TI 2000-038/3
Tinbergen Institute Discussion Paper

The Right Man for the Job

Coen N. Teulings
Pieter A. Gautier

Department of General Economics, Faculty of Economics, Erasmus University Rotterdam, and Tinbergen Institute

Tinbergen Institute

The Tinbergen Institute is the institute for economic research of the Erasmus Universiteit Rotterdam, Universiteit van Amsterdam and Vrije Universiteit Amsterdam.

Tinbergen Institute Amsterdam
Keizersgracht 482
1017 EG Amsterdam
The Netherlands
Tel.: +31.(0)20.5513500
Fax: +31.(0)20.5513555

Tinbergen Institute Rotterdam
Burg. Oudlaan 50
3062 PA Rotterdam
The Netherlands
Tel.: +31.(0)10.4088900
Fax: +31.(0)10.4089031

The right man for the job*

Coen N. Teulings[†] and Pieter A. Gautier[‡]

August 23, 2002

Abstract

This paper describes a search model with a continuum of worker and job types, free entry, and transferable utility. We apply a second order Taylor expansion to characterize the equilibrium, derive the "cost of search" and show that it is decreasing in the substitutability of worker types. This cost of search is then decomposed into three components: unemployment, vacancy costs and mismatch. Our contact technology rules out congestion effects between different worker types and therefore exhibits increasing returns to scale. One third of those increasing returns in contacts are shown to be absorbed by firms and workers being more choosy. The resulting equilibrium is not efficient. Unemployment benefits can reduce the loss by serving as a search subsidy. Numerical simulations of the model show that our Taylor expansions are quite accurate.

Keywords: assignment, search, unemployment

JEL codes E24, J21, J23

*We are grateful to three anonymous referees, Jaap Abbring, Daron Acemoglu, Orazio Attanasio, Jim Heckman, Ken Judd, Dale Mortensen, George Neumann, Chris Pissarides, Robert Shimer, Lones Smith, Gerard Van den Berg, and participants at seminars at Chicago, MIT, the Tinbergen Institute, Uppsala, Warwick, the World Congress of the Econometric Society in Seattle, the EALE/SOLE meeting in Milan and the SED meeting in Stocholm for their comments and suggestions.

[†]Erasmus University Rotterdam and Tinbergen Instititute, teulings@few.eur.nl.

[‡]Erasmus University Rotterdam and Tinbergen Institute, gautier@tinbergen.nl

1 Introduction

Two related strands have coexisted in the theoretical labor literature with relatively little interaction. On the one hand, there are the search and matching models, e.g. Diamond (1982a), Mortensen (1982), and Pissarides (2000). On the other hand, there is a literature that studies the problem of assigning heterogeneous workers to heterogeneous jobs, see for example Tinbergen (1956), Rosen (1974), Sattinger (1975), and Teulings (1995). Since heterogeneity of demand and supply is one of the main reasons for search, the integration of both strands is a natural way to go.

Recently, there have been some attempts to construct search models with ex ante heterogeneous agents and transferable utility, see e.g. Sattinger (1995), Marimon and Zilibotti (1999) and Shimer and Smith (2000, 2001a). However, the equilibrium of the latter can only be characterized by numerical simulation[1] while the first two papers achieve an analytical solution by making very specific assumptions. This state of affairs makes this literature rather esotheric, with limited applicability to empirical and policy analyses.

The present paper offers a methodology to measure the distortions in models with two sided heterogeneity and search frictions in a similar way as Harberger triangles quantify price distortions. Irrespective of the underlying shapes of the demand and supply curves, Harberger triangles provide second-order approximations to these losses. Similarly, we apply a Taylor expansion that yields rules of thumb for the size of search frictions that apply independently of functional forms. In both approaches, the elasticities of demand for various worker types play a crucial role in the calculation of these effects: the smaller the substitutability between workers types, the larger the cost of suboptimal assignment.

For the search part of the paper, Pissarides (1990) and Shimer and Smith (2000) are our benchmark. Like in Pissarides, utility is transferable, wages are set by Nash bargaining and a free entry condition determines the supply of vacancies. As in Shimer and Smith (2000) we have ex ante heterogeneous workers, and make a distinction between the mechanical contact process and the endogenous matching decision where a contact results in a match if the value of the match exceeds the sum of outside options, i.e. continued search.

[1] This can not easily be solved by applying particular functional forms because of the *corner problem*, see section 3.1. Marimon and Zilibotti (1999) avoid this problem by assuming that workers and jobs type lie on the same circle.

For the assignment problem, we apply the continuous-type comparative advantage framework of Teulings (1995, 2002). In this structure, worker types are characterized by a single index, referred to as the skill level. Likewise, jobs are characterized by their complexity level. Both indices are continuous. Better skilled workers have a comparative advantage in more complex jobs. The comparative advantage structure provides a completely natural reason for search because each worker type has its own "best" job type, where her comparative advantages are best utilized. In contrast, models with universally "good" jobs would not survive a free entry condition for vacancies because only "good" jobs would be created and other jobs would disappear.[2]

Compared to the complexity of search models, the Walrasian assignment model is simple. First order conditions, reflecting the point of tangency between cost and revenue functions, provide a solid structure to the market equilibrium. The first small step the researcher sets outside this Walrasian Utopia brings him into deep trouble. When there are search frictions, equilibrium is no longer reflected by this point of tangency. The cost function falls below the revenue function, and the value of search is now equal to the area enveloped by both functions. Instead of a single condition on the first derivatives of these functions, the evaluation of the integral requires all higher order derivatives to be taken into account now.

This interpretation alludes to a straightforward idea. Perhaps we can gain insight in a world with search frictions if we would add only the second order term to the Walrasian equilibrium. It is this idea that is investigated in the present paper. Our approach is not just a mathematical device. It has a number of important economic implications. First, taking the limit to the most efficient search technology eliminates higher order effects and makes the model converge to the Walrasian equilibrium. Second, elasticities of substitution are governed by the second derivative of cost functions. The same second derivative governs our second order Taylor expansion. Intuitively, the less easily firms can substitute between worker types, the more important will be a precise assignment of workers to jobs. Therefore, for a given contact technology, it takes more time to find a suitable partner when factors of production are less substitutable. Our analysis based on Taylor expansions provides a formal characterization of the relationship between the

[2] A typical example of this type of model is Shimer and Smith's (2001a) analysis of the constrained efficiency of a decentralized search process. In their analysis, bad types can expropriate good types since they have a comparative advantage in search. This expropriation causes the inefficiency.

degree of substitutability of worker types and the size of search frictions.

An important difference between our model and that of Shimer and Smith (2000) is that we include commodity markets for the output of each job type. When jobs are heterogeneous, so is their output, and hence prices differ over the output of each job type. When we assume these outputs to be perfect substitutes in consumption (or any other application), then output is effectively homogeneous. However, we show that, strictly speaking, our Taylor expansions do not apply in that case because then the equilibrium vacancy distribution is degenerate. Vacancies are opened in a small number of job types only. Under imperfect substitution, changes in commodity prices will offset this tendency of vacancies to cluster at particular values of the complexity index. Some imperfection in the substitutability of the output of various job types is therefore crucial for the existence of a well behaved, continuous equilibrium.

We are able to derive a simple, intuitive characterization of the *cost of search* for a particular worker type, defined as the surplus of value added in the optimal assignment relative to the reservation wage of that worker type. This cost of search depends on the scale of the market, the replacement rate, the cost of vacancies, and on the elasticity of substitution between worker types. Quite remarkably, the elasticity of the cost of search with respect to those factors does not depend on a specific functional form of the production function. It is fully determined by the order of the first non-vanishing terms in the Taylor expansions of the three crucial equations of the model. Interestingly, the implied elasticity of one minus the replacement rate is roughly consistent with the empirical evidence of Meyer (1990). We also show that the unemployment and vacancy rates are linearly related to the cost of search. Finally, we decompose the cost of search into three components: the cost of unemployment, the cost of maintaining vacancies and the productivity loss due to sub-optimal assignment. When search surpluses are shared equally between workers and firms (the bargaining power of the workers being 1/2), each of the three components accounts for one third of the cost. Again, the one third due to sub-optimal assignment follows immediately from a second order Taylor expansion of log value added in various job types. These results demonstrate the value of our Taylor expansion methodology.

A potential risk in our approach is that the ignored higher order terms are so large that we do not learn anything from the first non-vanishing terms. Therefore, we confront our approximations with the numerical solution of the model. Our approximations do surprisingly well for search frictions that generate reasonable values of unemployment.

We expect that these relations provide a useful starting point for empirical research and policy analysis, some of which we do in follow up papers. In terms of the empirical implications, the magnitude of the cost of search can be estimated from simple cross section data on wages by comparing dense areas like cities with non dense areas. By viewing cities as large scale areas that specialize in search intensive activities, the model offers an explanation for why people cluster in cities, despite the higher cost of living (Teulings and Gautier, 2002). A related prediction is that search frictions are larger in the tails of the skill distribution. Another implication of the model is that in a world with frictions, wages are concave in worker skill and job complexity. In Gautier and Teulings (2002), we explore this idea to derive the cost of frictions by including higher order terms of worker and job characteristics in a standard wage equation.

The paper is organized as follows. Section 2 presents the model. We start with the characterization of the equilibrium of the assignment of workers to jobs in the Walrasian benchmark. This benchmark will be the starting point of our Taylor expansions in Section 3. Next we introduce search frictions. Section 3 starts with some groundwork, needed for the characterization of the search equilibrium. We proof existence and differentiability. The latter is necessary for the application of Taylor expansions. Then, we discuss the use of second order Taylor expansions to evaluate the integrals over the matching sets. Finally, we discuss the applied matching technology in somewhat greater detail. Our technology implies increasing returns to scale while most research points in the direction of constant returns. We offer some arguments why constant returns are theoretically problematic in a random search environment and how our assumption can be squared with the empirical evidence. Section 4 applies the results of section 3 to analyze efficiency and optimal unemployment insurance issues and makes a precise decomposition of the cost of search. In Section 5, we confront our Taylor expansions with numerical solutions of the model. The simulations are specified such that we can closely track empirical estimates of all key parameters. Our approximations of the equilibrium are precise for realistic values of the unemployment rate. Finally, Section 6 concludes.

2 Structure of the economy

2.1 Assignment in the Walrasian benchmark

The most natural point to apply a Taylor expansion of a search equilibrium is the Walrasian equilibrium. We therefore start with a discussion of this Walrasian benchmark. Since most concepts carry over to the model with frictions, this discussion makes this model easier to understand. The discussion is limited to the essentials, see Teulings (1995, 2002) for details.

Workers and jobs are characterized by a single index, referred to as the skill level s and the job-complexity level c respectively. Both indices vary continuously, so that there exists an infinitum of worker and job types:

$$\begin{aligned} s &\in [s^-, s^+] \\ c &\in [c^-, c^+],\ c^- > 0 \end{aligned}$$

In the Walrasian benchmark, firms can open jobs of a particular c-type at zero cost. There is free entry of firms, which drives profits down to zero. Apart from this, there are no other factors of production in the economy. Workers supply a fixed amount of labor and receive no utility of leisure. Their utility depends only on the consumption of a single composite commodity (to be discussed below). Let $F(s, c)$ be the productivity of worker type s in job type c. We make four assumptions on $F(s, c)$:

1. $F(s, c)$ is twice differentiable;
2. $F_s(s, c) > 0$: absolute advantage of better skilled workers in any job type c;
3. $F(s, c)$ is log supermodular: better skilled workers have a comparative advantage in more complex jobs. Log supermodularity is a necessary condition for comparative advance;[3]

[3] Consider the case of supermodularity instead of log supermodularity $F(s, c) = sc$. The profit of a firm of type c is then equal to $P(c)\, cs - R(s)$, where $R(s)$ is the reservation wage of a type s worker. In a Walrasian world, actual wages are equal to reservation wages. Firms choose the c-type that maximizes $P(c)\, c$. There are two cases to consider, either, $P(c)\, c$ has a maximum or not. In the first case, firms open vacancies of that c-type only and other c-type commodities are not produced in equilibrium. Hence, there is no assignment problem. In the second case, all job types are equally profitable: $P(c)\, c = A$ (some constant independent of c). Then, the value of c is irrelevant for the firm. Firms' profits are equal to $As - R(s)$. Hence, the profit function does not depend on c and the assignment of worker types to job types is irrelevant. This result breaks down by imposing log supermodularity.

4. $\log F(s,c)$ is multiplicatively separable in s and c.

The first three assumptions are crucial for our results but the last one is added just for convenience. It does not drive the main conclusions of our analysis, but allows for a more transparent presentation. The following functional form is the simplest one that is consistent with those assumptions:

$$f(s,c) \equiv sc \tag{1}$$

(throughout the paper, lower cases denote the log of the corresponding upper cases). Since we have not yet defined the units of measurement of s and c, this specification encompasses any function that is multiplicatively separable in s and c.[4]

The output of a c-type job is traded at commodity markets with commodity prices $P(c)$, which are determined endogenously. The equilibrium assignment $s(c)$ maximizes the firms profits on a job of type c. Applying the zero profit condition: $P(c)\, e^{s(c)c} - R[s(c)] = 0$, the first order condition for the optimal assignment can be written as:

$$\begin{aligned} r'[s(c)] &= c > c^- > 0 \\ r''(s) &> 0 \\ r''[s(c)]s'(c) &= 1 \end{aligned} \tag{2}$$

The first line establishes formally that wages are an increasing function of s. It has a simple interpretation: $r'(s)$ measures the relative "price" of an additional unit of the skill index; c measures the relative productivity gain of an additional unit of the skill index in a job of type c. In equilibrium, both are equal. The second order condition (line 2) tells us that the (log) cost of hiring a worker with an additional unit of skill is increasing while the log returns are constant. The first line of (2) applies identically for all c. Hence, its first derivative must also apply (line 3). In combination with the second order condition, it follows that $s'(c) > 0$. This is what one would expect under comparative advantage: better skilled workers end up in more complex jobs.

[4] Alternatively, suppose $f(s,c) = \bar{s}(s)\,\bar{c}(c)$, where $\bar{s}(.)$ and $\bar{c}(.)$ are differentiable and strictly increasing. Then, we can just as well use $\bar{f}(\bar{s},\bar{c}) = \bar{s}\bar{c}$. Some commentators suggested that this specification implies increasing returns to scale. However, s and c are just indices that can be transformed at will and do not allow a scale interpretation. E.g. nothing would be changed by specifiying $f(s,c) = \sqrt{sc}$. Note that the restriction $c^- > 0$ guarantees that $F_s(s,c) > 0$, as has been previously assumed.

Since the zero profit condition also applies identically for all c, its first derivative must also apply. Hence (the effect via $s(c)$ drops out by the envelope theorem):

$$\begin{aligned} -p'(c) &= s(c) \\ -p''(c) &= s'(c) = 1/r''[s(c) > 0] \end{aligned} \tag{3}$$

Hence $p''(c) < 0$. Since $s(c)$ is strictly increasing, it has a well defined inverse function, denoted $c(s)$. Just as $s(c)$ is the profit maximizing s type for a c type job, $c(s)$ is the wage maximizing c type for an s type worker. The equilibrium assignment of worker to job types in the Walrasian equilibrium can therefore be expressed as a one-to-one correspondence between s and c.[5]

The c-type commodities are combined into the composite consumption good by a CES technology (with the standard Cobb Douglas extension for the special case $\eta = 1$):

$$\exp\left[\frac{\eta-1}{\eta}y^o\right] = \int_{c^-}^{c^+} \exp\left[\frac{\eta+1}{\eta}\underline{q}(c) + \frac{\eta-1}{\eta}y(c)\right] dc \tag{4}$$

where y^o denotes log aggregate output of the composite consumption good, $\eta \in [0,\infty]$ is the elasticity of substitution and $\underline{q}(c)$ is a twice differentiable function of weights of each type c in consumption with $\underline{q}"(c) < 0$, $\underline{q}'(c^-) = -s^-$, $\underline{q}'(c^+) = -s^+$, $\int_{c^-}^{c^+} \exp\left[(\eta+1)\,\underline{q}(c)\right] = 1$ (exogenous functions will be underlined throughout the paper). We take the price of the consumption good as the numeraire. From (4) we can derive the demand for commodity type c:

$$y(c) - y^o - \underline{q}(c) = -\eta[p(c) - \underline{q}(c)] \tag{5}$$

Two cases deserve special attention. In the case of perfect substitution ($\eta = \infty$), prices are effectively exogenous since $p(c) = \underline{q}(c)$. The assumption $\underline{q}"(c) < 0$ is consistent with the result that $p"(c) < 0$ for all commodities to be produced. Since $p'(c) = -s(c)$, the assumptions $\underline{q}'(c^-) = -s^-$, $\underline{q}'(c^+) = -s^+$ imply that all commodity types are produced and that there is a $c(s)$ for each s. In the case of a Leontieff technology ($\eta = 0$), the distribution of output per job type is exogenous $y(c) - y^o = \underline{q}(c)$.

