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General equilibrium in economies with in¯nite dimensional commodity spaces: a truncation approach

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Abstract

Mostly in¯nite dimensional economies can be considered limits of ¯nite dimensional economies, in particular when we think of time or product di®erentiation. We investigate conditions under which sequences of quasi-equilibria in ¯nite dimensional economies converge to a quasi-equilibrium in the in¯nite dimensional economy. It is shown that convergence indeed occurs if the usual continuity assumption concerning the preference relations for ¯nite dimensional commodity spaces is slightly modi¯ed.

key words: general equilibrium, truncation, in¯nite dimensional commodity space

1 Introduction

By now there is an abundant literature on the economic theory of competitive general equilibrium in economies with an in¯nite dimensional commodity space. Mas-Colell and Zame (1991) and Aliprantis et al. (1989) provide excellent surveys. The in¯nite dimensionality is brought about in a variety of ways. One can think of an in¯nite horizon, an in¯nite number of di®erentiated commodities (Mas-Colell (1975)) and of uncertainty (as one of the motivations in Bewley (1972)). The present paper is best understood in the context of economies with an in¯nite time domain. Such economies were also the main motivation for the seminal work of Peleg and Yaari (1970) and Bewley (1972). But our model allows for alternative interpretations as well. According to Boyd and McKenzie (1993), Hicks (1939) was the ¯rst to recognize that commodities delivered at di®erent instants of time should be considered as di®erent commodities. The limit of an economy when the number of periods goes to in¯nity, then is an economy with an in¯nite dimensional commodity space. Balasko (1997c) refers to Debreu (1959) and Malinvaud (1972) to illustrate that the choice of a ¯xed ¯nite horizon is problematic. Debreu (1959) argues: "there are conceptual di±culties in postulating a predetermined instant beyond which all economic activity either ceases or is outside the scope of the analysis". Malinvaud (1972) stipulates: "Also, we may prefer unlimited future time to choosing a ¯nite number of dates".

There are several ways to tackle the existence of a general equilibrium in an economy with an in¯nite dimensional commodity space. One approach is the Negishi approach. It is used in e.g. Van Geldrop and Withagen (1990 and 1996), Keyzer (1991) and Ginsburgh and Keyzer (1997). It exploits the fact that under the appropriate assumptions a general equilibrium is Pareto e±cient. Then Pareto e±cient alocations are calculated after assigning weights to the individual agents (¯nite in number). If the set of feasible allocations is compact, the problem of ¯nding a Pareto e±cient allocation is solvable in principle. The point then is to ¯nd weights such that the corresponding (shadow) budget constraints of all agents are satis¯ed. The resulting shadow prices are the equilibrium prices in the in¯nite dimensional economy. The advantage of this approach is that the problem essentially reduces to a ¯nite dimensional problem. A technique often employed in this approach is optimal control theory, which is warranted because the relationships involved are mostly represented by functions having nice properties.

Another way of attack is pursued by Peleg and Yaari (1970) and was generalized by Aliprantis et al. (1987). Basically it employs the well-known result due to Debreu and Scarf (1963) that the core of a replicating economy converges to the equilibrium allocation in a competitive economy. This pathway to existence is pursued by a.o. Boyd and McKenzie (1993) in their work on general consumption sets (see also Sun and Kusumoto (1997)). As stressed by Shannon (1997) these approaches require the existence of Pareto e±cient allocations and the non-emptyness of the core, which might be problematic in the presence of market imperfections.

A third route to existence starts from considering equilibria in truncated economies and to exploit their properties. This is done by Balasko et al. (1980) and Burke (1988) for pure exchange economies, by Van Geldrop et al. (1991) and Van Geldrop and Withagen (1999) for an economy with natural exhaustible resources. Also Bewley (1972) in his seminal paper used a limit argument starting from equilbria in finite dimensional economies.

Whatever method is used in order to prove the existence of a general equilibrium in an economy with an infinite dimensional commodity space, certain assumptions have to be made, some of which closely resemble assumptions made for models in a finite dimensional setting. Examples of such assumptions found in the literature include: consumption sets are bounded from below, the set of feasible allocations is compact (in some topology), initial endowments are interior, preference relations are monotonic and the like. One could argue that such assumptions are made to guarantee the existence of an equilibrium in the finite time analogue of the infinite time economy. Indeed, it usually doesn't require too much imagination to conceive of the economy under study as the limit of finite time economies. Some authors are quite explicit in this respect. Boyd and McKenzie (1993) for example put forward: "This is a limiting form of the futures economy of Hicks". In particular of course in the truncation approach first the existence of equilbria in the finite dimensional economy is to be established.

The next step in this experiment of thought would be to argue that investigating the existence of equilibria in infinite horizon economies makes sense only if the existence of each analogue finite horizon equilibrium is warranted. This suggests to study the conditions one has to impose on finite horizon economies equilibria in order to deduce the existence of an equilibrium in the infinite horizon economy. The present paper provides such an approach. Moreover, it offers a generalization of the work already done on truncation in several respects. First, to our knowledge most existing studies only consider pure exchange economies, whereas we also include production. Furthermore, our assumptions with respect to preferences and consumption sets are quite general. In contrast with recent work by Balasko (1997a, 1997b) we allow for rather general preference relations, at least we do not impose a constant rate of time preference.

We shall consider quasi-equilibria rather than equilibria because, as MasColell and Zame (1991, p. 1855) state 'the conditions which guarantee that the two notions coincide are entirely parallel to the well-understood, finite dimensional case'. In particular, we assume the existence of quasi-equilibria in each finite horizon economy (not caring about the conditions that have to hold for existence). The distinctive feature of the present study is that we subsequently show that under a rather mild condition with respect to the continuity of preferences, in addition to monotonicity, there exists a quasi-equilibrium in the infinite horizon economy, with a sublinear price system, if the sequence of equilibrium allocations in the finite horizon economies has a limit. This continuity condition includes a continuity condition used in an important paper by Prescott and Lucas (1972). Our result is obtained without imposing topological properties such as closedness on the consumption sets or the production sets. However, this of course does not mean that such assumptions can be abandoned in general; they are needed to prove existence in the truncated economy, from which we depart. Moreover we do not need boundedness of the set of feasible allocations a priori. The method used to obtain the main result is simple however. It only requires basic mathematical analysis.

This article is organized as follows. In Section 2 we present the model and discuss the assumptions made. We also state the main theorem. The theorem is proved in Section 3. The conclusions are given in Section 4.

