criterion may help explain some observed behavior towards risk and, eventually, for making investment decisions. As a special case of (2), assume that the functional form of the value function to be maximized is given by

$$V(W_0, U(P), D(P)) \tag{60}$$

where V increases in U and decreases in D, for any given W_0 and P.

For any initial capital of W_0 and strategy P, the *total* inherent reward and risk of the investment are naturally defined as $W_0U(P)$ and $W_0D(P)$, respectively. It bears repeating that the relative attractiveness of alternative strategies does not change with W_0 since $Z(P)$ is independent of W_0.

3.1 Inherent Analysis and Prospect Theory

Prospect theory, developed by Kahneman and Twersky (1979), is a theory of choice under uncertainty in which value is assigned to gains and losses rather than to final assets. A typical form of the objective functions discussed in the literature of prospect theory can be given as follows (see Benartzi and Thaler, 1995).

$$V(W|\underline{W}) = \begin{cases} (W - \underline{W})^a & W \geq \underline{W} \\ -\gamma(\underline{W} - W)^b & W < \underline{W} \end{cases} \tag{61}$$

where $a > 0$, $b > 0$, and $\gamma \geq 1$. Experiments suggest that investors are "loss averse" (for more discussion see, e.g., Benartzi and Thaler [1995]), which translates into the condition $\gamma \geq 1$.

The foundation of prospect theory is based more on observed behavior, however. There has been so far no normative justification for the theory. Interestingly, our development of inherent reward and risk up to Theorem 5 can be seen as offering a normative foundation for prospect theory.

The forms of the gain and loss functions derived in Theorem 5 coincide with the upper part and lower part of the objective function of a prospect investor (with a restriction that the neutral return be set at zero). Without the subsequent properness axiom (Axiom 8), I could have stopped there and defined the more general (but less applicable) forms of inherent reward and risk measures. To see this, let $\overline{U}(P)$ and $\overline{D}(P)$ denote, respectively, the *general inherent reward* and *general inherent risk* measures that can be derived without the Properness Axiom. They are defined as

$$\begin{aligned}\overline{U}(P) &= E[\max(r - r_0, 0)^a; P]\\ \overline{D}(P) &= E[\max(r_0 - r, 0)^b; P].\end{aligned}$$

If an investor maximizes the expected objective function as given in (61), then a necessary condition is that the *general inherent reward-to-risk ratio*, defined as $\overline{Z} = \overline{U}/\overline{D}$, be maximized. The reason is that maximization of the expected value of (61) implies that for any level of $\overline{D}$ the inherent reward $\overline{U}$ be maximized (or that for any level of $\overline{U}$ the inherent risk $\overline{D}$ be minimized) under any given feasibility condition.

I would rather maintain, however, that the more special U and D measures developed in this paper are more suitable for characterizing the inherent reward and risk of investments. They are not only simpler in computations, but are also consistent with Axiom 8 – an axiom that can avoid the extreme judgements about events with infinitesimally small chances or consequences that only marginally affect one's *current* wealth level.

3.2 Inherent Analysis and the Allais Paradox

Let us look at the following example that is adapted from Machina (1982).

Example 1 *(Allais Paradox).*

Game 1: Choose either of the following two strategies:

$$S_1: \quad \$1,000,000$$

$$S_2: \quad (0.1)\$5,000,000 \oplus (0.89)\$1,000,000 \oplus (0.01)\$0$$

Game 2: Choose either of the following two strategies:

$$S_3: \quad (0.1)\$5,000,000 \oplus (0.9)\$0$$

$$S_4: \quad (0.11)\$1,000,000 \oplus (0.89)\$0$$

If the investor is an expected utility maximizer, his choice should be either S_1 in Game 1 and S_4 in Game 2, or S_2 in Game 1 and S_3 in Game 2. It has been found, however, that the majority of the individuals asked to rank such strategies prefer S_1 to S_2 in Game 1, and prefer S_3 to S_4 in Game 2 (see more discussions in, e.g., Allais, 1953, 1979; Morrison, 1967; Raiffa, 1968; and DeGroot, 1970 [pp. 92-94]).

The inherent dominance criterion may help explain this paradox. In order to compute the inherent reward and risk of the strategies presented in these two games, a reasonable neutral benchmark V_0 need be specified. Since these games are presented to individuals without assuming any cost and there exists a choice (S_1) that gives a sure return of \$1 million, it is plausible to assume that $V_0 \in (0,1)$ (in million dollars). For one thing, it is difficult to imagine that one can be indifferent or even consider it a loss when receiving a million-dollar gift; thus $V_0 < 1$. Likewise, it is also difficult to imagine that one can be indifferent or even consider it a gain when coming out of such games empty handed; thus $V_0 > 0$. Consequently (in million dollars),

$$U_1 = 1 - V_0, \qquad D_1 = 0, \qquad Z_1 = \infty$$

$$U_2 = 0.1(5 - V_0) + 0.89(1 - V_0), \quad D_2 = 0.01V_0, \quad Z_2 < \infty$$

$$U_3 = 0.1(5 - V_0), \quad D_3 = 0.9V_0, \quad Z_3 = \frac{1}{9}(\frac{5}{V_0} - 1)$$
$$U_4 = 0.11(1 - V_0), \quad D_4 = 0.89V_0 \quad Z_4 = \frac{11}{89}(1 - V_0).$$

It is easy to see that for all $V_0 \in (0, 1)$, $Z_1 > Z_2$ and $Z_3 > Z_4$. In other words, the inherent dominance criterion resolves this Allais paradox in this example. Alternatively, the observation of individual choices in this kind of experiment supports the assumption that people prefer a higher inherent reward-to-risk ratio.

3.3 Inherent Risk vs. Standard Deviation and Put Option Price

In this subsection, I compare the inherent risk measure with standard deviation (e.g., Sharpe, 1966) or put option price as risk measures (e.g., Bodie, 1995; Zou, 1997). Suppose there is an asset such that investing 1 dollar today will yield $1 + u$ dollars with probability p or $1 + d$ dollars with probability $1 - p$ tomorrow, where $u > 0 > d$. For simplicity assume that investors can borrow and lend at a zero risk-free interest rate, i.e., $r_0 = 0$. I look at the risk measures first, and then at the measures involving both reward and risk.

The expected return (or risk premium) on this security is

$$E(r) = pu + (1 - p)d.$$

The standard deviation, put option price, and the inherent risk of this asset are given as follows, where in the price of the put option the parameter x denotes the strike price of the option minus 1.

The standard deviation: $\sigma = (u - d)\sqrt{p(1 - p)}$.

The price of put option: $P(x) = \frac{u}{u-d}(x - d)$.

The inherent risk: $D = -(1 - p)d$.