[5] Contrary to for example Shimer and Smith (2001a), there is no such thing as a universally "good" or "bad" job in this model. For each worker type s there exists one perfect match $c(s)$ that maximizes the joint surplus of the worker and the firm by making optimal use of her comparative advantages. Any other assignment, whether it is to a more or to a less complex job, would yield a lower value added $P(c)\,e^{sc}$.

Finally, let L be the size of the labor force and let $\underline{l}(s)$ be the density function of s; $\underline{l}(s)$ is twice differentiable. Then, equilibrium on commodity markets requires:

$$y(c) - y^o = \ln L + \ln \underline{l}[s(c)] + s(c)c + \ln s'(c) \tag{6}$$

The left hand side measures product demand for type c, the right hand side measures product supply, both in logs. The latter is equal to total supply plus the log density of $s(c)$ plus log productivity of type $s(c)$ in job type c plus the log Jacobian $\ln ds/dc = \ln s'(c)$.

The differentiability of the distribution of labor supply, $\underline{l}(s)$, and commodity demand, $\underline{q}(c)$, guarantees that $s(c)$ is differentiable. We can now define the Walrasian equilibrium:

Definition *The Walrasian equilibrium is defined as a quintet* $\{r(s), p(c), s(c), y(c), y^o\}$ *that solves the equations (2)-(6) and the boundary conditions* $s(c^-) = s^-$ *and* $s(c^+) = s^+$.

A crucial variable for our second order Taylor expansions is the second derivative of the reservation wage function. Since $r''[s(c)] = 1/s'(c)$, or equivalently $r''[s] = c'(s)$, it is a measure for job heterogeneity. The higher $r''(s)$, the more variation there exists in job complexity per unit of s. Hence, $r''(s)$ is the main determinant of the elasticities of substitution and complementarity between skill types (Teulings, 2002). The higher $r''(s)$ is, the more heterogeneous jobs are and the less easily substitutable workers are. Basically, when $r''(s)$ is high, a worker's productivity at his second best job drops relatively sharply compared to his first best job.

The empirical implications of the model are invariant to a linear transformation of s, since we have not yet defined the units of measurement of c. Any linear transformation of s can be absorbed by an opposite transformation of c and a redefinition of commodity prices $p(c)$. Since $r''(s)$ is affected by a linear transformation of s, it is unsuitable as a summary statistic for the degree of substitutability of worker types. Teulings (2002) therefore introduces the *complexity dispersion parameter*: $\gamma(s) \equiv r''(s)/r'(s)^2$. As can be checked easily, this parameter is invariant to a linear transformation of s.[6] It will also show up in the expression for the magnitude of search frictions.

[6] Consider the linear transformation $s^+ = \sigma_0 + \sigma_1 s$. For the calculation of $\gamma(s^+) = r"[s^+(s)]/r'[s^+(s)]^2$, the parameter σ_0 drops out by differentiation and the parameter σ_1 cancels in the numerator and denominator.

2.2 Adding search frictions

Search frictions force workers and firms to be less choosy than in a Walrasian world because it may take a long time span of non-production till the optimal matching partner is found. The matching set of worker type s is no longer a single point $c(s)$, but will be a larger subset of the domain of c, and mutatis mutandis the same for a firm of type c. As a consequence, some workers will be unemployed and some firms will have vacancies. All other assumptions are maintained. In particular, the commodity market does not exhibit search frictions. First, we introduce some additional notation. Let $h(s)$ denote the density of unemployed workers of type s per unit of labor supply L. Hence, $u(s) \equiv \frac{h(s)}{\underline{l}(s)}$ is the unemployment rate for workers of type s and the aggregate unemployment rate satisfies $u \equiv \int_{s^-}^{s^+} h(s)ds$. We denote the density of vacancies of type c per unit of labor supply by $g(c)$. The total number of vacancies per unit of labor supply follows then from $v \equiv \int_{c^-}^{c^+} g(c)dc$ while the total number of vacancies is Lv.

Maintaining a vacancy is costly. This cost is independent of the job type and equal to $K > 0$ units of the composite consumption good per period of time. Think of those costs as advertisement costs. A free entry condition for vacancies drives the asset value of a vacancy to zero in equilibrium. Let $B \geq 0$ denote unemployment benefits or the value of leisure, although the former interpretation is not fully consistent with the model, since we ignore the funding of these benefits. Nevertheless, we will loosely refer to the ratio of B to reservation wages as the replacement rate.

Search frictions enter the model by a simple linear contact rate $\lambda_{i \to j}$ for worker (job) type i to run into job (worker) type j:

$$\begin{aligned} \lambda_{s \to c} &\equiv \lambda^* L g(c) \\ \lambda_{c \to s} &\equiv \lambda^* L h(s) \end{aligned} \tag{7}$$

where λ^* is a technology parameter which measures the efficiency of the matching process. For notational convenience we define $\lambda \equiv \lambda^* L$. We can interpret λ then as the relevant scale of the labor market. Matches are destroyed at an exogenous rate δ. Further, we assume both workers and firms to be risk neutral.

Since we allow for bargaining over the match surplus, any match with a value that exceeds the sum of the outside options of worker and firm is acceptable. Let $R(s)$ be the reservation wage for type s and let $m_c(s)$ and $m_s(c)$ be the subsets of c and s for which this condition is satisfied. These subsets are determined by the condition that the match surplus is positive:

$$P(c)F(s,c) - R(s) > 0 \Longleftrightarrow c \in m_c(s) \Leftrightarrow s \in m_s(c) \tag{8}$$

or equivalently, in logs: $x(s,c) > 0$, where the log match surplus $x(s,c) \equiv p(c)+sc-r(s)$; $x(s,c)$ has the neat interpretation of being the relative surplus of value added above the reservation wage. From these definitions, the value of search for a worker and an employer respectively, can be expressed as:

$$R(s) = B + \frac{\lambda}{\rho+\delta}\int_{m_c(s)} g(c)\left[W(s,c) - R(s)\right] dc \tag{9}$$

$$K \geq \frac{\lambda}{\rho+\delta}\int_{m_s(c)} h(s)\left[P(c)F(s,c) - W(s,c)\right] ds \tag{10}$$

where $W(s,c)$ is the wage of a worker of type s who is employed at a job type c, and ρ is the discount rate. K can never be smaller than the right hand side of equation (10) due to the free entry condition. If equation (10) holds with equality, vacancies of a particular type c are opened. If not, no vacancies are opened and that particular commodity is not produced in equilibrium. This can happen in two cases only. First, no vacancies are opened for all c-types. This is the trivial equilibrium, where all workers are permanently unemployed and collect the value of leisure B. We rule out this case by assumption[7]. Second, no vacancies might be opened for some c-types. This can happen only when $\eta = \infty$. For any other value of η, $P(c)$ will go to infinity when the number of vacancies $g(c)$ goes to zero, so that either all c-types are produced or none. Wages are set by a simple Nash bargaining rule over the match surplus. Hence:

$$W(s,c) = \beta P(c)F(s,c) + (1-\beta)R(s) \tag{11}$$

where $0 < \beta < 1$, denotes the workers' bargaining power. Substituting (11) in (9) and (10) yields:

$$R(s) = B + \frac{\beta\lambda}{\rho+\delta}\int_{m_c(s)} g(c)\left[P(c)F(s,c) - R(s)\right] dc \tag{12}$$

$$K \geq \frac{(1-\beta)\lambda}{\rho+\delta}\int_{m_s(c)} h(s)\left[P(c)F(s,c) - R(s)\right] ds \tag{13}$$

[7] A sufficient condition for a non-trivial equilibrium can be found in the working paper version of this paper (TI 00-038/3).

Since we consider a stationary economy, the number of workers finding a job must equal the number loosing their job:

$$\delta\left[\underline{l}(s)-h(s)\right]=\lambda\, h(s)\int_{m_c(s)} g(c)\, dc \tag{14}$$

Physical output per job type can be derived from the inflow of new workers of type s times their productivity in job type c times the expected duration of the employment relation $1/\delta$:

$$\frac{Y(c)}{L}=\frac{\lambda}{\delta}g(c)\int_{m_s(c)} h(s)F\left(s,c\right)ds \tag{15}$$

Definition *A steady state search equilibrium is defined as an octet* $\{r(s), h\left(s\right), p(c), g\left(c\right), y\left(c\right), m_c(s), m_s(c), y^0\}$ *solving the equations (4), (5),(8) and (12)-(15).*

Equations (12)-(15) reveal the difficulty in the characterization of the equilibrium. These equations require the calculation of integrals of which both the integrand and the integration boundaries are endogenous. It is precisely this complexity which has prohibited progress in this type of models. Therefore, we attack this problem in the next section by using second order Taylor expansions. Before we can do so, we have to establish existence and differentiability of the functions that characterize equilibrium.

3 Analysis of the equilibrium

3.1 General characteristics

Proposition 1 *An equilibrium exists for the search model of Section 2.1 and 2.2 with* $B=0$ *and* $\eta=1$.

The proof can be found in Appendix 1. We have been able to construct an existence proof only for the Cobb Douglas case, $\eta=1$. The proof proceeds along the same lines as in Shimer and Smith (2001), with a number of non-trivial extensions to deal with the endogeneity of the supply of jobs for each c type in our model. While $\underline{l}\left(s\right)$ provides a natural upperbound for $u\left(s\right)$, and hence $h\left(s\right)$, no such upperbound is available for $g\left(c\right)$, as is required for the application of the Schauder fixed point theorem, see Stokey and Lucas (1989). The advantage of the Cobb Douglas technology is that value shares for each c-type are constant. This feature enables us to provide an upper bound for $R\left(s\right), \forall s$

and a lower bound for $P(c), \forall c$. An important step in the proof is that we are able to derive a strictly positive lower bound for $u(s), \forall s$. Unemployment for a particular skill type can never be below that lower bound, since then there are too few job seekers to make the opening of vacancies profitable, even when all reservation wages are equal to zero.

Next, we establish a number of characteristics of the equilibrium that are crucial for our Taylor expansions to apply. First, the concepts of $c(s)$ and $s(c)$ applied in Section 2.1 for the analysis of the Walrasian equilibrium are generalized to the analysis of search equilibria:

$$\begin{aligned} c(s) &\equiv \widetilde{c} \mid \{x(s,\widetilde{c}) > x(s,c), \forall c\} \\ s(c) &\equiv \widetilde{s} \mid \{x(\widetilde{s},c) > x(s,c), \forall s\} \end{aligned} \tag{16}$$

Hence, $c(s)$ is the (set of) value(s) c that maximizes $x(s,c)$ for a particular s, and mutatis mutandis the same for $s(c)$. In the Walrasian equilibrium, the zero profit condition is equivalent to $x[s(c),c] = 0$. For any other worker types, $x(s,c) < 0$, so that $s(c)$ is unique. A similar argument applies to $c(s)$. The subsequent proposition establishes (among other things) that a number of characteristics of $c(s)$ and $s(c)$ that apply in the Walrasian equilibrium carry over to search equilibria:

Proposition 2 *Consider the search model discussed in Section 2.1 and 2.2 with $\eta \in [0,\infty)$. The equilibrium satisfies the following conditions:*

1. *$x(s,c)$ is strictly concave in both its arguments,*

2. *$m_c(s)$ and $m_s(c)$ are strictly convex and $c(s)$ and $s(c)$ are unique,*

3. *$r(s)$, $p(c)$, and $x(s,c)$ are differentiable everywhere and twice differentiable almost everywhere, with $r'(s) > 0$, $r''(s) > 0$, $p''(c) < 0$,*

4. *let $c^-(s)$ be the lower bound of $m_c(s)$ and let $c^+(s)$ be the upper bound; then, whenever there is an interior solution, $c^-(s)$ is differentiable with $c^{-'}(s) > 0$, and the same holds for $c^+(s)$,*

5. *$h(s)$ and $g(c)$ are differentiable.*

The proof of the proposition can be found in Appendix 2. We are able to proof directly from the Bellman equations for the worker and the firm that $r(s)$ is convex and $p(c)$ is concave. This implies that the surplus function $x(s,c)$ is concave in both its arguments. The proof is much simpler than a similar derivation in Shimer and Smith (2000), basically because we impose a priori the sufficient condition for concavity of the matching set (log supermodularity of $F(s,c)$). From this result, all other results follow more or less automatically. Only the proof that the upper and lower bound of the matching sets in the s,c space are upward sloping requires a special treatment of the extreme cases (s^-,c^-) and (s^+,c^+), analogous to Shimer and Smith (2000).

Part 3 of the proposition states that the characteristics of the Walrasian equilibrium, $r'(s) > 0, r''(\dot{s}) > 0$, and $p''(c) < 0$, carry over to the search equilibrium.[8] The conditions on the second derivatives imply that $c(s)$ and $s(c)$ are unique, since $x(s,c)$ has only a single maximum in each of its arguments. Obviously, $x[s(c),c] > 0$ for all c since vacancies of all c-types exist, and mutatis mutandis the same is true for $x[s,c(s)]$.

Part 3 also implies that $x_c[s,c(s)] = 0$ and $x_s[s(c),c] = 0$ for any interior value of $c(s)$ and $s(c)$ since $x(s,c)$ is differentiable. Hence, the following relations generalize from the Walrasian equilibrium to search equilibria, compare equations (2) and (3): $-p'[c(s)] = s$, $-p''[c(s)]\,c'(s) = 1$, $r'[s(c)] = c$, and $r''[s(c)]\,s'(c) = 1$. Contrary to the Walrasian equilibrium, however, $c(s)$ is not necessarily the inverse of $s(c)$. Define the inverse function of $s(c)$: $s[d(s)] \equiv s$.[9] The following result provides an intuition for why $s(c)$ is in general not the inverse of $c(s)$.

Corollary 1 *Condition for $c(s)$ being equal to $d(s)$*

$$d(s) = c(s) \Leftrightarrow \frac{d\,x[s,c(s)]}{ds} = 0 \tag{17}$$

This result follows directly from the definition of $c(s)$, $x_c[s,c(s)] = 0$. Hence, equation (17) applies if $x_s[s,c(s)] = 0$. By the definition of $s(c)$, this is the case when $s = s[c(s)]$, or equivalently, when $d(s) = c(s)$. In the Walrasian equilibrium, $d\,x[s,c(s)]/ds$ is indeed zero, since $x[s,c(s)] = 0$ for all s due to the zero profit condition. In the search equilibrium, $x[s,c(s)]$ might very well differ between worker types.

[8] $r'(s)$ is non-differentiable only at the points where either $c^-(s)$ or $c^+(s)$ switches from an interior to a boundary solution. Since both functions are upward sloping (see Part 3) and since $c^-(s^+) < c^+$ and $c^+(s^-) > c^-$, this happens only once for both functions. A similar argument applies for the $p'(c)$.

[9] By the definition of $d(s)$, $r'(s) = d(s)$ and $r''(s) = d'(s)$.

Part 4 shows that an equilibrium exhibits positive assortative matching, the upper and lower bounds of the matching sets are upward sloping, so that "on average" better skilled workers are matched to more complex jobs. Moreover, these upper and lower bounds are differentiable.[10]

The situation is depicted graphically in Figure 1-3. Figure 1 shows $x(s,c)$ which reaches a maximum when the job is occupied by a type $s(c)$ worker. In the Walrasian case, all type c jobs are matched with type $s(c)$ workers because the loci of value added and reservation wages are tangent only there. In the search equilibrium, all s for which $x(s,c) > 0$ belong to the matching set $m_s(c)$. Figure 2 shows the same picture for a given worker type, s. Here, $c(s)$ maximizes $x(s,c)$ and all c for which $x(s,c) > 0$ belong to her matching set $m_c(s)$. Figure 3 shows the functions $s(c)$, $c(s)$, $c^-(s)$ and $c^+(s)$ in the s,c-space. Obviously, the maxima $s(c)$ and $c(s)$ are in between the upper and the lower bound.[11]

Corollary 2 *Proposition 2 does not apply for* $\eta = \infty$.

The intuition for Corollary 2 is that the proof of Proposition 2 requires the application of equation (5) to establish the continuity of $y(c)$. However, for $\eta = \infty$, this equation is reduced to $p(c) = \underline{q}(c)$ and hence $y(c)$ drops out. This potential discontinuity carries over to $g(c)$. There is economic intuition for this result. Consider equation (13). If vacancies for all c-types have to be opened, this equation must hold with equality for all c-types. Consider type $\bar{c}$. Suppose that initially $\bar{c}$ is the only type of job that is opened in some small neighborhood $[\bar{c} - \Delta, \bar{c} + \Delta]$. For all other c types in this neighborhood, equation (13) is satisfied with equality at best (since otherwise, some firm would have opened vacancies of that other c type, pushing up reservation wages and driving vacancies of type $\bar{c}$ out of the market by the mechanism discussed below). Consider an employer who is considering to open a vacancy $\bar{c} + h$, h being small. The matching set for this vacancy will almost completely overlap with that of type $\bar{c}$. Since no other vacancies than type $\bar{c}$ are open, reservation wages for the skill types $s \in m_s(\bar{c})$ are fully determined by the

[10] A referee wondered why our result are so much different from Burdett and Coles (1997), who find a strict segmentation of the market, implying the non-differentiability of the boundary functions. This difference is due to the fact that they assume non-transferable utility while we assume transferable utility.