2 The infinite dimensional economy

To describe the model of the infinite dimensional economy, we first introduce some notation. The set $\mathbb{N} = \{1, 2, \ldots, \}$ is the set of all positive integers and $\mathbb{R}$ is the set of all real numbers. The vector space $\mathbb{R}^{\mathbb{N}}$ is the set of all functions x from $\mathbb{N}$ into $\mathbb{R}$, assigning an element $x(s) \in \mathbb{R}$ to any integer $s \in \mathbb{N}$. So, any $x \in \mathbb{R}^{\mathbb{N}}$ is a vector (sequence) of real numbers of infinite length. With $\geq$ we denote the natural ordering in $\mathbb{R}^{\mathbb{N}}$

$$x \geq y \Leftrightarrow x(s) \geq y(s) \text{ for all } s \in \mathbb{N}.$$

By $\mathbb{R}^{\mathbb{N}}_+$ we denote the positive cone of $\mathbb{R}^{\mathbb{N}}$ related to $\geq$, so

$$\mathbb{R}^{\mathbb{N}}_+ = \{x \in \mathbb{R}^{\mathbb{N}} \mid \text{ for all } s \in \mathbb{N} : x(s) \geq 0\}.$$

Moreover, we write $x > y$ if $x \geq y$ and $x \neq y$, and $x \gg y$ if $x(s) > y(s)$ for all $s \in \mathbb{N}$. The vector with all components equal to zero is denoted by $\underline{0}$. For each $t \in \mathbb{N}$ and $x \in \mathbb{R}^{\mathbb{N}}$ the vector $Q_t x \in \mathbb{R}^{\mathbb{N}}$ denotes the projection of x on the set $\{y \in \mathbb{R}^{\mathbb{N}} \mid y(s) = 0 \text{ for all } s \geq t\}$, i.e.

$$Q_t x(s) = \begin{cases} x(s) & \text{for } 0 < s < t, \\ 0 & \text{for } s \geq t. \end{cases}$$

So, for given vector $x \in \mathbb{R}^{\mathbb{N}}$, the projection $Q_t x$ is obtained by setting the s-th component of $Q_t x$ equal to zero for all $s \geq t$. Clearly $x \geq y$ implies $Q_t x \geq Q_t y$ for all $t \in \mathbb{N}$. Furthermore, for $X \subset \mathbb{R}^{\mathbb{N}}$, we define $Q_t(X) = \{y \in \mathbb{R}^{\mathbb{N}} | \exists x \in X : y = Q_t(x)\}$. Observe that for any $X \subset \mathbb{R}^{\mathbb{N}}$ it holds that $Q_1(X) = \{\underline{0}\}$. Although $Q_t x$ can be seen as a truncation of the vector x, and so $Q_t(X)$ as a truncation of the set X, each vector $Q_t x$ is still a vector in the infinite dimensional space $\mathbb{R}^{\mathbb{N}}$, and so is $Q_t(X)$ a set in this infinite dimensional space. In order to get truncations in a finite dimensional space we also define the truncation $\overline{Q}_t$, with for $x \in \mathbb{R}^{\mathbb{N}}$, $\overline{Q}_t x$ the $(t-1)$-dimensional vector in $\mathbb{R}^{t-1}$ defined by

$$\overline{Q}_t x(s) = Q_t x(s) = x(s), \quad s = 1, \ldots, t-1,$$

and for $X \subset \mathbb{R}^{\mathbb{N}}$, $\overline{Q}_t(X) = \{y \in \mathbb{R}^{t-1} | \exists x \in X : y = \overline{Q}_t(x)\}$. Finally, for $p \in \mathbb{R}^{\mathbb{N}}_+$ and $z \in \mathbb{R}^{\mathbb{N}}$ we define

$$p[z] = \lim_{t \to \infty} \sup \sum_{s=1}^{t} p(s)z(s).$$

Observe that $p[z]$ can be $\pm\infty$. Also, when z lies in the positive cone $\mathbb{R}^{\mathbb{N}}_+$, $p[z]$ reduces to

$$p[z] = \sup_{t \in \mathbb{N}} \sum_{s=1}^{t} p(s)z(s) = \sum_{s=1}^{\infty} p(s)z(s),$$

and for $z \in \mathbb{R}^{\mathbb{N}}$ and $t \in \mathbb{N}$, $p[Q_t z]$ becomes

$$p[Q_t z] = \sum_{s=1}^{t} p(s)z(s).$$

The infinite dimensional economy, denoted by E, has the vector space $\mathbb{R}^{\mathbb{N}}$ as the infinite dimensional commodity space. Throughout this paper the leading example is an economy in which a vector x of commodities denotes the quantities of a single commodity at an infinite number of periods, i.e. s denotes the time index and $x(s)$ denotes the quantity at time s, $s \in \mathbb{N}$. It was already remarked in the Introduction that the model is most easily interpreted in such a dynamic setting. In this interpretation the notion of the commodity space might then seem a little bit odd, because it allows for only one marketed commodity per period of time. However it is easily seen that the analysis is not affected at all if there are markets for some arbitrary number of commodities in each period of time. This would just require a simple rearrangement.

The economy E is assumed to contain a finite number H of consumers, labelled by $h = 1, \ldots, H$ and a finite number F of producers, labelled by $f = 1, \ldots, F$. With a slight abuse of notation, we also use H and F to denote the set of consumers and producers respectively. Each consumer $h \in H$ is characterized by an initial endowment $\omega_h \in \mathbb{R}^{\mathbb{N}}_+$, a consumption set $X_h \subset \mathbb{R}^{\mathbb{N}}$, and a preference relation, denoted by $\succ_h$, on X_h. With respect to the preference relation $\succ_h$ on X_h of consumer h we mean with $x \succ_h y$ that x is strictly preferred to y. Each producer $f \in F$ is characterized by his production set $Y_f \subset \mathbb{R}^{\mathbb{N}}$. With respect to the projections of the consumption and production sets we make the following assumption.

Assumption 2.1
For the infinite dimensional economy E the following holds:

(i) **Consumption sets** For each $h \in H$, the consumption set X_h is a subset of the positive cone $\mathbb{R}_+^{\mathbb{N}}$ and satisfies $Q_t(X_h) \subset X_h$ for all $t \in \mathbb{N}$.

(ii) **Production sets** For each $f \in F$, the production set Y_f satisfies $Q_t(Y_f) \subset Y_f$ for all $t \in \mathbb{N}$.