Example 2 *(Superiority of the inherent dominance criterion).*

As a numerical example, consider 3 assets all having 50% chance of "up" or "down". They differ only in the magnitude of the "ups". Asset 1 will yield 60 percent, Asset 2 will yield 40 percent, and Asset 3 only 20 percent. All the assets lose 20 percent when the state is "down" (see Figure 1).

Clearly, both the standard deviation and the put option price give us distorted measures of risk (see Table 1). By these measures, Asset 1 is the most risky and Asset 3 the least risky. All the three assets however, have the same probability of losing 20%, which is only correctly reflected in the inherent-risk measure D.

The shortcoming of volatility and option prices as risk measures can be further demonstrated when we change the magnitude of downside losses as well (see Figure 2). Now, suppose that Asset 1 remains the same as in Figure 1. Instead of losing 20% when the state is down, Asset 2 will lose 22% and Asset 3 will lose 24%, with the same probability of 0.5.

In Figure 2, it is clear that the inherent risk of Asset 3 is the largest, followed by Asset 2, and Asset 1 has the least risk among the three alternatives. There is no doubt that such ranking of risks should be agreed upon by all the rational players who prefer more to less. Table 2, however, shows the irrational rankings that would be arrived at if one choose the standard deviation or the put option price to measure risk.

Of course, looking at risk alone is not sufficient for ranking investment opportunities. I now incorporate the reward in my analysis. The following can be easily verified.

The call option price: $C(x) = \frac{-d}{u-d}(u-x)$.

The inherent reward $U = pu$.

The Mean-Variance Reward-to-Risk Ratio (Sharpe ratio): $S_p = \frac{E(r)-r_0}{\sigma} = \frac{pu+(1-p)d}{(u-d)\sqrt{p(1-p)}}$.

The Inherent Reward-to-Risk Ratio: $Z = \frac{U}{D} = -\frac{pu}{(1-p)d}$.

Example 3 *(Inherent dominance measure and Sharpe ratio)*

Consider the following two investment alternatives:

$$\begin{aligned} A &: (0.99)\$3 \oplus (0.01)(-\$1) \\ B &: (0.98)\$100,000,000 \oplus (0.02)\$0 \end{aligned}$$

Which one should an investor prefer? Obviously B. However,

$$\begin{aligned} Z_A &= 0.99(3)/(0.01) = 297.0 < Z_B = \infty \\ S_A &= 2.96/0.397 = 7.46 > S_B = 98/14.0 = 7 \end{aligned}$$

Thus, according to the Sharpe ratio one should choose A instead of B. This example highlights the problem with variance or standard deviation as a risk measure. Despite the fact that Asset B has no downside risk and much upside potential, its standard deviation grows as fast as its mean. In fact one can add as many zeros as one wishes on the upside potential in B and the Sharpe ratio of B will remain at 7. The Z ratio, on the other hand, shows the attractiveness of B correctly.

Example 4 *(Inherent dominance measure and skewness)*

Comparing further the Z measure and the Sharpe measure S, I have the following observation.

(1) For any two assets A and B whose returns follow binomial processes having a zero skewness, the inherent dominance measure $Z_A \geq Z_B$ if and only if the Sharpe measure $S_A \geq S_B$;

(2) For any two assets A and B whose returns follow binomial processes having the same mean and standard deviation but different non-negative skewness, the inherent dominance measure

$Z_A > Z_B$ if and only if asset A has a strictly higher skewness than asset B. In other words, the mean-variance preserving skewness is favorably ranked by measure Z, even though the Sharpe measure has no power to detect such differences in skewness.

To show that these statements hold, let μ denote the asset's mean return. The skewness of an asset's return is defined as

$$\begin{aligned} s &= \frac{p(u-\mu)^3 + (1-p)(d-\mu)^3}{\sigma^2} \\ &= (1-2p)(u-d) \end{aligned}$$

Clearly, s is a monotonically decreasing function of p, $s > 0$ if and only if $p < 0.5$ and $s = 0$ if and only if $p = 0.5$. Thus, substituting $p = 0.5$ into the S measure and the Z measure gives

$$S = \frac{u+d}{u-d} \text{ and } Z = -\frac{u}{d}$$

It follows that

$$S = 1 - \frac{2}{1+Z}$$

or that S and Z are positively related. This completes the proof of assertion (1).

Now fix μ and σ and regard u and d as functions of p that are determined from the following equations:

$$\begin{aligned} \mu &= pu + (1-p)d \\ \sigma &= (u-d)\sqrt{p(1-p)} \end{aligned}$$

Differentiation yields

$$\frac{\partial u}{\partial p} = \frac{d-u}{2p} \text{ and } \frac{\partial d}{\partial p} = \frac{d-u}{2(1-p)}$$

It follows that

$$\frac{\partial Z}{\partial p} = -\frac{(u+d)\mu}{2(1-p)^2 d^2} < 0.$$

Since s is a monotonically decreasing function of p, Z must be monotonically increasing in s.

3.4 Inherent Dominance and Stochastic Dominance

Let two cumulative distributions G and F be given on $\mathbb{R}$, representing two alternative investment strategies, respectively. It is well known that G dominates F in the sense of first-order stochastic dominance if and only if for all $x \in \mathbb{R}$,

$$G(x) \leq F(x), \tag{62}$$

with strict inequality for some x. It follows that

$$U_G - U_F = \int_r^{\infty} (t-r)d(G(t) - F(t)) = -\int_r^{\infty} [G(t) - F(t)]dt \geq 0, \tag{63}$$

$$D_G - D_F = \int_{-\infty}^{r} (r-t)d(G(t) - F(t)) = \int_{-\infty}^{r} [G(t) - F(t)]dt \leq 0, \tag{64}$$

with at least one strict inequality. Consequently, $Z_G > Z_F$. That is, among all strategies that can be ranked by the first-order stochastic dominance, the measure Z ranks them consistently with a higher Z indicating better dominance.

This observation can be further extended to the second-order stochastic dominance. G dominates F in the sense of second-order stochastic dominance if and only if for all $x \in \mathbb{R}$,

$$\int_{-\infty}^{x} G(t)dt \leq \int_{-\infty}^{x} F(t)dt \tag{65}$$

with strict inequality for some x. It follows that $D_G - D_F \leq 0$ [see (64)]. Taking the limit as x goes to infinity in (65) yields $\int_{-\infty}^{\infty} G(t)dt \leq \int_{-\infty}^{\infty} F(t)dt$. As a result,

$$U_G - U_F - (D_G - D_F) = \int_{-\infty}^{\infty} F(t)dt - \int_{-\infty}^{\infty} G(t)dt \geq 0. \tag{66}$$

Thus,

$$\begin{aligned} Z_G - Z_F &= \frac{1}{D_G D_F}(U_G D_F - U_F D_G) & (67) \\ &= \frac{1}{D_G D_F}[U_G(D_F - D_G) + D_G(U_G - U_F)] & (68) \\ &\geq \frac{1}{D_G D_F}(U_G - D_G)(D_F - D_G) \geq 0 & (69) \end{aligned}$$

where in (67) at least one inequality holds strictly.