[11] Note that for any interior solution, $c^-(s)$ is the inverse of $s^+(c)$ and $c^+(s)$ is the inverse of $s^-(c)$, since they all solve the equation $x(s,c) = 0$.

equivalent of equation (12):

$$R(s) = B + \frac{\beta\lambda}{\rho+\delta} G\left[P(\bar{c}) F(s,\bar{c}) - R(s)\right]$$

where G is the number of vacancies of type $\bar{c}$ (note that $g(c)$ is degenerate in this example). Hence, the only variable that is left to let equation (13) be satisfied with equality for type $\bar{c}+h$ is $P(\bar{c}+h)$. That works for a finite η, but not for $\eta=\infty$, since then $P(c)$ is fully determined by $\underline{q}(c)$. Hence, it is only by coincidence that equation (13) is satisfied with equality for type $\bar{c}+h$ in that case. In our numerical simulations, we find indeed this clustering of vacancies at a small number of c-types. The only stable equilibrium seems to be a complete segmentation of the market, where only a limited number of c types are produced in equilibrium, $c_1, c_2, c_3, ..$, with consecutive but non-overlapping matching sets, $s^+(c_1) = s^-(c_2), s^+(c_2) = s^-(c_3), ...$ Proposition 2 shows that even a slight imperfection in the substitutability between commodity types (a finite η) resolves the issue, since then $g(c)$ is continuous. This result shows the importance of modelling commodity markets explicitly.[12]

Corollary 3 *Convergence to the Walrasian equilibrium:*

$$\begin{aligned}\lim_{\lambda\to\infty} r(s)_{|\lambda} &= r_W(s)\\ \lim_{\lambda\to\infty} p(c)_{|\lambda} &= p_W(s)\end{aligned}$$

where the suffix W denotes the values applying in the Walrasian equilibrium

This can be seen by considering equation (13). For $\lambda\to\infty$, the integral $\int h(s)\left[P(c)F(s,c) - R(s)\right] ds$ must vanish. Since by definition, $s(c)\in m_s(c)$, this implies that either $h[s(c)]$ or $P(c)F[s(c),c] - R[s(c)]$ must vanish. In fact, both do. Hence, $\lim_{\lambda\to\infty} P(c)F[s(c),c] - R[s(c)] = 0$, or equivalently, $\lim_{\lambda\to\infty} s(c)c + p(c) - r[s(c)] = 0$, for all c. This is the zero profit condition for the Walrasian equilibrium. Since equation (13) applies with equality and identically for all c for a finite η, its first derivative also applies:

$$0 = \int_{m_s(c)} h(s) P(c) F(s,c)\left[p'(c) + s\right] ds$$

[12] Clustering will never occur in the Walrasian equilibrium. The intuition is that in that case, matching sets are defined by a one-to-one correspondence $s(c)$. Hence, there is a reservation wage $r[s(c)]$ for each c that can be used to let the zero profit condition be satisfied. In a search equilibrium, matching sets of neighboring c types overlap, invalidating this one-to-one correspondence.

Since $\lim_{\lambda\to\infty} s(c)\,c + p(c) - r[s(c)] = 0$, the matching set $m_s(c)$ converges to the single point $s(c)$. Hence, $p'(c) = -s(c)$ and equation (3) applies. Differentiating the zero profit condition with respect to c and using (3) yields equation (2).

3.2 Using Taylor expansions for the integrals

Since the calculation of the integrals in equation (12) and (13) requires the evaluation of $x(s,c)$ around its maximum, it is a straightforward idea to approximate this function by a parabola, that is, by a second order expansion. The validity of this approximation requires this maximum $x[s, c(s)]$ to be small. Since $x[s, c(s)]$ equals zero in the Walrasian case, our approximation applies close to the competitive equilibrium, that is, for $\lambda \to \infty$. The following proposition provides approximations of the integrals in these equations:

Proposition 3 *Consider the search model discussed in Section 2.1 and 2.2 with $\eta \in [0,\infty)$. The following approximations hold:*

1. *for any s with interior solutions to $c^+(s)$ and $c^-(s)$:*

$$\begin{aligned}\left\{\int_{m_c(s)} g(c)\left[\frac{P(c)\,F(s,c)}{R(s)} - 1\right] dc\right\}^2 &= \frac{32}{9} g\,[c(s)]^2\, c'(s)\,\{x^*(s) + o\,[x^*(s)]\}^3 \\ \left\{\int_{m_c(s)} g(c)\,dc\right\}^2 &= 8g\,[c(s)]^2\, c'(s)\,\{x^*(s) + o\,[x^*(s)]\}\end{aligned}$$

where $x^*(s) \equiv x[s, c(s)]$;

2. *for any c with interior solutions to $s^+(c)$ and $s^-(c)$:*

$$\left\{\int_{m_s[c]} h(s) R(s)\left[\frac{P(c)\,F(s,c)}{R(s)} - 1\right] ds\right\}^2 = \frac{32}{9} h\,[s(c)]^2\, R\,[s(c)]^2\, s'(c)\,\{x^o(c) + o\,[x^o(c)]\}^3$$

where $x^o(c) \equiv x[s(c), c]$.

The full proof of the proposition can be found in Appendix 3. We provide simple intuitions for the main steps of the proof by means of Figure 2. We concentrate on the first equation of Part 1, the equation in Part 2 can be derived by a similar argument. In the first step, we apply an approximation of the integrand: $g(c)\left[\frac{P(c)F(s,c)}{R(s)} - 1\right] = g(c)\left[e^{x(s,c)} - 1\right] \cong g(c)\,x(s,c)$. This approximation applies, like all our approximations, for small $x(s,c)$. In the second step, we approximate the domain of integration. We find the two roots of the equation $x(s,c) = 0$ by a second order Taylor expansion of $x(s,c)$ around its maximum $x^*(s)$. By construction, $x_c[s,c(s)] = 0$. Hence, $x[s,c(s) \pm z] - x^*(s) \cong \frac{1}{2} x_{cc}[s,c(s)]\, z^2$. Define Δ as $x[s,c(s)+\Delta] = 0$. Then, $\Delta \equiv c^+(s) - c(s) \cong c(s) - c^-(s) \cong \sqrt{-2x^*(s)/x_{cc}[s,c(s)]}$.

In the third step, the integral $\int_{-\Delta}^{\Delta} g[c(s)+z]\, x[s,c(s)+z]\, dz$ is approximated by $g[c(s)] \int_{-\Delta}^{\Delta} x[s,c(s)+z]\, dz$. This step is crucial, as it allows us to ignore variation in $g(c)$ and focus completely on the variation in $x(s,c)$. This step is allowed by a combination of two arguments related to a Taylor expansion of $g(c)$ around $c(s)$. First, the first order effect drops out since the two effects on the right and the left cancel exactly. Second, the magnitude of the second order effect $g''[c(s)]/g[c(s)]$ is small compared to that of $x_{cc}[s,c(s)]/x^*(s)$ since $x[s,c(s) \pm \Delta] \cong 0$.

In the final step, we integrate $\int_{-\Delta}^{\Delta} x[s,c(s)+z]\, dz$ by approximating $x[\cdot]$ by a parabola. The surface of the rectangle 2(A+B) in Figure 2 equals $2\Delta x^*(s) = 2\sqrt{2c'(s)x^*(s)^3}$, where we apply $x_{cc}[s,c(s)] = p''[c(s)] = -c'(s)^{-1}$. Two thirds of this surface is below the parabola. This step has an interesting extension, which will play an important role in the analysis of the efficiency of search equilibria in Section 4. When a type s worker is employed, the average surplus of her value added relative to her reservation wage can be approximated by $\frac{2}{3}x^*(s)$. By a complementary argument, the average loss of her value added relative to the maximum value added is $\frac{1}{3}x^*(s)$.

Most steps in the proof of Proposition 3 require the differentiability of various functions which has been proven in Proposition 2. Since this proof excludes the case $\eta = \infty$, see Corollary 2, we have to exclude that case here again.

Proposition 3 is only applicable for the values of s for which $c^+(s)$, $c^-(s)$, $s^+[d(s)]$, and $s^-[d(s)]$ have interior solutions. If they do not have interior solutions, $x(s,c)$ does not have to be zero at the boundaries of the integration interval. As shown in Figure 3, this happens in the North-East and the South-West corner of the s,c space. We refer to this problem as the *corner problem*. The smaller search frictions, the tighter the band of the matching sets around $c(s)$, and hence the smaller the subset of the domain of s for which this problem applies. In the limiting case of a search equilibrium close to the Walrasian optimum, we can ignore the corner problem. Moreover, the relative importance of these corners is small if the skill distribution is unimodal with little probability mass in the tails (i.e. the normal distribution).

3.3 The search equilibrium

The relations derived in Proposition 3 can be used to obtain a Taylor expansion for equations (12), (13), and (14) that applies for small search frictions, that is, the limit of $\lambda \to \infty$:

Proposition 4 *Consider the search model discussed in Section 2.1 and 2.2 with $\eta \in [0, \infty)$. Then, for any s with interior solutions to $c^+ (s)$ and $c^- (s)$:*

$$\begin{aligned} x^*(s) &= \left[\frac{1}{2} Q^2 B^* (s)^2 K^* (s)^2 \underline{l}(s)^{-2} c'(s)\right]^{1/5} + o\left[x^* (s)\right] && (18) \\ \frac{\rho+\delta}{\delta} u(s) &= \frac{2}{3} \beta B^* (s)^{-1} \left\{x^*(s) + o\left[x^* (s)\right]\right\} && (19) \\ \frac{\rho+\delta}{\delta} v(s) &= \frac{2}{3} (1-\beta) K^*(s)^{-1} \left\{x^*(s) + o\left[x^* (s)\right]\right\} && (20) \end{aligned}$$

where $Q \equiv \frac{9}{8} \frac{(\rho+\delta)^2}{\delta \beta (1-\beta) \lambda}$, $B^*(s) \equiv 1 - \frac{B}{R(s)}$, $K^*(s) \equiv \frac{K}{R(s)}$, $v(s) \equiv \frac{g[c(s)] c'(s)}{\underline{l}(s)}$

The proof of this proposition is delegated to Appendix 4. Apart from the substitution of the integrals in Proposition 3 and some rearrangement, two further steps are made in the derivation of these relations. First, we apply the standard approximation that the unemployment rate is small relative to unity: $\frac{u(s)}{1-u(s)} \cong u(s)$, or equivalently: $\frac{h(s)}{\underline{l}(s)-h(s)} \cong \frac{h(s)}{\underline{l}(s)}$. By equation (19), $u(s)$ is proportional to $O\left[x^* (s)\right]$. Hence, this approximation can be done at zero loss of approximation order, since the term $o\left[x^* (s)\right]$ had to be included for the approximation error in the integral anyway, see Proposition 3. Second, contrary to the Walrasian equilibrium, $s(c)$ is not the inverse of $c(s)$ in a search equilibrium, see Corollary 1. We are able to bound the differences $d'(s) - c'(s)$ and $x^o\left[d(s)\right] - x^*(s)$[13] to be of order $o\left[x^* (s)\right]$. Again, these approximations can therefore be included at zero loss of approximation order. The intuition for this result is simple. Recall that in the Walrasian equilibrium $s(c)$ is the inverse of $c(s)$, and hence $d(s) = c(s)$. When search frictions are small, the equilibrium is close to the Walrasian one and the difference between $d(s)$ and $c(s)$ is also small. This step allows us to integrate Part 1 and Part 2 of Proposition 3.

[13] Which is equal to $x\left[s, d(s)\right] - x\left[s, c(s)\right]$ by definition.

Finally, note that $B^*(s)$, $K^*(s)$ and $c'(s)$ depend on $r(s)$ and are therefore endogenous. One can approximate these functions by their Walrasian value by Corollary 3.

Proposition 4 provides a convenient characterization of a search equilibrium by simple relations for $x^*(s)$, the rate of unemployment $u(s)$, and the number of vacancies per unit of labor supply of type s, $v(s)$.[14] $x^*(s)$ measures the relative difference between value added of a type s worker when assigned to the optimal job and the reservation wage. In a Walrasian equilibrium, this difference would be zero. Hence, we refer to $x^*(s)$ as the *cost of search*. Proposition 4 shows unemployment and vacancies to be linearly related to the cost of search. The intuition for these linear relations is that agents respond to an increase in search frictions along two channels. First, they spend more time in the search state which results in an increase in unemployment and vacancies. Second, they reduce their reservation match quality, which yields an increase in the relative difference between maximum value added and the reservation wage, $x^*(s)$.

We have written the relations for unemployment and vacancies in a way that makes them easy to interpret. The factor $\frac{\rho+\delta}{\delta}$ accounts for the net discounted cost of unemployment (vacancies) for a job seeker (firm). Since a job seeker has to invest in an unemployment spell before being able to reap its benefits, this factor is greater than one. We use these relations in the next section to decompose the cost of search into three components, the cost of unemployment, the cost of maintaining vacancies, and the loss of output due to suboptimal assignment.

$x^*(s)$ depends on a composite parameter Q, reflecting a number of factors. First, the cost of waiting for a better match relative to the expected duration of the match, $\frac{(\rho+\delta)^2}{\delta}$, enters positively, because agents accept a lower match quality when the cost of waiting rises. Second, the distribution of bargaining power matters, due to hold up problems. The higher β, the larger the worker's expected return to search is and the greater therefore her willingness to invest in search. This explains why the model breaks down if we attribute the whole surplus either to the worker ($\beta = 1$) or to the firm ($\beta = 0$). In that case, the other side has no incentives for search, so nobody will enter the market. It also explains why the cost of search $x^*(s)$ are smallest for $\beta = \frac{1}{2}$. We return to this issue in Section 4.2. Finally, the size of the labor market, λ, enters negatively. This reflects the increasing returns to scale (IRS) in the contact technology.

[14] The multiplication by $c'(s)$ in the definition of $v(s)$ is the Jacobian, transforming the distribution of vacancies per job type c to a distribution per worker type s.

The cost of search depends positively on one minus the replacement rate $B^{*}(s)$ and the ratio of the cost of maintaining a vacancy relative to reservation wages $K^{*}(s)$. These effects reflect the notion that match quality falls when the cost of unemployment and the cost of maintaining a vacancy go up. The distribution of labor supply $\underline{l}(s)$ enters negatively. This effect is the same as the effect of λ and is again due to the IRS in the contact technology. Whether one increases the size of a particular segment of the market s, either by a general increase in supply, raising λ, or by a concentration of total supply in that particular segment, raising $\underline{l}\,(s)$, is irrelevant.[15] In both cases match quality goes up by the returns to scale in contact technology. The only thing that matters is the product of both factors, $\lambda \underline{l}\,(s)$.[16] For unimodal distributions, the density in both tails will be lower. Then, by IRS, search frictions are larger in the tails, simply by the returns to scale in contact technology.

Finally, the dispersion in job complexity per unit of the skill distribution, $c^{'}(s)$, increases the cost of search. There is an alternative way to interpret this effect. As discussed in Section 2.1, any linear transformation of the scale of measurement of s can be offset by a compensating transformation in c and $p\,(c)$ without changing the empirical implications of the model. It is therefore more instructive to write $x^{*}\,(s)$ as a function of log wages instead of s, since that yields an expression that does not depend on arbitrary choices of units of measurement. Applying this transformation to equation (18) yields:

$$\widehat{x}^{*}(r)^{5}=\frac{81}{128}Q^{2}\widehat{B}^{*}\,(r)^{2}\,\widehat{K}^{*}\,(r)^{2}\,\underline{f}(r)^{-2}\widehat{\gamma}(r) \tag{21}$$

where $\underline{f}(r)$ is the density function of log reservation wages r and where $\widehat{x}^{*}\,[r\,(s)] \equiv x^{*}\,(s)$, and mutatis mutandis the same for B^{*}, K^{*} and γ; $\gamma\,(s) \equiv r"\,(s)\,/r'\,(s)^{2} = c'\,(s)\,/c\,(s)^{2}$ is the complexity dispersion parameter discussed in Section 2.1.[17] The complexity dispersion

[15] A referee wondered why the punctual value of the density function matters and not its value along the whole matching set of a particular $c\,(s)$ job-type. The reason is that we consider the limiting case where $x^{*}\,(s)$ is small. Hence, matching sets are small, too. Then, by the differentiability of $\underline{l}\,(s)$, its punctual value is a good approximation of its values over the matching set.

[16] An interesting question is what happens with the matching rate for type s workers when $\underline{l}(s)$ increases? This depends on the value of η. When $\eta \to \infty$, an increase in $\underline{l}(s)$ will raise the supply of $c(s)$ jobs. Hence, the matching rate for type s workers will increase by a standard thick market argument. At the other extreme, when $\eta = 0$, we have to account for general equilibrium effects. The increase in $l(s)$ decreases wages for those s-types. This widens their matching sets. However, this effect is insufficient to offset the increased labor supply effect.