Observe that these assumptions imply that $\underline{0} \in X_h$, $h \in H$, and $\underline{0} \in Y_f$, $f \in F$, so it requires the possibility of zero consumption and zero production. The assumptions (i) and (ii) further imply the possibility of truncation, i.e. for any t it holds that any feasible consumption vector $x_h \in X_h$, respectively any feasible production vector $y_f \in Y_f$, remains feasible when all quantities as from period t are replaced by zero. For a consumption vector this means that for any t free disposal of all consumption as from period t is feasible. For the producers the assumption needs some more discussion, because the assumption implies that for any feasible y and any $y(t) < 0$ there exists a feasible $\bar{y}$ with $\bar{y}(t) = 0$, i.e. it is always possible to replace a nonzero quantity of input by zero input. This seems to be questionable. However, observe that the assumption only says that the producer can always decide to do nothing as from some future instant of time. What is excluded here is the occurrence of negative external effects over time. For example, present production does not cause future pollution which may harm consumers. So, if the input $y(t) < 0$ is replaced by the zero input $\bar{y}(t) = 0$, then it may cause zero outputs at any later period. When $y(t) \geq 0$ the assumption says that any production plan as from period t can be replaced by free disposal at t and zero input and output as from period $t + 1$.

We now consider the so called truncated economies $\bar{E}^T$, $T = 1, 2, \ldots$, related to E. In the truncated economy $\bar{E}^T$, each consumer $h \in H$ is characterized by the initial endowment $\bar{\omega}_h^T = \bar{Q}_{T+1}\omega_h \in \mathbb{R}_+^T$, the consumption set $\overline{X}_h^T = \bar{Q}_{T+1}(X_h) \subset \mathbb{R}^T$, and the preference relation $\succsim_h^T$ on $\overline{X}_h^T$ defined as follows. For a T-dimensional vector $\bar{x}$ in $\overline{X}_h^T$, define the infinite dimensional vector $x \in Q_{T+1}(X_h)$ by $x(s) = \bar{x}(s)$ if $s \leq T$ and $x(s) = 0$ if $s > T$. Since $Q_{T+1}(X_h) \subset X_h$ according to (i) of Assumption 2.1, the restriction of $\succsim_h$ to $Q_{T+1}(X_h)$ is well-defined. To define $\succsim_h^T$ on $\overline{X}_h^T$, let $\bar{x}$ and $\bar{y}$ be any pair of two T-dimensional vectors in $\overline{X}_h^T$, and x and y the corresponding infinite dimensional vectors in $Q_{T+1}(X_h)$. Then

$$\bar{x} \succsim_h^T \bar{y} \text{ if and only if } x \succsim_h y.$$

Each producer $f \in F$ is characterized by the production set $\overline{Y}_f^T = \bar{Q}_{T+1}(Y_f) \subset \mathbb{R}^T$. The economies $\bar{E}^T$ are finite in the sense that the corresponding commodity spaces $\bar{Q}_{T+1}(\mathbb{R}^{\mathbb{N}})$ are finite dimensional.

Definition 2.1 Feasible Allocation

Let $T \in \mathbb{N}$ be given. Then a feasible allocation in the finite dimensional truncated economy $\bar{E}^T$ is a collection of commodity bundles $\bar{x}_h^T \in \bar{X}_h^T$, $h \in H$, and $\bar{y}_f^T \in \bar{Y}_f^T$, $f \in F$, such that

$$\sum_{h \in H} \bar{x}_h^T = \sum_{h \in H} \bar{\omega}_h^T + \sum_{f \in F} \bar{y}_f^T.$$

So, a collection of commodity bundles specifying a consumption bundle for each consumer and a production bundle for each producer is feasible if each consumption bundle belongs to its truncated consumption set, each production bundle to its truncated production set and the total (truncated) consumption equals the total (truncated) initial endowment plus the total (truncated) production. We now define a quasi-equilibrium, being a feasible allocation and a price vector such that each consumer minimizes his expenditures and each producer maximizes her profit.

Definition 2.2 Quasi-equilibrium in finite dimensional economy

A quasi-equilibrium for the truncated finite dimensional economy $\bar{E}^T$ is a feasible allocation $\bar{x}_h^T \in \bar{X}_h^T$, $h \in H$, $\bar{y}_f^T \in \bar{Y}_f^T$, $f \in F$ and a price vector $\bar{p}^T \in \bar{Q}_{T+1}(\mathbb{R}^N_+)$, $\bar{p}^T \neq \underline{0}$, such that

(i) for all $h \in H$ and for all $x \in \bar{X}_h^T$: $x \succ_h^T \bar{x}_h^T \Rightarrow \sum_{s=1}^T \bar{p}^T(s)x(s) \geq \sum_{s=1}^T \bar{p}^T(s)\bar{x}_h^T(s)$ (expenditure minimization),

(ii) for all $f \in F$ and for all $y \in \bar{Y}_f^T$: $\sum_{s=1}^T \bar{p}^T(s)y(s) \leq \sum_{s=1}^T \bar{p}^T(s)\bar{y}_f^T(s)$ (profit maximization).

Usually expenditure minimization is defined by stipulating that when a bundle is at least as good as the equilibrium bundle, it is at least as expensive as the equilibrium bundle. In the we state this condition for bundles strict preferred to the equilibrium bundle in order to avoid the necessity to introduce more notation. Moreover the two definitions are equivalent when preferences are monotonic, as assumed in Assumption 2.4 below.

Instead of stating the well-known conditions for the existence of (quasi-)equilibria in finite commodities, we directly assume that each truncated economy $\bar{E}^T$ admits a quasi-equilibrium.

Assumption 2.2

For each $T \in \mathbb{N}$, the truncated finite dimensional economy $\bar{E}^T$ has a quasi-equilibrium with price vector $\bar{p}^T \in \bar{Q}_{T+1}(\mathbb{R}^N_+)$ with $\bar{p}^T(1) > 0$.

Any finite horizon quasi-equilibrium naturally entails a non-zero price vector by their definition. Here we postulate that the price of the first commodity is positive. We have to make such an assumption explicitly because it cannot be excluded that in a quasi-equilibrium with horizon T only the price at T is positive. If we would start from an equilibrium, not a quasi-equilibrium, then this assumption can be dropped. In the assumption we postulate the existence of a quasi-equilibrium in every finite horizon economy. Notice that what we actually need is that such equilibria exist for economies with sufficiently large horizons.