In other words, among all investment strategies that can be ranked by the second-order stochastic dominance, the measure Z ranks them consistently with a higher Z indicating a better strategy. The mean-variance criterion, on the other hand, does not satisfy this property as clearly shown in the previous examples. These examples confirm that the inherent dominance measure Z is a criterion superior to both that of stochastic dominance and mean-variance. Indeed, variance (or standard deviation) measures only how realized returns on an asset fluctuate around their *own* mean, not a neutral-return benchmark.[16] And the stochastic dominance criteria are difficult to parameterize and incapable of ranking many interesting investment alternatives.

3.5 Performance evaluation

The inherent dominance measure Z can also be easily applied to measuring the performance of investment strategies. If a fund manager claims that he has better absolute performance than

[16]In other words, variance or standard deviation can be misleading as a risk measure *per se*, as well as for ranking investment alternatives whose return distributions are not symmetric. The mean-variance analysis may be suitable only for very short trading intervals (see Samuelson, 1970; and Samuelson and Merton, 1975).

another manager, for example, he must be able to show a higher Z. This criterion arises since if he realized a lower Z on his fund than the other fund, at least some investors will prefer the other fund from the stochastic dominance property of the Z measure. Moreover, since the Z measure is defined without any restriction on the return distributions, it can be used to rank performance of investment decisions involving superior information or dynamic strategies (cf. criticism of the use of SML by Dybvig and Ross, 1975a, 1975b), and of strategies involving the use of derivative securities. Thus, the inherent dominance theory offers a new approach for comparing portfolio performance. Long-term performance can now be evaluated against the universal benchmark of risk-free returns, rather than against any subjectively chosen benchmark. I discuss further performance evaluation issues in Zou (2000).

4 COMPUTATION OF INHERENT REWARD AND RISK

The advantage of the mean-variance analysis is its simplicity. The inherent analysis, admittedly, may involve complex computations since the inherent measures depend on the entire distribution and not just on the first two moments of a distribution. Fortunately, thanks to the development of option pricing models, much of the computational work can be easily built on the existing models. In this section, I restrict attention to the binomial models and normal distributions of returns only. For sake of simplicity, I neglect dividends or storage costs of assets.

4.1 Binomial process of returns

Consider a buy-and-hold strategy which lasts for n periods. Let R_0 denote the gross risk-free rate of return (1 plus the interest rate) per period, and assume that the return on this investment follows a binomial process such that the gross return in each period is either u with probability p or d with probability $1-p$. Let P denote the return distribution of this strategy at the end of n periods. Then,

$$
\begin{aligned}
U(P) &= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \max(0, u^j d^{n-j} - R_0^n), \\
D(P) &= \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \max(0, R_0^n - u^j d^{n-j}).
\end{aligned}
$$

These measures are derived simply by substituting the true probability p for the risk-neutral probability in the binomial option pricing models, and then multiplying by R_0^n to yield the end-of-period expected payoffs of a call option for U and a put option for D. The strike price of these options is R_0^n.

Since the binomial option pricing techniques have been extensively explored, and any distribution can be approximated by a binomial process (the inter-temporal returns not necessarily stationary nor identical, see, e.g., Rubinstein, 1994), computation programs are available that greatly facilitate the computation of inherent reward and risk of investment strategies.

4.2 Normal distribution of returns

Now let us turn to the case where investment returns are normally distributed. As to be expected, the Black-Scholes option pricing formula helps us easily derive closed-form expressions for the inherent reward and risk measures.

Assume that the investment returns are normally distributed with mean μT and standard deviation $\sigma\sqrt{T}$ over the holding period $[0, T]$, where μ and σ are the annualized and continuously compounded expected return and standard deviation, and T is the length of the holding period measured in years. Let r_0 denote the continuously compounded annual risk-free interest rate, and let P denote the end-of-horizon distribution of the investment returns. It can be shown that

$$U(P) = e^{\mu T}N(d_1) - e^{r_0 T}N(d_2) \tag{70}$$

$$D(P) = e^{r_0 T}N(-d_2) - e^{\mu T}N(-d_1) \tag{71}$$

where

$$d_1 = (\frac{\mu - r_0}{\sigma} + \frac{\sigma}{2})\sqrt{T}; \quad d_2 = (\frac{\mu - r_0}{\sigma} - \frac{\sigma}{2})\sqrt{T}.$$

Because the derivation sheds light on the difference between the inherent measures and the option prices, I sketch a proof for the expression of $D(P)$ as follows.[17]

A European put option on an asset with an exercise price equal to X can be described as the present value of the option in a fictitious risk-neutral economy. Let $V(0)$ and $V(T)$ denote the underlying asset's current price and price at T, respectively, and suppose that the risk-free rate were μ instead of r_0. A *fictitious* value for this *imagined* put option $P_{r0=\mu}$ can then be obtained.

$$P_{r0=\mu} = e^{-\mu T}E(max[0, X - V_T]) = e^{-\mu T}XN(-d_2) - V_0 N(-d_1) \tag{72}$$

where

$$d_1 = \frac{ln(\frac{V_0}{X}) + (\mu + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \text{ and } d_2 = d_1 - \sigma\sqrt{T} \tag{73}$$

Substituting $V_0 e^{r_0 T}$ for X in (72)-(73), and noting that $D(P) = e^{\mu T}P_{r0=\mu}/V_0$, yields the expression in (71). The expression in (70) for the inherent reward, $U(P)$, can be derived in the same way.

[17]Note that in the derivation, there is no need to assume that the actual risk-free rate $r_0 = \mu$; I only use the option pricing formula and take a short cut to the derivation of the expression of $D(P)$.

5 INHERENT ANALYSIS OF PORTFOLIOS AND INHERENT EFFICIENCY

I now extend the previous development of inherent measures to a portfolio context. I investigate the issues of diversification, introduce the concept of inherent efficiency of an investment strategy or portfolio, and examine the implications regarding equilibrium asset prices if there exist inherently efficient portfolios.