[17] The denominator of $\widehat{\gamma}(r)$ comes in as the Jacobian from a transformation of variable: $\underline{l}\,(s) = \underline{f}\,[r(s)]\,r'(s)$.

parameter is inversely related to the elasticities of substitution between worker types. Hence, the lower the substitutability between worker types, the higher the cost of search. There is a simple intuition for this result. The more easy worker types can be substituted between job types, the lower the productivity loss due to suboptimal matching is, and hence, the wider matching sets are.[18] Also, the expected unemployment duration falls since workers become less choosy.

A remarkable conclusion from Proposition 4 is that the elasticities of the relations discussed above do not depend on any of the model's parameters. Most variables, like one minus the replacement rate, relate to $x^*(s)$ by a $\frac{2}{5} = 0.4$ elasticity. This result does not depend on the specific form of the production function $F(s, c)$; the only requirements are its log supermodularity and its differentiability. Then, the elasticity results follow solely from the order of the first non-vanishing terms in the approximation of the integrals in Proposition 3: an order 3 in the integral for the worker's Bellman equation minus an order 1 in the worker's flow equilibrium equation plus an order 3 for the firm's Bellman equation. This underscores the usefulness of Taylor expansions.

This general rule applies also to the elasticity of unemployment with respect to one minus the replacement rate. This elasticity is -0.6. Meyer (1990) finds an elasticity of unemployment with respect to the benefit level of up to about minus one for the United States. His source of variation is mainly structural variation in legislation between states. Hence, we feel comfortable to interpret his estimate as reflecting the elasticity of equilibrium unemployment. This estimate is consistent with our model when the replacement rate is 0.60. The model implies that the detrimental effect on unemployment goes up with every percent further increase in B.

Proposition 4 provides a relation for $x^*(s)$ but not for reservation wages. When search frictions do not have a general equilibrium effect on commodity prices, $p(c)_{|\lambda} = p_W(c)$ (and hence on the optimal assignment $c(s)$) then reservation wages follow immediately from the definitional relation: $r(s)_{|\lambda} = r_W(s) - x^*(s)$: reservation wages are equal to their value in the Walrasian equilibrium minus the cost of search. Hence, Proposition 4 would allow an approximation of $r(s)_{|\lambda}$ up to an order $o[x^*(s)]$. Regrettably, the identity

[18] In the limit, when worker types become perfect substitutes ($\gamma \to 0$), the model converges to the Pissarides (2000) model with IRS. The Taylor expansions do not work in that case because the corner problem issues that we discussed at the end of the previous section can then no longer be ignored. We need heterogeneity for the scale elasticity to reduce when moving from the contact to the matching rate.

$p(c)_{|\lambda} = p_W(c)$ applies only when $\eta = \infty$, since then commodity prices are effectively exogenous: $p(c) = \underline{q}(c)$. For all other values of η, search frictions do have general equilibrium effects on prices. An evaluation of these effects is outside the scope of this paper, see Teulings and Gautier (2002) for an analysis.

Proposition 1 proofs existence, but not uniqueness. How should we interpret the approximate characterization of the equilibrium in Proposition 3 and 4, if we cannot proof its uniqueness? We offer a number of arguments. First, multiplicity might not be an issue. In the homogeneous workers and firms version of this model, the equilibrium is unique.[19] Obviously, heterogeneity complicates the issue. However, the absence of congestion effects seems to reduce the likelihood of multiplicity, since the opening of a vacancy of one type does not directly reduce the profitability of opening a vacancy of another type.[20] Second, the Taylor expansions in Proposition 3 apply generally. Third, multiplicity might be a higher order phenomenon that either falls within the approximation term $o[x^*(s)]$ or occurs only outside the domain where our approximations do a decent job. Finally, the approximations in Proposition 4 track the numerical solutions presented in Section 5 closely. These simulations are only based on a limited subset of the parameter space, but the least that we can say is that in that subset Proposition 4 works fine.

3.4 Congestion effects and IRS

A standard search model a la Pissarides (1990) is typically characterized by two types of search externalities, congestion and thick market effects. The congestion effect reflects the negative externality that a job seeker imposes on other job seekers by reducing their chances to run into a vacancy. The thick market effect reflects the positive externality that this same job seeker imposes on firms by increasing their chances to meet a job seeker. In a model with heterogeneous types, where not every contact yields a match, congestion effects have an unpleasant implication. A low skilled immigrant will find it harder to find a hamburger job when a bunch of Harvard graduates enter the market, simply because each additional job seeker reduces all other job seekers' chance to meet

[19] Using the stripped down notation of the Appendix 1, a homogeneous model reads: $R = \chi\theta v(P - R)$, $1 = \psi u(P - R)$, $1 - u = \theta uv$, $Y = \theta uvm$. The solution for u reads: $u = \frac{1}{2\psi}\left[1 - \chi \pm \sqrt{(1-\chi)^2 + 4\psi\chi}\right]$, which has a single positive root.

[20] For example, Shimer and Smith's (2001b) result that there is no steady state equilibrium, but a limit cycle seems to be contingent on the presence of congestion effects.

a particular type of vacancy. We find this hard to believe.[21] We therefore specified the matching technology in equation (7) such that it exhibits no congestion effects. The only technology that satisfies this requirement reads:

$$m = \lambda \int_{s^-}^{s^+} h(s)ds \int_{c^-}^{c^+} g(c)\, dc = \lambda\, u\, v \tag{22}$$

where m is the flow of contacts between workers and firms. The contact rate of a job seeker, $m/u = \lambda v$, is independent of the number of other job seekers on the market, and mutatis mutandis the same for vacancies, which indeed rules out congestion effects. The consistency of this technology with equation (7) is checked easily. The matching elasticities with respect to u and v are both equal to one, hence their sum is two, which is twice as large as in the constant returns to scale (CRS) specification of Pissarides (2000). Hence, our specification implies IRS. It is worth noting that economists are in good company when using this proportional contact technology. It is equivalent to the standard model of the velocity of chemical reactions in gasses used in chemistry. In a reaction involving two molecules, doubling pressure doubles the reaction speed. From that perspective, the proportional contact technology seems quite natural. Most of the empirical evidence finds CRS however, see Petrongolo and Pissarides (1999) for a survey. How do we square our assumption of no congestion effects and IRS with this evidence?[22] We offer three arguments.

First, the empirical research refers to the number of realized matches, while our technology refers to the number of contacts between workers and firms, or even better, potential contacts. Not all contacts yield a match. Agents respond to the greater efficiency of the contact process not only by reducing their search spells, but also by becoming

[21] Two referees pointed out an alternative approach: a model of directed search, where job seekers/firms can select the pool from which to sample offers. Since low skilled immigrants and Harvard graduates are likely to sample from different pools, their supply would not interact. However, this approach raises the issue of how these pools are formed. Obviously, the heterogeneity of the search problem could be completely resolved by introducing an infinite number of small, but homogeneous pools. With CRS in contact technology, there is nothing to stop the economy from moving in that direction, since the scale is irrelevant by definition. Again, IRS is needed so that agents come to face a trade off between homogeneity and scale in the formation of pools. A formal analysis of this issue is a fruitful avenue for future research. Shimer (2001) studies the effects of directed search in a different setting with universally best jobs. In such a setting CRS can give non trivial equilibria.

[22] Not all the evidence rejects IRS. Yashiv (1996) finds IRS in the Israelian matching function and Shimer (1999) gives demographical evidence for the US that supports the thick market externality arguments. Burdett et al. (1994) argue that IRS are obscured by aggregation and frequency bias in the data.

choosier (resulting in a better match quality). Hence, IRS in the contact technology does not translate one-to-one into IRS in realized matches. This issue cannot be resolved by using data on contacts, or referrals in the wording of Berman (1997), since when a Wall Street stock broker sees a "help wanted" sign in a Hamburger restaurant, he is unlikely to report this event as a referral. This type of self selection explains why so few job offers get rejected. Disappointing as it is, there is no direct way to establish the returns to scale elasticity from any existing data set. An important goal of our analysis is to establish the size of the effect of an increased contact rate on the match quality and on the number of accepted matches, so that our assumptions can be tested indirectly. Proposition 4 allows for the calculation of the returns to scale in the matching process. The elasticities of the cost of search $x^*(s)$ and hence unemployment $u(s)$ and vacancies $v(s)$ with respect to λ are equal to 0.4. Hence, a 1 % increase in total labor supply increases the total number of vacancies and unemployed by (1-0.4)% = 0.6%. By the flow equilibrium and the constancy of the separation rate δ, the number of matches varies proportionally to λ (up to a factor $1 - u(s) = 1 - O[x^*(s)]$). A 0.6 % increase in both inputs in the matching process yields therefore a 1 % increase in the number of matches. The returns to scale elasticity can then be calculated to be equal to $\frac{1}{0.6} = 1.66$. Hence, one third of the increasing returns in the contact technology are absorbed by a greater choosiness of job seekers and firms.

Second, most existing estimates of returns to scale are based on cyclical variation in the number of vacancies and unemployed, not on true variations in the scale of the market. This cyclical variation is affected by all kinds of out-of-equilibrium processes, which usually are not fully modelled. We suspect that this variation does therefore not adequately reflect differences in the scale of a labor market, leading to a downwardly biased estimate of the scale elasticity.

Finally, one has to consider: what is scale? Obviously, saying that scale matters is not the same as saying that the US labor market with 200 million inhabitants is more efficient than that of the Netherlands with only 15 million inhabitants. A more useful way to analyze the effect of scale on the efficiency of the search process is to interpret it as the density of the labor market, for example the number of people per square mile. Then, a comparison between, for example, Manhattan and Wyoming, or equivalently, metropoles and small villages, offers a much better testing ground for returns to scale than intertemporal variation or a comparison between countries.

It should be noted that our Taylor expansion methodology is also applicable in a model including congestion effects. The only difference is that there are a number of additional integrals that need to be evaluated by this methodology.

4 The analysis of the equilibrium

4.1 The constrained planner's optimum

The decentralized search equilibrium is efficient if it generates the same net discounted output as achieved by a social planner, taking search frictions as given. In a homogeneous worker and job type CRS world, constrained efficiency can be achieved when the Hosios (1990) condition is satisfied: the workers' share of the match surplus is equal to his marginal contribution to the matching process. The efficient outcome maximizes the discounted value of aggregate output net of the cost of vacancies and the foregone value of leisure for workers. In Appendix 5, we show that in a stationary equilibrium, optimality requires that:

$$\bar{R}\,(s) = B + \frac{\lambda}{\rho+\delta}\int_{\overline{m}_c(s)} g(c)\left[\bar{P}(c)F(s,c) - \bar{R}\,(s)\right]dc \tag{23}$$

$$K = \frac{\lambda}{\rho+\delta}\int_{\overline{m}_s(c)} \bar{h}(s)\left[\bar{P}(c)F(s,c) - \bar{R}\,(s)\right]ds \tag{24}$$

where $\bar{R}\,(s)$ and $\bar{P}(c)$ are respectively the shadow price for an unemployed worker and a commodity of type c, $\bar{h}(s)$ and $\overline{g}(c)$ are unemployment and vacancies in the social optimum, and $\overline{m}_s(c)$ and $\overline{m}_c(s)$ are the optimal matching sets defined by a condition similar to (8). The only difference between (23) and (24) on the one hand and (12) and (13) on the other hand is that β and $1-\beta$ are replaced by unity in the latter set of equations. Only then, workers and firms receive the full rewards of their marginal contribution to the matching process. There is a simple intuition for this result. In Pissarides' constant returns to scale world, a job seeker entering the labor market imposes a positive externality on employers (since their contact rate goes up) and a negative externality on other job seekers (since their contact rate goes down). In our increasing returns world, there is no negative externality, since the contact rate for workers is independent of the number of other job seekers that are wandering around. We should award workers the full surplus of the match to reward them for the positive externality they impose on employers and mutatis mutandis the same argument applies for employers. Hence, there is insufficient output to reward both sides by their marginal contribution to the contact process, as is standard in an IRS world.

The straightforward question is then whether a market equilibrium is efficient if we replace our assumption of IRS in matching technology by CRS and impose the Hosios condition. Shimer and Smith (2001a) have analyzed this question in a slightly different setup, where they treat workers and jobs symmetrically. In particular, they have a fixed type distribution instead of our assumption of free entry on the vacancy side. They find that good workers and jobs search too little and bad workers search too much. The reason is that the match surplus is shared between both players, so that in a match of a bad and a good worker, the bad worker appropriates part of the surplus that would go to the good worker in a Walrasian world. Clearly, this mechanism depends on the presence of congestion effects. Without congestion effects, good workers could simply ignore the bad workers and wait for a good partner. With congestion effects, the large share of bad workers in the pool of job seekers reduces the chance of a good worker to find a good partner. The absence of congestion in our model explains why this problem does not show up here. On top of that, the free entry condition applied in this paper might affect Shimer and Smith's conclusions. The distinction between good and bad jobs plays a crucial role in their analysis. However, free entry implies that there are no good or bad jobs (all yield zero profit in equilibrium).[23] We leave this issue for future analysis.[24]

4.2 The composition of the efficiency loss

The welfare loss compared to the Walrasian equilibrium can be decomposed into three components: the lost production due to unemployment, the cost of maintaining vacancies and the cost of suboptimal assignment. The first component can be calculated from the

[23] Shimer and Smith analyzed the case of free entry with K differing between job types (not reported in their paper). This heterogeneity is required in their model since otherwise firms would open "good" jobs only. With these assumptions, "good" jobs should be subsidized for optimality. In our model, K can be constant due to the logsupermodularity of $F(s,c)$, so that there are no "good" or "bad" jobs.

[24] Davis (1995) studies a model where good jobs are more costly to create and produce more output. Since firms pay the full creation cost, there are too little good jobs when workers get part of the surplus. However, efficiency in the size of the vacancy pool requires the Hosios condition to be satisfied, which attributes workers part of the surplus. This mechanism is absent in our analysis, since K is constant across job types.

unemployment rate $u(s)$ for type s times the cost of unemployment relative to the reservation wage $B^*(s)$ times a discount factor $\frac{\rho+\delta}{\delta}$ to account for the fact that the investment in search precedes the flow of returns from the employment relation. Hence, by (19) this component is equal to

$$\frac{\rho+\delta}{\delta}u(s)B^*(s) = \frac{2}{3}\beta x^*(s).$$

The second component can be calculated from the vacancy rate $v(s)$ of type s, multiplied by the relative cost of a vacancy, $K^*(s)$, and the discount factor $\frac{\rho+\delta}{\delta}$. By (20) this is equal to,

$$\frac{\rho+\delta}{\delta}v(s)K^*(s) = \frac{2}{3}(1-\beta)\,x^*(s).$$

The cost of suboptimal assignment follows directly from the result discussed in Section 3.2 that the average loss relative to the optimal Walrasian allocation is ${}^{1}/_{3}x^*(s)$. Adding up the three components yields the total cost:

$$Loss_{Wal} \simeq \frac{1}{3}\left[2\beta + 2(1-\beta) + 1\right]x^*(s) = x^*(s) \tag{25}$$

As expected, the total welfare loss is equal to the cost of search. All components vary proportional to $x^*(s)$. The relative importance of unemployment and vacancies in the cost of search are independent of $B^*(s)$ and $K^*(s)$. The ratio between both cost types is fully determined by the bargaining power parameter β. This is due to the fact that firms keep investing in vacancies till the cost of keeping the vacancy open is equal to their expected share in the future surpluses from an employment relation. Similarly, workers adjust their reservation wages such that the share in the expected surplus from search is equal to their reservation wage. When $\beta = \frac{1}{2}$, the decomposition takes an extremely simple form: all three components account for one third of the efficiency loss.

4.3 Efficiency loss due to inadequate incentives

Alternatively, we can decompose the efficiency loss into a purely mechanical loss of search frictions and a loss due to inadequate incentives. Only the latter component can be eliminated by a social planner, see Section 4.1. The mechanical loss can be found by an equivalent of Proposition 4 now using the equations (23) and (24) instead of the Bellman equations (12) and (13). This procedure is equivalent to replacing the factor $\beta(1-\beta)$ in the denominator of Q by unity in equation (18). Hence, the mechanical cost of search are a share $\beta^{0.4}(1-\beta)^{0.4}$ of the total cost of search frictions, $x^*(s)$. The efficiency loss due to inadequate incentives can then be calculated as the difference between the full cost of search and the mechanical loss:

$$Loss_{SP} \simeq \left[1 - \beta^{0.4}(1-\beta)^{0.4}\right] x^{*}(s) \tag{26}$$

This loss is minimized by setting $\beta = 0.5$, which mimics the conclusion of Section 3.3. For $\beta = 0.5$, $1 - \beta^{0.4}(1-\beta)^{0.4} = 0.43$. Hence, about half of the cost of search are due to inadequate incentives. The inefficiency due to the lack of incentives for $\beta = 0.5$ can be offset by quadrupling the size of the labor market (λ). In a similar vein, unemployment and vacancies change by: $\beta^{-0.6}(1-\beta)^{0.4}$, and $\beta^{0.4}(1-\beta)^{-0.6}$, respectively. The ratio of the social planners level of unemployment and the decentralized equilibrium level of unemployment is $\beta^{-0.6}(1-\beta)^{0.4}$, which is 1.15 for $\beta = 0.5$. Social planner's unemployment is therefore higher than unemployment in the decentralized market equilibrium. There is too low a reward for search activities.