We now depart from the existence of a quasi-equilibrium for any finite dimensional economy by defining for any truncated finite dimensional economy $\overline{E}^T$ a corresponding truncated infinite dimensional economy E^T. To do so, for all $h \in H$, define $\omega_h^T = Q_{T+1}\omega_h \in \mathbf{R}_+^T$, $X_h^T = Q_{T+1}(X_h) \subset \mathbf{R}^T$, with the preference relation $\succsim_h$ on X_h restricted to X_h^T, and, for all $f \in F$, define $Y_f^T = Q_{T+1}(Y_f) \subset \mathbf{R}^T$. Observe that these truncated infinite dimensional economies correspond to the truncated finite dimensional economies in the sense thay any finite T-dimensional vector is extended with an infinite number of zero components, i.e. any finite dimensional vector, say $\overline{q}^T$, is extended to a vector q^T by setting $q^T(s) = \overline{q}^T(s)$ for $s \leq T$ and $q^T(s) = 0$ for $s > T$. Then the following corollary is straightforward.

Corollary 2.1

Let the consumption bundles $\overline{x}_h^T \in \overline{X}_h^T$, $h \in H$, production bundles $\overline{y}_f^T \in \overline{Y}_f^T$, $f \in F$, and price vectors $\overline{p}^T \in \overline{Q}_{T+1}(\mathbf{R}_+^N)$ with $\overline{p}^T(1) > 0$ be a quasi-equilibrium for the truncated finite dimensional economy $\overline{E}^T$. Then the corresponding collection of vectors $x_h^T \in X_h^T$, $h \in H$, $y_f^T \in Y_f^T$, $f \in F$, and price vector $p^T \in Q_{T+1}(\mathbf{R}_+^N)$ is a quasi-equilibrium for the truncated infinite dimensional economy, i.e.

(i) $\sum_{h \in H} x_h^T = \sum_{h \in H} \omega_h^T + \sum_{f \in F} y_f^T$ (feasibility),

(ii) for all $h \in H$ and for all $x \in X_h^T$: $x \succsim_h x_h^T \Rightarrow p^T[x] \geq p^T[x_h^T]$ (expenditure minimization),

(iii) for all $f \in F$ and for all $y \in Y_f^T$: $p^T[y] \leq p^T[y_f^T]$ (profit maximization).

Observe that the last condition can be replaced by: for all $f \in F$ and for all $y \in Y_f$: $p^T[y] \leq p^T[y_f^T]$, since all prices $p^T(s) = 0$ for $s > T$.

Corollary 2.1 says that the assumption of the existence of a quasi-equilibrium for any truncated finite dimensional economy implies the existence of a quasi-equilibrium for the corresponding truncated infinite dimensional economy and hence the existence conditions for the finite economy imply existence of quasi-equilibrium for the corresponding truncated infinite dimensional economy. However, instead of stating these existence conditions, in this paper we want to concentrate on the existence of a quasi-equilibrium in the infinite dimensional economy under the assumption that each truncated infinite dimensional economy admits a quasi-equilibrium. In particular we want to show that under certain (additional) assumptions on the consumer and producer characteristics, for any sequence of quasi-equilibria allocations of the truncated infinite dimensional economies, it holds that the pointwise limit, if exists, is a quasi-equilibrium allocation of the infinite dimensional economy. Therefore we only state the assumptions needed to prove that the limit of the sequence of truncated infinite dimensional equilibria is indeed a quasi-equilibrium of the infinite dimensional economy. Some of these assumptions are also standard assumptions for the existence of quasi-equilibrium in a finite dimensional economy. However, it should be stressed that additional, but well-known assumptions with respect to preferences, endowments and technologies are needed, to establish the existence of quasi-equilibria in the finite dimensional economies.

The next assumption has to be made on the consumption sets in addition to (i) of Assumption 2.1.

Assumption 2.3
For each $h \in H$, the consumption set X_h satisfies the following condition: if $x \in X_h$; $\hat{x} \in X_h$ and $x \leq \hat{x}$, then $\alpha x + (1 - \alpha)\hat{x} \in X_h$ for all $0 \leq \alpha \leq 1$.

Assumption 2.1, part (i) and Assumption 2.3 on the consumption sets are rather innocuous. Note that we do not require convexity here, although it may be a necessary assumption for the existence of a quasi-equilibrium. However, observe that (i) of Assumption 2.1 implies that $\underline{0} \in X_h$ by taking $t = 1$. Hence, together with Assumption 2.3 this implies that $\alpha x \in X_h$ for all $0 \leq \alpha \leq 1$, if $x \in X_h$, i.e. for any feasible consumption vector x the scale of consumption can be decreased arbitrarily. More generally, (i) of Assumption 2.1 and Assumption 2.3 imply that for all t the scale of consumption from period $t+1$ and onwards can be decreased arbitrarily. However, it should be observed that a decrease in the scale of consumption at some period $t + 1$ leads to a decrease of consumption at all periods after $t + 1$ at the same scale.

The preference relations are assumed to satisfy the following conditions.

Assumption 2.4
For each $h \in H$, the preference relation $\succ_h$ satisfies

(i) **Continuity** Let x^n; $n \in \mathbb{N}$, be an infinite sequence of elements in X_h, and let $x \in X_h$, $b \in X_h$ and $\hat{x} \in X_h$ be such that $x^n \leq b$ for all $n \in \mathbb{N}$, $\lim_{n \to \infty} x^n(s) = x(s)$ for all $s \in \mathbb{N}$, and $\hat{x} \succ_h x$. Then there exist $t_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that $Q_t\hat{x} \succ_h x^n$ for all $t \geq t_0$ and $n \geq n_0$.

(ii) **Monotonicity** For each pair x; $\hat{x} \in X_h$ with $\hat{x} > x$ it holds $\hat{x} \succ_h x$,

The assumption states that for every $t \in \mathbb{N}$, the preferences are monotone and upper hemicontinuous for the restriction of the preference relation to the projection space $Q_t(X_h)$.

It should be noticed that the continuity assumption is very similar to the one made by Prescott and Lucas (1972). They require that if $x, z \in X_h$ and $x \succ_h z$ then $Q_t x \succ_h z$ for all t sufficiently large. We require somewhat more because their consumption sets are L_∞. The preferences are assumed to display monotonicity in the strong sense: a bundle is strictly preferred to another bundle if the former one is greater with at least one strictly larger item. In particular this implies that for every $x \gg \underline{0}$ in $\mathbb{R}^{\mathbb{N}}$ it holds that $x \succ_h Q_{t+1}x \succ_h Q_t x$ for all $t \in \mathbb{N}$. Some of the restrictions on the preferences implied by part (ii) of Assumption 2.4 on monotonicity together with part (i) of Assumption 2.1, saying that $Q_t(X_h) \subset X_h$ for all $t \in \mathbb{N}$, are illustrated in the next example.