Definition 6 (Inherent Efficiency) *A distribution $P \in \mathcal{P}$ is inherently efficient if and only if $P \succeq Q$ for all distributions $Q \in \mathcal{P}$; that is, if and only if $Z(P) \geq Z(Q)$ for all $Q \in \mathcal{P}$.*

It is useful to also define inherent efficiency on some constrained distribution class of $\mathcal{P}$. For instance, not all distributions in $\mathcal{P}$ may be attained with feasible strategies $S \in \Omega$. Let $\mathcal{P}_0 \subseteq \mathcal{P}$ be the class of distributions in $\mathcal{P}$ that are attainable in the sense that

$$\mathcal{P}_0 = \{P_S \in \mathcal{P} | \exists S \in \Omega, S : 1 \rightarrow P_S \in \mathcal{P}\}$$

I assume here that if $P \in \mathcal{P}_0$ and $Q \in \mathcal{P}_0$, then for all $\phi \in [0,1]$, $\phi P + (1-\phi)Q \in \mathcal{P}_0$.[18] It is clear that the definition of feasible strategies defines a correspondence from the strategy space Ω onto the attainable return distribution space $\mathcal{P}_0$. For convenience, thus, I shall occasionally directly call a return distribution in $P \in \mathcal{P}_0$ a strategy.

Definition 7 (Constrained Inherent Efficiency) *A strategy $S \in \Omega$ with return distribution $P_S \in \mathcal{P}_0$ is inherently efficient within $\mathcal{P}_0$ if and only if $P_S \succeq Q$ for all distributions $Q \in \mathcal{P}_0$; in other words, if and only if $Z(P_S) \geq Z(Q)$ for all $Q \in \mathcal{P}_0$.*

[18]This assumption can be made stronger by allowing for leverage and short selling, of course, in which case $\phi \in \Re$.

For notational convenience, I consider here strategies whose returns are continuously distributed. Analysis of discrete distributions is similar and I leave the details to the reader. Let two arbitrary strategies A and B be given, let $x = r_A - r_0$ and $y = r_B - r_0$ denote the random excess returns over the next period of A and B, respectively, and let $f(x, y)$ denote the joint density function of x and y. The marginal density functions will be denoted by $f_A(x)$ and $f_B(y)$, respectively. I also allow limited liability to prevent excess returns from falling below -100%, thus $x \in [-1, \infty)$ and $y \in [-1, \infty)$.[19]

Consider portfolio $P = \phi A + (1 - \phi)B$ with $0 < \phi < 1$. That is, the return on portfolio P will be $\phi x + (1 - \phi)y$ if the realized joint returns of A and B are x and y. Letting $c = \phi/(1 - \phi)$, it can be verified that

$$
\begin{aligned}
D(P) &= -\int_{\phi x+(1-\phi)y\leq 0} [\phi x + (1-\phi)y] f(x,y)dxdy \qquad (74)\\
&= -\phi \int_{-1}^{\frac{1}{c}} x[\int_{-1}^{-cx} f(y|x)dy] f_A(x)dx - (1-\phi)\int_{-1}^{c} y[\int_{-1}^{-\frac{y}{c}} f(x|y)dx] f_B(y)dy \qquad (75)\\
&= -\phi \int_{-1}^{\frac{1}{c}} xF_B(-cx|x) f_A(x)dx - (1-\phi)\int_{-1}^{c} yF_A(-\frac{y}{c}|y) f_B(y)dy, \qquad (76)
\end{aligned}
$$

where $F_A(\cdot|y)$ and $F_B(\cdot|x)$ are the conditional cumulative distributions (of the excess returns) for the two strategies. First note that if x and y are perfectly positively correlated, then $y = a + bx$ with $b > 0$. Arbitrage will ensure that $a = 0$, thus $y = bx$. In this case, $F_A(-\frac{y}{c}|y) = 1$ if $y < 0$ and $= 0$ if $y > 0$. Similarly, $F_B(-cx|x) = 1$ if $x < 0$ and $= 0$ if $x > 0$. As a result,

$$D(P) = -\phi \int_{-1}^{0} xf_A(x)dx - (1-\phi)\int_{-1}^{0} yf_B(y)dy \qquad (77)$$

[19] I am thus using forward prices (current price multiplied by $1 + r_0$) to determine limited liability. Otherwise I can require the returns and not excess returns to be greater or equal to -1. This is not essential, however, since the subsequent analysis can be directly extended to any lower bound of the return variables.

$$= \phi D(A) + (1-\phi)D(B) \tag{78}$$

Lemma 10 (Sub-additivity) *Let portfolio $P = \phi A + (1-\phi)B$ with $0 < \phi < 1$ be given. Then $D(P) < \phi D(A) + (1-\phi)D(B)$ if and only if x and y are not perfectly positively correlated. That is, inherent risk can be reduced through diversification among imperfectly or negatively correlated strategies.*

Proof: Suppose that x and y are imperfectly correlated. Note that $F_B(-cx|x)$ and $F_A(-\frac{y}{c}|y)$ are bounded and monotonic functions of y and x respectively, and $\lim_{x\to-1} F_B(-cx|x) = F_B(c|-1^+)$, $\lim_{y\to-1} F_A(-\frac{y}{c}|y) = F_A(\frac{1}{c}|-1^+)$ exist and $\lim_{x\to\infty} F_B(-cx|x) = \lim_{y\to\infty} F_A(-\frac{y}{c}|y) = 0$. It follows from the Second Mean Value Theorem that there exist $\xi \in (-1, \frac{1}{c})$ and $\eta \in (-1, c)$ such that

$$\begin{aligned}
D(P) &= -\phi F_B(c|-1^+)\int_{-1}^{\xi} x f_A(x)dx - (1-\phi)F_A(\frac{1}{c}|-1^+)\int_{-1}^{\eta} y f_B(y)dy && (79)\\
&\leq -\phi F_B(c|-1^+)\int_{-1}^{0} x f_A(x)dx - (1-\phi)F_A(\frac{1}{c}|-1^+)\int_{-1}^{0} y f_B(y)dy && (80)\\
&\leq -\phi \int_{-1}^{0} x f_A(x)dx - (1-\phi)\int_{-1}^{0} y f_B(y)dy && (81)\\
&= \phi D(A) + (1-\phi)D(B) && (82)
\end{aligned}$$

where at least one of the inequalities must strictly hold. □

Theorem 10 *Let strategy $B \in \mathcal{P}_0$ be given and let r_B denote the random return under strategy B. If B is inherently efficient on $\mathcal{P}_0$, then for all feasible strategies $A \in \mathcal{P}_0$ with (random) return denoted r_A,*

$$E(r_A - r_0|r_B \geq r_0)E(r_B - r_0|r_B \leq r_0) \geq E(r_B - r_0|r_B \geq r_0)E(r_A - r_0|r_B \leq r_0) \tag{83}$$

where equality holds if short selling A is feasible.