Burdett (1979), Diamond (1981), and Marimon and Zilibotti (1999) have shown that unemployment compensation can increase welfare, even when agents are risk neutral, by decreasing mismatch. For an analysis of this issue, we have to account for the funding of B. Suppose we pay unemployment benefits from an insurance premium that is proportional to earned wages. When we define $B^{*}(s)$ relative to the net reservation wage, this will have no further impact on the model, since the difference between the gross and the net value of $r(s)$ is a constant. However, our loss function, equation (25), has to take into account that the cost of unemployment relative to the reservation wage is no longer equal to $B^{*}(s)$ but to unity since unemployment benefits need to be financed while the value of leisure does not. Hence, the optimal unemployment benefit minimizes

$$\frac{1}{3}\left[2B^{*}(s)^{-1}\beta + 3 - 2\beta\right] x^{*}(s) = Q^{o}(s)\left[2B^{*}(s)^{-1}\beta + 3 - 2\beta\right] B^{*}(s)^{0.4}$$

where all terms that do not depend on $B^{*}(s)$ are collected in $Q^{o}(s)$ in the second expression, see equation (18). Hence, the optimal unemployment benefit satisfies:

$$B^{*}(s) = \frac{3\beta}{3-2\beta} \tag{27}$$

The optimal net replacement rate, $B/R(s) = 1 - B^{*}(s)$ is therefore a negative function of the bargaining power of workers. This fits the intuition. When workers have a strong bargaining position, it does not make sense to give them an even better outside option. If a social planner could set both β and $B^*(s)$ jointly, he would make workers as choosy as possible by setting $B(s) \to R(s)$. At the same time he would stimulate vacancy supply by rewarding the employers with almost the full share of the match surplus, hence $\beta \to 0$. However, the implementation of this scheme would require that the social planner is able to manipulate the bargaining process and that he is able to set B independently for each s type.

5 Simulations

5.1 Methodology

We check the quality of our approximations of $x^*(s)$ based on Proposition 4 and equation (21) by comparing them with numerical simulations of the search equilibrium. As a benchmark, we seek a specification of $\underline{l}(s)$ and $\underline{q}(c)$ that yields a realistic shape of the wage distribution and that provides an analytical solution for the Walrasian equilibrium (i.e. $\lambda = \infty$). In particular, we consider an equilibrium where the complexity dispersion parameter is constant along the domain of s: $\gamma(s) = \gamma$. We apply a logarithmic transform of the skill and complexity indices s and c in the presentation of our results:

$$
\begin{aligned}
s^* &= -\ln(-s), s^+ \leq 0 \\
c^* &= \ln c
\end{aligned}
$$

The advantage of these transformed indices is that in the Walrasian equilibrium, the optimal assignment $c^*(s^*)$ and the log reservation wage function $r(s^*)$ are linear. We use a normal distribution with standard deviation σ for log wages. Due to the linear relation between log wages and the transformed skill variable s^*, the latter is also distributed normally in the Walrasian equilibrium. Appendix 6 discusses these issues in greater detail.[25]

We use realistic values for all parameters, so that our simulations give a reasonable impression of what our model would imply empirically. Teulings and Vieira (1998) review

[25] Our numerical simulations are based on a grid for s^* and c^* ranging from minus three till plus three times their standard deviation. We divide the domain of both variables in 100 intervals per standard deviation, yielding a matrix of 601×601. We have three iteration loops. (1) calculate $R(s)$ conditional on $p(c)$ and $g(c)$, see (12), (2) calculate $h(s)$ and $y(c)$ conditional on $p(c)$, $g(c)$ and $R(s)$, see (14) and (15), (3) adjust $g(c)$ to satisfy (13).

the value for the complexity dispersion parameter implied by estimation results for the Netherlands, Portugal, and the United States. They conclude that its value is in the range 2.5-5. We apply $\gamma = 3$ and $\gamma = 5$. Empirically, the standard deviation of log wages, σ, is in the range of $0.30 - 0.60$ for most OECD countries. We apply a value of 0.60 in our simulations. The values for the other parameter are $\delta = 0.15$, $\beta = 0.40$, $\rho = 0.10$. For $B^*(s)$ and $K^*(s)$, we apply their values in the search equilibrium for the evaluation of the quality of the approximation using $B = 0.09$ and $K = 0.50$.[26]

Our simulations serve two goals. First, they give information on the accuracy of our Taylor expansions for various levels of λ. We set $\eta = 0$ for this purpose. Second, the simulations document the clustering of vacancies when $\eta = \infty$ as discussed by Corollary 2 in Section 3.1.

5.2 The accuracy of the Taylor expansions for $\eta = 0$

Table 1 shows aggregate outcomes for different values of λ and γ. We reduce λ by a factor 4 in every simulation, starting from a high value of 2500. We present the standard deviation of log reservations wages, the relative output loss compared to the Walrasian optimum[27], the unemployment rate and the number of vacancies per unit of labor supply. According to Proposition 4, unemployment and vacancies should all increase linearly with x^* (s^*). Furthermore, each reduction of λ by a factor 4 should increase x^* by a factor $4^{0.4} = 1.74$, see equation (18). Table 1 provides strong support for these implications. Based on the value of unemployment, $\lambda = 156$ is a reasonable value ($u \cong 5\%$). This implies that the average worker has $\lambda v \cong 15$ contacts a year. The cost of search and unemployment are smaller when $\gamma = 3$ than when $\gamma = 5$ for all values of λ. Finally, search frictions lead to a substantial increase in wage dispersion, due to general equilibrium effects on task prices .

Table 2 gives the simulation results for values of s^* with each column covering one standard deviation of the skill distribution. We present the unemployment rate $u(s^*)$, the log reservation wage, the maximum surplus, and the forecasting errors in unemployment

[26]The value of δ corresponds to an average job duration of 7 years. The value of β squares with the evidence by Abowd and Lemieux (1993). By a proper normalization, the mean wage equals unity in the Walrasian equilibrium. Hence, a worker earning a wage twice the standard deviation of 0.60 below the mean, faces a replacement rate of $0.09 \exp(-1.20) = 30$ %. In a model including capital, K can be viewed as the cost of the unused capital stock. For the mean wage of unity, $K = 0.5$ implies that capital accounts for one third of value added $(\frac{0.5}{1+0.5})$.

[27]log output under Walras$-$ log[output in the numerical equilibrium $uB-$ vK]

when applying the analytical approximations. The approximations are very accurate. For small search frictions the model almost exactly mimics the simulation outcomes. But also for large search frictions ($\lambda = 156$), the error is less than 7%, even for the extreme worker types.[28] Our approximations work fine except when unemployment rates are very high, which occurs for low values of λ and or for for low skill types (more than 2 standard deviations from the mean). The high unemployment rate for low skilled workers is due to the high replacement rate that these workers face: $1 - B^{*}(s)$. The less precise approximation results can be traced back immediately to the fact that we assume $\frac{u(s)}{1-u(s)} \approx u(s)$, which obviously does not work well for high unemployment levels, say $u(s) > 20\%$.

The pattern of $x^{*}(s^{*})$ along the domain of s^{*} is U-shaped. By the linear relation between $x^{*}(s^{*})$ and $u(s^{*})$, this pattern is repeated in the unemployment rate by type. This phenomenon is due to equation (18), where three factors vary with s: B^{*}, K^{*}, and $\underline{l}$. In our simulations, the latter factor dominates. The skill density is the highest around the mean and the lowest in the tails of the distribution. A low density translates in a large $x^{*}(s^{*})$ due to IRS. Hence, search frictions are largest in the tails of the distribution. These frictions depress reservation wages in both tails of the skill distribution relative to the Walrasian benchmark. Also, they raise the unemployment rate in the tails. The general equilibrium effect offsets the effect on wages, at least for the high s^{*} types. Again, our approximations are very precise. For large search frictions ($\lambda = 156$), our approximation error is less than 2% for the median worker. As we move to the tails, the approximations become less precise but for the worker types located at one s.d. from the mean, the error is still only 7%.

Figures 4 and 5 plot the matching sets and $c^{*}(s^{*})$ in the s^{*}, c^{*}-space for $\lambda = 2500$ and 39 and for $\eta = 0$. The Walrasian benchmark is represented by the diagonal. Those Figures reveal that that larger search frictions lead to wider matching sets. Figure 6 plots $P(c)$ for the case $\lambda = 2500$ relative to its value in the Walrasian equilibrium. Prices are above their Walrasian value in both corners, in particular for the upper tail. This reflects the general equilibrium effects. Obviously, the substantial price increases in the upper

[28] For simulations with $\gamma = 3$ (available upon request), we obtained approximation errors of similar order of magnitude. We find that in particular the low skilled workers benefit from better substititability of worker types. This is due to general equilibrium effects in the reservation wage.

tail pushes up reservation wages.[29]

5.3 Vacancy clustering for $\eta = \infty$

Simulations for $\eta = \infty$ show that no well behaved equilibrium exists and that vacancies cluster together, see Corollary 2 in Section 3.1. Figure 7 shows that when frictions are small, $\lambda = 2500$, there is no evidence of clustering. The matching sets in Figure 7 are very similar to those in Figure 4. However, when frictions become larger, $\lambda = 39$, we clearly see waves appearing in the boundaries of the matching sets in Figure 8.

Figure 9 portrays the output for various job types c for $\lambda = 2500$ relative to that in the Walrasian equilibrium. For a broad range of jobs in the middle of the distribution, their ratio is close to unity. Only in the corners, clustering is clearly observable. For $\lambda = 39$, see Figure 10, search frictions are so important that the process of clustering is clearly manifest along the whole domain of c^*.

6 Final Remarks

The use of Taylor expansions for the characterization of the equilibrium in a search model with two side heterogeneity has been shown to be fruitful. We obtained a precise approximation of the cost of search and made a decomposition of this cost into its three components, unemployment, vacancies and the cost of misassignment. Perhaps, the most striking result of this approach is that the elasticities of the cost of search with respect to its explanatory variables hold for any log-super modular production function and are independent of its precise form. This elasticity, which is 0.4 for most variables, is fully determined by the order of the first non-vanishing term in the Taylor expansion. For example, increasing the replacement rate by 1% leads to a 0.4% increase in the cost of search. Furthermore, we have been able to relate formally the substitutability of worker types to the cost of search: the worse the substitutability between types, the higher the cost of search are.

The analytical expression for the cost of search in terms of observables generates many testable predictions on unemployment, vacancies, and wages, both within a single economy

[29] We have also run simulations with different values of the exogenous values which gave similar good approximations. Changing the values of β showed for example that the losses due to search are indeed minimized for $\beta = 0.5$. We do not report them for space consideration.

and between economies of various scale. For example, one implication is that wages are concave in both worker and job type, where the degree of concavity can be related to the substitutability between worker types and the magnitude of the cost of search. In Gautier and Teulings (2002), we give empirical evidence for this concavity.

We have deliberately ruled out congestion effects in the model, for reasons discussed in Section 3.4, although the implied IRS contact technology seems to contradict most of empirical evidence. We have however shown that one third of the IRS is absorbed by increased selectivity in the acceptance of matches by workers and firms. This points to a more general mechanism similar to the standard problem of self selection in empirical research. In equilibrium, IRS will be fully exploited. Activities which are most search intensive because they either require scarce worker types or a large variety of worker types, will be undertaken in dense areas with a comparative advantage in search. We provide empirical evidence for that in Teulings and Gautier (2002). All those endogenous responses tend to reduce observed returns to scale. However, the absence of congestion effects is no prerequisite for the applicability of our Taylor expansion methodology. It can be generalized easily to a model that allows for congestion effects. The analytical characterization of the cost of search can help to derive empirical predictions that can discriminate between models with and without congestion effects.

7 Literature

Abowd, J.M. and T. Lemieux (1993) The effects of product market competition on collective bargaining agreements: the case of foreign competition in Canada, *Quarterly Journal of Economics*, 108, 983-1014.

Berman E. (1997) Help wanted, job needed: Estimates of a matching function from employment service data, *Journal of Labor Economics*, vol 15, no. 1, part 2. January 1997, 251-292.

Burdett K. (1979) Unemployment insurance payments as a search subsidy: a theoretical analysis, *Economic Inquiry*, 42, 333-43.

Burdett K. and M.G. Coles (1997) Marriage and class, *Quarterly Journal of Economics*, 112, 141-68.

Burdett K. and M.G. Coles (1999) Long-Term partnership formation: Marriage and employment *Economic Journal* 109, 307-334.

Burdett K., M.G. Coles and J.C. Van Ours (1994) Temporal aggregation bias in stock-flow models, *CEPR Discussion Paper* 967.

Davis, S.J. (1995) The quality distribution of jobs and the structure of wages in search equilibrium, *mimeo*, University of Chicago.

Diamond P.A. (1981) Mobility costs, frictional unemployment, and efficiency, *Journal of Political Economy*, 89, 798-812.

Diamond P.A. (1982a) Wage determination and efficiency in search equilibrium, *Review of Economic Studies*, 49, 217-227.

Diamond P.A. (1982b) Aggregate demand management in search equilibrium, *Journal of Political Economy*, 90, 881-894.

Gautier P.A. and C.N. Teulings (2002) How large are search frictions? *Mimeo Tinbergen Institute.*

Hosios, A.J. (1990) On the efficiency of matching and related models of search and unemployment, *Review of Economic Studies 57,* 279-298.

Marimon R. and F. Zilibotti (1999) Unemployment vs. mismatch of talents: reconsidering unemployment benefits, *Economic Journal* 109 266-291.

Meyer B. (1990) Unemployment insurance and unemployment spells, *Econometrica* 58, 757-782.

Mortensen D. (1982) Property rights and efficiency in mating, racing and related games, *American Economic Review*, 72, 968-979.

Petrongolo B. and C.A. Pissarides (1999) Looking into the black box: A survey of the matching function, working paper LSE.

Pissarides C.A. (2000) *Equilibrium unemployment theory*, 2nd edition, MIT press., Cambridge MA.

Rosen, S. (1974) Hedonic prices and implicit markets: product differentiation in pure competition", *Journal of Political Economy* , 82, 34-55.

Sattinger M. (1975) Comparative advantage and the distribution of earnings and abilities, *Econometrica*, 43, 455-68.

Sattinger M. (1995) Search and the efficient assignment of workers to jobs, *International Economic Review*, 36, 283-302.

Shimer R. (1999) The impact of young workers on the aggregate labor market, *NBER Working Paper No.*7306, Cambridge MA.

SHIMER R. (2001) The assignment of workers to jobs in an economy with coordination frictions, *mimeo*, Princeton.

SHIMER R. AND L. SMITH (2000) Assortative matching and search, *Econometrica*, 68, 343-370.

SHIMER R. AND L. SMITH (2001a) Matching, Search, and Heterogeneity, *Advances in Macroeconomics (BE Journals in Macroeconomics)* 1(1): Article 5.

SHIMER R. AND L. SMITH (2001b) Non-stationary Search, *Mimeo.*

SMITH L. (1995) Cross-sectional dynamics in a two-sided matching model, mimeo MIT.

TEULINGS C.N. (1995) The wage distribution in a model of the assignment of skills to jobs, *Journal of Political Economy*, 280-315.

TEULINGS (2002) Comparative advantage, relative wages, and the accumulation of human capital, Tinbergen Institute discussion paper 02-081/3.

TEULINGS C.N. AND P.A. GAUTIER (2002) Search and the city, mimeo, Tinbergen Institute discussion paper 061/3.

TEULINGS C.N. AND J. VIEIRA (1998) Urban versus rural return to human capital in Portugal: A Cook-Book Recipe for applying assignment models, *Tinbergen Discussion Paper*, 98-095/3.

TINBERGEN J. (1956) On the theory of income distribution, *Weltwirtschaftliches Archiv*, 77, 156-173.

YASHIV E. (1996) The determinants of equilibrium unemployment, *mimeo*, department of economics, Tel Aviv University.