Example 2.1

Consider the (utility) function $u_h\colon X_h \to \mathbb{R}$ given by

$$u_h(x) = \sum_{t=1}^{\infty} \frac{1}{2^t} \ln x(t)$$

and define $\succ_h$ by $x \succ_h y$ if and only if $u_h(x) > u_h(y)$. Then $\succ_h$ does not satisfy the monotonicity condition because $u_h(x)$ is not defined when some of the components of x are equal to zero and hence neither $Q_t x$ can be compared to $Q_{t+1}x$, nor two elements x, y in $Q_t(X_h)$ with $x > y$. Also in case of

$$u_h(x) = \prod_{t=1}^{\infty} x(t)^{\frac{1}{2^t}}$$

the monotonicity assumption is not satisfied because for any t we have $u_h(x) = 0$ for all $x \in Q_t(X_h)$.

The final assumption with respect to the initial endowments entails that there is a consumption bundle with positive consumption in all coordinates, which is feasible for all consumers. Arrow and Hahn (1971, p.65) call this a "surely innocuous proposition". For the paper at hand it is important in the construction of the normalized prices. By A_h we denote the subset of X_h consisting of the consumption bundles feasible for consumer h when all other consumers consume nothing, i.e.

$$A_h = \{x \in X_h \mid \exists y_f \in Y_f,\ f \in F : x \le \sum_{f \in F} y_f + \sum_{h \in H} \omega_h\}.$$

The conditions imposed on the consumption sets X_h and the production sets Y_f guarantee that for each $h \in H$ also the subset A_h of X_h satisfies (i) of Assumption 2.1 and Assumption 2.3. Additionally, we postulate the following assumption, which says that together with the initial endowments a strictly positive bundle of commodities can be produced.

Assumption 2.5

There exists a vector $a \in \cup_{h \in H} A_h$, with $a(s) > 0$ for all $s \in \mathbb{N}$.

To state the central result of the paper, we first have to introduce the concept of quasi-equilibrium in an infinite dimensional economy.

Definition 2.3 Quasi-equilibrium in infinite dimensional economy
A quasi-equilibrium for the infinite dimensional economy $\mathcal{E}$ is a collection of commodity bundles $x_h \in X_h$, $h \in H$, a collection of production bundles $y_f \in Y_f$, $f \in F$ and a price vector $p \in \mathbb{R}_+^{\mathbb{N}}$ with $p \neq \underline{0}$, such that

(i) $\sum_{h \in H} x_h = \sum_{h \in H} \omega_h + \sum_{f \in F} y_f$ (feasibility),

(ii) for all $h \in H$ and for all $x \in X_h$: $x \succ_h x_h \Rightarrow p[x] \geq p[x_h]$ (expenditure minimization),

(iii) for all $f \in F$ and for all $y \in Y_f$: $p[y] \leq p[y_f]$ (profit maximization).

We are now ready to state the main result

Theorem 2.2
Let $\mathcal{E}$ be an infinite dimensional economy satisfying Assumptions 2.1, 2.3, 2.4, 2.5. For $T \in \mathbb{N}$, let $x_h^T \in X_h^T$, $h \in H$, $y_f^T \in Y_f^T$, $f \in F$, and $p^T \in Q_{T+1}(\mathbb{R}_+^{\mathbb{N}})$ with $p^T(1) > 0$ be a quasi-equilibrium for $\mathcal{E}^T$. Furthermore, let there exist consumption bundles $x_h \in X_h$, $h = 1, \ldots, H$, and production bundles $y_f \in Y_f$, $f = 1, \ldots, F$, such that

$$\lim_{T \to \infty} x_h^T(s) = x_h(s) \; ; \quad s \in \mathbb{N};$$

$$\lim_{T \to \infty} y_f^T(s) = y_f(s) \; ; \quad s \in \mathbb{N};$$

and, for every $h \in H$, let there exists $b_h \in X_h$ and $c_h \in A_h$, such that

$$x_h^T \leq b_h; \text{ for all } T \in \mathbb{N} \text{ and } x_h < c_h.$$

Then there is a price vector $p \in \mathbb{R}_+^{\mathbb{N}}$, such that the allocation x_h, $h = 1, \ldots, H$ and y_f, $f = 1, \ldots, F$, together with the price vector p is a quasi-equilibrium for the infinite dimensional economy $\mathcal{E}$. Furthermore it holds that

(i) for all $h \in H$: $\lim_{T \to \infty} p^T[x_h^T] = p[x_h] < \infty$,

(ii) for all $h \in H$: $\lim_{T \to \infty} p^T[\omega_h^T] = p[\omega_h] < \infty$,

(iii) for all $f \in F$: $\lim_{T \to \infty} p^T[y_f^T] = p[y_f] < \infty$.

The proof of the theorem is given in the next section. The theorem states that there exists a price vector in $\mathbb{R}^N$ that sustains the pointwise limits of quasi-equilibrium allocations of the truncated economies as a quasi-equilibrium allocation of the infinite dimensional horizon economy. Moreover, the values of the commodity bundles in the truncated economy quasi-equilibria converge to the values of the commodity bundles in the quasi-equilibrium of the infinite dimensional economy. It should be noticed that a quasi-equilibrium exists for each truncated infinite dimensional economy, because of Assumption 2.2 and Corollary 2.1. Further it should be observed that it is assumed that the sequences of the consumption bundles x_h^T, $h \in H$ and production bundles y_f^T, $f \in F$ of the quasi-equilibrium allocations in the truncated infinite dimensional economies E^T, $T \in \mathbb{N}$, are assumed to have a pointwise limit in respectively X_h, $h \in H$ and Y_f, $f \in F$. It is easy to give conditions which guarantee point-wise convergence. For example, if the equilibrium allocations in the finite dimensional economies are uniformly bounded or if the production sets are uniformly bounded, we get the desired result. We have refrained from making such assumptions in order to be as general as possible: limits may exist even if the assumptions mentioned above are not satisfied.