Proof: If B is inherently efficient, then by definition it must have the highest inherent dominance measure. That is, for any strategy $A \in \mathcal{P}_0$, and any portfolio $P = \phi A + (1-\phi)B$,

$$Z(B) \geq \max Z(P) \quad \forall \phi \in \mathbb{R}$$

Again, let $x = r_A - r_0$ and $y = r_B - r_0$ denote the excess returns of A and B. Let $F(x, y)$ denote the joint cumulative distribution of (x, y). The inherent reward and risk of this portfolio are given by

$$\begin{aligned} U(P) &= \int_{\phi x + (1-\phi)y \geq 0} [\phi x + (1-\phi)y] dF(x, y), && (84) \\ D(P) &= -\int_{\phi x + (1-\phi)y \leq 0} [\phi x + (1-\phi)y] dF(x, y). && (85) \end{aligned}$$

It can be verified that (after dropping those terms that converge to zero as $\phi \to 0$)

$$\begin{aligned} \frac{\partial U(P)}{\partial \phi}|_{\phi=0} &= \int_{y \geq 0} (x - y) dF(x, y), && (86) \\ \frac{\partial D(P)}{\partial \phi}|_{\phi=0} &= \int_{y \leq 0} (y - x) dF(x, y). && (87) \end{aligned}$$

By definition,

$$\begin{aligned} \int_{y \geq 0} x dF(x, y) &= E(x|y \geq 0)[1 - F_B(0)], \\ \int_{y \leq 0} x dF(x, y) &= E(x|y \leq 0) F_B(0), \\ \int_{y \geq 0} y dF(x, y) &= E(y|y \geq 0)[1 - F_B(0)] = U(B), \\ \int_{y \leq 0} y dF(x, y) &= E(y|y \leq 0) F_B(0) = -D(B). \end{aligned}$$

Thus, B being inherently efficient implies that

$$\begin{aligned} & \frac{\partial U(P)}{\partial \phi}|_{\phi=0} D(B) - \frac{\partial D(P)}{\partial \phi}|_{\phi=0} U(B) \\ = \;& [E(x|y \geq 0)(1 - F_B(0)) - U(B)] D(B) + [E(x|y \geq 0) F_B(0) + D(B)] U(B) \\ = \;& [E(x|y \geq 0)(1 - F_B(0))] D(B) - [E(y|y \geq 0) F_B(0)] U(B) \leq 0 && (88) \end{aligned}$$

where equality holds if $\phi < 0$ is allowed, i.e., if short selling asset A is feasible. Since $U(B) = E(y|y \geq 0)[1 - F_B(0)]$ and $D(B) = -E(y|y \leq 0)F_B(0)$, it follows that (83) must hold. □

An immediate corollary of Theorem 10 is a capital asset pricing model that depicts a simple relationship between the inherent reward and inherent risk of assets and an inherently efficient portfolio.

Corollary 1 *If there exists an inherently efficient portfolio (or strategy) $B \in \mathcal{P}_0$, and short selling of assets is feasible for assets in Ω, then for all strategies $A \in \mathcal{P}_0$,*

$$U(A) = D(A) + \beta_z[U(B) - D(B)] \tag{89}$$

where

$$\beta_z = \frac{E[(r_0 - r_A)|r_B \leq r_0]}{E[(r_0 - r_B)|r_B \leq r_0]}$$

This is an interesting relationship between all strategies in Ω and one particular inherently efficient strategy in Ω in terms of their inherent reward and risk.

Substituting M (the market portfolio) for B in (89) yields the following capital asset pricing relationship that first appeared in Bawa and Lindenberg (1977).

Corollary 2 *If there exists a market portfolio M which is inherently efficient, then for any asset A,*

$$E(r_A) - r_0 = \beta[E(r_M) - r_0] \tag{90}$$

where

$$\beta = \frac{E[(r_0 - r_A)|r_M \leq r_0]F_M(r_0)}{D_M}$$

This corollary follows directly from the fact that $E(r|P) - r_0 = U(P) - D(P)$. I shall explore the many implications of (83) further in Zou (2000), and examine how it works in a complete capital market.

6 CONCLUSION

This paper presents a new paradigm (*inherent reward and risk paradigm*) for investment analysis under uncertainty. Assuming that there are inherent investment reward and risk that can be analyzed objectively, I take a normative approach to investigate how such reward and risk should be perceived and measured – for any well defined investment strategy. By formulating a system of axioms for *common (objective) judgement*, I derive the existence of a unique (ratio scale) pair of inherent reward (U) and inherent risk (D) measures. These measures are simply the expected (absolute) values of the investment's positive excess returns (gains) and negative excess returns (losses). I then define the inherent reward-to-risk ratio, $Z = U/D$, that can be used to consistently rank all well defined investment strategies in terms of inherent dominance. This inherent dominance criterion is shown to be consistent with the first-order stochastic dominance criterion and the no-arbitrage principle. As a result, I establish a normative foundation for the analysis of investment reward and risk, and for the more general purpose of making investment decisions under uncertainty.

A by-product is Theorem 4, which can be regarded as a generalization of von Neumann-Morgenstern expected utility theory that allows the functional forms of utility of gains and utility of losses to vary as the investor's benchmark and/or investment horizon changes. It thus contributes a normative foundation to prospect theory, in which similar investment objectives are frequently assumed.

The inherent reward and risk paradigm is shown to produce fruitful results. Apart from being able to resolve some problems in the expected utility paradigm, it also enjoys computational simplicity thanks to the development of option pricing models. Most of the known results in the mean-variance paradigm find a parallel expression here, such as inherent risk diversification, the separation theorem, inherent efficiency (v.s. mean-variance efficiency), the capital asset pricing model, and so on. All these results are generally improved in the inherent reward and risk paradigm, though. They now hold for all feasible distributions of returns (with finite means) and no longer depend on the specific assumptions concerning the return distributions (e.g., normal). The difficult issue of performance evaluation becomes a simple matter of comparing the inherent reward-to-risk ratio, the Z measure (or its variations, see Zou (2000)).

Many issues remain to be investigated. I list here just a few thoughts to conclude the paper. What are further implications of the capital asset pricing model derived under the inherent efficiency hypothesis? How to test the model and what empirical results do we expect?[20] Can we extend the inherent theory to situations where multiple sources of risk are involved? What are the intertemporal properties of an asset's or dynamic strategy's inherent reward and risk? Are the inherent analysis and results robust to heterogeneous beliefs and asymmetric information? What will be the optimal investment decisions in an inherently efficient market? And so on ...

7 REFERENCES

Allais, M. "Le Comportement de l'Homme Rationel devant le Risque, Critique des Postulates et Axiomes de l'Ecole Americaine." *Econometrica*, 21, 1953, pp. 503-546.

Allais, M. " The So-Called Allais Paradox and Rational Decisions Under Uncertainty." Part V of *Expected Utility Hypothesis and the Allais Paradox*, ed. by M. Allais and O. Hagen. Dordrecht, Holland: D Reidel Publishing Company, 1979.