A Appendices

A.1 Proof of Proposition 1

Simplification of the notation

L can be normalized to unity without loss of generality. The model is written in a simplified form for the sake of convenience:

$$\begin{aligned}
R(s) &= \chi\theta \int_{c^-}^{c^+} m(s,c)\, g(c) \left[P(c)\, e^{sc} - R(s)\right] dc, \forall s && (28)\\
1 &= \psi \int_{s^-}^{s^+} m(s,c)\, h(s) \left[P(c)\, e^{sc} - R(s)\right] ds, \forall c && (29)\\
\underline{l}(s) - h(s) &= \theta\, h(s) \int_{c^-}^{c^+} m(s,c)\, g(c)\, dc, \forall s && (30)\\
Y(c) &= \theta g(c) \int_{s^-}^{s^+} m(s,c)\, h(s) e^{sc} ds, \forall c && (31)\\
P(c) &= \underline{Q}(c)^2 \left(\frac{Y(c)}{Y^o}\right)^{-1}, \forall c && (32)\\
Y^o &= \exp\left[\int_{c^-}^{c^+} \underline{Q}(c)^2 \ln Y(c)\, dc\right] && (33)
\end{aligned}$$

where $\chi \equiv \frac{\beta\delta}{\rho+\delta}$, $\psi \equiv \frac{(1-\beta)\lambda}{(\rho+\delta)K}$, $\theta \equiv \frac{\lambda}{\delta}$, where $\underline{Q}(c) \equiv \exp\left[\underline{q}(c)\right]$ and where $m[s, c, R(s), P(c)]$ is an indicator function taking the value unity if $P(c)\, F(s,c) - R(s) \geq 0$ and zero otherwise. We omit the final two arguments of $m(\cdot)$ for the sake of convenience. Equation (29) applies with equality since we impose the feasibility constraint and since $\eta < \infty$.

Lemma 1: A lower bound for $u(s)$: $u(s) > u^-, \forall s$

Equation (29) implies:

$$P(c) \int_{s^-}^{s^+} m(s,c)\, h(s) e^{sc}\, ds > \frac{1}{\psi}$$

Then, equation (31) requires:

$$\frac{\theta}{\psi} g(c) < Y(c)\, P(c) \leq \underline{Q}^{+2}$$

where $\underline{Q}^+ \equiv \sup\left[\underline{Q}(c)\right]$. The last inequality follows from equation (32). Substitution in equation (30) yields:

$$u(s) = \left[1 + \theta \int_{c^-}^{c^+} m(s,c)\, g(c)\, dc\right]^{-1} > \left[1 + \psi\left(c^+ - c^-\right) \underline{Q}^{+2}\right]^{-1} \equiv u^-$$

Q.E.D.

Definition of the mapping

Define: $V(c) \equiv \underline{Q}(c)^{-2}\, Y(c) = Y^o P(c)^{-1}$. The equations (32) and (31) and the definition of $V(c)$ are used to eliminate $P(c)$, $Y(c)$, and $g(c)$. The model can be rewritten as a mapping of $[R, V, u, Y^o]$ into itself:

$$\begin{aligned}
TR(s) &= \frac{\chi}{1+\chi\bar{G}_{R,V,Y^o,u}} \int_{c^-}^{c^+} \bar{g}(c)_{R,V,Y^o,u} \max\left[Y^o e^{sc}, V(c) R(s)\right] dc, \forall s \\
TV(c) &= \frac{\psi}{1+\psi H_u} \int_{s^-}^{s^+} \underline{l}(s)\, u(s) \max\left[Y^o e^{sc}, V(c) R(s)\right] ds, \forall c \\
Tu(s) &= \max\left\{\left[1+\int_{c^-}^{c^+} m(s,c)\, \bar{g}(c)_{R,V,u} V(c)\, dc\right]^{-1}, u^-\right\}, \forall s \\
TY^o &= \min\left\{\exp\left[\int_{c^-}^{c^+} \underline{Q}(c)^2 \left[2\underline{q}(c) + \ln V(c)\right] dc\right], Y^o_W\right\}
\end{aligned}$$

where Y^o_W is the output in the Walrasian case and where:

$$\begin{aligned}
\bar{g}(c)_{R,V,Y^o,u} &\equiv \frac{\underline{Q}(c)^2}{\int_{s^-}^{s^+} m(s,c)\, \underline{l}(s)\, u(s) e^{sc} ds} \\
\bar{G}_{R,V,Y^o,u} &\equiv \int_{c^-}^{c^+} \bar{g}(c)_{R,V,Y^o,u}\, dc \\
H_u &\equiv \int_{s^-}^{s^+} \underline{l}(s)\, u(s) ds
\end{aligned}$$

A search equilibrium is a fixed point of T for which $Y^o < Y^o_W$ and $u(s) > u^-, \forall s$. Since $Y^o = Y^o_W$ can only be attained when $u(s) = 0, \forall s$ and since $u(s) > u^-, \forall s$ by Lemma 1, a point for which $Y^o = Y^o_W$ and $u(s) = u^-$ can never be a fixed point of this mapping. Hence, any fixed point of T is a search equilibrium. Note that the specification of equation (29) includes the trivial equilibrium with $Y^o = 0$, since then $V(c) = 0$, satisfying $TV(c) = V(c)$. Hence, the existence proof does not invoke the feasibility constraint (48).

The proof proceeds in two steps. First, we apply a contraction mapping argument to $Tu(s)$, proving the existence of a unique equilibrium $u(s)_{R,V,Y^o}$ conditional on R, V, Y^o. Second, we apply the Schauder fixed point theorem to proof existence of a fixed point for $T[R, V, Y^o]$.

Step 1: Existence and uniqueness of u_{R,V,Y^o}

The subscripts R, V, Y^o are omitted for notational convenience. We apply a log transformation of the unemployment rate $w(s) \equiv \ln u(s)$. Then, we must proof existence and uniqueness of a fixed point of the mapping:

$$Ww(s) = -\ln\left[1 + A_w(s)\right]$$

where:

$$A_w(s) \equiv \min\left[\int_{c^-}^{c^+} m(s,c)\,B_w(c)^{-1}\,dc, A^+\right]$$
$$B_w(c) \equiv \frac{V(c)}{\underline{Q}(c)^2}\int_{s^-}^{s^+} m(s,c)\,\underline{l}(s)\,u(s)\,e^{sc}ds$$

and where $A^+ \equiv \frac{1-u^-}{u^-}$. By construction, $Ww(s) \geq \ln u^-$. $Ww(s)$ is a contraction mapping if $\sup_s |Ww_1 - Ww_2| < k\Omega$, with $k<1$ and where $\Omega \equiv \sup_s |w_1 - w_2|$. We have:

$$|B_1 - B_2| \leq \left(e^{\Omega}-1\right)B_2 \Rightarrow$$
$$\left|B_1^{-1} - B_2^{-1}\right| = \left|\frac{B_1 - B_2}{B_1 B_2}\right| \leq \left(e^{\Omega}-1\right)B_1^{-1} \Rightarrow$$
$$|A_1 - A_2| \leq \left(e^{\Omega}-1\right)A_1 \Rightarrow$$
$$|Ww_1 - Ww_2| = \left|\ln\left(\frac{1+A_2}{1+A_1}\right)\right| \leq \ln\left(1+\frac{|A_1-A_2|}{1+A_1}\right) \leq \frac{A_1}{1+A_1}\Omega$$

where A_i is short hand notation for $A_{w_i}(s)$, and mutatis mutandis the same for B_i.We have: $\sup\left[\frac{A_1}{1+A_1}\right] \leq \frac{A^+}{1+A^+} < 1$. Q.E.D.

Step 2: Existence of R, V, Y^o using $u = u_{R,V,Y^o}$

Since the model is invariant to a linear transform of s, see Section 2.1, we can normalize s^+ to zero without loss of generality. Hence $\sup_{s,c} e^{sc} = \exp(s^+c^+) = 1$ and $Y_W^o \leq 1$. Let R and V be families of Lipschitz functions on $[s^-, 0]$ and $[c^-, c^+]$ respectively, such that $|R(s+h) - R(s)| \leq k_R h$, where $k_R \equiv c^+ Y_W^o$, and $|V(c+h) - V(c)| \leq k_P h$, where $k_P \equiv -s^- Y_W^o$. Furthermore, let $R, V, Y^o \in [0, Y_W^o]$. Hence, the domain of R, V, Y^o is non-empty, closed, bounded and convex. Clearly, since $\sup_{sc} e^{sc} = 1$ and $Y_W^o \leq 1$, $TR, TV, TY^o \in [0, Y_W^o]$. The application of the Schauder fixed point theorem on the mapping $T: R, V, Y^o \rightarrow R, V, Y^o$ requires the families R and V to be equicontinuous and T itself to be continuous.

Equicontinuity of R and V

$$\begin{aligned} TR(s+h) - TR(s) &< \sup_c \left\{\max\left[Y^o e^{(s+h)c}, V(c)\,R(s+h)\right] - \max\left[Y^o e^{sc}, V(c)\,R(s)\right]\right\} \\ &\leq \sup_c \left\{\max\left[Y^o e^{(s+h)c}\left(e^{hc}-1\right), V(c)\left\{R(s+h) - R(s)\right\}\right]\right\} \\ &\leq \max\left[Y_W^o\left(1 - e^{-hc^+}\right), R(s+h) - R(s)\right] \leq k_R h \end{aligned}$$

A family of Lipschitz functions with the same modulus is equicontinuous. The same argument applies to TV.

Continuity of $T: R, V, Y^o \rightarrow R, V, Y^o$

The proof of the continuity of TR in R and TV in V is exactly similar to Shimer and Smith (2000: 366). Since R and V enter symmetrically in TR (and the same in TV), these proofs are also similar, and mutatis mutandis the same for Y^o. The proof of the continuity of TY^o in V is straightforward. Q.E.D.

A.2 The proof of Proposition 2

Some notation and the continuity of $R(s)$, $P(c)$ and $x(s,c)$

Let the function $r^0(s,m)$ be defined implicitly by:

$$R^0(s,m) \equiv B + \frac{\lambda\beta}{\rho+\delta}\int_m g(c)\left[P(c)F(s,c) - R^0(s,m)\right] dc \tag{34}$$

where $R^0(s,m) \equiv \exp\left[r^0(s,m)\right]$; $p^0(c,m)$ is defined similarly. Hence:

$$r(s) = \max_m r^0(s,m) = r^0\left[s, m_c(s)\right]$$

The continuity of $R(s)$ and hence $r(s)$ follows from the continuity of $F(s,c)$ and the fact that type $s \pm h$ can always mimic the matching $m_c(s)$ of type s. The same argument applies to $P(c)$. The differentiability of $x(s,c)$ follows immediately. Q.E.D.

Lemma 1: $r_s^0(s,m) > 0, r_{ss}^0(s,m) \geq 0, p_{cc}^0(c,m) \leq 0$, with strict inequality if m is not single valued.

Partially differentiating (34) with respect to s, dividing by $R^0(s,m)$, and differentiating again yields:

$$\begin{aligned} r_s^0(s,m) &= \frac{\lambda\beta}{\rho+\delta}\int_m g(c)\left[c\frac{P(c)F(s,c)}{R^0(s,m)} - r_s^0(s,m)\right] dc \\ r_{ss}^0(s,m) &= \frac{\lambda\beta}{\rho+\delta}\int_m g(c)\left\{c\left[c - r_s^0(s,m)\right]\frac{P(c)F(s,c)}{R^0(s,m)} - r_{ss}^0(s,m)\right\} dc \end{aligned} \tag{35}$$

Since $g(c)\,c\frac{P(c)F(s,c)}{R^0(s,m)} > 0$, the first equation implies $r_s^0(s,m) > 0$. Rewriting the first equation and substitution of equation (34) yields:

$$r_s^0(s,m)B = \frac{\lambda\beta}{\rho+\delta}\int_m \left[c - r_s^0(s,m)\right] g(c)P(c)F(s,c)dc \geq 0$$

Define the expectation E[.] and variance operator V[.] for c over the support m with the density function $g(c)P(c)F(s,c)/G$, where $G = \int_m g(c)P(c)F(s,c)\,dc$. By the last equation:

$$\mathrm{E}[c] = \left(1 + \frac{\rho+\delta}{\lambda\beta G}B\right) r_s^0(s,m) \geq r_s^0(s,m)$$

Hence:

$$\begin{aligned} &\int_m c\{c - r_s^0(s,m)\}\frac{g(c)P(c)F(s,c)}{R^0(s,m)}dc \\ \geq\; &\int_m c\{c - \mathrm{E}[c]\}\frac{g(c)P(c)F(s,c)}{R^0(s,m)}dc = \frac{G}{R^0(s,m)}\mathrm{V}[c] \geq 0 \end{aligned}$$

Hence, by the second equation of (35), $r_{ss}^0(s,m) \geq 0$. If m is not single valued, then strict inequality applies, since $\mathrm{V}[c] > 0$. The proof of $p_{cc}^0(c,m) \leq 0$, with strict inequality if m is not single valued, is straightforward. Q.E.D.

Part 1: the strict convexity of $m_c(s)$ and $m_s(c)$ and the uniqueness of $c(s)$ and $s(c)$

The strict convexity of the matching sets is proven by the strict concavity of $x(s,c)$ in s. This requires:

$$x(s+h,c) - 2x(s,c) + x(s-h,c) = -[r(s+h) - 2r(s) + r(s-h)] < 0$$

Since $r^0(s,m) \leq r^0[s,m_c(s)] = r(s)$:

$$r(s+h) - 2r(s) + r(s-h) \geq r^0[s+h,m_c(s)] - 2r^0[s,m_c(s)] + r^0[s-h,m_c(s)] \geq 0$$

where the last inequality follows from Lemma 1, $r^0_{ss}(s,m) \geq 0$. Equality requires $r^0_{ss}(s,m) = 0$. We show that $r^0_{ss}(s,m) = 0$ is inconsistent with $m_c(s)$ being single valued, which contradicts Lemma 1. Since $x[s,c(s)]$ maximizes $x(s,c)$ and by the strict concavity of $x(s,c)$, $x[s,c(s)] - x[s,c(s)-h] \geq 0$ and $x[s,c(s)+h] - x[s,c(s)] \leq 0$ for any $h > 0$. By assumption, $x[s,c(s)] \geq 0$, since otherwise $m_c(s) = \emptyset$. Hence, if $x[s,c(s)+h] - 2x[s,c(s)] + x[s,c(s)-h] = 0$, then $x[s,c(s)+h] = x[s,c(s)] = x[s,c(s)+h] \geq 0$, and therefore, $c(s)+h \in m_c(s)$ and $c(s)-h \in m_c(s)$. Hence, $m_c(s)$ is not single valued, contradicting the condition for $r^0_{ss}(s,m) = 0$. $x(s,c)$ is therefore strictly concave in s. A similar argument establishes strict concavity in c.[30] Hence, $m_s(c)$ and $m_c(s)$ are convex and the maxima $c(s)$ and $s(c)$ are unique. Q.E.D.

Lemma 2: the continuity of $c^-(s), c^+(s), s^-(c)$ and $s^+(c)$

For any interior solution of $c^-(s)$, $x[s,c^-(s)] = x[s-h,c^-(s-h)] = 0$. By the continuity of $r(s)$:

$$\lim_{h\to 0} x[s-h,c(s-h)] = \lim_{h\to 0} x\left[s,c^-(s-h)\right] = 0$$

We shall proof that the latter equality holds only if $c^-(s)$ is continuous. Since $m_c(s)$ is not single valued, $c^-(s) < c(s)$. By the concavity of $x(s,c)$ and the definition of $c(s)$ as its maximum, $x(s,c) - x(s,c-h) > 0$ for any $c < c(s)$ and for any $h > 0$ and hence for $c = c^-(s)$. A similar argument applies to $c^+(s), s^-(c)$ and $s^+(c)$. Q.E.D.

Lemma 3: the continuity of $h(s)$ and $g(c)$

By the exogeneity of labor supply and by equation (14), $h(s)$ is bounded: $0 < h(s) < \underline{l}(s)$. Hence, $\int_{m_s(c)} h(s) F(s,c)\, ds$ is continuous by the continuity of $s^-(c)$ and $s^+(c)$. Moreover, this integral must be strictly positive for equation (13) to have a solution. By equation (5) and the continuity of $\underline{q}(c)$ and $p(c)$, $y(c)$ is continuous. Hence, $g(c)$ is continuous by equation (15). The continuity of $g(c)$ implies the continuity of $h(s)$ by equation (14) and the continuity of $c^-(s)$ and $c^+(s)$. Q.E.D.

Part 2: the twice differentiability of $r(s)$ and $p(c)$, $r'(s) > 0, r"(s) > 0, p"(c) < 0, x_{ss}(s,c) < 0, x_{cc}(s,c) < 0$

[30] We use that $Y(c) > 0$ for a finite η, so that equation (13) holds with equality for all c.