Observe further that it is also assumed that for every consumer h the sequence x_h^T of consumption vectors is bounded from above by some vector $b_h \in X_h$ and that its pointswise limit vector x_h is bounded by a feasible consumption bundle c_h which is larger in at least one component than the bundle x_h. These assumptions are necessary to apply the continuity property if the preference relation. From the monotonicity as stated in part (ii) of Assumption 2.4 the latter implies that $c_h \succ_h x_h$, that is the candidate equilibrium bundle in the infinite dimensional economy. If the assumption would be strengthened by requiring the existence of a feasible consumption bundle that is greater in each item, then the monotonicity as assumed in (ii) of Assumption 2.4 could be relaxed to weak monotonicity. A second alternative would be to maintain the weak condition in the theorem that c_h is only larger in at least one item, and to impose weak monotonicity by adding strict quasi-concavity of the preference relation.

3 The Proof

Proof of Theorem 2.2.

We proceed by a sequence of steps.

1. First we show that for all $h \in H$ the value $p^T[x_h^T]$ is bounded from above for T large enough. This result plays a central role in the next steps. Fix $h \in H$. From the continuity as stated in part (i) of Assumption 2.4 it follows that there are $\bar{t}$ and $\bar{T}$ such that

$$Q_t c_h \succ_h x_h^T \text{ for all } t \geq \bar{t}, \text{ and } T \geq \bar{T}. \tag{1}$$

So, with $\tau = \max[$[illegible]$]$, it follows that

$$Q_\tau c_h \succsim_h x_h^T \text{ for all } T \geq \tau. \tag{2}$$

Now, for any $T \in \mathbb{N}$, we normalize the prices of the quasi-equilibrium price vector p^T of the truncated infinite dimensional economy E^T by

$$\sum_{s=1} p^T(s) = 1, \tag{3}$$

which normalization is possible because $p^T(1) > 0$ for all $T \in \mathbb{N}$. By definition of quasi-equilibria, x_h^T satisfies the expenditure minimization property. For all $T \geq \tau$, it holds that $Q_\tau c_h \in X_h^T$. It follows from relation (2) that

$$p^T[x_h^T] \leq p^T[Q_\tau c_h]. \tag{4}$$

Since by definition $Q_\tau c_h(s) = 0$ for $s \geq \tau$, this implies by the normalization (3) that

$$p^T[x_h^T] \leq \sum_{s=1} p^T(s) Q_\tau c_h(s) \leq \max\{c_h(s) | s = 1, \ldots, \tau\}, \quad \text{for all } T \geq \tau, \tag{5}$$

which shows that for all $T \geq \tau$, $p^T[x_h^T]$ is bounded from above by $\max\{c_h(s) | s = 1, \ldots, \tau\}$. In the following we define

$$M = \sum_{h \in H} \max\{c_h(s) | s = 1, \ldots, \tau\}.$$

2. Second we derive a price vector $p \in \mathbb{R}^N$, which will be shown to be the quasi-equilibrium price vector. To do so, take some $T \geq \tau$ and consider the given quasi-equilibrium allocation x_h^T, $h \in H$, y_f^T, $f \in F$ and the normalized quasi-equilibrium price vector $p^T \in Q_{T+1}(\mathbb{R}^N_+)$. From the feasibility of the quasi-equilibrium allocation it follows that

$$\sum_{h \in H} p^T[\omega_h] + \sum_{f \in F} p^T[y_f^T] = \sum_{h \in H} p^T[x_h^T]. \tag{6}$$

By definition $\omega_h \in \mathbb{R}^N_+$ and thus $p^T[\omega_h] \geq 0$, $h \in H$. By (ii) of Assumption 2.1 we have that $\underline{0} \in Y_f$ and thus $\underline{0} \in Y_f^T$ and hence $p^T[y_f^T] \geq 0$, $f \in F$, because y_f^T maximizes profit on Y_f^T. Since, by inequality (5), $p^T[x_h^T] \leq \gamma_h$, it follows that

$$\sum_{h \in H} p^T[\omega_h] \leq M \text{ and } \sum_{f \in F} p^T[y_f^T] \leq M. \tag{7}$$

Now, let $a \in A_h$ for some h. Then $a \leq \sum_{f \in F} \bar{y}_f + \sum_{h \in H} \omega_h$ for some $\bar{y}_f \in Y_f$, $f \in F$ and thus

$$0 \leq p^T[a] \leq \sum_{f \in F} p^T[\bar{y}_f] + \sum_{h \in H} p^T[\omega_h]. \tag{8}$$

Since $p^T(s) = 0$ for $s > T$ it follows that $p^T[\bar{y}_f] = p^T[Q_{T+1}\bar{y}_f] \leq p^T[y_f^T]$, because y_f^T maximizes profit on Y_f^T and $Q_{T+1}\bar{y}_f \in Y_f^T$. Together with inequalities (7) it follows that

$$0 \leq p^T[a] \leq 2M; \text{ for all } T \geq \tau. \tag{9}$$

By Assumption 2.5 there exists a vector $\hat{a} \in \cup_{h \in H} A_h$ with $\hat{a}(s) > 0$ for all $s \in \mathbb{N}$ and thus

$$p^T(s) \leq \frac{2M}{\hat{a}(s)}; \text{ for all } s \in \mathbb{N};\ T \geq \tau. \tag{10}$$

So the sequence $(p^T)_{T \geq \tau}$ is pointwise bounded in $\mathbb{R}_+^{\mathbb{N}}$ and hence by a diagonal argument there is a subsequence $(p^{T_k})_{k \in \mathbb{N}}$ having a pointwise limit $p \in \mathbb{R}_+^{\mathbb{N}}$, i.e. $\lim_{k \to \infty} p^{T_k}(s) = p(s)$, $s \in \mathbb{N}$. For convenience and without loss of generality, in the sequel we suppose that $\lim_{T \to \infty} p^T(s) = p(s)$, $s \in \mathbb{N}$. Due to the normalization (3) it holds that $\sum_{s=1}^{\tau} p(s) = 1$ and so $p > \underline{0}$.

3. We now show that p, x_h, $h \in H$ and y_f, $f \in F$, is a quasi-equilibrium for the infinite dimensional economy E, i.e. the price vector p and the allocation x_h, $h \in H$ and y_f, $f \in F$ satisfy the conditions (i)-(iii) of Definition 2.3.

First, since for each T the allocation x_h^T, $h \in H$ and y_f^T, $f \in F$ is a feasible allocation for E^T, the feasibility condition (i) is an immediate consequence of the pointwise convergence of these sequences to x_h, $h \in H$ and y_f, $f \in F$.