Allais, M. "The Foundations of a Positive Theory of Choice Involving Risk and a Criticism of the Postulates and Axioms of the American School." Part II of *Expected Utility Hypothesis and*

[20]See, e.g., Harlow and Rao (1989) and the references therein for empirical investigations of the CAPM model derived with risk being defined as a second-order lower-partial moment.

the Allais Paradox., ed. by M. Allais and O. Hagan. Dordrecht, Holland: D Reidel Publishing Co. 1979. (Translated from *Econometrie*, Colloques Internationaux du Centre national de la Recherche Scientifique. Vol. XL. Paris, 1953, pp. 257-332.

Arrow, K. J. “Essays in the Theory of Risk Bearing.” *Amsterdam: North Holland*, 1974.

Artzner, Philippe, Freddy Delbaen, and Jean-Marc Eber, “Coherent Measure of Risk.” *Journal of Mathematical Finance*, vol. 9, no. 3, July 1999, 203-228.

Bawa, Vijay S. “Optimal Rules for Ordering Uncertain prospects.” *Journal of Financial Economics*, 2, 1975, pp. 95-121.

Bawa, Vijay S. “Admissible Portfolios For All Individuals.” *The Journal of Finance*, vol. XXXI, no. 4, September 1976, pp. 1169-1183.

Bawa, Vijay S. and Lindenberg, Eric B. “Capital market Equilibrium in A Mean-Lower Partial moment Framework.” *Journal of Financial Economics*, 5, 1977, pp. 189-200.

Bawa, Vijay S. “Safety-First, Stochastic Dominance, and Optimal Portfolio Choice.” *Journal of Financial and Quantitative Analysis*, June 1978, pp. 255-269.

Benartzi, Shlomo, and Richard H. Thaler. “Myopic Loss Aversion and the Equity Premium Puzzle.” *Quarterly Journal of Economics*, February, 1997, pp. 73-92.

Bernardo, Antonio E., and Olivier Ledoit, “Gain, Loss, and Asset Pricing.” *Journal of Political Economy*, vol. 108, no. 1, 2000, pp. 144-172.

Black, Fischer, and Myron Scholes. “The Pricing of Options and Corporate Liabilities.” *Journal of Political Economy*, vol. 81, no. 3, May/June 1973, pp. 637-54.

Bodie, Zvi. "On the Risk of Stocks in the Long Run." *Financial Analysts Journal*, vol.51, no. 3, May/June 1995, pp. 18-22.

Cass, David and Stiglitz, Joseph E. "The Structure of Investor preferences and Asset Returns, and Separability in Portfolio Allocation: A Contribution to the Pure Theory of Mutual Funds." *Journal of Economic Theory*, 2, 1970, pp. 122-160.

Debreu, G. "Theory of Value: An Axiomatic Analysis of General Equilibrium." *New Haven: Yale University Press*, 1959.

DeGroot, Morris H., *Optimal Statistical Decisions*, McGraw-Hill, Inc. New York, 1970.

Diamond, P., and J. E. Stiglitz. "Increase in Risk and in Risk Aversion." *Journal of Economic Theory*, 8, 1974, pp. 337-360.

Domar, E.V., and R.A. Musgrave. "Proportional Income Taxation and Risk-Taking." *Quarterly Journal of Economics*, vol. 58, May, 1944, pp. 389-422.

Dybvig, Philip H. "Distributional Analysis of Portfolio Choice." *Journal of Business*, vol. 61, no. 3, 1988, pp. 369-393.

Dybvig, Philip H. "Inefficient Dynamic Portfolio Strategies or How to Throw Away a Million Dollars in the Stock Market." *Review of Financial Studies*, Vol. 1, no. 1, (1988), pp. 67-88.

Dybvig, Philip H. and Jr. Ingersoll, Jonathan E. "Mean-Variance Theory in Complete Markets." *Journal of Business*, vol. 55, no. 2, 1982, pp. 233-251.

Dybvig, Philip H. and Ross, Stephen A. "Differential Information and Performance Measurement Using a Security Market Line." *The Journal of Finance*, vol. XL, no. 2, June 1985a, pp. 383-399.

Dybvig, Philip H. and Ross, Stephen. "The Analytics of Performance Measurement Using a Security market Line." *The Journal of Finance*, vol. XL, no. 2, June 1985b, pp. 401-416.

Ellsberg, D. "Risk, Ambiguity and the Savage Axioms,", *Quarterly Journal of Economics*, 75, pp. 643-669.

Elton, Edwin J. "Expected Return, Realized Return, and Asset Pricing Tests." *The Journal of Finance*, vol. LIV, no. 4, August 1999, pp. 1199-1220.

Fishburn, Peter C. "Preference-Based Definition of Subjective Probability," *Annals of mathematical Statistics*, 38, 1967, pp. 1605-1617.

Fishburn, Peter C. "Mean-Risk Analysis with Risk Associated with Below-Target Returns" *The American Economic Review*, vol. 67, no. 2, March 1977, pp. 116-126.

Fishburn, Peter C. "On the Theory of Ambiguity", *International Journal of Information and Management Science*, 2, 1991, pp. 1-16.

Friedman, M., and L. J. Savage. "The Utility Analysis of Choices Involving Risk." *Journal of Political Economy*, 56, 1948, pp. 279-304.

Hadar, J., and W. Russell. "Rules for Ordering Uncertain Prospects." *American Economic Review*, 59, 1969, pp. 25-34.

Hagen, O. "Towards a Positive Theory of Preference Under Risk." *Expected Utility Hypothesis and the Allais Paradox* ed. By M. Allais and O. hagen. Dordrecht, Holland: D. Reidel Publishing Company, 1979, pp. 271-302.

Harlow, W.V., and Ramesh K.S. Rao. "Asset Pricing in a Generalized Mean-lower Partial Moment Framework: Theory and Evidence." *Journal of Financial and Quantitative Analysis*, (September) 1989, pp. 285-311.

Harlow, W.V. "Asset Allocation in a Downside-risk Framework." *Financial Analysts Journal*, vol. 47, no. 5, September/October 1991, pp. 28-40.

Herstein, I. N., and J. Milnor, "An Axiomatic Approach to Measurable Utility." *Econometrica*, 21, 1953, pp. 291-297.

Hewitt, Edwin, and Karl Stromberg. *Real and Abstract Analysis*, Springer Verlag, 1955.

Hicks, J.R., *Value and Capital.* Oxford University Press, Fair Lawn, N.J., 1939.

Kahneman, Daniel, and Amos Twersky. "Prospect Theory: An Analysis of Decision under Risk." *Econometrica*, vol. 47, no.2, March 1979, pp. 263-291.