By Lemma 3, we can apply Leibniz rule to equation (12) and (13). Then, by Lemma 1 and since $x[s, c^-(s)] = x[s, c^+(s)] = 0$ for any interior solution of $c^-(s)$ and $c^+(s)$, we have:

$$r'(s) = r_s^0[s, m_c(s)] > 0$$

Totally differentiating equation $x[s, c^-(s)] = 0$ and since $x_c[s, c^-(s)] \neq 0$, see the argument in the proof of Lemma 3, shows that $c^-(s)$ is differentiable for any interior solution, with $c^{-'}(s) = -\frac{x_s[s,c^-(s)]}{x_c[s,c^-(s)]}$. The same applies to $c^+(s)$. Hence, for interior solutions to $c^-(s)$ and $c^+(s)$:

$$r"(s) = r_{ss}^0[s, m_c(s)] - 2\left\{g\left[c^+(s)\right]\frac{x_s[s, c^+(s)]^2}{x_c[s, c^+(s)]} - g\left[c^-(s)\right]\frac{x_s[s, c^-(s)]^2}{x_c[s, c^-(s)]}\right\}$$

Since $c^+(s) > c(s)$ and hence $x_c[s, c^+(s)] < 0$ (and reverse for $x_c[s, c^-(s)]$), the term in curly brackets is negative. Hence, by Lemma 1, $r"(s) > 0$ and hence $x_{ss}(s, c) < 0$. Whenever matching sets are bounded by the upper support for c, $c^{+'}(s) = 0$, so that the first term in curly brackets drops out. Mutatis mutandis the same applies to the lower support of c. Hence, $r'(s)$ is not differentiable only at the transition points between interior and exterior solutions. A symmetric argument establishes the twice differentiability of $p(c)$, with $p"(c) < 0$ and hence $x_{cc}(s, c) < 0$. Q.E.D.

Part 3: $c^{-'}(s) > 0$ and $c^{+'}(s) > 0$ for any interior solution

Since $s^-(c)$ is the inverse of either $c^+(s)$ or $c^-(s)$, and vice versa for $s^+(c)$, it is sufficient to prove that $s^{-'}(c) > 0$ and $s^{+'}(c) > 0$. Since $s^{-'}(c) = -x_c[s^-(c), c]/x_s[s^-(c), c]$, we must prove that $x_c[s^-(c), c]$ and $x_s[s^-(c), c]$ are of opposite sign. By the definition of $s(c)$, $x_s(s, c) > 0$ for any $s < s(c)$ and hence for $s^-(c)$; therefore we have to proof $x_c[s^-(c), c] < 0$. By the definition of $c(s)$, $x_c(s, c) < 0$ for any $c > c(s)$. We therefore have to proof that for any interior $s^-(c)$:

$$c > c\left[s^-(c)\right] \tag{36}$$

Since $c'(s) > 0$, $c(s)$ has a well defined inverse for any interior solution, which is defined as $t(c)$, with $t'(c) > 0$. For any interior solution, inequality (36) can therefore be written as:

$$t(c) > s^-(c)$$

which holds if we can show that $t(c) \in m_s(c)$ for all c. The problem here is to rule out cases as sketched in Figure 11 where either (s^-, c^-) or (s^+, c^+) are not an element of the matching set because $t(c)$ is not defined along the full support of c. We have: $x_c[t(c), c] = 0$. Hence, we must proof that $\forall c \exists s \in m_s(c)$ such that $x_c(s, c) = 0$. Consider equation (13). Since this equation applies identically, its first derivative with respect to c must apply. Since $x_c(s, c) = s + p'(c)$:

$$0 = \int_{m_s(c)} P(c)\, F(s, c)\, x_c(s, c)\, ds$$

which provides the proof by the continuity of $x_c(s,c)$ in s.[31] A similar argument proves $s^{+'}(c) > 0$. Q.E.D.

Part 4: the differentiability of $h(s)$ and $g(c)$

Having established the differentiability of $c^+(s), c^-(s), s^+(c)$ and $s^-(c)$, we can repeat the argument in Lemma 3 to establish the differentiability of $h(s)$ and $g(s)$. Q.E.D.

A.3 The proof of Proposition 3

Part 1: the integrals in equation (12) and (14)

We have:

$$\int_{m_c(s)} g(c) \left[\frac{P(c) F(s,c)}{R(s)} - 1 \right] dc = \int_{m_c(s)} g(c) \left[e^{x(s,c)} - 1 \right] dc$$

By Proposition 2, $g(c)$ is differentiable, $x_c[s, c(s)] = 0$, and $x_{cc}[s, c(s)] = p''[c(s)] = -1/c'(s)$. Define $z \equiv c - c(s)$, $\Delta(s) \equiv c^+(s) - c(s)$, and $\Delta^-(s) \equiv c^-(s) - c(s)$. Then, by a second order Taylor expansion of the integrand around $c(s)$:

$$\int_{m_c(s)} g(c) \left[e^{x(s,c)} - 1 \right] dc = g \int_{\Delta^-}^{\Delta} \left[1 + \frac{g'}{g} z + o(z) \right] \left[E \left(1 - \frac{1}{2c'} z^2 + o(z^2) \right) - 1 \right] dz \tag{37}$$

where $E \equiv e^x$ with $x \equiv x[s, c(s)]$ and where g and g' are evaluated at $c(s)$ and where c' denotes $c'(s)$. The arguments of all functions are suppressed for convenience. By a second order Taylor expansion of $x(s,c)$ around $c(s)$ (again, the first order term drops out) and since $x[s, c(s) + \Delta(s)] = x[s, c(s) + \Delta^-(s)] = 0$:

$$\begin{aligned} x &= \frac{1}{2c'} \Delta^2 + o(\Delta^2) \\ \Delta^2 &= 2c' [x + o(x)] \end{aligned} \tag{38}$$

and the same for Δ^-. Hence:

$$\Delta^2 + o(\Delta^2) = \Delta^{-^2} + o\left(\Delta^{-^2}\right) \Rightarrow \Delta - \Delta^- = o(\Delta)$$

Substituting these results in equation (37) yields:

[31] Note that this argument applies only because K does not depend on c. Alternatively, when K would depend on c, we can prove Part 4 for the case $B = 0$, starting from $c^-(s)$, applying the inverse of $s(c)$, and showing that this inverse is element of $m_c(s)$ by differentiating the Bellman equation for the worker. When K depends on c and $B > 0$, we cannot rule out the case that (s^-, c^-) does not belong the matching set. The interpretation is that low skilled workers have a comparative advantage in search.

$$
\begin{aligned}
& g\int_{\Delta^-}^{\Delta}\left[1+\frac{g'}{g}z+o\left(z\right)\right]\left[E\left(1-\frac{1}{2c'}z^2+o\left(z^2\right)\right)-1\right]dz \\
= \; & g\left[\left(E-1\right)\left(z+o(z)\right)-\frac{1}{2c'}E\left(\frac{1}{3}z^3+o\left(z^3\right)\right)\right]_{z=\Delta^-}^{\Delta} \\
= \; & \frac{1}{2c'}g\left[\frac{2}{3}z^3\right]_{z=-\Delta^-}^{\Delta}+o\left(\Delta^3\right)=\frac{2}{3c'}g\Delta^3+o\left(\Delta^3\right)
\end{aligned}
$$

Squaring the final expression and applying (38) yields the first equation of Part 1. By a similar argument:

$$
\begin{aligned}
\int_{m_c(s)} g(c)\,dc &= g\int_{\Delta^-}^{\Delta}\left[1+\frac{g'}{g}z+o\left(z\right)\right]dz \\
&= g\left[z\right]_{z=-\Delta^-}^{\Delta}+o\left(\Delta\right)=2g\Delta+o\left(\Delta\right)
\end{aligned}
$$

Squaring the final expression and applying (38) yields the second equation. Q.E.D.

Part 2: the integral in equation (13)

The proof is exactly similar to that of the first equation of Part 1, integrating over s instead of c, and replacing $g\left(c\right)$ by the composite function $\left[h\left(s\right)R\left(s\right)\right]$. Q.E.D.

A.4 The proof of Proposition 4

Step 1: Substitution of the integrals of Proposition 3

We square the equations (12), (13), and (14), where equation (13) is evaluated at $c=d\left(s\right)$, so that $s\left(c\right)=s\left[d\left(s\right)\right]=s$ and that $s'\left(c\right)=d'\left[s(c)\right]^{-1}$. Substitution of the expressions for the integrals from Proposition 3 and using the definitions of $B^*\left(s\right)$ and $K^*\left(s\right)$ in Proposition 4 yields:

$$
\begin{aligned}
B^*\left(s\right)^2 &= \frac{32}{9}\left(\frac{\lambda\beta}{\rho+\delta}\right)^2 g\left[c(s)\right]^2 c'\left(s\right)\left\{x^*\left(s\right)+o\left[x^*\left(s\right)\right]\right\}^3 \qquad (39)\\
\left[\underline{l}(s)-h(s)\right]^2 &= 8\left(\frac{\lambda}{\delta}\right)^2 h(s)^2 g\left[c(s)\right]^2 c'\left(s\right)\left\{x^*\left(s\right)+o\left[x^*\left(s\right)\right]\right\} \\
K^*\left(s\right)^3 &= \frac{32}{9}\left(\frac{\lambda(1-\beta)}{\rho+\delta}\right)^2 h(s)^2 d'\left(s\right)^{-1}\left\{x^+\left(s\right)+o\left[x^+\left(s\right)\right]\right\}^3
\end{aligned}
$$

where $x^+\left(s\right)\equiv x^o\left[d\left(s\right)\right]$.

Step 2: $x^+\left(s\right)=x^*\left(s\right)+o\left[x^{*'}\left(s\right)\right]$

Since $x^*\left(s\right)\equiv x\left[s,c\left(s\right)\right]$ and since $x_c\left[s,c\left(s\right)\right]=0$, $x^{*'}\left(s\right)=x_s\left[s,c\left(s\right)\right]$. Since $x_s\left[s,d\left(s\right)\right]=0$ and $x_{sc}\left(s,c\right)=1$, a Taylor expansion of $x_s\left(s,c\right)$ with respect to c in $c\left(s\right)$ yields:

$$
x^{*'}\left(s\right)=x_s\left[s,c\left(s\right)\right]=x_{sc}\left[s,c\left(s\right)\right]\left[c\left(s\right)-d\left(s\right)\right]=c\left(s\right)-d\left(s\right) \qquad (40)
$$

A Taylor expansion of $x(s,c)$ in $c(s)$, using the definitions of $x^*(s)$ and $x^+(s)$, $x_c[s,c(s)] = 0$, and equation (40) yields:

$$\begin{aligned} x^+(s) &= x^*(s) + \frac{1}{2}x_{cc}[s,c(s)][c(s)-d(s)]^2 + o\left[[c(s)-d(s)]^2\right] \\ &= x^*(s) + o\left[x^{*'}(s)\right] \end{aligned} \tag{41}$$

Q.E.D.

Step 3: Simplification of the system (39)

We drop the argument s from all functions for the sake of convenience. From equation (40): $d' = c' - x^{*"}$. By the equations (39) and (41) and the definition of Q in Proposition 4 we have:

$$\begin{aligned} \frac{1}{2}\left(\frac{QB^*K^*}{\underline{l}-h}\right)^2\left(c' - x^{*"}\right) &= (x^* + o[x^*])^2\left(x^* + o[x^*] + o\left[x^{*'}\right]\right)^3 \\ \frac{h}{\underline{l}-h} &= \frac{2}{3}\frac{\delta\beta}{\rho+\delta}B^{*-1}(x^* + o[x^*]) \\ \frac{g[c(s)]\sqrt{c'(c'-x^{*"})}}{\underline{l}-h} &= \frac{2}{3}\frac{\delta(1-\beta)}{\rho+\delta}K^{*-1}(x^* + o[x^*]) \end{aligned} \tag{42}$$

Note that the equations in (42) are differential equations in s.

Step 4: $x^{*'}(s) = O[x^*(s)]$ solves equation (42)

Observe that $\lambda Q = \frac{9}{8}\frac{(\rho+\delta)^2}{\delta\beta(1-\beta)}$ does not depend on λ. Hence:

$$\lim_{\lambda\to\infty}\lambda^2 x^{*^5} = \frac{1}{2}\left(\frac{\lambda QB^*K^*}{\underline{l}-h}\right)^2 c'$$

solves first the differential equation (42) for $\lim_{\lambda\to\infty}$, since differentiating this solution with respect to s shows that $x^{*'} = x^{*"} = O[x^*]$. Q.E.D.

Step 5: $\underline{l}(s) - h(s) = \underline{l}(s)\{1 + O[x^*(s)]\}$

Inspection of the second equation of (42) proofs this result directly. Q.E.D.

Proof of Proposition 4:

Substitution of Step 4 and 5 in equation (42) yields Proposition 4. Q.E.D.

A.5 The planner's problem

Since the optimization problem is dynamic, all policy functions are extended with a time index.[32] A bar on top of a function denotes its value in the social planner's optimum.

[32] Since matching sets are also time dependent, $s \in m_s(c,t)$ does not necessarily imply $s \in m_s(c,t+h)$, $h > 0$. For example, in Shimer and Smith (2001b) the optimal policy exhibits cycles, since the planner periodically destroys a subset of matches to change the composition of the searching population. We simplify the social planner's problem by ruling out this type of separations a priori.

Let $\bar{m}(s,c,t)$ be an indicator function with $\bar{m}(s,c,t)=1 \Leftrightarrow s \in \bar{m}_s(c,t)$ and $\bar{m}(s,c,t)=0 \Leftrightarrow s \notin \bar{m}_s(c,t)$. Define $J(s,c,t)$ as the net discounted value of the future output of all matches of worker type s and job type c which start at time t:

$$\begin{aligned} J(s,c,t) &\equiv \lambda \bar{m}(s,c,t)\,\bar{h}(s,t)\bar{g}(c,t)\tilde{P}(c,t)F(s,c) \\ \tilde{P}(c,t) &\equiv \int_t^\infty \bar{P}(c,\tau)e^{-(\rho+\delta)\tau}d\tau \end{aligned}$$

The social objective function reads:

$$\max \int_0^\infty e^{-\rho t}\left(\int_{s^-}^{s^+}\int_{c^-}^{c^+} J(s,c,t)\,dcds + B\int_{s^-}^{s^+}\bar{h}(s,t)\,ds - K\int_{c^-}^{c^+}\bar{g}(c,t)\,dc\right)dt$$

subject to the dynamic constraint:

$$\dot{\bar{h}}(s,t)=\delta\left[\underline{l}(s)-\bar{h}(s,t)\right]-\lambda\,\bar{h}(s,t)\int_{c^-}^{c^+}\bar{m}(s,c,t)\,\bar{g}(c,t)\,dc \tag{43}$$

Assume that there exists a stationary optimum and restrict our attention to this optimum.[33] Hence, we can suppress the time dependence of variables in the notation and we can simplify $\widetilde{P}(c,t)=(\rho+\delta)^{-1}\bar{P}(c,t)$. The current value Hamiltonian with the current value Lagrange multiplier $\widetilde{R}(s)$ for the dynamic constraint (43) reads:

$$\begin{aligned} H &= \int_{s^-}^{s^+}\int_{c^-}^{c^+} J(s,c)\,dcds + B\int_{s^-}^{s^+}\bar{h}(s)\,ds - K\int_{c^-}^{c^+}\bar{g}(c)\,dc \\ &\quad + \int_{s^-}^{s^+}\widetilde{R}(s)\left\{\delta\left[\underline{l}(s)-\bar{h}(s)\right]-\lambda\,\bar{h}(s)\int_{c^-}^{c^+}\bar{m}(s,c)\,\bar{g}(c)\,dc\right\}ds \end{aligned}$$

First, consider the contribution of vacancies of type c. The objective function can be written as the integral of $H^c(c)$ over c plus a term that does not depend on $\bar{g}(c)$, where $H^c(c)$ is defined as:

$$H^c(c) \equiv \int_{s^-}^{s^+} J(s,c) - \lambda\bar{m}(s,c)\,\bar{h}(s)\,\bar{g}(c)\widetilde{R}(s)\,ds - K\bar{g}(c)$$

The first order condition for $\bar{g}(c)$ is:

$$\frac{\partial H^c(c)}{\partial \bar{g}(c)}=0 \tag{44}$$

[33] As demonstrated by Shimer and Smith (2001b), see the previous note, this assumption is by no means innocuous. An optimal policy might be cyclical. This issue falls outside the scope of this paper.

Next, consider the rules for optimal matching. The objective function can be written as the integral of $H^s(s)$ over s plus a term that does not depend on $\bar{h}(s)$ and $\widetilde{R}(s)$, where $H^s(s)$ is defined as:

$$H^s(s) \equiv \int_{c^-}^{c^+} J(s,c)\,dc + B\bar{h}(s) + \widetilde{R}(s)\left\{\delta\left[\underline{l}(s) - \bar{h}(s)\right] - \lambda\,\bar{h}(s)\int_{c^-}^{c^+} \bar{m}(s,c)\,\bar{g}(c)\,dc\right\}$$

where $\bar{h}(s)$ is the state variable and $\bar{m}(s,c)$ is the control variable. In the steady state, where $\dot{\widetilde{R}}(s) = 0$, the costate equation and the first order condition imply:

$$\begin{aligned} \frac{\partial H^s(s)}{\partial \bar{h}(s)} &= \rho\widetilde{R}(s) \\ \bar{m}(s,c) &= \begin{cases} 1 \text{ if } F(s,c)\,\widetilde{P}(c) \geq \widetilde{R}(s) \\ 0 \text{ if } F(s,c)\,\widetilde{P}(c) < \widetilde{R}(s) \end{cases} \end{aligned} \tag{45}$$

Equation (23) and (24) follow from (44) and (45).