Second we show the expenditure minimization. For some $h \in H$, let $\bar{x}_h \in X_h$ such that $\bar{x}_h \succsim_h x_h$. Because of part (i) of Assumption 2.4 there exist $\bar{t} \in \mathbb{N}$ and $\bar{T} \in \mathbb{N}$ such that for all $t \geq \bar{t}$ and $T \geq \bar{T}$ it holds that

$$Q_t\bar{x}_h \succsim_h x_h^T$$

and hence by the expenditure minimization of x_h^T in the quasi-equilibrium of the truncated economy E^T

$$p^T[Q_t\bar{x}_h] \geq p^T[x_h^T]; \text{ for all } T \geq t.$$

So, setting $t = \bar{t}$ and taking limits for $T \to \infty$ we get

$$p[\bar{x}_h] \geq p[Q_{\bar{t}}\bar{x}_h] \geq p[x_h];$$

which shows that x_h satisfies the expenditure minimization condition.

Third, we show that profit maximization holds. For some $f \in F$, let $\bar{y}_f \in Y_f$. By definition of Y_f^T we have that for all $t \in \mathbb{N}$ and all $t \leq T$ it holds that $Q_t(Y_f) \subset Y_f^T$ and thus $Q_t\bar{y}_f \in Y_f^T$. By the profit maximization of y_f^T in the quasi-equilibrium of E^T it follows that for all $T \in \mathbb{N}$,

$$p^T[Q_t\bar{y}_f] \leq p^T[y_f^T]; \text{ for all } T \geq t.$$

Letting $T \to \infty$ we get that

$$p[Q_t \bar{y}_f] \leq p[y_f]; \text{ for all } t \in \mathbb{N}$$

and hence $p[\bar{y}_f] \leq p[y_f]$.

4. It remains to prove the assertions (i)-(iii) of Theorem 2.2.
4a. We first show the boundedness of the values of the commodity bundles x_h and ω_h, $h = 1, \ldots, H$, and $y_f \in Y_f$, $f = 1, \ldots, F$. Since x_h, $h = 1, \ldots, H$, and $y_f \in Y_f$, $f = 1, \ldots, F$, is a quasi-equilibrium allocation, we have that x_h is feasible for consumer h, i.e. $x_h \in A_h$. Then, by (i) of Assumption 2.1, $Q_t x_h \in A_h$ for all $t \in \mathbb{N}$. So, it follows from using inequality (9) that

$$p[Q_t x_h] = \lim_{T \to \infty} p^T[Q_t x_h] \leq 2M.$$

Hence

$$p[x_h] = \lim_{t \to \infty} p[Q_t x_h] \leq 2M.$$

Since $\omega_h \in A_h$, by the same reasoning it follows that also $p[\omega_h] \leq 2M$. Because of feasibility we have that

$$\sum_{f \in F} y_f = \sum_{h \in H} x_h - \sum_{h \in H} \omega_h \leq \sum_{h \in H} x_h;$$

since $\omega_h \in \mathbb{R}^N_+$ for all $h \in H$. Hence,

$$p[\sum_{f \in F} y_f] \leq p[\sum_{h \in H} x_h] \leq 2HM.$$

Because of (ii) of Assumption 2.1 we have that $\underline{0} \in Y_f$ for all $f \in F$ and thus it follows from the profit maximization condition that $p[y_f] \geq 0$ for all $f \in F$. Therefore

$$p[y_f] \leq 2HM; \text{ for all } f \in F;$$

which shows the boundedness of all commodity bundles.
4b. Finally we show that the values of the bundles in the truncated quasi-equilibria converge to the values of the bundles in the quasi-equilibrium of the infinite dimensional economy.

Let some $h \in H$ be given. To prove that $\lim_{T \to \infty} p^T[x^T_h] = p[x_h]$, we first show that for any $\epsilon > 0$ there exists $t_1(\epsilon) \in \mathbb{N}$ and $T_1(\epsilon) \in \mathbb{N}$ such that

$$\sum_{s > t_1(\epsilon)} p^T(s) x^T_h(s) < \epsilon; \text{ for all } T \geq T_1(\epsilon). \qquad (11)$$

By the assumptions of the theorem, there exist $c_h \in A_h$ such that $c_h > x_h$. By Assumption 2.3 and the definition of A_h it follows that also $\alpha c_h + (1 - \alpha)x_h \in A_h$ for all $0 < \alpha < 1$. Since $\alpha c_h + (1 - \alpha)x_h > x_h$, part (ii) of Assumption 2.4 implies that $\alpha c_h + (1 - \alpha)x_h \succ_h x_h$, so that by part (i) of Assumption 2.4 and $x_h^T \leq b_h$, $T \in \mathbb{N}$, it follows that there exists $t(\alpha) \in \mathbb{N}$ and $T(\alpha) \in \mathbb{N}$ such that

$$Q_t(\alpha c_h + (1 - \alpha)x_h) \succ_h x_h^T; \text{ for all } t \geq t(\alpha) \text{ and } T \geq T(\alpha):$$

Since for all $\bar{x}_h \in X_h$, $Q_{t+1}\bar{x}_h \in X_h^T$ for all $T \geq t$, the expenditure minimization property of x_h^T in the quasi-equilibrium of the truncated economy E^T implies that for all $t+1 \geq t(\alpha)$ and $T \geq \max[t; T(\alpha)]$ it holds that

$$p^T[Q_{t+1}(\alpha c_h + (1 - \alpha)x_h)] \geq p^T[x_h^T] = \sum_{s>t} p^T(s)x_h^T(s) + p^T[Q_{t+1}x_h^T];$$

and so

$$\sum_{s>t} p^T(s)x_h^T(s) \leq p^T[Q_{t+1}(\alpha c_h + (1 - \alpha)x_h)] - p^T[Q_{t+1}x_h^T]$$

$$\leq \alpha p^T[Q_{t+1}c_h] + p^T[Q_{t+1}(x_h - x_h^T)]: \quad (12)$$

Since $c_h \in A_h$, we know from inequality (9) that $p^T[c_h] \leq 2M$ for all $T \geq \tau$. Now, for given $\epsilon > 0$, take $\alpha = \frac{\epsilon}{4M}$. Then, for all $T \geq \tau$ and $t \in \mathbb{N}$,

$$0 \leq \alpha p^T[Q_{t+1}c_h] \leq \alpha p^T[c_h] \leq \frac{1}{2}\epsilon: \quad (13)$$

Now, take $t_1(\epsilon) = \max[\tau; t(\frac{\epsilon}{4M})]$. Then it follows from inequalities (12) and (13) that for all $T \geq \max[t_1(\epsilon); T(\frac{\epsilon}{4M})]$

$$\sum_{s>t_1(\epsilon)} p^T(s)x_h^T(s) \leq \frac{1}{2}\epsilon + p^T[Q_{t_1(\epsilon)+1}(x_h - x_h^T)]: \quad (14)$$