Kimball, Miles S. "Standard Risk Aversion." *Econometrica*, vol.61, no.3, May 1993, pp. 589-611.

Knight, F.H., *Risk, Uncertainty and Profit.* New York: Houghton Mifflin 1921 (*Reprints of Economic Classics*, New York, 1964).

Kreps, D.M., "A Representation Theorem for 'Preference for Flexibility'." *Econometrica*, 47, 1979, pp. 565-577.

Levy, Haim, and Yoram Kroll. "Stochastic Dominance with Riskless Assets." *Management Science*, December 1976, pp. 743-777.

Lintner, J. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets." *Review of Economics and Statistics*, 47, Feb. 1965a, 13-37.

Lintner, J. "Security Prices, Risk, and Maximal Gains from Diversification." *Journal of Finance*, 20, Dec. 1965b, pp. 587-615.

MacCrimmon, K. R., and S. Larsson. "Utility Theory: Axioms versus 'Paradoxes'." *Expected Utility Hypothesis and the Allais Paradox* ed. by M. Allais and O. Hagen. Dordrecht, Holland: D. Reidel Publishing Company, 1979.

Machina, M. J. "Expected Utility Analysis without the Independence Axiom." *Econometrica* 50, 1982, 277-323.

Machina, M. J. "Dynamic Consistency and Non-Expected Utility Models of Choice Under Uncertainty." *Journal of Economic Literature* XXVII, 1989, pp. 1622-1668.

Malinvaud, E. "Note on von Neumann-Morgenstern's Strong Independence Axiom." *Econometrica*, 20, 1952, 679.

Markowitz, Harry. "Portfolio Selection." *Journal of Finance* 7, no. 1, 1952, pp. 77-91.

Markowitz, Harry. *Portfolio Selection: Efficient Diversification of Investments*, New York: Wiley, 1959.

Marschak, J. "Rational Behavior, Uncertain Prospects, and Measurable Utility." *Econometrica*, 18, 1950, pp. 111-141. ("Errata," *Econometrica*, 18, 1950, 312.)

Mondigliani, Franco, and Leah Mondigliani. "Risk-Adjusted Performance." *Journal of Portfolio Management*, Winter 1997, pp. 45-54.

Mondigliani, Franco, and M.H. Miller. "The Cost of Capital, Corporation Finance and the Theory of Investment." *American Economic Review*, 48, June 1958, pp. 261-297.

Morgan, J.P. *RiskMetrics Monitor*, 4th Quarter, 1995.

Morrison, D. G. "On the Consistency of Preferences in Allais's Paradox." *Behavioral Science*, 12,1967, pp. 373-383.

Moskowitz, H. "Effects of Problem Representation and Feedback on Rational Behavior in Allais and Morlat-Type problems." *Decision Science*, 5, 1974, pp. 225-242.

Mossin, Jan. "Equilibrium in a Capital Asset Market." *Econometrica*, 34, 1966, pp. 768-783.

Mossin, Jan. "Optimal Multiperiod Portfolio Policies." *Journal of Business*, 41, (1968), pp. 122-136.

Peters, Hans and Wakker, Peter. "Independence of Irrelevant Alternatives and Revealed Group Preference." *Econometrica*, vol. 59, no. 6, November 1991, pp. 1787-1801.

Porter, R. Burr. "Semivariance and Stochastic Dominance: A Comparison." *American Economic Review*, vol. 64, no. 1, March 1974, pp. 200-204.

Pratt, John W. and Zeckhauser, Richard J. "Proper Risk Aversion." *Econometrica*, vol. 55, no. 1, January 1987, pp. 143-154.

Pratt, J. W. "Risk Aversion in the Small and in the Large." *Econometrica*, 32, 1964, pp. 122-136.

Pruitt, J.W. "Pattern and Level of Risk in Gambling Decisions." *Psychological Review*, vol 69, May 1962, pp. 187-201.

Rietz, Thomas A. "The Equity Premium-A Solution." *Journal of Monetary Economics*, 22, 1988, pp. 117-131.

Ross, Stephen A. "Adding Risks: Samuelson's Fallacy of Large Numbers Revisited." em Journal of Financial and Quantitative Analysis, vol.34, no. 3, September 1999, pp. 323-339.

Ross, Stephen A. "Mutual Fund Separation in Financial Theory-The Separating Distributions." *Journal of Economic Theory*, 17, 1978, pp. 254-286.

Rothschild, M., and J. Stiglitz. "Increasing Risk: I. A Definition." *Journal of Economic Theory*, 2, 1970, pp. 225-243. ("Addendum," *Journal of Economic Theory*, 5, 1972, 306.)

Rothschild, M., and J. Stiglitz. "Some Further Results on the Measurement of Inequality." *Journal of Economic Theory*, 6, 1973, pp. 188-204.

Roy, A.D. "Safety-First and Holding of Assets." *Econometrica*, Vol. 20, 1952. pp. 431-449.

Rubinstein, Mark. "An Aggregation Theorem for Securities markets." *Journal of Financial Economics*, 1, 1974, pp. 225-244.

Rubinstein, Mark. "Implied Binomial Trees." *Journal of Finance*, Vol. LXIX, No.3, July, 1994, pp. 771-818.

Samuelson, Paul A. "The St. Petersburg Paradox as a Divergent Double Limited." *International Economic Review*, 1, 1960, pp. 31-37.

Samuelson, Paul A. "Risk and Uncertainty: A Fallacy of Large Numbers." *Scientia*, vol. 57. no. 6, April/May 1963, pp.1-6.

Samuelson, P. A. "Utility, Preference, and Probability." abstract of a paper given before the conference on "Les Fondements et Applications de la Theorie du Risque en Econometrie." Paris, May, 1952. Reprinted as Chapter 13 in *Collected Scientific Papers of Paul A. Samuelson*, Volume I ed. by J. Stiglitz. Cambridge. Mass.: MIT Press, 1966.

Samuelson, Paul A. "Lifetime Portfolio Selection by Dynamic Stochastic Programming." *Review of Economic Statistics*, (August 1969), pp. 239-246.

Samuelson, Paul A. "The Fundamental Approximation Theorem of Portfolio Analysis in terms of Means, Variances and Higher Moments." *Review of Economics Studies*, 37, 1970, pp. 537-42.

Samuelson, Paul A. "The Long-term Case for Equities." *Journal of Portfolio Management*, (Fall 1994), pp. 15-24.

Sarin, Rakesh and Wakker Peter. "A Simple Axiomatization of Nonadditive Expected Utility." *Econometrica*, vol. 60, no. 6, November 1992, pp. 1255-1272.