A.6 The respecification in terms of s^* and c^*

We look for a specification of $\underline{q}(c)$ which satisfies the assumption of a constant complexity dispersion parameter. It is convenient to start with the case $\eta = \infty$, since then: $p_W(c) = \underline{q}(c)$. The following specification satisfies our requirement:

$$\underline{q}(c) = \frac{1}{\gamma}\ln(\gamma c) \tag{46}$$

where $\gamma > 0$ and $c^- > 0$. Equation (46) applies identically, and hence its first derivative. Then, using $p'_W[c(s)] = -s$, we have $p'_W[c(s)] = \frac{1}{\gamma\, c(s)} = -s$, or $c(s) = -\frac{1}{\gamma s}$, for $s^+ < 0$. Integrating $r'_W(s) = c_W(s)$ and applying the zero profit constraint $p_W(c) + sc - r_W(s) = 0$ yields the locus of log reservation wages in a Walrasian world:

$$r_W(s) = -\frac{1}{\gamma}\ln(-s)$$

It is convenient to transform both s and c: $s^* \equiv -\ln(-s)$ and $c^* \equiv \ln c$. This yields linear relations for $c^*_W(s^*)$ and $r_W(s^*)$:

$$\begin{aligned} c^*_W(s^*) &= s^* - \ln\gamma \\ r_W(s^*) &= \frac{1}{\gamma}s^* \end{aligned} \tag{47}$$

For $\eta \neq \infty$, $\underline{q}(c)$ can be derived straightforwardly from equation (5).

B Figures

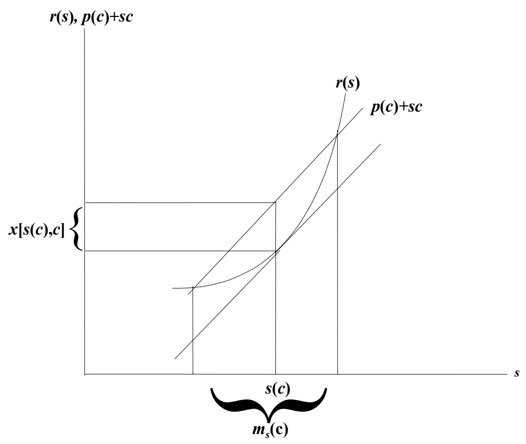

Figure 1: Walras versus search frictions for a job type c

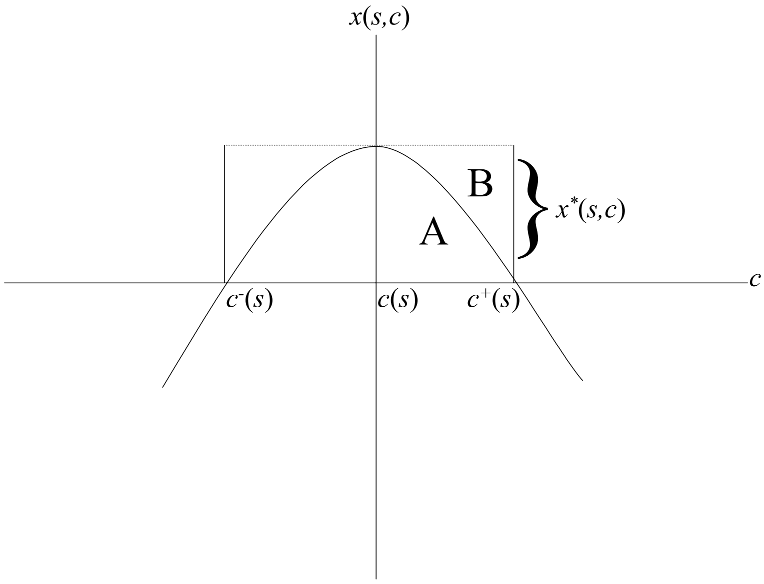

Figure 2: Taylor approximation of the surplus for a given worker type s

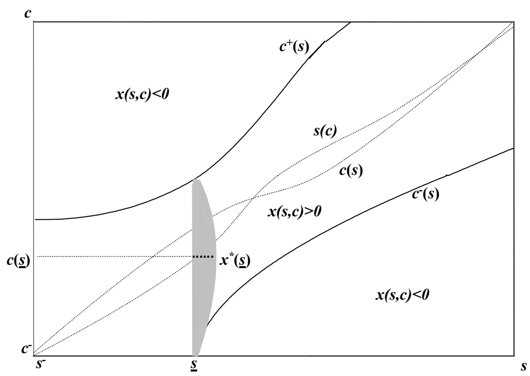

Figure 3: The aggregate equilibrium

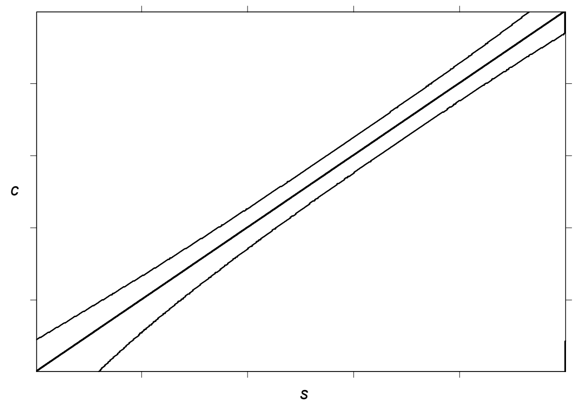

Figure 4: Matching sets for $\lambda = 2500$ and $\eta = 0$

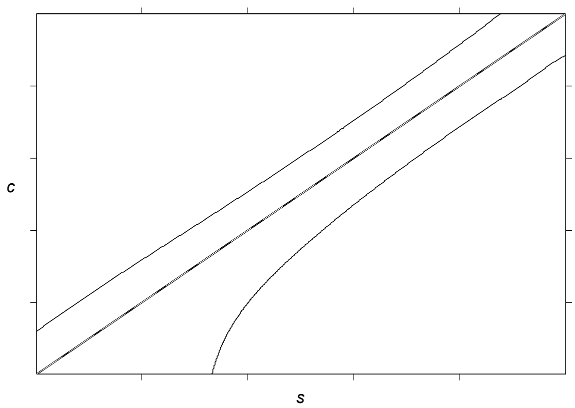

Figure 5: Matching Sets for $\lambda = 39$, $\eta = 0$

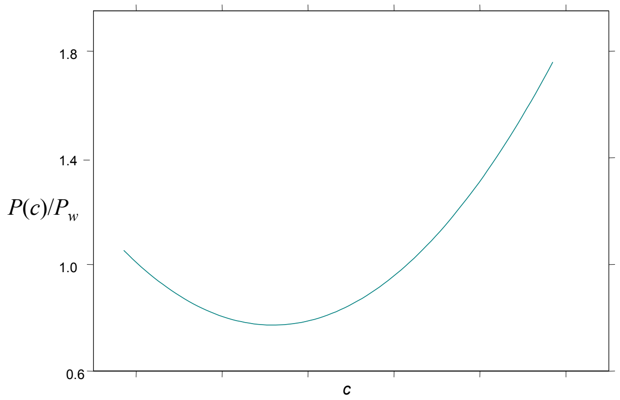

Figure 6: Price relative to Walrasian equilibrium for $\lambda = 2500$ and $\eta = 0$.

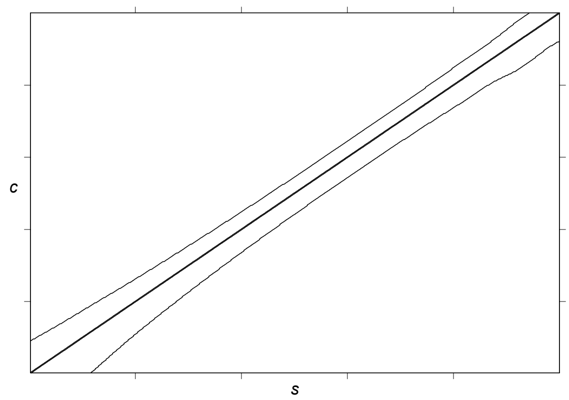

Figure 7: Matching sets for $\lambda = 2500$ and $\eta = \infty$

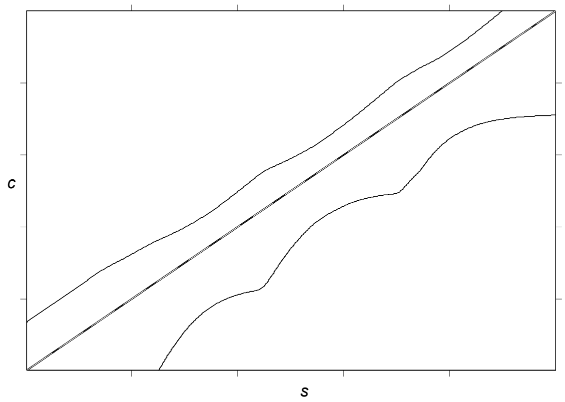

Figure 8: Matching Sets for $\lambda = 39$, $\eta = \infty$

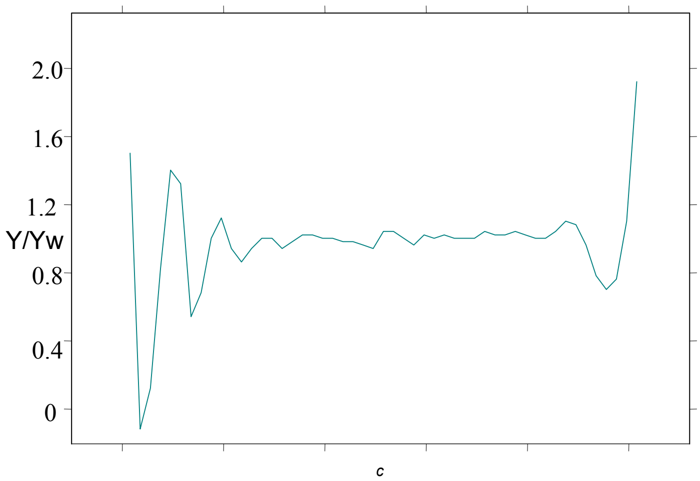

Figure 9: Output relative to Walras for $\lambda = 2500$, $\eta = \infty$

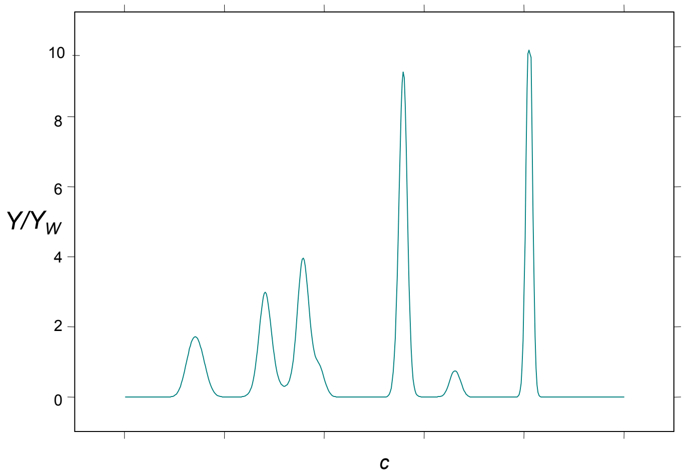

Figure 10: Output relative to Walras for $\lambda = 39$, $\eta = \infty$

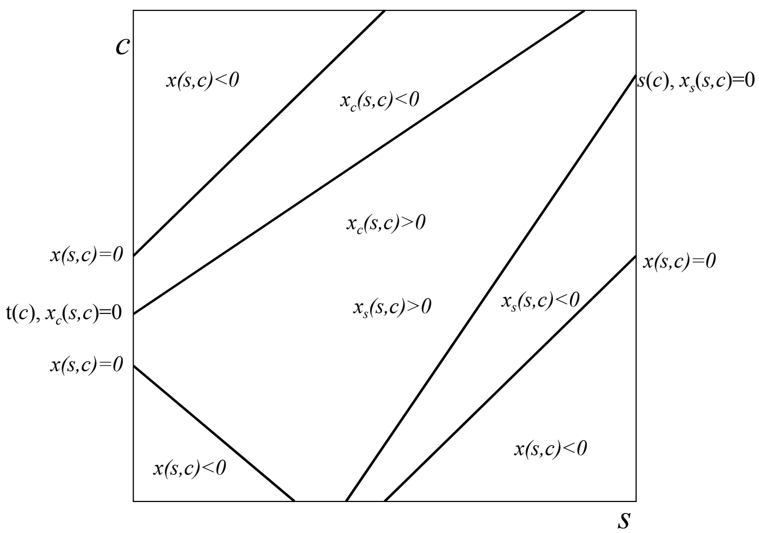

Figure 11: The case where $t\,(c)$ is not defined along the full support of c.

C Tables

Table 1: Aggregate outcomes for different values of λ and γ

γ	λ	$2,500$	625	156	39
5	stdev r	0.653	0.685	0.732	0.780
	loss (in %)	5.6	9.6	16.8	28.9
	u (in %)	1.6	3.0	5.7	11.2
	g (in %)	3.8	6.4	10.6	16.5
3	stdev r	0.627	0.649	0.679	0.714
	loss (in %)	4.9	8.8	15.1	26.0
	u (in %)	1.4	2.5	4.7	9.2
	g (in %)	3.5	5.8	9.6	15.2

Note: Simulations for $\eta = 0$, loss is output loss (in logs) due to search frictions

Table 2: Simulation results for different worker skill groups, $\eta = 0$, $\gamma = 5$

λ	s^*	-2 sd	-1 sd	mean	+1 sd	+2 sd
∞	$r(s^*)$	-1.2	-0.6	0	0.6	1.2
2500	$u(s^*)$ (in %)	5.3	1.9	1.1	1.0	1.4
	$r(s^*)$	-1.48	-0.77	-0.10	0.55	1.22
	$x(s^*)$ (in %)	18.4	9.0	6.1	5.8	8.5
	error $x(s^*)$ %	-8.21	-3.86	0.30	-1.68	2.83
	error $u(s^*)$ %	-1.17	0.16	0.65	0.41	0.58
625	$u(s^*)$ (in %)	10.8	3.5	2.0	1.8	2.5
	$r(s^*)$	-1.64	-0.88	-0.18	0.51	1.21
	$x(s^*)$ (in %)	31.2	16.0	10.8	10.5	14.7
	error $x(s^*)$ %	-12.21	-4.58	-0.82	1.15	2.07
	error $u(s^*)$ %	-3.9	-1.2	-0.8	-1.1	-0.6
156	$u(s^*)$ (in %)	23.0	6.9	3.7	3.3	4.3
	$r(s^*)$	-1.89	-1.08	-0.31	0.43	1.20
	$x(s^*)$ (in %)	52.2	28.6	19.6	18.6	26.1
	error $x(s^*)$ %	-14.19	-7.27	-1.94	0.27	3.35
	error $u(s^*)$ %	-6.31	2.15	2.34	3.88	4.44
39	$u(s^*)$ (in %)	45.6	14.6	7.4	6.1	7.7
	$r(s^*)$	-2.14	-1.39	-0.55	0.28	1.15
	$x(s^*)$ (in %)	80.1	49.8	35.5	34.1	45.5
	error $x(s^*)$ %	-14.65	-13.30	-5.65	-0.44	1.53
	error $u(s^*)$ %	-30.44	0.73	3.18	3.83	4.14

Note: sd is standard deviation, skill distribution is normal.

D Sufficient conditions for a non-trivial equilibrium (not to be included in the paper)

When the value of leisure B and/or the cost of a vacancy K are too high, the matching set of some of the worker types will be empty because the value of output does not exceed the value of leisure or alternatively, their reservation wage is too high for firms to recover their vacancy costs when matched with them. We present a sufficient condition for non-empty matching sets for all worker types, for the case $\eta < \infty$[34]. For a sufficient condition, consider the case where the skill distribution is such that all worker types are of the lowest skill type s^{-}. This is the worst possible case due to the fact that high skilled workers have an absolute advantage at all job types so that their value of search is more likely to exceed B. Consider the limiting case for which equation (13) is satisfied with equality subject to the constraints: $R(s^{-}) > B$. and $0 \leq h(s^{-}) \leq 1$. This limiting case sets $R(s^{-}) = B$ and $h(s^{-}) = 1$. Hence,

$$K < \frac{(1-\beta)\lambda}{\rho+\delta}\left[P(c)F(s^{-},c) - B\right]$$

which has to apply for all c. Hence, $P(c)\,F(s^{-},c)$ has to be constant in the critical case, say $P(c)\,F(s^{-},c) = A$. An expression for A can be derived from the cost function per unit of output that goes with the production function (4). Hence, the sufficient condition for all s types to have non-empty matching sets reads:

$$\begin{aligned} K &< \frac{(1-\beta)\,\lambda}{\rho+\delta}\,[A-B] \qquad (48)\\ A^{1-\eta} &\equiv \int_{c^{-}}^{c^{+}} \exp\left[(\eta+1)\,\underline{q}\,(c) + (\eta-1)\,s^{-}c\right] dc \end{aligned}$$

[34] The condition for $\eta = \infty$ follows from a simple generalization of the subsequent argument.