By the pointwise limit convergence of x_h^T to x_h it holds that

$$\lim_{T \to \infty} p^T(s)(x_h(s) - x_h^T(s)) = 0; \text{ for all } s \in \mathbb{N};$$

so that there exists $\hat{T}$ such that for all $T \geq \hat{T}$ it holds that

$$|p^T[Q_{t_1(\epsilon)+1}(x_h - x_h^T)]| < \frac{1}{2}\epsilon: \quad (15)$$

From inequalities (14) and (15) it follows that

$$\sum_{s>t_1(\epsilon)} p^T(s)x_h^T(s) \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon; \text{ for all } T > T_1(\epsilon); \quad (16)$$

with $T_1(\epsilon) = \max[t_1(\epsilon); T(\frac{\epsilon}{4M}); \hat{T}]$, which shows assertion (11). Notice that this assertion also holds for any $\hat{t}_1 > t_1(\epsilon)$ and $T_1 > \max[\hat{t}_1; T_1(\epsilon)]$.

From the boundedness of $p[x_h]$ it follows that there exists $t_2(\epsilon)$ such that

$$\sum_{s>t_2(\epsilon)} p(s)x_h(s) \le \epsilon. \qquad (17)$$

Observe that the left hand sides of (16) and (17) are nonnegative. So, for $t(\epsilon) = \max[t_1(\epsilon); t_2(\epsilon)]$ and $T > \max[t_2(\epsilon); T_1(\epsilon)]$ it follows from (16) and (17) that

$$\begin{aligned}
\left|p[x_h] - p^T[x_h^T]\right| &= \left|\sum_{s=1}^{t(\epsilon)} (p(s)x_h(s) - p^T(s)x_h^T(s)) + \sum_{s>t(\epsilon)} p(s)x_h(s) - \sum_{s>t(\epsilon)} p^T(s)x_h^T(s)\right| \\
&\le \left|\sum_{s=1}^{t(\epsilon)} (p(s)x_h(s) - p^T(s)x_h^T(s))\right| + \left|\sum_{s>t(\epsilon)} p(s)x_h(s)\right| + \left|\sum_{s>t(\epsilon)} p^T(s)x_h^T(s)\right| \\
&\le \left|\sum_{s=1}^{t(\epsilon)} (p(s)x_h(s) - p^T(s)x_h^T(s))\right| + 2\epsilon.
\end{aligned}$$

Since $\lim_{T\to\infty} (p(s)x_h(s) - p^T(s)x_h^T(s)) = 0$ for all $s \in \mathbb{N}$ there is $T_2(\epsilon)$ such that

$$\left|\sum_{s=1}^{t(\epsilon)} (p(s)x_h(s) - p^T(s)x_h^T(s))\right| < \epsilon$$

for all $T > T_2(\epsilon)$. So, for all $T > \max[t_2(\epsilon); T_1(\epsilon); T_2(\epsilon)]$, it holds that

$$\left|p[x_h] - p^T[x_h^T]\right| < 3\epsilon.$$

Letting $\epsilon \to 0$ it follows that

$$\lim_{T\to\infty} p^T[x_h^T] = p[x_h].$$

It remains to show the convergence of the profits and the values of the initial endowments. From the feasibility condition of the quasi-equilibria we have for all $T \in \mathbb{N}$ and all $s \in \mathbb{N}$ that

$$\sum_{h\in H} x_h^T(s) = \sum_{h\in H} \omega_h^T(s) + \sum_{f\in F} y_f^T(s)$$

and therefore

$$\sum_{h\in H}\sum_{s>t} p^T(s)x_h^T(s) = \sum_{h\in H}\sum_{s>t} p^T(s)\omega_h^T(s) + \sum_{f\in F}\sum_{s>t} p^T(s)y_f^T(s).$$

For $\varepsilon > 0$, let $\hat{t}_1(\varepsilon) \in \mathbb{N}$ and $\hat{T}_1(\varepsilon) \in \mathbb{N}$ be such that assertion (16) is satisfied for all $h \in H$. Then, it follows for all $T > T_1(\varepsilon)$ that

$$\sum_{h \in H} \sum_{s > \hat{t}_1(\varepsilon)} p^T(s)\omega_h^T(s) + \sum_{f \in F} \sum_{s > \hat{t}_1(\varepsilon)} p^T(s) y_f^T(s) = \sum_{h \in H} \sum_{s > \hat{t}_1(\varepsilon)} p^T(s) x_h^T(s) \le H\varepsilon. \qquad (18)$$

Clearly, for all $t \in \mathbb{N}$ and $h \in H$, $\sum_{s>t} p^T(s)\omega_h^T(s) \geq 0$. If for some $T \in \mathbb{N}$, $t \in \mathbb{N}$ and $f \in F$, $\sum_{s>t} p^T(s) y_f^T(s) < 0$, then we would have $p^T[Q_t y_f^T] > p^T[y_f^T]$, contradicting profit maximization in E^T because $Q_t y_f^T \in Y_f^T$. So, for all $T \in \mathbb{N}$ $t \in \mathbb{N}$ and $f \in F$ we have that $\sum_{s>t} p^T(s) y_f^T(s) \geq 0$. So, with inequality (18) it follows that for all $T > \hat{T}_1(\varepsilon)$

$$0 \le \sum_{s > \hat{t}_1(\varepsilon)} p^T(s)\omega_h(s) < H\varepsilon; \ h \in H$$

and

$$0 \le \sum_{s > \hat{t}_1(\varepsilon)} p^T(s) y_f^T(s) < H\varepsilon; \ f \in F.$$

Consequently, following the same approach as with the consumers' expenditures,

$$\lim_{T \to \infty} p^T[\omega_h^T] = p[\omega_h]; \ h \in H$$

and

$$\lim_{T \to \infty} p^T[y_f^T] = p[y_f]; \ f \in F.$$

Q.E.D.

4 Concluding remarks

We have derived conditions guaranteeing that the sequences of general quasi-equilibria in finite horizon economies converge to a general quasi-equilibrium in the corresponding infinite horizon economy. Basically all that is required is the existence of limits of the finite horizon equilibrium allocations and a rather straightforward extension of the usual continuity assumption with respect to the preference relation.

Our approach is a generalization of earlier work by Van Geldrop et al. (1991) who use specific production and utility functions. The advantage of our approach seems to be that it is analytically rather straightforward and allows for a nice economic interpretation.

5 References

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