Savage, L.J., *The Foundations of Statistics*, New York: John Wiley & Sons, 1954. (New York: Dover Publications, 1972).

Schmeidler, D. "Subjective Probability and Expected Utility without Additivity." *Econometrica*, 57, 1989, pp. 571-587.

Sharpe, William F. "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk." *Journal of Finance*, 19, 1964, pp. 423-439.

Sharpe, William F. "Mutual Fund Performance." *Journal of Business*, January 1966, pp. 119-138.

Shelley, Marjorie K. "Gain/Loss Asymmetry in Risky Intertemporal Choice" *Organizational Behavior and Human Decision Processes*, 59, 1994, pp. 124-159.

Simon, Herbert A. "A Behavioral Model of Rational Choice." In *Models of Thought.*, 7-19, New Haven Conn.:Yale University Press, 1979[1955].

Simon, Herbert A. "Rational Choice and the Structure of the Environment." In *Models of Thought.*, 20-28, New Haven Conn.:Yale University Press, 1979[1956].

Stone, B.K., "A General Class of Three Parameter Risk Measures." *Journal of Finance*, vol. 28, June 1973, pp. 675-685.

Tobin, J. "Liquidity Preference as Behavior Toward Risk." *Review of Economic Studies*, 25, 1958, pp. 65-85.

Tversky, A. "Utility Theory and Additivity Analysis of Risky Choices." *journal of Experimental Psychology*, 75, 1967a, pp. 27-36.

Tversky, A. "Additivity, Utility, and Subjective Probability." *Journal of Mathematical Psychology*, 4, 1967b, pp. 175-201.

Tversky, A. "Intransitivity of Preferences." *Psychological Review*, 76, 1969, pp. 31-48.

Tversky, A. "A Critique of Expected Utility Theory: Descriptive and Normative Considerations." *Erkenntnis*, 9, 1975, pp. 163-173.

Tversky, Amos and Wakker, Peter. "Risk Attitudes and Decision Weights." *Econometrica*, vol.63, no. 6, November 1995, pp. 1255-1280.

Von Neumann, J. and O. Morgenstern "Theory of games and Economic Behavior." *Princeton: Princeton University Press*, 1944. Second Edition, 1947. Third Edition, 1953.

Wakker, Peter and Deneffe, Daniel. "Eliciting Von Neumann-Morgenstern Utilities When Probabilities Are Distorted or Unknown." *management Science*, vol. 42, no. 8, August 1996, pp. 1131-1150.

Wakker, Peter. "Savage's Axioms Usually Imply Violation of Strict Stochastic Dominance." *Review of Economic Studies*, 60, 1993, pp. 487-493.

Williams, Joseph T. "Capital Asset Prices with Heterogeneous Beliefs." *Journal of Financial Economics*, 5, 1977, pp. 219-239.

Zou, Liang. "Investments with Downside Insurance and the Issue of Time Diversification." *Financial Analyst Journal*, July/August 1997, pp. 73-79.

Zou, Liang. "Inherent Reward and Risk (Part II): Valuation of Investment Strategies and Contingent Claims." Working Paper, Finance Group, University of Amsterdam, 2000.

	Upside Prifit (p=0.5)	Downside Loss (1-p=0.5)	σ	P	D
Asset 1	60%	-20%	0.40	0.15	0.10
Asset 2	40%	-20%	0.30	0.13	0.10
Asset 3	20%	-20%	0.20	0.10	0.10

Table 1: Ranking of Risk with Different Measures (constant inherent risk)

	Upside Prifit (p=0.5)	Downside Loss (1-p=0.5)	σ	P	D
Asset 1	60%	-20%	0.40	0.15	0.10
Asset 2	40%	-22%	0.31	0.14	0.11
Asset 3	20%	-24%	0.22	0.11	0.12

Table 2: Ranking of Risk with Different Measures (increasing inherent risk)

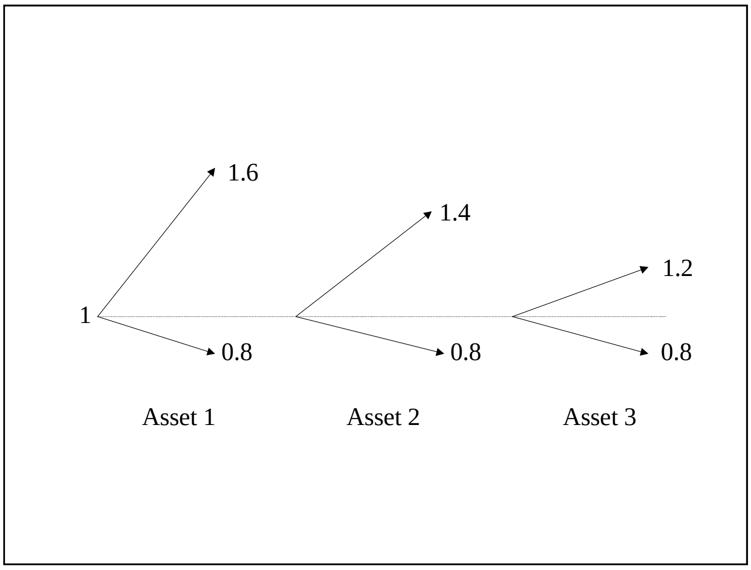

Figure 1: Assets with the same downside risk but different upside potentials.

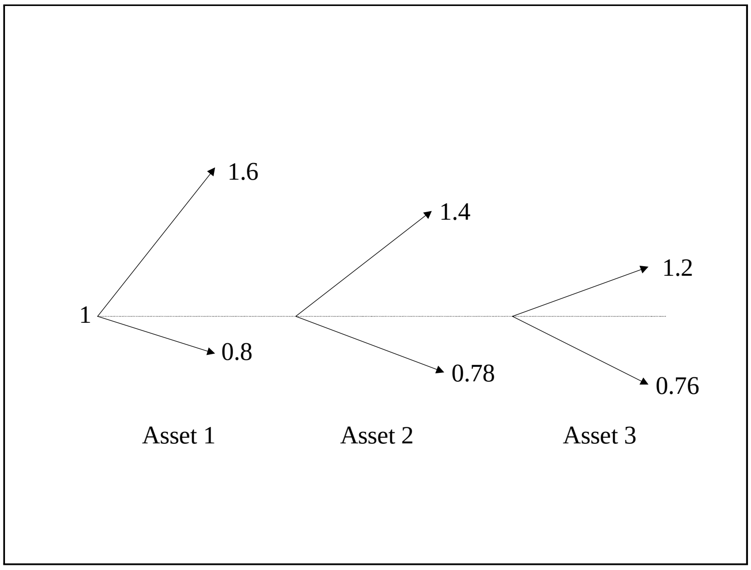

Figure 2: Assets with different downside risk and upside potentials.